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L^2 BOUNDEDNESS OF THE CAUCHY TRANSFORM IMPLIES L^2 BOUNDEDNESS OF ALL CALDERÓN-ZYGMUND OPERATORS ASSOCIATED TO ODD KERNELS

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Abstract

Let μ be a Radon measure on $\mathbb{C}$ without atoms. In this paper we prove that if the Cauchy transform is bounded in $L^2(\mu)$, then all 1-dimensional Calderón-Zygmund operators associated to odd and sufficiently smooth kernels are also bounded in $L^2(\mu)$.

1. Introduction

We say that $k(\cdot,\cdot)\colon \mathbb{C}^2 \setminus \{(x,y) \in \mathbb{C}^2 : x = y\} \to \mathbb{C}$ is a 1-dimensional Calderón-Zygmund kernel if there exist some constants $C > 0$ and η, with $0 < \eta \leq 1$, such that the following inequalities hold for all $x, y \in \mathbb{C}$, $x \neq y$:

$$(1.1)\qquad \begin{aligned} &|k(x,y)| \leq \frac{C}{|x-y|}, && \text{and} \\ &|k(x,y) - k(x',y)| \leq \frac{C|x-x'|^\eta}{|x-y|^{1+\eta}} && \text{if } |x-x'| \leq |x-y|/2. \end{aligned}$$

Given a positive or complex Radon measure μ, we define

$$(1.2)\qquad T\mu(x) := \int k(x,y)\, d\mu(y), \quad x \in \mathbb{C} \setminus \operatorname{supp}(\mu).$$

We say that T is a Calderón-Zygmund operator (CZO) with kernel $k(\cdot,\cdot)$. The integral in the definition may not be absolutely convergent if $x \in$

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$\operatorname{supp}(\mu)$. For this reason, we consider the following ε-truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon \mu(x) := \int_{|x-y|>\varepsilon} k(x,y)\, d\mu(y), \quad x \in \mathbb{C}.$$

Observe that now the integral on the right hand side converges absolutely if, for instance, $|\mu|(\mathbb{C}) < \infty$.

Given a fixed positive Radon measure μ on $\mathbb{C}$ and $f \in L^1_{\mathrm{loc}}(\mu)$, we denote

$$T_\mu f(x) := T(f\, d\mu)(x) \quad x \in \mathbb{C} \setminus \operatorname{supp}(f\, d\mu),$$

and

$$T_{\mu,\varepsilon} f(x) := T_\varepsilon(f\, d\mu)(x).$$

The last definition makes sense for all $x \in \mathbb{C}$ if, for example, $f \in L^1(\mu)$. We say that T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

In this paper we will consider kernels of the form $k(x,y) := K(x-y)$, where K is an odd (i.e. $K(x) = K(-x)$ for all $x \in \mathbb{C} \setminus \{0\}$), $\mathcal{C}^\infty$ function defined in $\mathbb{C} \setminus \{0\}$ such that

$$|x|^{1+j} |\nabla^j K(x)| \in L^\infty(\mathbb{C}) \quad \text{for all } j = 0, 1, 2, \dots. \tag{1.3}$$

A basic example of Calderón-Zygmund operator with this type of kernel is the Cauchy transform. This is the operator $\mathcal{C}$ associated to the Cauchy kernel. That is,

$$\mathcal{C}\mu(x) := \int \frac{1}{y-x}\, d\mu(y), \quad x \in \mathbb{C} \setminus \operatorname{supp}(\mu).$$

The related operators $\mathcal{C}_\varepsilon$, $\mathcal{C}_\mu$, and $\mathcal{C}_{\mu,\varepsilon}$ are defined as above, in the particular case $k(x,y) = 1/(y-x)$.

The main result of the paper is the following.

Theorem 1.1. *Let μ be a Radon measure without atoms on $\mathbb{C}$. If the Cauchy transform $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$, then any 1-dimensional Calderón-Zygmund operator T_μ associated to a kernel of the form $k(x,y) := K(x-y)$, with $K(\cdot)$ odd, $\mathcal{C}^\infty$, and satisfying* (1.3)*, is also bounded on $L^2(\mu)$.*

If μ coincides with the 1-dimensional Hausdorff measure on a 1-dimensional Ahlfors-David regular set E, this result was already known. Recall that $E \subset \mathbb{C}$ is 1-dimensional Ahlfors-David regular (AD regular) if there exists some constant $C > 0$ such that

$$C^{-1} r \le \mathcal{H}^1(B(x,r) \cap E) \le Cr \quad \text{for } x \in E,\ 0 < r \le \operatorname{diam}(E), \tag{1.4}$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure. In this case, Mattila, Melnikov and Verdera [**MMV**] proved that the $L^2(\mu)$ boundedness of the Cauchy transform implies that E is uniformly rectifiable in the sense of David and Semmes (see [**DS2**]) and then, from the results in [**DS1**] and [**DS2**], one deduces that all Calderón-Zygmund operators with antisymmetric kernel are also bounded in $L^2(\mu)$.

For some Cantor sets in $\mathbb{C}$ and μ equal to the natural probability measure associated to them, it has been shown in [**MaT**] that Theorem 1.1 (and even its natural generalization to higher dimensions) also holds. Let us also mention that J. Verdera explained us [**Ve**] a simple argument for the proof of Theorem 1.1 in the particular case where $k(x,y) = K(x-y)$ is a homogeneous kernel smooth enough (say $\mathcal{C}^3$) outside the origin.

A fundamental tool for the proof of Theorem 1.1 is the corona decomposition obtained in [**To4**] for measures with linear growth and finite curvature (see the next section for the precise meaning of these notions) in order to show that analytic capacity is invariant, up to multiplicative estimates, under bilipschitz mappings. The technique of corona decomposition goes back to Carleson's proof of the corona theorem, and has been extensively used by David and Semmes in [**DS1**] and [**DS2**] in their pioneering study of uniformly rectifiable sets.

To prove Theorem 1.1, following [**Se**], we will split the Calderón-Zygmund operator T into different operators K_R, each one associated to a *tree* of the corona decomposition (see Sections 5 and 6). Each operator K_R is bounded because on each tree the measure μ can be approximated by arc length on an Ahlfors-David regular curve. Moreover, in a sense, the different operators K_R behave in a quasiorthogonal way. The *quasiorthoganility arguments* that appear in the present paper are inspired in part by [**MaT**].

The obtention in [**To4**] of the corona construction mentioned above relies heavily on the relationship between the Cauchy transform and curvature of measures (see (2.3)). In higher dimensions, a notion analogous to curvature useful to study the L^2 boundedness of Riesz transforms has not been found yet, and perhaps it does not exist (see [**Fa**]). As a consequence, the arguments in the present paper don't have an easy generalization to the case of Riesz transforms in higher dimensions.

The plan of the paper is the following. In Section 2 we state some preliminary definitions and results that will be used in the rest of the paper. In Section 3 we describe the corona decomposition of [**To4**] for measures with linear growth and finite curvature. In the subsequent section, we show how Theorem 1.1 is equivalent to the technical Lemma 4.1 (the Main Lemma). The rest of the paper is devoted to the proof of this lemma, with the exception of Section 12, which includes a slightly more general version of Theorem 1.1.

2. Preliminaries

A positive Radon measure μ is said to have linear growth if there exists some constant C_0 such that $\mu(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$. It easily seen that such a measure satisfies the following estimate for all $x \in \mathbb{C}$, $r > 0$, $\alpha > 0$:

$$\int_{|x-y|>r} \frac{1}{|x-y|^{1+\alpha}}\, d\mu(y) \leq \frac{C_0\, C_\alpha}{r^\alpha}, \tag{2.1}$$

where C_α depends only on α. This inequality will be used often in this paper. It can be proved splitting the domain of integration into annuli $\{x \in \mathbb{C} : 2^k r < |y-x| \leq 2^{k+1} r\}$, $k \geq 0$, for example.

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $R(x,y,z) = \infty$, $c(x,y,z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we set $c(x,y,z) = 0$. For a positive Radon measure μ, we define the *curvature of* μ as

$$c^2(\mu) = \iiint c(x,y,z)^2\, d\mu(x)\, d\mu(y)\, d\mu(z). \tag{2.2}$$

The notion of curvature of measures was introduced by Melnikov [**Me**] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible of the recent advances in connection with analytic capacity (see [**Lé**], [**Da3**] and [**To3**], for example).

The relationship between the Cauchy transform and curvature of measures was found by Melnikov and Verdera [**MeV**]. They proved that if μ has linear growth, then

$$\|\mathcal{C}_\varepsilon \mu\|^2_{L^2(\mu)} = \frac{1}{6} c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})), \tag{2.3}$$

where $c^2_\varepsilon(\mu)$ is an ε-truncated version of $c^2(\mu)$ (defined as in the right hand side of (2.2), but with the triple integral over $\{(x,y,z) \in \mathbb{C}^3 : |x-y|, |y-z|, |x-z| > \varepsilon\}$), and $O(\mu(\mathbb{C}))$ is an extra term satisfying $|O(\mu(\mathbb{C}))| \leq C\mu(\mathbb{C})$, where the constant C depends only on the linear growth constant C_0. Moreover, there is also a strong connection (see [**Pa**]) between the notion of curvature of measures and the β's from Jones' travelling salesman theorem [**Jo**]. The relationship with Favard length is an open problem (see Section 6 of the excellent survey paper [**Matt**], for example).

If T is a CZO, we denote

$$T_*\mu(x) := \sup_{\varepsilon>0} |T_\varepsilon \mu(x)|,$$

and if $f \in L^1_{\text{loc}}(\mu)$, we set $T_{\mu,*}f(x) := T_*(f\,d\mu)(x)$.

In the paper, by a square we mean a square with sides parallel to the axes. Moreover, we assume the squares to be half closed - half open. The side length of a square Q is denoted by $\ell(Q)$. Given a square Q and $a > 0$, aQ denotes the square concentric with Q with side length $a\ell(Q)$. The average (linear) density of a Radon measure μ on Q is

$$\theta_\mu(Q) := \frac{\mu(Q)}{\ell(Q)}. \tag{2.4}$$

A square $Q \subset \mathbb{C}$ is called 4-dyadic if it is of the form $[j2^{-n}, (j+4)2^{-n}) \times [k2^{-n}, (k+4)2^{-n})$, with $j, k, n \in \mathbb{Z}$. So a 4-dyadic square with side length $4 \cdot 2^{-n}$ is made up of 16 dyadic squares with side length 2^{-n}. We will work quite often with 4-dyadic squares.

Given a square Q (which may be non dyadic) with side length 2^{-n}, we denote $J(Q) := n$. Given $a, b > 1$, we say that Q is (a, b)-doubling if $\mu(aQ) \leq b\mu(Q)$. If we don't want to specify the constant b, we say that Q is a-doubling.

Remark 2.1. If $b > a^2$, then it easily follows that for μ-a.e. $x \in \mathbb{C}$ there exists a sequence of (a, b)-doubling squares $\{Q_n\}_n$ centered at x with $\ell(Q_n) \to 0$ (and with $\ell(Q_n) = 2^{-k_n}$ for some $k_n \in \mathbb{Z}$ if necessary).

As usual, in the paper the letter 'C' stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as C_1, retain its value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

3. The corona decomposition

This section deals with the corona construction obtained in [**To4**]. In the next theorem we will introduce a family Top of 4-dyadic squares (the top squares) satisfying some precise properties. Given any square $Q \in$ Top, we denote by Stop(Q) the subfamily of the squares $P \in$ Top satisfying

(a) $P \cap 3Q \neq \varnothing$,

(b) $\ell(P) \leq \frac{1}{8}\ell(Q)$,

(c) P is maximal, in the sense that there doesn't exist another square $P' \in$ Top satisfying (a) and (b) which contains P.

We also denote by $Z(\mu)$ the set of points $x \in \mathbb{C}$ such that there does not exist a sequence of $(70, 5000)$-doubling squares $\{Q_n\}_n$ centered at x with $\ell(Q_n) \to 0$ as $n \to \infty$, so that moreover $\ell(Q_n) = 2^{-k_n}$ for some $k_n \in \mathbb{Z}$. By the preceding remark we have $\mu(Z(\mu)) = 0$.

The set of good points for Q is defined as

$$G(Q) := 3Q \cap \operatorname{supp}(\mu) \setminus \left[Z(\mu) \cup \bigcup_{P \in \operatorname{Stop}(Q)} P \right].$$

Given two squares $Q \subset R$, we set

$$\delta_\mu(Q, R) := \int_{R_Q \setminus Q} \frac{1}{|y - x_Q|} \, d\mu(y),$$

where x_Q stands for the center of Q, and R_Q is the smallest square concentric with Q that contains R. See [**To1**, Lemma 2.1] for some properties dealing with the coefficients $\delta_\mu(Q, R)$.

Theorem 3.1 (The corona decomposition)**.** *Let μ be a Radon measure supported on $E \subset \mathbb{C}$ such that $\mu(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$ and $c^2(\mu) < \infty$. There exists a family* Top *of* 4*-dyadic* $(16, 5000)$*-doubling squares (called* top squares*) which satisfy the packing condition*

$$\sum_{Q \in \operatorname{Top}} \theta_\mu(Q)^2 \mu(Q) \lesssim \mu(E) + c^2(\mu), \tag{3.1}$$

and such that for each square $Q \in$ Top *there exists a C_1-AD regular curve Γ_Q such that:*

(a) $G(Q) \subset \Gamma_Q$.

(b) *For each $P \in \operatorname{Stop}(Q)$ there exists some square $\widetilde{P}$ containing P such that $\delta_\mu(P, \widetilde{P}) \leq C\theta_\mu(Q)$ and $\widetilde{P} \cap \Gamma_Q \neq \varnothing$.*

(c) *If P is a square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \varnothing$ or there is another square $P' \in \operatorname{Stop}(Q)$ such that $P \cap P' \neq \varnothing$ and $\ell(P') \leq \ell(P)$, then $\mu(P) \leq C\, \theta_\mu(Q)\, \ell(P)$.*

Moreover, Top *contains one* 4*-dyadic square R_0 such that $E \subset R_0$.*

In the theorem, that Γ_Q is C_1-AD regular means that the AD regularity constant in the definition (1.4) is $\leq C_1$. Notice that C_1 does not depend on Q.

For the reader's convenience, before going on we will make some comments on Theorem 3.1. Roughly speaking, the theorem describes how the support of a measure μ with linear growth and finite curvature can be approximated by a collection of AD regular curves Γ_Q, where each Γ_Q is associated to a square belonging to a family called Top. Condition (3.1) means that, in a sense, the family Top (and so the collection of curves) is not too big.

The statements (a) and (b) say that on each square $3Q$, with $Q \in$ Top, the support of μ can be approximated by Γ_Q up to the scale of the stopping squares in $\operatorname{Stop}(Q)$. More precisely, in (a) one states that μ-almost every point $x \in 3Q$ which does not belong to any stopping square in $\operatorname{Stop}(Q)$ lies on Γ_Q. The statement (b) means that each square from $\operatorname{Stop}(Q)$ is quite close to Γ_Q, in a sense. The coefficient $\delta_\mu(P, \widetilde{P})$ appearing in (b) measures how different is $\widetilde{P}$ from P. For example, if μ is a 1-dimensional AD regular measure [i.e. $\mu(B(x,r)) \approx r$ for $x \in \operatorname{supp}(\mu)$, $0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu))$], then the condition $\delta_\mu(P, \widetilde{P}) \leq C\theta_\mu(Q)$ implies that $\ell(P) \approx \ell(P')$, assuming $P \cap \operatorname{supp}(\mu) \neq \varnothing$. Thus in this case $\operatorname{dist}(P, \Gamma_Q) \leq C\ell(P)$ (like in the geometric corona construction of [**DS1**] and [**DS2**]).

Finally, (c) is a technical statement about the densities of the squares P which intersect $3Q$. It asserts that if one of these squares P is smaller than Q and larger than some stopping square $P' \in \operatorname{Stop}(Q)$ which intersects P (in a sense, this means that P has an intermediate size between Q and the squares in $\operatorname{Stop}(Q)$), then $\theta(P) \leq C\theta_\mu(Q)$. This statement is easier to understand if one also considers the good points in $G(Q)$ as stopping squares from Q with zero side length.

4. The Main Lemma

In order to reduce the technical difficulties, to prove Theorem 1.1 we will assume that the kernel $k(x,y)$ of T is uniformly bounded in L^∞, and all our estimates will be independent of the L^∞ norm of $k(x,y)$. See [**Ch**, p. 109] or [**To2**, eq. (44)], for example. In this case, the definition of $T\mu(x)$ makes sense for all $x \in \mathbb{C}$ when μ is compactly supported. Of course, all the estimates will be independent of the L^∞ norm of $k(x,y)$. The proof of Theorem 1.1 in full generality follows from this particular instance by a standard smoothing procedure.

We will prove the following result.

Lemma 4.1 (**Main Lemma**). *Let μ be a Radon measure on $\mathbb{C}$, compactly supported, with linear growth and finite curvature, and let T_μ be a 1-dimensional CZO with antisymmetric kernel. We have*

$$\|T_\mu 1\|^2_{L^2(\mu)} \lesssim \mu(\mathbb{C}) + c^2(\mu).$$

Let us see that Theorem 1.1 follows easily from this lemma and the $T(1)$ Theorem. Indeed, if the Cauchy transform is bounded with respect to μ and μ has no atoms, then μ has linear growth (see [**Da2**, Proposition III.1.4]) and $c^2(\mu_{|Q}) \leq C\mu(Q)$ for any square $Q \subset \mathbb{C}$, by (2.3) with $\mu_{|Q}$ instead of μ. If we apply Lemma 4.1 to the measure $\mu_{|Q}$ we get

$$\int_Q |T_\mu \chi_Q|^2 \, d\mu \lesssim \mu(Q) + c^2(\mu_{|Q}) \lesssim \mu(Q).$$

Then by the $T(1)$ Theorem of Nazarov, Treil and Volberg for non doubling measures [**NTV1**], T_μ is bounded on $L^2(\mu)$.

Remark 4.2. Actually, the $T(1)$ Theorem in [**NTV1**], for a general antisymmetric CZO, asserts that if

$$\int_{2Q} |T_\mu \chi_Q|^2 \, d\mu \lesssim \mu(Q)$$

for all squares $Q \subset \mathbb{C}$, then T_μ is bounded in $L^2(\mu)$. However, using ideas such as the ones in [**To2**, Remark 7.1 and Lemma 7.3] this condition can be weakened, so that it is enough to assume that

$$\int_Q |T_\mu \chi_Q|^2 \, d\mu \lesssim \mu(2Q)$$

for all squares $Q \subset \mathbb{C}$ in order to show that T_μ is bounded in $L^2(\mu)$.

The rest of the paper is devoted to the proof of Lemma 4.1. So the measure μ that we consider now is assumed to be compactly supported and to have linear growth and finite curvature.

5. Translation of dyadic lattices and adaptation of the corona decomposition

Later on we will have to average some estimates over dyadic lattices obtained by translation of the usual dyadic lattice $\mathcal{D}$. This fact requires the introduction of suitable variants of the family Top appearing in Theorem 3.1 better adapted to the translated dyadic lattices. Let us remark that the technique of averaging over different dyadic lattices is not new, and it has been used, for example, in [**GJ**], [**NTV1**], [**NTV2**], etc.

We proceed now to introduce the necessary new notation. Suppose that $\operatorname{supp}(\mu)$ is contained in a square with side length 2^N. For $\omega \in [0, 2^{N+1})^2 =: \Omega$, let $\mathcal{D}(\omega)$ be the dyadic lattice obtained by translating the lattice $\mathcal{D}$ on $\mathbb{C}$ by the vector w, that is to say, $\mathcal{D}(\omega) := \mathcal{D} + \omega$.

Recall that, by Theorem 3.1, μ has a corona decomposition in terms of a family, called Top, of 4-dyadic (with respect to $\mathcal{D}$) 16-doubling squares. To adapt this decomposition to the dyadic lattice $\mathcal{D}(\omega)$, for each fixed ω, we proceed as follows. We say that a square $Q \in \mathcal{D}(\omega)$ belongs to the family Top_ω if there exists some square $Q' \in \mathrm{Top}$ such that $Q \cap Q' \neq \varnothing$ and $\ell(Q') = \ell(Q)$. Notice that $Q \subset 3Q'$, $Q' \subset 3Q$, and for each fixed $Q' \in \mathrm{Top}$ there are at most four squares $Q \in \mathrm{Top}_\omega$ such that $Q \cap Q' \neq \varnothing$ and $\ell(Q') = \ell(Q)$. Since Q' is doubling it easily follows that $\theta_\mu(3Q) \approx \theta_\mu(Q')$, and then

$$\sum_{Q \in \mathrm{Top}_\omega} \theta_\mu(3Q)^2 \mu(Q) \lesssim \sum_{Q' \in \mathrm{Top}} \theta_\mu(Q')^2 \mu(3Q') \lesssim \mu(\mathbb{C}) + c^2(\mu). \tag{5.1}$$

Given $Q \in \mathrm{Top}_\omega$, we denote by $\mathrm{Stop}_\omega(Q)$ the family of *maximal* (and thus disjoint) squares P in Top_ω with $P \subsetneq Q$. Notice that the rule used to define $\mathrm{Stop}_\omega(Q)$ is different from the one used for $\mathrm{Stop}(Q)$. Finally we let $\mathrm{Tree}_\omega(Q)$ be the class of squares in $\mathcal{D}(\omega)$ contained in Q, different from Q, which are not proper subsquares of any $P \in \mathrm{Stop}_\omega(Q)$. This notation is inspired by the one in [**AHMTT**].

6. Decomposition of $T\mu$ with respect to the adapted corona decomposition and strategy of the proof of Main Lemma 4.1

For each fixed ω, to estimate $\|T\mu\|_{L^2(\mu)}$ we will decompose $T\mu$ using the dyadic lattice $\mathcal{D}(\omega)$ and the corresponding corona decomposition adapted to $\mathcal{D}(\omega)$. Now we will describe this decomposition, and at the end of the current section we will describe the global strategy for the proof of Main Lemma 4.1.

Let ψ be a non negative radial $\mathcal{C}^\infty$ function such that $\chi_{B(0,1)} \leq \psi \leq \chi_{B(0,3/2)}$. For each $n \in \mathbb{Z}$, set $\psi_n(z) := \psi(2^n z)$ and $\varphi_n := \psi_n - \psi_{n+1}$, so that each function φ_n is non negative and supported on $B\left(0, \frac{3}{2}2^{-n}\right) \setminus B(0, 2^{-n-1})$, and moreover we have

$$\sum_{n \in \mathbb{Z}} \varphi_n(x) = 1$$

for any $x \in \mathbb{C} \setminus \{0\}$. Given an antisymmetric Calderón-Zygmund kernel $k(x, y)$ and its associated CZO, T, for each $n \in Z$ we denote

$$T_n\mu(x) := \int \varphi_n(x - y)\, k(x, y)\, d\mu(y). \tag{6.1}$$

For each $Q \in \mathcal{D}(\omega)$, we set

$$T_Q\mu := \chi_Q T_{J(Q)}\mu,$$

where $J(Q)$ stands for the integer such that $\ell(Q) = 2^{-J(Q)}$. We also denote $\mathcal{D}_n(\omega) := \{Q \in \mathcal{D}(\omega) : \ell(Q) = 2^{-n}\}$. We have

$$T\mu = \sum_{n\in\mathbb{Z}} T_n\mu = \sum_{n\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_n(\omega)} T_Q\mu = \sum_{R\in\mathrm{Top}_\omega} \sum_{Q\in\mathrm{Tree}_\omega(R)} T_Q\mu + {\sum_Q}' T_Q\mu,$$

where the last sum runs over a finite (at most) number of squares $Q \in \mathcal{D}(\omega)$, with $\ell(Q) \approx \operatorname{diam}(\operatorname{supp}(\mu))$. So we have

$$\Big\| {\sum_Q}' T_Q\mu \Big\|_{L^2(\mu)} \leq {\sum_Q}' \|T_Q\mu\|_{L^2(\mu)} \lesssim \mu(\mathbb{C})^{1/2},$$

because it is immediate to check that $\|T_Q\|_{L^2(\mu),L^2(\mu)} \leq C$.

Hence to prove Lemma 4.1 we only have to estimate the $L^2(\mu)$ norm of the term $\sum_{R\in\mathrm{Top}_\omega} \sum_{Q\in\mathrm{Tree}_\omega(R)} T_Q\mu$. For $R \in \mathrm{Top}_\omega$, we set

$$K_R\mu := \sum_{Q\in\mathrm{Tree}_\omega(R)} T_Q\mu.$$

We have

$$\Big\| \sum_{R\in\mathrm{Top}_\omega} K_R\mu \Big\|^2_{L^2(\mu)} = \sum_{R\in\mathrm{Top}_\omega} \|K_R\mu\|^2_{L^2(\mu)} + \sum_{Q,R\in\mathrm{Top}_\omega : Q\neq R} \langle K_Q\mu, K_R\mu\rangle. \tag{6.2}$$

The first term on the right hand side of (6.2) will be estimated in Section 7 exploiting the idea that on each $\mathrm{Tree}_\omega(R)$, with $R \in \mathrm{Top}_\omega$, the measure μ can be approximated quite well by some measure absolutely continuous with respect to the arc length on Γ_R whose Radon-Nikodym density is $\lesssim \theta_\mu(3R)$. We will use the fact that $T_{\mathcal{H}^1_{\Gamma_R}}$ (this is the CZO with the same kernel as T_μ, with μ interchanged with $\mathcal{H}^1_{\Gamma_R}$) is bounded on $L^2(\mathcal{H}^1_{\Gamma_R})$, because Γ_R is an AD regular curve. Then we will prove that $\|K_R\mu\|^2_{L^2(\mu)} \lesssim \theta_\mu(3R)^2\mu(3R)$.

We will deal with the second term on the right side of (6.2) in Sections 8, 9 and 10. For this term we will use quasi-orthogonality arguments. In the sum we only have to consider pairs of squares $Q, R \in \mathrm{Top}_\omega$ with $Q \cap R \neq \varnothing$. Suppose, for example, that $Q \subsetneq R$. One should think that, because of the antisymmetry of the kernel $k(x, y)$, the functions $K_Q\mu$ are "close" to have zero μ-mean, while $K_R\mu$ are smooth functions on $Q \subset R$. See the beginning of Section 8 for more precise information.

7. Estimate of $\sum_{R\in\mathrm{Top}_\omega}\left\|K_R\mu\right\|^2_{L^2(\mu)}$

In this section we will prove the following lemma, by renormalization and comparison with the arc length measure on Γ_R.

Lemma 7.1. *For each $R\in\mathrm{Top}_\omega$ we have*

$$\left\|K_R\mu\right\|^2_{L^2(\mu)}\lesssim\theta_\mu(3R)^2\mu(3R).$$

Observe that from this lemma and (5.1) we get

$$\sum_{R\in\mathrm{Top}_\omega}\left\|K_R\mu\right\|^2_{L^2(\mu)}\lesssim\sum_{R\in\mathrm{Top}_\omega}\theta_\mu(3R)^2\mu(3R)\lesssim\mu(\mathbb{C})+c^2(\mu).$$

7.1. Regularization of the stopping squares. Given a fixed $R\in\mathrm{Top}_\omega$, let $R_1\in\mathrm{Top}$ be such that $R\cap R_1\neq\varnothing$ and $\ell(R)=\ell(R_1)$. Let $\Gamma_R:=\Gamma_{R_1}$ be the AD regular curve satisfying (a) and (b) in Theorem 3.1. For technical reasons, we need to introduce a regularized version of the family $\mathrm{Stop}(R_1)$ (or $\mathrm{Stop}_\omega(R)$) that we will denote by $\mathrm{Reg}_\omega(R)$. First we set

$$d_R(x):=\inf_{Q\in\mathrm{Stop}(R_1)}\left\{\mathrm{dist}(x,Q)+\ell(Q),\ \mathrm{dist}(x,G(R_1))\right\}. \tag{7.1}$$

For each $x\in 3R\setminus G(R_1)$, let Q_x be a dyadic square from $\mathcal{D}_\omega$ containing x such that

$$\frac{d_R(x)}{20}<\ell(Q_x)\leq\frac{d_R(x)}{10}.$$

Then, $\mathrm{Reg}_\omega(R)$ is a maximal (and thus disjoint) subfamily of $\{Q_x\}_{x\in 3R}$. Notice that these squares need not be contained in R.

Lemma 7.2. (a) $\bigcup_{Q\in\mathrm{Reg}_\omega(R)}Q\subset\bigcup_{Q\in\mathrm{Stop}(R_1)}Q$.

(b) *If $P,Q\in\mathrm{Reg}_\omega(R)$ and $2P\cap 2Q\neq\varnothing$, then $\ell(Q)/2\leq\ell(P)\leq 2\ell(Q)$.*

(c) *If $Q\in\mathrm{Reg}_\omega(R)$ and $x\in Q$, $r\geq\ell(Q)$, then $\mu(B(x,r)\cap 3R)\lesssim\theta_\mu(3R)r$.*

(d) *For each $Q\in\mathrm{Reg}_\omega(R)$, there exists some square $\widetilde{Q}$ which contains Q such that $\delta_\mu(Q,\widetilde{Q})\lesssim\theta_\mu(3R)$ and $\frac{1}{2}\widetilde{Q}\cap\Gamma_R\neq\varnothing$.*

The proof of this lemma follows by standard arguments. See [**To4**, Lemma 8.2], for example.

The properties (a), (c), and (d) stated in Lemma 7.2 ensure that the nice properties fulfilled by the squares in $\mathrm{Stop}(R_1)$ or $\mathrm{Stop}_\omega(R)$ (stated in Theorem 3.1 or which are direct consequence of it) also hold for the squares in $\mathrm{Reg}_\omega(R)$. The main advantage of the family $\mathrm{Reg}_\omega(R)$ over $\mathrm{Stop}(R_1)$ and $\mathrm{Stop}_\omega(R)$ is due to the property (b), which says that the size of two squares from $\mathrm{Reg}_\omega(R)$ is similar if they are close one to each other. This is a technical property which will be useful for the estimates below, and that the squares from $\mathrm{Stop}(R_1)$ and $\mathrm{Stop}_\omega(R)$ don't satisfy (in general). This is why we can think of $\mathrm{Reg}_\omega(R)$ as a regularized version of $\mathrm{Stop}(R_1)$ or $\mathrm{Stop}_\omega(R)$. Let us remark that this "regularization" technique has already been used by David and Semmes (see [**DS1**], for example).

7.2. The suppressed operators T_Φ. Given a Calderón-Zygmund kernel $k(x,y)$ and a non negative Lipschitz function Φ with Lipschitz constant ≤ 1, following [**NTV2**], we define another kernel $k_\Phi(x,y)$ as follows:

$$k_\Phi(x,y) := k(x,y)\frac{|x-y|^2}{|x-y|^2+\Phi(x)\Phi(y)}.$$

It turns out that $k_\Phi(x,y)$ is a Calderón-Zygmund kernel whose constants (in the definition (1.1)) are bounded above independently of Φ (see [**NTV2**], and also [**Pr**] for example). We denote by T_Φ the Calderón-Zygmund operator associated to the kernel $k_\Phi(x,y)$.

One should think of T_Φ as a kind of smooth ε-truncation of T, with $\varepsilon = \Phi(x)$ (see Lemma 7.3 below). The operator T_Φ will act as a smooth version of K_R, choosing $\Phi(x)$ appropriately. Notice that, unlike T_Φ, K_R is not smooth enough to be a CZO.

We also define the maximal operator

$$M_\Phi^r\nu(x) = \sup_{r\geq\Phi(x)}\frac{|\nu|(B(x,r))}{r}.$$

In particular,

$$M_\Phi^r(f\,d\mu)(x) = \sup_{r\geq\Phi(x)}\frac{1}{r}\int_{B(x,r)}|f|\,d\mu.$$

Lemma 7.3. *For any complex Radon measure ν on $\mathbb{C}$, the following properties hold:*

(a) *For all $x,y\in\mathbb{C}$, $x\neq y$:*

$$|k_\Phi(x,y)| \leq \min\left(\frac{1}{\Phi(x),\Phi(y)}\right).$$

(b) *For all $x\in\mathbb{C}$ and $\varepsilon\geq\Phi(x)$,*

$$|T_{\Phi,\varepsilon}\nu(x) - T_\varepsilon\nu(x)| \lesssim M_\Phi^r\nu(x).$$

(c) *For all $x\in\mathbb{C}$ and $\varepsilon\leq\Phi(x)$,*

$$|T_{\Phi,\varepsilon}\nu(x) - T_{\Phi,\Phi(x)}\nu(x)| \lesssim M_\Phi^r\nu(x),$$

where $T_{\Phi,\Phi(x)}\nu$ is the $\Phi(x)$-truncated version of T_Φ.

Proof: The estimates in this lemma are similar to the ones proved for the Cauchy kernel in [**NTV2**]. For the sake of completeness, we will show the detailed arguments (at least, for (b) and (c)). For the statement (a), we refer to [**Pr**].

The estimate (b) follows easily. A straightforward calculation shows that

$$|k_\Phi(x,y)-k(x,y)| \lesssim \frac{\Phi(x)\Phi(y)}{|x-y|\big(|x-y|^2+\Phi(x)\Phi(y)\big)}$$

$$\lesssim \frac{\Phi(x)\big(|x-y|+\Phi(x)\big)}{|x-y|^3}.$$

Then, for $\varepsilon \geq \Phi(x)$, using (2.1) (with $|\nu|$ instead of μ) we get

$$|T_{\Phi,\varepsilon}\nu(x)-T_\varepsilon\nu(x)| \lesssim \int_{|x-y|>\varepsilon} \frac{\Phi(x)}{|x-y|^2}\,d|\nu|(y)+\int_{|x-y|>\varepsilon}\frac{\Phi(x)^2}{|x-y|^3}\,d|\nu|(y)$$

$$\lesssim \left(\frac{\Phi(x)}{\varepsilon}+\frac{\Phi(x)^2}{\varepsilon^2}\right)\sup_{r\geq\varepsilon}\frac{|\nu|(B(x,r))}{r} \lesssim M^r_\Phi\nu(x).$$

The statement (c) is a direct consequence of (a): for $\varepsilon \leq \Phi(x)$ we have

$$|T_{\Phi,\varepsilon}\nu(x)-T_{\Phi,\Phi(x)}\nu(x)| = \left|\int_{\varepsilon<|x-y|\leq\Phi(x)} k_\Phi(x,y)\,d\nu(y)\right|$$

$$\lesssim \frac{|\nu|(B(x,\Phi(x)))}{\Phi(x)} \lesssim M^r_\Phi\nu(x). \qquad \square$$

Given a fixed $R \in \mathrm{Top}_\omega$, we choose the following function Φ:

$$\Phi(x) := \frac{1}{20}\,d_R(x), \tag{7.2}$$

where $d_R(x)$ has been defined in (7.1). Notice that $\Phi(x)$ is Lipschitz with Lipschitz constant $\leq 1/20 < 1$.

Lemma 7.4. *Let $R \in \mathrm{Top}_\omega$ and Φ defined in* (7.2)*. We have*

(a) *If $x \in Q$ for some $Q \in \mathrm{Stop}_\omega(R)$, then $\Phi(x) \leq 3\ell(Q)/20$.*

(b) *If $x \in R \setminus \bigcup_{Q\in\mathrm{Stop}_\omega(R)} Q$, then $\Phi(x) = 0$.*

(c) *If $x \in Q$ for some $Q \in \mathrm{Reg}_\omega(R)$, then $\Phi(x) \geq 2\ell(Q)/5$.*

(d) *For all $x \in 3R$ and $r \geq \Phi(x)$, we have*

$$\mu(B(x,r)\cap 3R) \leq C_2\theta_\mu(3R)\,r. \tag{7.3}$$

Proof: The statements in this lemma are a direct consequence of the definition of the family $\mathrm{Reg}_\omega(R)$ and of the properties shown in Lemma 7.2. First we show (a). Take $x \in Q$, with $Q \in \mathrm{Stop}_\omega(R)$. Let $Q' \in \mathrm{Top}$ be such that $Q \cap Q' \neq \varnothing$ and $\ell(Q) = \ell(Q')$. We cannot assure that $Q' \in \mathrm{Stop}(R_1)$ because of the different rules used to define $\mathrm{Stop}(R_1)$ and $\mathrm{Stop}_\omega(R)$. We know that either $\ell(Q) = \ell(Q') \geq \ell(R)/4$ or there exists some $P \in \mathrm{Stop}(R_1)$ (which may coincide with Q') that contains Q'. In the first case we have $d_R(x) \leq \ell(R)/2 \leq 2\ell(Q)$, which implies (a). In the latter case (i.e. when there exists some $P \in \mathrm{Stop}(R_1)$ such that $P \supseteq Q'$), from the definitions it is easily seen that indeed we have $P = Q'$ and then $d_R(x) \leq 3\ell(Q)$ because $\mathrm{dist}(x, Q') \leq 2\ell(Q)$, which is equivalent to (a).

The assertion (b) follows in an analogous way. We leave the details for the reader.

To prove (c), consider $x \in Q$, with $Q \in \mathrm{Reg}_\omega(R)$. By definition, there exists some $y \in Q$ such that $d_R(y) \geq 10\ell(Q)$. Since $d_R(\cdot)$ is 1-Lipschitz, we infer that $d_R(x) \geq 8\ell(Q)$ which is equivalent to (b).

Consider now $x \in 3R$ and $r \geq \Phi(x)$. If $x \notin \bigcup_{Q\in\mathrm{Reg}_\omega(R)} Q$ then (7.3) holds for all $r > 0$. Suppose now that there exists some $Q \in \mathrm{Reg}_\omega(R)$ such that $x \in Q$. From (b) we deduce that $5r/2 \geq \ell(Q)$. By Lemma 7.2 (c) we have

$$\mu(B(x, 5r/2) \cap 3R) \leq C_2 \theta_\mu(3R) \frac{5r}{2},$$

which yields (d). □

Lemma 7.5. *For $R \in \mathrm{Top}_\omega$ and $x \in R$ we have*

$$|K_R\mu(x)| \leq T_{\Phi,\mu,*}\chi_{3R}(x) + C\theta_\mu(3R).$$

In this statement $T_{\Phi,\mu,*}\chi_{3R}(x)$ stands for

$$T_{\Phi,\mu,*}\chi_{3R}(x) := \sup_{\varepsilon>0} |T_{\Phi,\varepsilon}(\chi_{3R}\, d\mu)(x)|,$$

where $T_{\Phi,\varepsilon}$ is the ε-truncated version of T_Φ.

Proof: Consider $Q \in \mathrm{Stop}_\omega(R)$ such that $x \in Q$. Then we have

$$\begin{aligned} |K_R\mu(x)| &= |T_{J(Q)}\mu(x) - T_{J(R)}\mu(x)| \\ &\leq \left| \int_{|x-y|>\ell(Q)} k(x,y)\chi_{3R}(y)\, d\mu(y) \right| + C\theta_\mu(3R). \end{aligned}$$

By Lemma 7.4 (a), $\Phi(x) \leq \ell(Q)$, and then by Lemma 7.3 (b) we get

$$\begin{aligned}\left|\int_{|x-y|>\ell(Q)} k(x,y)\chi_{3R}(y)\,d\mu(y)\right| &= |T_{\ell(Q)}(\chi_{3R}\,d\mu)(x)| \\ &\leq |T_{\Phi,\ell(Q)}(\chi_{3R}\,d\mu)(x)| + CM^r_\Phi(\chi_{3R}d\mu)(x) \\ &\leq T_{\Phi,\mu,*}\chi_{3R}(x) + C\theta_\mu(3R). \qquad \square\end{aligned}$$

Our next objective consists in showing that $T_{\Phi,\mu,*}$ is bounded on $L^2(\mu|3r)$ with norm $\leq C\theta_\mu(3R)$. From this result and the preceding lemma, we will deduce Lemma 7.1. First, in next subsection we will study the $L^2(\mu|3r)$ boundedness of $T_{\Phi,\mu}$ (see Lemma 7.9). In Subsection 7.4 we will deal with the operator $T_{\Phi,\mu,*}$.

7.3. L^2 boundedness of $T_{\Phi,\mu}$. Before proving the $L^2(\mu|3r)$ boundedness of $T_{\Phi,\mu}$ with norm $\leq C\theta_\mu(3R)$, we need to prove the following result.

Lemma 7.6. *Let $R \in \mathrm{Top}_\omega$ and Φ defined as in (7.2). Consider the measure $\sigma = \theta_\mu(3R)\,d\mathcal{H}^1_{|\Gamma_R}$. Then $T_{\Phi,\sigma}$ is bounded from $L^p(\sigma)$ into $L^p(\mu|3R)$, for $1 < p < \infty$, with norm $\leq C_p\theta_\mu(3R)$, with C_p depending only on p. Also, $T_{\Phi,\sigma}$ is bounded from $L^1(\sigma)$ into $L^{1,\infty}(\mu|3R)$, with norm $\leq C\theta_\mu(3R)$.*

This result is only a slight variant of the one obtained by G. David in [**Da1**, Proposition 5]. However, we will prove it for the sake of completeness. The first step consists in proving next lemma.

Lemma 7.7. *Let $R \in \mathrm{Top}_\omega$ and Φ defined as in (7.2). For any $0 < s \leq 1$, the following inequality holds:*

(7.4)

$$T_{\Phi,*}(f\,d\mathcal{H}^1_{\Gamma_R})(x) \leq C_s\Big[M^r_\Phi\big(T_*(f\,d\mathcal{H}^1_{\Gamma_R})^s\,d\mathcal{H}^1_{\Gamma_R}\big)(x)^{1/s} + M^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x)\Big],$$

for any $x \in 3R$, with C_s depending on s.

Proof: We will show that $|T_{\Phi,\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x)|$ is bounded above by the right hand side of (7.4), for every $\varepsilon > 0$. By Lemma 7.3 (c), it is enough to consider the case $\varepsilon \geq \Phi(x)$. Moreover, to prove (7.4) we may assume $\varepsilon \geq \frac{9}{10}\,\mathrm{dist}(x,\Gamma_R)$, since otherwise we have $T_{\Phi,\varepsilon}\nu(x) = T_{\Phi,\varepsilon_0}\nu(x)$ with $\varepsilon_0 = \frac{9}{10}\,\mathrm{dist}(x,\Gamma_R)$.

So we assume $\varepsilon \geq \max\big(\Phi(x), \frac{9}{10}\,\mathrm{dist}(x,\Gamma_R)\big)$. In this situation, we have

$$\mathcal{H}^1(B(x,2\varepsilon)\cap\Gamma_R) \gtrsim \varepsilon. \tag{7.5}$$

Let us check that

$$|T_{\Phi,\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x)| \le |T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(y)| + CM^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x) \tag{7.6}$$

for every $y \in B(x,2\varepsilon)\cap\Gamma_R$. Indeed, since $\varepsilon \ge \Phi(x)$, by Lemma 7.3 (b) we get

$$|T_{\Phi,\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x)| \le |T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(x)| + CM^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x).$$

We put

$$\begin{aligned}|T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(x)| \le\ & |T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(x) - T_{4\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x)| \\ & + |T_{4\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x) - T_{\mathbb{C}\setminus B(x,4\varepsilon)}(f\,d\mathcal{H}^1_{\Gamma_R})(y)| \\ & + |T_{\mathbb{C}\setminus B(x,4\varepsilon)}(f\,d\mathcal{H}^1_{\Gamma_R})(y) - T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(y)| \\ & + |T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(y)|.\end{aligned} \tag{7.7}$$

We have

$$\begin{aligned}|T_{4\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x) - T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(x)| &\lesssim \frac{1}{\varepsilon}\int_{B(x,4\varepsilon)} |f|\,d\mathcal{H}^1_{\Gamma_R} \\ &\lesssim M^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x).\end{aligned}$$

On the other hand, by standard estimates we obtain

$$\begin{aligned}|T_{4\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x) - T_{\mathbb{C}\setminus B(x,4\varepsilon)}(f\,d\mathcal{H}^1_{\Gamma_R})(y)| &\lesssim \sup_{r\ge 4\varepsilon}\frac{1}{r}\int_{B(x,r)} |f|\,d\mathcal{H}^1_{\Gamma_R} \\ &\lesssim M^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x).\end{aligned}$$

It is also straightforward to check that the third term on the right hand side of (7.7) is bounded above by $CM^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x)$. So (7.6) holds.

From (7.6), for any $0 < s \le 1$ we derive

$$|T_{\Phi,\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x)|^s \le |T_\varepsilon(f\,d\mathcal{H}^1_{\Gamma_R})(y)|^s + CM^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x)^s.$$

If we average this estimate with respect to $y \in B(x,2\varepsilon)\cap\Gamma_R$ and we use (7.5), we obtain

$$\begin{aligned}|T_{\Phi,\varepsilon}(f\,d\mathcal{H}^1_{\Gamma_R})(x)|^s \lesssim \frac{1}{2\varepsilon}\int_{y\in B(x,2\varepsilon)\cap\Gamma_R} & T_*(f\,d\mathcal{H}^1_{\Gamma_R})(x)^s d\mathcal{H}^1_{\Gamma_R} \\ & + M^r_\Phi(f\,d\mathcal{H}^1_{\Gamma_R})(x)^s.\end{aligned}$$

Exponentiating by $1/s$, (7.4) follows. □

Proof of Lemma 7.6: First we show that M^r_Φ (to be more precise, perhaps we should write $M^r_{\Phi,\sigma}$ instead of M^r_Φ now) is bounded from $L^1(\sigma)$ into $L^{1,\infty}(\mu_{|3R})$ with norm $\leq C\theta_\mu(3R)$ and from $L^p(\sigma)$ into $L^p(\mu_{|3R})$ with norm $\leq C_p\theta_\mu(3R)$, for $1 < p \leq \infty$. Notice that for $p = \infty$ this follows easily from the definitions. Interpolation with the weak $(1,1)$ estimate then yields the result for $1 < p < \infty$.

In fact, we will prove a slightly stronger result than the weak $(1,1)$ estimate. Consider the following maximal operator:

$$N^r(f\,d\sigma)(x) := \sup \frac{1}{r}\int_{B_r} |f|\,d\sigma,$$

where the supremum is taken over all the balls B_r of radius r which contain x such that $\mu(5B_r) \leq C_4\theta_\mu(3R)5r$ [the constant C_2 is the same as in (7.3)]. Notice that

$$M^r_\Phi(f\,d\sigma)(x) \leq N^r(f\,d\sigma)(x) \quad \text{for all } f \in L^1_{\text{loc}}(\sigma),\ x \in 3R.$$

Let $\lambda > 0$ be fixed. We want to estimate the μ-measure of

$$\Omega_\lambda := \{x \in 3R : N^r(f\,d\sigma)(x) > \lambda\}.$$

By the $5r$-covering theorem of Vitali, there exists a family of disjoint balls $\{B_{r_i}\}_{i\in I}$ such that

$$\Omega_\lambda \subset \bigcup_{i\in I} 5B_{r_i} \cap 3R,$$

with $\mu(5B_{r_i} \cap 3R) \leq C_4\theta_\mu(3R)5r_i$ for each ball B_{r_i}. Then we have

$$\begin{aligned}\mu(\Omega_\lambda) &\leq \sum_i \mu(5B_{r_i} \cap 3R) \leq 5C_4\theta_\mu(3R)\sum_i r_i\\ &\lesssim \frac{\theta_\mu(3R)}{\lambda}\sum_i \int_{B_{r_i}\cap 3R} |f|\,d\sigma \lesssim \frac{\theta_\mu(3R)}{\lambda}\int_{\Omega_\lambda} |f|\,d\sigma.\end{aligned} \tag{7.8}$$

Thus N^r and thus M^r_Φ are bounded from $L^1(\sigma)$ into $L^{1,\infty}(\mu_{|3R})$ with norm $\leq C\theta_\mu(3R)$.

For $1 < p < \infty$, by inequality (7.4), the boundedness of M^r_Φ from $L^p(\sigma)$ into $L^p(\mu_{|3R})$ with norm $\leq C\theta_\mu(3R)$ and the $L^p(\sigma)$ boundedness of $T_{\sigma,*}$ imply the boundedness of $T_{\Phi,\sigma,*}$ from $L^p(\sigma)$ into $L^p(\mu)$ with norm $\leq C\theta_\mu(3R)$.

Let us deal with the weak $(1,1)$ case. From (7.4) (with $s=1/2$) we deduce

$$\begin{aligned}\mu\{x\in 3R : T_{\Phi,*}(f\,d\sigma)(x)>\lambda\}\\ &\leq \mu\{x\in 3R : M^r_\Phi\big(T_*(f\,d\sigma)^{1/2}d\sigma\big)(x)>C_3\theta_\mu(3R)\lambda^{1/2}\}\\ &\quad+\mu\{x\in 3R : M^r_\Phi(f\,d\sigma)(x)>C_4\lambda\},\end{aligned}$$

with $C_3, C_4>0$. The last term on the right hand side is bounded above by $C\theta_\mu(3R)\|f\|_{L^1(\sigma)}/\lambda$ because of the weak $(1,1)$ inequality obtained for M^r_Φ. To estimate the first term on the right hand side we will use (7.8) and we will apply Kolmogorov's inequality: we denote

$$\Omega_0 := \{x\in 3R : N^r\big(T_*(f\,d\sigma)^{1/2}d\sigma\big)(x)>C_3\theta_\mu(3R)\lambda^{1/2}\},$$

and then we get

$$\begin{aligned}\mu\{x\in 3R : M^r_\Phi\big(T_*(f\,d\sigma)^{1/2}\big)(x)>C_3\theta_\mu(3R)\lambda^{1/2}\}\leq\mu(\Omega_0)\\ &\lesssim \frac{1}{\lambda^{1/2}}\int_{\Omega_0}|T_*(f\,d\sigma)|^{1/2}\,d\sigma\\ &\lesssim \frac{1}{\lambda^{1/2}}\mu(\Omega_0)^{1/2}\|T_*(f\,d\sigma)\|^{1/2}_{L^{1,\infty}(\sigma)}\\ &\lesssim \frac{\theta_\mu(3R)^{1/2}}{\lambda^{1/2}}\mu(\Omega_0)^{1/2}\|f\|^{1/2}_{L^1(\sigma)}.\end{aligned}$$

Thus $\mu(\Omega_0)\lesssim\theta_\mu(3R)\|f\|_{L^1(\sigma)}/\lambda$, and we are done. □

The following result is probably well known, but we will also prove it for completeness.

Proposition 7.8. *Let ν be a Radon measure on $\mathbb{C}$. Let S be an operator bounded on $L^2(\nu)$ with norm N_2. Suppose that S and its adjoint S^* are bounded from $L^1(\nu)$ into $L^{1,\infty}(\nu)$ with norm $\leq N_1$. Then, $N_2\leq CN_1$, where C is an absolute constant.*

Proof: For $1<p<\infty$, we denote by $N_p(S)$ and $N_p(S^*)$ the respective norms of S and S^* as operators in $L^p(\nu)$. By duality we obviously have $N_2(S)=N_2(S^*)=N_2$. By real interpolation we have $N_{4/3}(S)\leq CN_1^{1/2}N_2^{1/2}$, and similarly for S^*: $N_{4/3}(S^*)\leq CN_1^{1/2}N_2^{1/2}$. By duality, the last inequality yields $N_4(S)\leq CN_1^{1/2}N_2^{1/2}$. Then, by complex interpolation, $N_2(S)\leq N_{4/3}(S)^{1/2}N_4(S)^{1/2}\leq CN_1^{1/2}N_2^{1/2}$, which is equivalent to $N_2\leq C^{1/2}N_1$. □

Lemma 7.9. *For $R \in \mathrm{Top}_\omega$ and Φ defined as in (7.2), $T_{\Phi,\mu}$ is bounded on $L^2(\mu|3R)$ with norm $\leq C\theta_\mu(3R)$.*

By Proposition 7.8, to prove this lemma it is enough to show that $T_{\Phi,\mu}$ is bounded from $L^1(\mu|3R)$ into $L^{1,\infty}(\mu|3R)$ with norm $\leq C\theta_\mu(3R)$. A direct proof of the $L^2(\mu|3R)$ boundedness (by comparison of $\mu|3R$ with some appropriate measure supported on Γ_R) would be more difficult.

Proof: Let us show that $T_{\Phi,\mu}$ is bounded from $L^1(\mu|3R)$ into $L^{1,\infty}(\mu|3R)$ with norm $\leq C\theta_\mu(3R)$. Set $\{Q_i\}_{i\in I} := \mathrm{Reg}_\omega(R)$ and let $\{\widetilde{Q}_i\}_{i\in I}$ be μ-doubling squares such that, for each $i \in I$, Q_i is contained in and concentric with $\widetilde{Q}_i$, and moreover $\delta_\mu(Q_i, \widetilde{Q}_i) \leq C\theta_\mu(3R)$ and $\frac{1}{2}\widetilde{Q}_i \cap \Gamma_R \neq \varnothing$. Suppose that the order of the sequence $\{Q_i\}_{i\in I}$ is non increasing in size. We set $\chi_i := \chi_{Q_i \setminus \bigcup_{j=1}^{i-1} Q_j}$. For each $i \in I$ we define

$$\varphi_i(x) := \frac{\chi_{\widetilde{Q}_i \cap \Gamma_R}(x)}{\mathcal{H}^1(\widetilde{Q}_i \cap \Gamma_R)} \int \chi_i f \, d\mu.$$

We have

$$f = f\chi_{\mathbb{C}\setminus\bigcup_i Q_i} + \sum_i f\chi_i =: g + b.$$

By the properties of the squares $\{Q_i\}_i$, $\mathrm{supp}(\mu) \setminus \bigcup_i Q_i \subset \Gamma_R$, and by the Radon-Nikodym theorem,

$$\mu_{|\mathbb{C}\setminus\bigcup_i Q_i} = \eta \mathcal{H}^1_{\Gamma_R},$$

where η some function such that $0 \leq \eta \leq C\theta_\mu(3R)$. By Lemma 7.6, $T_{\mu_{|\mathbb{C}\setminus\bigcup_i Q_i},\Phi}$ is bounded from $L^1(\mu|\mathbb{C} \setminus \bigcup_i Q_i)$ into $L^{1,\infty}(\mu)$ with norm $\leq C\theta_\mu(3R)$ and so

$$\mu\{x : |T_{\Phi,\mu} g(x)| > \lambda\} \lesssim \frac{\theta_\mu(3R)}{\lambda} \int |g| \, d\mu \lesssim \frac{\theta_\mu(3R)}{\lambda} \int |f| \, d\mu. \tag{7.9}$$

To deal with the term corresponding to b, we put

$$b \, d\mu = \sum_i \big(f\chi_i \, d\mu - \varphi_i \, d\mathcal{H}^1_{\Gamma_R}\big) + \sum_i \varphi_i \, d\mathcal{H}^1_{\Gamma_R} =: \sum_i \nu_i + \sum_i \varphi_i \, d\mathcal{H}^1_{\Gamma_R}.$$

Observe that

$$\sum_i \int |\varphi_i| \, d\mathcal{H}^1_{\Gamma_R} = \sum_i \left| \int \chi_i f \, d\mu \right| \leq \int |f| \, d\mu.$$

By Lemma 7.6 we obtain

$$\mu\left\{x : T_\Phi\left(\sum_i \varphi_i\, d\mathcal{H}^1_{\Gamma_R}\right)(x) > \lambda\right\} \lesssim \frac{\theta_\mu(3R)}{\lambda}\left\|\sum_i \varphi_i\right\|_{L^1(\mathcal{H}^1_{\Gamma_R})} \tag{7.10}$$

$$\lesssim \frac{\theta_\mu(3R)}{\lambda}\|f\|_{L^1(\mu)}.$$

Now we turn our attention to $T_\Phi(\sum_i \nu_i)$. We set

$$\int\left|T_\Phi\left(\sum_i \nu_i\right)\right| d\mu \le \sum_i \int_{\mathbb{C}\setminus 2\widetilde{Q}_i} |T_\Phi \nu_i|\, d\mu + \sum_i \int_{2\widetilde{Q}_i} |T_\Phi \nu_i|\, d\mu. \tag{7.11}$$

Since $\int d\nu_i = 0$, for $x \notin 2\widetilde{Q}_i$, by standard estimates, we get

$$|T_\Phi \nu_i(x)| \lesssim \frac{\ell(\widetilde{Q}_i)\|\nu_i\|}{|x - z_i|^2},$$

where z_i is the center of Q_i. Thus, by (2.1),

$$\int_{\mathbb{C}\setminus 2\widetilde{Q}_i} |T_\Phi \nu_i|\, d\mu \lesssim \|\nu_i\| \int_{\mathbb{C}\setminus 2\widetilde{Q}_i} \frac{\ell(\widetilde{Q}_i)}{|x - z_i|^2}\, d\mu(x)$$

$$\lesssim \theta_\mu(3R)\|\nu_i\| \le C\theta_\mu(3R)\int |\chi_i f|\, d\mu.$$

To estimate the last integral in (7.11) we set

$$\int_{2\widetilde{Q}_i} |T_\Phi \nu_i|\, d\mu \le \int_{2\widetilde{Q}_i} |T_\Phi(\varphi_i\, d\mathcal{H}^1_{\Gamma_R})|\, d\mu$$

$$+ \int_{2\widetilde{Q}_i \setminus 2Q_i} |T_\Phi(\chi_i f\, d\mu)|\, d\mu$$

$$+ \int_{2Q_i} |T_\Phi(\chi_i f\, d\mu)|\, d\mu =: I_1 + I_2 + I_3.$$

To deal with I_1 we take into account that φ_i is an L^2 function and $\widetilde{Q}_i$ is doubling, and then we apply Lemma 7.6:

$$I_1 \lesssim \mu(\widetilde{Q}_i)^{1/2}\left(\int |T_\Phi(\varphi_i\, d\mathcal{H}^1_{\Gamma_R})|\, d\mu\right)^{1/2} \lesssim \mu(\widetilde{Q}_i)^{1/2}\|\varphi_i\|_{L^2(\theta_\mu(3R)\mathcal{H}^1_{\Gamma_R})}$$

$$\lesssim \mu(\widetilde{Q}_i)^{1/2}\theta_\mu(3R)^{1/2}\frac{\|\chi_i f\|_{L^1(\mu)}}{\mathcal{H}^1(2\widetilde{Q}_i \cap \Gamma_R)} \lesssim \theta_\mu(3R)\|\chi_i f\|_{L^1(\mu)}.$$

For I_2 we use the brutal estimate

$$|T_\Phi(\chi_i f\, d\mu)(x)| \lesssim \frac{\|\chi_i f\|_{L^1(\mu)}}{|x-z_i|} \quad \text{if } x \in 2\widetilde{Q}_i \setminus Q_i$$

and we obtain

$$I_2 \lesssim \|\chi_i f\|_{L^1(\mu)} \int_{2\widetilde{Q}_i\setminus Q_i} \frac{1}{|x-z_i|}\, d\mu$$

$$= \delta_\mu(2Q_i, 2\widetilde{Q}_i)\|\chi_i f\|_{L^1(\mu)} \lesssim \theta_\mu(3R)\|\chi_i f\|_{L^1(\mu)}.$$

Finally, for I_3 we use that if $x \in Q_i$ then $\Phi(x) \gtrsim \ell(Q_i)$, and so by Lemma 7.3 (a), $|T_\Phi(\chi_i f\, d\mu)(y)| \lesssim \|\chi_i f\|_{L^1(\mu)}/\ell(Q_i)$ for any $y \in 2Q_i$. Thus,

$$I_3 \lesssim \|\chi_i f\|_{L^1(\mu)} \frac{\mu(2Q_i)}{\ell(Q_i)} \lesssim \theta_\mu(3R)\|\chi_i f\|_{L^1(\mu)}.$$

Therefore,

$$\int \left| T_\Phi\left(\sum_i \nu_i\right)\right| d\mu \lesssim \theta_\mu(3R)\|f\|_{L^1(\mu)}.$$

From this estimate, (7.9), and (7.10), we deduce that $T_{\Phi,\mu}$ is bounded from $L^1(\mu|3R)$ into $L^{1,\infty}(\mu|3R)$ with norm $\leq C\theta_\mu(3R)$. □

7.4. L^2 boundedness of $T_{\Phi,\mu,*}$.

Lemma 7.10. *For $R \in \mathrm{Top}_\omega$ and Φ defined as in (7.2), $T_{\Phi,\mu,*}$ is bounded on $L^2(\mu|3R)$ with norm $\leq C\theta_\mu(3R)$.*

This result is a direct consequence of a Cotlar type inequality proved by Nazarov, Treil and Volberg. To state it we need to introduce some notation. Given a Radon measure σ on $\mathbb{C}$ and $f \in L^1_{\mathrm{loc}}(\sigma)$, we set

$$\widetilde{M}_\sigma f(x) = \sup_{r>0} \frac{1}{\sigma(B(x,3r))} \int_{B(x,r)} |f|\, d\sigma,$$

and

$$\widetilde{M}_{\sigma,3/2} f(x) = \sup_{r>0} \left(\frac{1}{\sigma(B(x,3r))} \int_{B(x,r)} |f|^{3/2}\, d\sigma\right)^{2/3}.$$

The following lemma has been proved in [**NTV2**, Theorem 5]. See also [**Vo**, Theorem 11.1].

Lemma 7.11 ([**NTV2**, Theorem 5]). *Let σ be a Radon measure on $\mathbb{C}$, and let S be a Calderón-Zygmund operator with kernel $s(x,y)$ which satisfies* (1.1) *with constant C_S. For a fixed $C_0>0$, let*

$$\mathcal{R}(x)=\sup\{r>0:\sigma(B(x,r))>C_0 r\},\quad \text{for } x\in\mathbb{C}.$$

Suppose that the kernel $s(x,y)$ satisfies

$$|s(x,y)|\leq \min\left(\frac{1}{\mathcal{R}(x)},\frac{1}{\mathcal{R}(y)}\right)$$

for all $x,y\in\mathbb{C}$. Then there is an absolute constant A such that the following Cotlar type inequality holds for any $f\in L^1_{\rm loc}(\sigma)$ and $x\in\mathbb{C}$:

$$S_*(f\,d\sigma)(x)\leq A\,C_S\Big[\widetilde{M}_\sigma(S(f\,d\sigma))(x)+\big(C_0+\|S_\sigma\|_{L^2(\sigma),L^2(\sigma)}\big)\widetilde{M}_{\sigma,3/2}f(x)\Big].$$

To prove Lemma 7.10 from the preceding Cotlar type inequality, we only have to take $S:=T_\Phi$, $\sigma:=\mu_{|3R}$ and $C_0:=C\theta_\mu(3R)$ and use Lemma 7.9.

7.5. Proof of Lemma 7.1. By Lemmas 7.5 and 7.10 we have

$$\begin{aligned}\big\|K_R\mu\big\|_{L^2(\mu)} &\leq \big\|\chi_R T_{\Phi,*}(\chi_{3R}\mu)\big\|_{L^2(\mu)}+C\theta_\mu(3R)\mu(3R)^{1/2}\\ &\lesssim \theta_\mu(3R)\mu(3R)^{1/2}.\end{aligned}$$

□

8. Estimate of $\sum_{Q,R\in\mathrm{Top}_\omega:Q\neq R}\langle K_Q\mu,K_R\mu\rangle$

Since $\mathrm{supp}(K_Q\mu)\cap\mathrm{supp}(K_R\mu)=\varnothing$ unless $Q\subset R$ or $R\subset Q$, we have

$$\sum_{Q,R\in\mathrm{Top}_\omega:Q\neq R}\langle K_Q\mu,K_R\mu\rangle=2\,\mathrm{Re}\sum_{Q,R\in\mathrm{Top}_\omega:Q\subsetneq R}\langle K_Q\mu,K_R\mu\rangle. \tag{8.1}$$

As explained at the end of Section 6, to estimate the sums in (8.1) we want to use some kind of quasi-orthogonality argument. We would be very happy if the functions $K_Q\mu$ had mean value zero, because in this case we would use the smoothness of $K_R\mu$ on Q to conclude that $\langle K_Q\mu,K_R\mu\rangle$ becomes very small as $\ell(Q)/\ell(R)\to 0$.

However, the functions $K_Q\mu$ don't have mean value zero. Indeed, recall that

$$K_Q\mu=\sum_{M\in\mathrm{Tree}_\omega(R)}T_M\mu, \tag{8.2}$$

and also that $T_M\mu=\chi_M T_n\mu$, with $n=J(M)$. Then, by the antisymmetry of the kernel of T_n we have

$$\int\chi_M T_n(\chi_M\,\mu)\,d\mu=0. \tag{8.3}$$

Unfortunately, from this fact we cannot infer that $T_M\mu$, and so $K_R\mu$, have mean value zero.

Nevertheless, the identity (8.3) becomes more useful if we replace M by a larger square (keeping n fixed). Let us explain why. Consider a square $S \in \Delta_m(\omega)$, with $m \ll n$, and then by antisymmetry we have again

$$\int_S T_n(\chi_S\, \mu)\, d\mu = 0.$$

Therefore,

$$\begin{aligned}\sum_{M\in\mathcal{D}_n(\omega):M\subset S} \int T_M\mu\, d\mu &= \int_S T_n(\mu)\, d\mu \\ &= \int_S T_n(\chi_S\, \mu)\, d\mu + \int_S T_n(\chi_{\mathbb{C}\setminus S}\, \mu)\, d\mu \\ &= \int_S T_n(\chi_{\mathbb{C}\setminus S}\, \mu)\, d\mu.\end{aligned}$$

Now recall that the kernel of T_n is $\varphi_n(x-y)\, k(x,y)$, and $\varphi_n(x-y) = 0$ if $|x-y| \geq 2^{-n+1}$. As a consequence,

$$\int_S T_n(\chi_{\mathbb{C}\setminus S}\, \mu)\, d\mu = \int_S T_n(\chi_{U_{2^{-n+1}}(\partial S)}\mu)\, d\mu = -\int_{U_{2^{-n+1}}(\partial S)} T_n(\chi_S\mu)\, d\mu,$$

where $U_{2^{-n+1}}(\partial S)$ is the 2^{-n+1}-neighborhood of ∂S, which is a very thin tubular neighborhood (we are assuming that $\ell(S) \gg 2^{-n}$). We obtain

$$\left|\sum_{M\in\mathcal{D}_n(\omega):M\subset S} \int T_M\mu\, d\mu\right| \leq \|T_n\chi_S\|_{L^\infty(\mu)}\mu(U_{2^{-n+1}}(\partial S)),$$

and we should expect $\mu(U_{2^{-n+1}}(\partial S)) \ll \mu(S)$ quite often (i.e. averaging with respect to dyadic lattices).

In order to implement this idea (or a variant of it), we operate as follows. Given $Q, R \in \mathrm{Top}_\omega$ with $Q \subsetneq R$, there exists some $P \in \mathrm{Stop}_\omega(R)$ which contains Q. When we replace the functions $K_Q\mu$ by sums like the one on the right hand side of (8.2), for each fixed $R \in \mathrm{Top}_\omega$, we get

$$\begin{aligned}\sum_{Q\in\mathrm{Top}_\omega:Q\subsetneq R} \langle K_Q\mu, K_R\mu\rangle &= \sum_{P\in\mathrm{Stop}_\omega(R)}\ \sum_{Q\in\mathrm{Top}_\omega:Q\subset P} \langle K_Q\mu, K_R\mu\rangle \\ &= \sum_{P\in\mathrm{Stop}_\omega(R)}\ \sum_{n>J(P)} \langle \chi_P T_n\mu, K_R\mu\rangle.\end{aligned}$$

To estimate the last sum we split each $P\in\operatorname{Stop}_\omega(R)$ into squares $S\in\mathcal{D}_m(\omega)$, where $m=m(J(P),n)$ is an intermediate value between n and $J(P)$ (for example, the integer part of the arithmetic mean of $J(P)$ and n). So we put

$$\begin{aligned}\sum_{Q\in\mathrm{Top}_\omega:Q\subsetneq R}\langle K_Q\mu,K_R\mu\rangle &= \sum_{P\in\mathrm{Stop}_\omega(R)}\sum_{n>J(P)}\sum_{S\in\mathcal{D}_m(\omega):S\subset P}\langle\chi_S T_n\mu,K_R\mu\rangle\\ &= \sum_{P\in\mathrm{Stop}_\omega(R)}\sum_{n>J(P)}\sum_{S\in\mathcal{D}_m(\omega):S\subset P}\langle\chi_S T_n(\chi_S\mu),K_R\mu\rangle\\ &\quad+\sum_{P\in\mathrm{Stop}_\omega(R)}\sum_{n>J(P)}\sum_{S\in\mathcal{D}_m(\omega):S\subset P}\langle\chi_S T_n(\chi_{\mathbb{C}\setminus S}\mu),K_R\mu\rangle\\ &=: A+B.\end{aligned}\tag{8.4}$$

We will estimate the terms A and B separately. To deal with A we will take into account that $\chi_S T(\chi_S\mu)$ has mean value 0 and $K_R\mu$ is a smooth function on $S\subset P$. To deal with B we will use the fact that $\chi_S T_n(\chi_{\mathbb{C}\setminus S}\mu)$ vanishes out of a thin tubular neighborhood of ∂S, as explained above, whose average (with respect to $\omega\in\Omega$) μ-measure is small.

9. Estimate of A in (8.4)

In the following lemma, given $Q,R\in\mathrm{Top}_\omega$ such that $Q\subsetneq R$, we denote by R_Q the square from $\operatorname{Stop}_\omega(R)$ which contains Q.

Lemma 9.1. *For each $\omega\in\Omega$ and $R\in\mathrm{Top}_\omega$, we have*

$$|A|=|A(\omega,R)|\lesssim\theta_\mu(3R)\sum_{Q\in\mathrm{Top}_\omega:Q\subsetneq R}2^{-|J(Q)-J(R_Q)|/2}\theta_\mu(3Q)\mu(Q).$$

Proof: Since $\int_S T_n(\chi_S\mu)\,d\mu=0$ (by the antisymmetry of T_n), we have

$$\langle\chi_S T_n(\chi_S\mu),K_R\mu\rangle=\int_S T_n(\chi_S\mu)(K_R\mu-K_R\mu(z_S))\,d\mu,$$

where z_S stands for the center of S. Notice that, on each stopping square P of R, the kernel of k_R of K_R coincides with a smooth truncation of $k(x,y)$ which satisfies the gradient condition

$$|k_R(x,y)-k_R(x',y)|\lesssim\frac{|x-x'|}{(\ell(P)+|x-y|)^2}.$$

Hence, using (2.1), for $x \in S$ one easily gets

$$|K_R\mu(x) - K_R\mu(z_S)| \lesssim \int_{3R} \frac{\ell(S)}{(\ell(P) + |z_S - y|)^2}\, d\mu(y) \lesssim \frac{\ell(S)}{\ell(P)} \theta_\mu(3R).$$

Thus, by the choice of $m = m(J(P), n)$,

$$\begin{aligned}\big|\langle \chi_S T_n(\chi_S\mu), K_R\mu\rangle\big| &\lesssim 2^{-|n-J(P)|/2}\theta_\mu(3R) \int_S |T_n(\chi_S\mu)|\, d\mu \\ &\lesssim 2^{-|n-J(P)|/2}\theta_\mu(3R) \int_S \theta_{\mu,n}(x)\, d\mu(x),\end{aligned}$$

where we have denoted $\theta_{\mu,n}(x) := \mu(B(x, 2^{-n+2}))/2^{-n+2}$. Summing on S we obtain

$$\sum_{S\in\mathcal{D}_m(\omega):S\subset P} |\langle \chi_S T_n(\chi_S\mu), K_R\mu\rangle| \lesssim 2^{-|n-J(P)|/2}\theta_\mu(3R) \int_P \theta_{\mu,n}(x)\, d\mu(x).$$

To estimate A, now we organize the sums in trees

$$\begin{aligned}|A| &\lesssim \theta_\mu(3R) \sum_{P\in \mathrm{Stop}_\omega(R)} \sum_{n>J(P)} 2^{-|n-J(P)|/2} \int_P \theta_{\mu,n}(x)\, d\mu(x) \\ &\lesssim \theta_\mu(3R) \sum_{P\in \mathrm{Stop}_\omega(R)} \sum_{n>J(P)} \sum_{M\in\mathcal{D}_n(\omega):Q\subset P} 2^{-|J(M)-J(P)|/2}\theta_\mu(3M)\mu(M) \\ &= \theta_\mu(3R) \sum_{P\in \mathrm{Stop}_\omega(R)} \sum_{Q\in \mathrm{Top}_\omega:Q\subset P} \sum_{M\in \mathrm{Tree}_\omega(Q)} 2^{-|J(M)-J(P)|/2}\theta_\mu(3M)\mu(M).\end{aligned}$$

In the last sum we have $\theta_\mu(3M) \le C\theta_\mu(3Q)$ and

$$\sum_{M\in \mathrm{Tree}_\omega(Q)} 2^{-|J(M)-J(P)|/2}\mu(M) \lesssim 2^{-|J(Q)-J(P)|/2}\mu(Q).$$

Therefore,

$$\begin{aligned}|A| &\lesssim \theta_\mu(3R) \sum_{P\in \mathrm{Stop}_\omega(R)} \sum_{Q\in \mathrm{Top}_\omega:Q\subset P} 2^{-|J(Q)-J(P)|/2}\theta_\mu(3Q)\mu(Q) \\ &= \theta_\mu(3R) \sum_{Q\in \mathrm{Top}_\omega:Q\subsetneq R} 2^{-|J(Q)-J(R_Q)|/2}\theta_\mu(3Q)\mu(Q).\end{aligned}$$

□

Now we need to introduce some additional notation. Given $R \in \mathrm{Top}_\omega$ and $k \geq 1$, we define a family of squares $\mathrm{Stop}^k_\omega(R) \subset \mathcal{D}_\omega$ inductively: we set $\mathrm{Stop}^1_\omega(R) := \mathrm{Stop}_\omega(R)$, and for $k \geq 2$,

$$\mathrm{Stop}^k_\omega(R) = \{Q : \exists\, \widetilde{R} \in \mathrm{Stop}^{k-1}_\omega(R) \text{ such that } Q \in \mathrm{Stop}(\widetilde{R})\}.$$

That is, $\mathrm{Stop}^k_\omega(R) = \mathrm{Stop}_\omega(\mathrm{Stop}^{k-1}_\omega(R))$.

Lemma 9.2. *We have*

$$\sum_{R\in\mathrm{Top}_\omega} |A(\omega,R)| \lesssim \sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)^2\mu(R).$$

Proof: From the preceding lemma we get

$$\begin{aligned}
&\sum_{R\in\mathrm{Top}_\omega} |A(\omega,R)| \\
&\lesssim \sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)\mu(R)^{1/2} \sum_{Q\in\mathrm{Top}_\omega : Q\subsetneq R} 2^{-|J(Q)-J(R_Q)|/2}\frac{\mu(Q)^{1/2}}{\mu(R)^{1/2}}\theta_\mu(3Q)\mu(Q)^{1/2} \\
&= \sum_{k\geq 1} 2^{-k/2} \sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)\mu(R)^{1/2} \sum_{Q\in\mathrm{Stop}^k_\omega(R)} \frac{\mu(Q)^{1/2}}{\mu(R)^{1/2}}\theta_\mu(3Q)\mu(Q)^{1/2}.
\end{aligned}$$

In the last identity we used that if $Q \in \mathrm{Stop}^k_\omega(R)$, then $|J(Q)-J(R_Q)| \geq k-1$. By Cauchy-Schwartz, for each $k \geq 1$ and every $R \in \mathrm{Top}_\omega$ we have

$$\begin{aligned}
\sum_{Q\in\mathrm{Stop}^k_\omega(R)} \frac{\mu(Q)^{1/2}}{\mu(R)^{1/2}}\theta_\mu(3Q)\mu(Q)^{1/2} &\leq \left(\sum_{Q\in\mathrm{Stop}^k_\omega(R)} \frac{\mu(Q)}{\mu(R)}\right)^{1/2} \\
&\qquad \times \left(\sum_{Q\in\mathrm{Stop}^k_\omega(R)} \theta_\mu(3Q)^2\mu(Q)\right)^{1/2} \\
&\leq \left(\sum_{Q\in\mathrm{Stop}^k_\omega(R)} \theta_\mu(3Q)^2\mu(Q)\right)^{1/2},
\end{aligned}$$

since $\mathrm{Stop}_\omega^k(R)$ is a family of disjoint dyadic squares, for each $k \geq 1$. Using Cauchy-Schwartz again, we obtain

$$\sum_{R\in\mathrm{Top}_\omega} |A(\omega,R)|$$

$$\lesssim \sum_{k\geq 1} 2^{-k/2} \sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)\mu(R)^{1/2} \left(\sum_{Q\in\mathrm{Stop}_\omega^k(R)} \theta_\mu(3Q)^2\mu(Q)\right)^{1/2}$$

$$\lesssim \sum_{k\geq 1} 2^{-k/2} \left(\sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)^2\mu(R)\right)^{1/2} \left(\sum_{R\in\mathrm{Top}_\omega}\sum_{Q\in\mathrm{Stop}_\omega^k(R)} \theta_\mu(3Q)^2\mu(Q)\right)^{1/2}$$

$$\lesssim \sum_{k\geq 1} 2^{-k/2} \sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)^2\mu(R) \approx \sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)^2\mu(R). \qquad \square$$

10. Estimate of B in (8.4)

Recall that, given $F \subset \mathbb{C}$, $U_\delta(F)$ stands for the δ-neighborhood of F. Also, for any $n \in \mathbb{Z}$, we denote

$$\partial\mathcal{D}_n(\omega) := \bigcup_{Q\in\mathcal{D}_n(\omega)} \partial Q.$$

Notice that $U_\delta(\partial\mathcal{D}_n(w))$ is the union of the tubular δ-neighborhoods of the vertical and horizontal lines of the grid which defines $\mathcal{D}_n(\omega)$.

Lemma 10.1. *For $R \in \mathrm{Top}_\omega$, $P \in \mathrm{Stop}_\omega(R)$, we have*

$$\left|\sum_{n>J(P)} \sum_{S\in\mathcal{D}_{m(P,n)}(\omega):S\subset P} \langle \chi_S T_n(\chi_{\mathbb{C}\setminus S}\mu), K_R\mu\rangle\right|$$

$$\lesssim \sum_{Q\in\mathrm{Top}_\omega:Q\subset P} \|K_R\mu\|_{L^2(\mu|Q)}\theta_\mu(3Q) \times \sum_{n>J(Q)} \mu\big(Q\cap U_{2^{-n+1}}(\partial\mathcal{D}_{m(P,n)}(\omega))\big)^{1/2}, \tag{10.1}$$

where we have denoted $m(P,n) := [(n+J(P))/2]$.

Proof: Observe that $\chi_S T_n(\chi_{\mathbb{C}\setminus S}\mu)$ vanishes out of

$$U_{2^{-n+1}}(\partial S) \subset U_{2^{-n+1}}(\partial \mathcal{D}_{m(P,n)}(\omega)),$$

and for $x \in M \cap U_{2^{-n+1}}(\partial S)$, $M \in \mathcal{D}_n(\omega)$, we have

$$|\chi_S T_n(\chi_{\mathbb{C}\setminus S}\mu)(x)| \lesssim \theta_\mu(3M).$$

Thus the left hand side of (10.1) is bounded above by

$$C \sum_{n\geq J(P)} \sum_{\substack{S\in\mathcal{D}_{m(P,n)}(\omega):\\ S\subset P}} \sum_{\substack{M\in\mathcal{D}_n(\omega):\\ M\subset S}} \theta_\mu(3M) \int_{U_{2^{-n+1}}(\partial\mathcal{D}_{m(P,n)}(\omega))\cap M} |K_R\mu|\, d\mu$$

$$\lesssim \sum_{Q\in\mathrm{Top}_\omega: Q\subset P} \theta_\mu(3Q) \sum_{M\in\mathrm{Tree}_w(Q)} \int_{U_{2\ell(M)}(\partial\mathcal{D}_{m(P,M)}(\omega))\cap M} |K_R\mu|\, d\mu$$

$$\lesssim \sum_{Q\in\mathrm{Top}_\omega: Q\subset P} \theta_\mu(3Q) \sum_{n>J(Q)} \int_{U_{2^{-n+1}}(\partial\mathcal{D}_{m(P,n)}(\omega))\cap Q} |K_R\mu|\, d\mu,$$

and so the lemma follows from Cauchy-Schwartz inequality. □

Given $\omega \in \Omega$, and $Q, R \in \mathrm{Top}_\omega$ with $Q \subsetneq R$, we denote

$$\mu_{\omega,Q,R} := \left(\sum_{n>J(Q)} \mu\big(Q \cap U_{2^{-n+1}}(\partial\mathcal{D}_{m(R_Q,n)}(\omega))\big)^{1/2}\right)^2. \tag{10.2}$$

Recall that R_Q is the square from $\mathrm{Stop}_\omega(R)$ which contains Q.

Lemma 10.2. *We have*

$$\sum_{R\in\mathrm{Top}_\omega} |B(\omega,R)| \lesssim \left(\sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)^2\mu(3R)\right)^{1/2} \times \sum_{k\geq 1}\left(\sum_{R\in\mathrm{Top}_\omega} \sum_{Q\in\mathrm{Stop}^k_\omega(R)} \theta_\mu(3Q)^2 \mu_{w,Q,R}\right)^{1/2}.$$

Proof: From the preceding lemma we get

$$\sum_{R\in\mathrm{Top}_\omega} |B(\omega,R)| \lesssim \sum_{R\in\mathrm{Top}_\omega} \sum_{Q\in\mathrm{Top}_\omega: Q\subsetneq R} \|K_R\mu\|_{L^2(\mu|Q)} \theta_\mu(3Q)\mu^{1/2}_{\omega,Q,R}$$

$$= \sum_{k\geq 1}\sum_{R\in\mathrm{Top}_\omega} \theta_\mu(3R)\mu(3R)^{1/2} \sum_{Q\in\mathrm{Stop}^k_\omega(R)} \frac{\|K_R\mu\|_{L^2(\mu|Q)}}{\theta_\mu(3R)\mu(3R)^{1/2}}\theta_\mu(3Q)\mu^{1/2}_{\omega,Q,R}.$$

By Cauchy-Schwartz and since, by Lemma 7.1, we have

$$\sum_{Q\in \mathrm{Stop}_\omega^k(R)} \|K_R\mu\|^2_{L^2(\mu|Q)} = \|K_R\mu\|^2_{L^2(\mu)} \lesssim \theta_\mu(3R)^2\mu(3R),$$

for each $k\geq 1$ and every $R\in \mathrm{Top}_\omega$ we get

$$\sum_{Q\in \mathrm{Stop}_\omega^k(R)} \frac{\|K_R\mu\|_{L^2(\mu|Q)}}{\theta_\mu(3R)\mu(3R)^{1/2}}\theta_\mu(3Q)\mu_{\omega,Q,R}^{1/2}$$

$$\leq \left(\sum_{Q\in \mathrm{Stop}_\omega^k(R)} \frac{\|K_R\mu\|^2_{L^2(\mu|Q)}}{\theta_\mu(3R)^2\mu(3R)}\right)^{1/2}\left(\sum_{Q\in \mathrm{Stop}_\omega^k(R)}\theta_\mu(3Q)^2\mu_{\omega,Q,R}\right)^{1/2}$$

$$\lesssim \left(\sum_{Q\in \mathrm{Stop}_\omega^k(R)}\theta_\mu(3Q)^2\mu_{\omega,Q,R}\right)^{1/2}.$$

By Cauchy-Schwartz again, we obtain

$$\sum_{R\in \mathrm{Top}_\omega} |B(\omega,R)|$$

$$\lesssim \sum_{k\geq 1}\sum_{R\in \mathrm{Top}_\omega}\theta_\mu(3R)\mu(3R)^{1/2}\left(\sum_{Q\in \mathrm{Stop}_\omega^k(R)}\theta_\mu(3Q)^2\mu_{\omega,Q,R}\right)^{1/2}$$

$$\leq \sum_{k\geq 1}\left(\sum_{R\in \mathrm{Top}_\omega}\theta_\mu(3R)^2\mu(3R)\right)^{1/2}\left(\sum_{R\in \mathrm{Top}_\omega}\sum_{Q\in \mathrm{Stop}_\omega^k(R)}\theta_\mu(3Q)^2\mu_{\omega,Q,R}\right)^{1/2},$$

and the lemma follows. □

The next step consists of averaging the estimates obtained in the preceding lemma for $\sum_{R\in \mathrm{Top}_\omega}|B(\omega,R)|$ with respect the probability measure p defined by the normalized Lebesgue measure on $\Omega=[0,2^{N+1})^2$.

Lemma 10.3. *We have*

$$\int_{\omega\in\Omega}\sum_{R\in \mathrm{Top}_\omega}|B(\omega,R)|\,dp(\omega)\lesssim \sum_{\widetilde{R}\in \mathrm{Top}}\theta_\mu(\widetilde{R})^2\mu(\widetilde{R}).$$

In this lemma and its proof, the symbol $\widetilde{\ }$ is written above the squares that belong to the family Top (which, in particular implies that they do not belong to $\mathcal{D}(\omega)$, in general).

Proof: By the preceding lemma we have

$$\int_\Omega \sum_{R\in \mathrm{Top}_\omega} |B(\omega,R)|\,dp(\omega) \lesssim \left(\sum_{\widetilde{R}\in \mathrm{Top}} \theta_\mu(\widetilde{R})^2\mu(\widetilde{R})\right)^{1/2}$$

(10.3)

$$\times \sum_{k\geq 1}\int_\Omega \left(\sum_{R\in \mathrm{Top}_\omega}\sum_{Q\in \mathrm{Stop}^k_\omega(R)} \theta_\mu(3Q)^2\mu_{w,Q,R}\right)^{1/2} dp(\omega).$$

We will show that the integral on the right side is bounded above by some constant times $2^{-k/2}\left(\sum_{\widetilde{R}\in \mathrm{Top}} \theta_\mu(\widetilde{R})^2\mu(\widetilde{R})\right)^{1/2}$.

For $k\geq 1$, and $Q\in \mathrm{Top}_\omega$ let $R^k_Q\in \mathrm{Top}_\omega$ be the square such that $Q\in \mathrm{Stop}^k_\omega(R)$ (in case that it exists). By Hölder's inequality, for each $k\geq 1$ the last integral in the preceding estimate is bounded above by

(10.4)

$$\left(\int_\Omega \sum_{R\in \mathrm{Top}_\omega}\sum_{Q\in \mathrm{Stop}^k_\omega(R)} \theta_\mu(3Q)^2\mu_{w,Q,R}\,dp(\omega)\right)^{1/2}$$

$$=\left(\int_\Omega \sum_{Q\in \mathrm{Top}_\omega}\theta_\mu(3Q)^2\left[\sum_{n>J(Q)}\mu(Q\cap U_{2^{-n+1}}(\partial\mathcal{D}_{m(R^{k-1}_Q,n)}(\omega)))^{1/2}\right]^2 dp(\omega)\right)^{1/2}$$

$$=: I_k^{1/2}.$$

For the first identity, look at the definition of $\mu_{\omega,Q,R}$ in (10.2) and observe that if $Q\subset R_Q$, where $R_Q\in \mathrm{Stop}_\omega(R)$ and $Q\in \mathrm{Stop}^k_\omega(R)$, then $Q\in \mathrm{Stop}^{k-1}_\omega(R_Q)$.

Notice that for every $\omega\in\Omega$

$$\sum_{Q\in \mathrm{Top}_\omega}\cdots \leq \sum_{\widetilde{Q}\in \mathrm{Top}}\ \sum_{\substack{Q\in \mathrm{Top}_\omega:\\ Q\cap\widetilde{Q}\neq\varnothing,\\ \ell(Q)=\ell(\widetilde{Q})}}\cdots,$$

and for each $\widetilde{Q}\in \mathrm{Top}$ there are finitely many squares $Q\in \mathrm{Top}_\omega$ which intersect Q with the same size as Q, and moreover in this case $\theta_\mu(3Q)\approx$

$\theta_\mu(\widetilde{Q})$. Thus we deduce

(10.5)

$$I_k \lesssim \sum_{\widetilde{Q}\in \mathrm{Top}} \theta_\mu(\widetilde{Q})^2$$

$$\times \sum_{\substack{Q\in \mathrm{Top}_\omega:\\ Q\cap\widetilde{Q}\neq\varnothing,\\ \ell(Q)=\ell(\widetilde{Q})}} \int_\Omega \left[\sum_{n>J(\widetilde{Q})} \mu\big(3\widetilde{Q}\cap U_{2^{-n+1}}(\partial\mathcal{D}_{m(R_Q^{k-1},n)}(\omega))\big)^{1/2}\right]^2 dp(\omega).$$

To deal with the term inside the integral we apply Cauchy-Schwartz:

(10.6)

$$\left[\sum_{n>J(\widetilde{Q})} \mu\big(3\widetilde{Q}\cap U_{2^{-n+1}}(\partial\mathcal{D}_{m(R_Q^{k-1},n)}(\omega))\big)^{1/2}\right]^2$$

$$\leq \sum_{n>J(\widetilde{Q})} 2^{(n-J(\widetilde{Q}))/10}\mu\big(3\widetilde{Q}\cap U_{2^{-n+1}}(\partial\mathcal{D}_{m(R_Q^{k-1},n)}(\omega)\big)\sum_{n>J(\widetilde{Q})} 2^{(-n+J(\widetilde{Q}))/10}$$

$$= C\sum_{n>J(\widetilde{Q})} 2^{(n-J(\widetilde{Q}))/10}\mu\big(3\widetilde{Q}\cap U_{2^{-n+1}}(\partial\mathcal{D}_{m(R_Q^{k-1},n)}(\omega)\big).$$

From the fact that $Q\in \mathrm{Stop}_\omega^{k-1}(R_Q^{k-1})$, we infer that $\ell(R_Q^{k-1})\geq 2^{k-1}\ell(Q)$. In other words, $J(R_Q^{k-1})\leq J(Q)-k+1$, and so

$$m(R_Q^{k-1},n)\leq \left[\frac{J(Q)+n-k+1}{2}\right] = \left[\frac{J(\widetilde{Q})+n-k+1}{2}\right].$$

This implies that

$$\partial\mathcal{D}_{m(R_Q^{k-1},n)}(\omega)\subset \partial\mathcal{D}_{\left[\frac{J(\widetilde{Q})+n-k+1}{2}\right]}(\omega).$$

Then, by (10.5) and (10.6) we get

(10.7)

$$I_k \lesssim \sum_{\widetilde{Q}\in \mathrm{Top}} \theta_\mu(\widetilde{Q})^2$$

$$\times \sum_{n>J(\widetilde{Q})} 2^{(n-J(\widetilde{Q}))/10} \int_\Omega \mu\Big(3\widetilde{Q}\cap U_{2^{-n+1}}(\partial\mathcal{D}_{\left[\frac{J(\widetilde{Q})+n-k+1}{2}\right]}(\omega)\Big)\, dp(\omega).$$

By Fubini it is easy to check that for every square $Q_0 \subset \mathbb{C}$ and all $\delta > 0$ and $m \in \mathbb{Z}$ we have

$$\int_\Omega \mu(Q_0 \cap U_\delta(\partial\mathcal{D}_m(\omega)))\, dp(\omega) \leq \frac{C\delta}{2^{-m}}\mu(Q_0).$$

Therefore, we deduce

$$\int_\Omega \mu\Big(3\widetilde{Q} \cap U_{2^{-n+1}}(\partial\mathcal{D}_{\left[\frac{J(\widetilde{Q})+n-k+1}{2}\right]}(\omega)\Big)\, dp(\omega)$$

$$\lesssim \frac{2^{-n}}{2^{[-J(\widetilde{Q})-n+k]/2}}\mu(3\widetilde{Q}) = 2^{[-n+J(\widetilde{Q})]/2}2^{-k/2}\mu(3\widetilde{Q}).$$

Now we plug this estimate into (10.7) and we take into account that $\widetilde{Q}$ is doubling:

$$I_k \lesssim \sum_{\widetilde{Q}\in \mathrm{Top}} \theta_\mu(\widetilde{Q})^2\mu(3\widetilde{Q}) \sum_{n\geq J(\widetilde{Q})} 2^{2(-n+J(\widetilde{Q}))/5}2^{-k/2}$$

$$\lesssim 2^{-k/2} \sum_{\widetilde{Q}\in \mathrm{Top}} \theta_\mu(\widetilde{Q})^2\mu(\widetilde{Q}).$$

By this estimate and (10.3), we are done. □

11. Proof of the Main Lemma 4.1

This is a straightforward consequence of Lemmas 7.1, 9.2 and 10.3. Indeed, remember that in Section 6 we showed that

$$\|T\mu\|^2_{L^2(\mu)} \lesssim \mu(\mathbb{C}) + \Bigg\|\sum_{R\in \mathrm{Top}_\omega} K_R\mu\Bigg\|^2_{L^2(\mu)}$$

$$= \mu(\mathbb{C}) + \sum_{R\in \mathrm{Top}_\omega} \|K_R\mu\|^2_{L^2(\mu)} + \sum_{Q,R\in \mathrm{Top}_\omega: Q\neq R} \langle K_Q\mu, K_R\mu\rangle.$$

By Lemma 7.1,

$$\sum_{R\in \mathrm{Top}_\omega} \|K_R\mu\|^2_{L^2(\mu)} \lesssim \sum_{\widetilde{R}\in \mathrm{Top}} \theta_\mu(\widetilde{R})^2\mu(\widetilde{R}),$$

and by Lemmas 9.2 and 10.3,

$$\int_\Omega \Bigg|\sum_{Q,R\in \mathrm{Top}_\omega: Q\neq R} \langle K_Q\mu, K_R\mu\rangle\Bigg|\, dp(\omega) \lesssim \sum_{\widetilde{R}\in \mathrm{Top}} \theta_\mu(\widetilde{R})^2\mu(\widetilde{R}).$$

Thus,

$$\|T\mu\|_{L^2(\mu)}^2 \lesssim \mu(\mathbb{C}) + \sum_{\widetilde{R}\in \mathrm{Top}} \theta_\mu(\widetilde{R})^2\mu(\widetilde{R}) \lesssim \mu(\mathbb{C}) + c^2(\mu). \qquad \square$$

12. A final remark

With some very minor changes in the arguments used for Theorem 1.1, one can prove the following (slightly) more general result.

Theorem 12.1. *Let μ be a Radon measure without atoms on $\mathbb{C}$. Suppose that the Cauchy transform $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$. Let $k(x,y)$ be an antisymmetric 1-dimensional Calderón-Zygmund kernel and let T be the associated Calderón-Zygmund operator (as in (1.2)). Suppose that, for any AD regular curve Γ, $T_{\mathcal{H}^1_\Gamma}$ is bounded on $L^2(\mathcal{H}^1_\Gamma)$, and its norm is bounded above by some constant depending only the AD regularity constant of Γ and the constants C, η appearing in (1.1). Then T_μ is also bounded on $L^2(\mu)$.*

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