RINGS WHOSE CLASS OF PROJECTIVE MODULES IS SOCLE FINE

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Abstract

A class $\mathcal{C}$ of modules over a unitary ring is said to be socle fine
if whenever $M, N \in \mathcal{C}$ with $\mathrm{Soc}(M) \cong \mathrm{Soc}(N)$ then $M \cong N$. In
this work we characterize certain types of rings by requiring a
suitable class of its modules to be socle fine. Then we study socle
fine classes of quasi-injective, quasi-projective and quasicontinu-
ous modules which we apply to find socle fine classes in special
types of noetherian rings. We also initiate the study of those rings
whose class of projective modules is socle fine.

1. Introduction

The notion of socle fine class of modules has been previously used in
the algebraic literature without an explicit formulation. Thus in [**5**, The-
orem 9.3.7, p. 166] it is proved that an algebra A is quasi-Frobenius if and
only if each principal A-module has a simple socle and, for any two non-
isomorphic principal A-modules P_1 and P_2, we have $\mathrm{Soc}(P_1) \not\cong \mathrm{Soc}(P_2)$.
Clearly, this last assertion is equivalent to the fact that the class of prin-
cipal A-modules is socle fine. Also, the result given in [**5**, Corollary 9.4.3,
p. 171] could be stated by proclaiming that any class of indecomposable
modules of the same finite length, over a uniserial algebra, is socle fine.
Other results in this vein appear in [**15**] where some socle fine classes are
found in the context of CEP-rings. More recently, Page and Zhou in [**23**,
Theorem 24, p. 2920] find a series of equivalent conditions characterizing
the socle fine character of certain natural classes. An explicit formulation
of the notion was given by A. Idelhadj and E. A. Kaidi in [**11**] where they

give a socle fine characterization of semiartinian rings. Continuing this philosophy, artinian and noetherian rings are characterized in [16] by socle fine classes. The interesting paper [12] contains also characterizations for V-rings and pseudo-Frobenius rings. Complementary literature on socle and radical fine classes can be found in [13], [17] and [10].

Throughout this work the word ring will mean a unitary (associative) ring and modules are understood as unitary (left) modules. If R is a ring and X a class or R-modules, we shall say that X is *socle fine* (respectively *radical fine*) whenever for any $M, N \in X$ we have $\mathrm{Soc}(M) \cong \mathrm{Soc}(N)$ (respectively $M/\mathrm{Rad}(M) \cong N/\mathrm{Rad}(N)$) if and only if $M \cong N$. We shall denote the injective hull of a module M by $E(M)$.

2. QI, QP and quasi-continuous modules

Our main references for quasi-injective (QI), quasi-projective (QP) and quasi-continuous modules are [22] and [1]. By introducing QI or QP modules into the scene, we obtain another characterization for semisimple rings:

Theorem 2.1. *For any ring R, the following assertions are equivalent:*

1) *R is semisimple.*

2) *The class of all QP modules is socle fine.*

3) *The class of all QI modules is socle fine.*

4) *The class $\mathcal{C}$ of all the finite direct sums of QP modules is socle fine.*

Proof: If R is semisimple the class of all R-modules is socle fine hence 1) implies 2), 3) and 4). Suppose 2). As R and $\mathrm{Soc}(R)$ are QP and $\mathrm{Soc}(R) = \mathrm{Soc}(\mathrm{Soc}(R))$ we have $R \cong \mathrm{Soc}(R)$ implying that R is semisimple. If 3) holds, since $\mathrm{Soc}(E(R)) = \mathrm{Soc}(R) = \mathrm{Soc}(\mathrm{Soc}(R))$ and both of $E(R)$ and $\mathrm{Soc}(R)$ are QI, we have $E(R) \cong \mathrm{Soc}(R)$ is semisimple and so R is also. Finally suppose 4). We shall prove that R is semisimple by showing that the class of the QP R-modules is closed under finite direct sums. Let $X_1, \ldots, X_n$ be QP R-modules, and $X = \oplus_i X_i$ be an element in $\mathcal{C}$, let $Y := \mathrm{Soc}(X)$ which is QP by its semisimple character. Since $X, Y \in \mathcal{C}$ and $\mathrm{Soc}(X) = \mathrm{Soc}(Y)$, we conclude $X \cong Y$ hence X is semisimple and therefore QP. Now the fact that the class of QP modules is closed under finite direct sum implies that R is semisimple (see [7] and [20]).

Let $\mathcal{C}$ be a socle fine class, M an element in $\mathcal{C}$ and $[M]$ the isomorphism class of M. Then the class $\mathcal{C} \cup [M]$ is also socle fine. Moreover if $\{M_i\}_{i \in I}$ is a collection of elements in $\mathcal{C}$, the class of R-modules $\mathcal{C} \cup (\cup_{i \in I}[M_i])$ is

socle fine. It is obvious that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two R-module socle fine classes such that $\forall\, X \in \mathcal{C}_1,\ \forall\, Y \in \mathcal{C}_2,\ \mathrm{Soc}(X) \not\cong \mathrm{Soc}(Y)$ then $\mathcal{C}_1 \cup \mathcal{C}_2$ is a socle fine class. $\square$

Theorem 2.2. *Let M be a quasi-continuous R-module and $\mathcal{C}$ the class of its direct summands. Then $\mathcal{C}$ is socle fine if and only if $\mathrm{Soc}(M)$ is essential in M.*

Proof: Let us suppose that $\mathcal{C}$ is socle fine. As $E(M) = E(\mathrm{Soc}(M)) \oplus W$ with $\mathrm{Soc}(W) = 0$ and M is quasi-continuous, by [**22**, Theorem 2.8, (4), p. 20] we have that $M = (M \cap E(\mathrm{Soc}(M))) \oplus (M \cap W)$, where $\mathrm{Soc}(M)$ is essential in $M \cap E(\mathrm{Soc}(M))$ and $\mathrm{Soc}(M) \cap W = 0$. Since $M \cap W$ is a direct summand of M with zero socle and $\mathcal{C}$ is socle fine, $M \cap W = 0$ and $\mathrm{Soc}(M)$ is essential in M. Let us suppose now that $\mathrm{Soc}(M)$ is essential in M. Let N be a direct summand of M. Then $\mathrm{Soc}(N)$ is essential in N hence $E(N) = E(\mathrm{Soc}(N))$. Therefore if $N_1, N_2 \in \mathcal{C}$, with $\mathrm{Soc}(N_1) \cong \mathrm{Soc}(N_2)$ we have that $E(N_1) \cong E(N_2)$ and by [**22**, Theorem 2.31, p. 34], we have $N_1 \cong N_2$. $\square$

Corollary 2.1. *Let R be a ring. Then R is a V-ring if and only if the class C of its indecomposable QI modules with essential socle is socle fine. The ring R is a noetherian V-ring if and only if the class D of its QI modules with essential socle is socle fine.*

Proof: Let R be a V-ring and M be an indecomposable QI R-module. Then $E(M)$ is also indecomposable (see [**22**, Theorem 2.8, p. 20]). Let S be a simple (hence injective) submodule of M. Then $E(M) = S$ and so $M = S$. By the previous corollary, the class C is socle fine. Reciprocally, if C is socle fine take a simple R-module S. Then $E(S)$ is indecomposable and therefore $S, E(S) \in C$. Since $\mathrm{Soc}(S) = \mathrm{Soc}(E(S))$ we have $S \cong E(S)$ and so S is injective. Thus R is a V-ring.

Suppose now that R is a noetherian V-ring. It is easy to prove that the indecomposable QI modules are semisimple hence they form a socle fine class D. On the other hand, if D is socle fine, by the proved previous part of this corollary R is a V-ring. Next we prove that any semisimple module is injective: let $M = \oplus_i S_i$ with each S_i a simple submodule. Then $M \in D$ and as $\mathrm{Soc}(E(M)) = \mathrm{Soc}(M) = M$ we deduce that $E(M)$ has an essential socle; then $M, E(M) \in D$ and $\mathrm{Soc}(M) = \mathrm{Soc}(E(M))$ hence $M \cong E(M)$ and so any semisimple module is injective. This implies that the class in [**16**, Theorem 3] is socle fine and therefore R is noetherian. $\square$

Corollary 2.2. *Let D be a noetherian domain with Krull dimension 1 and $\mathcal{C}$ any class of pairwise relatively injective torsion D-modules. Then $\mathcal{C}$ is socle fine.*

Proof: By Theorem 2.2 it suffices to prove that any element $M \in \mathcal{C}$ has essential socle. Since M is quasi-injective, each submodule of M is essential in a summand of M (see [**22**, Proposition 2.1, p. 18]). Then $M = T \oplus W$ where $\mathrm{Soc}(M)$ is essential in T (and $\mathrm{Soc}(W) = 0$). As $E(W)$ is a direct sum of indecomposable injective D-modules, applying the corollary of [**26**, Theorem 2.32, p. 53] each injective indecomposable module is of the form $E(D/P)$ with P a prime ideal of D. As $\mathrm{Soc}(W) = \mathrm{Soc}(E(W)) = 0$. Then each indecomposable component $E(D/P)$ of $E(W)$ has zero socle hence P is not maximal, and so, since the Krull dimesion of D is one, $P = 0$. Consequently $E(W) = \oplus_{i \in I} Q_i$, with $Q_i = Q(D)$ the field of fractions of D. Then $T(E(W)) = 0$ implying $T(W) = 0$. Furthermore $M = T(M) = T(T) \oplus T(W) = T(T) \subset T \subset M$, hence $M = T$, $W = 0$ and $\mathrm{Soc}(M)$ is essential in M. $\square$

Corollary 2.3. *If D is a Dedekind domain, the class of the indecomposable D-modules of the same (finite) length is socle fine.*

Proof: This corollary is a consequence of the structure theory of quasi-injective modules over Dedekind domains. We recall that, in this context, any QI module is either injective or a torsion D-module M such that for each nonzero prime ideal P, the P-primary component M_P of M is a direct sum of isomorphic modules each one of them being isomorphic to D/P^n ($n > 0$) or to $E(D/P)$ (see [**9**]). The indecomposable D-modules of finite length are torsion modules (see [**21**, Theorem 1, p. 49]), by the structure theory for finitely generated modules over a Dedekind domain, they have the form D/P^n with P a prime nonzero ideal and n agreeing with the length of D/P^n. As a consequence, these D-modules are QI. Moreover the D-module $(D/P^n) \oplus (D/Q^n)$ with $P \neq Q$, is also QI hence D/P^n and D/Q^n are relatively injective [**1**, Proposition 2.2, p. 15]. Then applying Corollary 2.2, the class of the indecomposable D-modules of the same finite length is socle fine and the corollary is proved. $\square$

Let D be a Dedekind domain and T a nonzero torsion D-module. Then T has a direct summand isomorphic either to $E(D/P)$ or to D/P^{n_P} for some maximal ideal P of D and $n_P \in \mathbb{N} - \{0\}$ (see [**18**]). By using this, it can be proved that M is a QP (quasi-projective) D-module if and only if M is either a projective module or a torsion D-module such that for each maximal ideal P, its P-primary component is a direct sum of modules all isomorphic to R/P^{n_p} for some $n_p \in \mathbb{N} - \{0\}$. It is easy to

prove (see [**1**, Exercise 18, p. 24]) that if D is a Dedekind domain and M a finitely generated torsion D-module then M is QI, if and only if M is QP, if and only if $M \cong (D/P_1^{n_1})^{m_1} \oplus (D/P_2^{n_2})^{m_2} \oplus \cdots \oplus (D/P_k^{n_k})^{m_k}$, where $P_1, P_2, \ldots, P_k$ are different maximal ideals of D, and $m_i, n_i \in \mathbb{N} - \{0\}$ for each $i = 1, 2, \ldots, k$.

Proposition 2.1. *Let D be a Dedekind domain. If $\mathcal{C}$ is a class of QP D-modules with nonzero socle and all of them with isomorphic radical, then $\mathcal{C}$ is socle fine. In particular any class of QI finitely generated torsion modules with isomorphic radicals is socle fine.*

Proof: We consider M and N two elements of the class $\mathcal{C}$ with isomorphic socles. By the previous paragraph

$$M \cong \bigoplus_{i \in I} (D/P_i^{m_i})^{(A_i)},$$

$$N \cong \bigoplus_{j \in J} (D/Q_j^{n_j})^{(B_j)}$$

where P_i and Q_j are maximal, and A_i, B_j are nonempty sets. By the fact that the socles are isomorphic we have the existence of a bijection $\sigma : I \to J$ such that $|A_i| = |B_{\sigma(i)}|$, $P_i = Q_{\sigma(i)}$ for all $i \in I$. On the other hand, after a suitable reordering, we can write $N \cong \bigoplus_{i \in I} (D/P_i^{n_i})^{(A_i)}$ therefore

$$\mathrm{Rad}(M) \cong \bigoplus_{i \in I} (D/P_i^{m_i-1})^{(A_i)},$$

$$\mathrm{Rad}(N) \cong \bigoplus_{i \in I} (D/P_i^{n_i-1})^{(A_i)}$$

and as their radicals are isomorphic their primary components are isomorphic also. Thus $(D/P_i^{m_i-1})^{(A_i)} \cong (D/P_i^{n_i-1})^{(A_i)}$. Futhermore their annihilators agree, that is to say, $P_i^{m_i-1} = P_i^{n_i-1}$ impliying $m_i = n_i$ for all i. $\square$

3. Rings whose class of projective modules is socle fine

It has been mentioned in the introduction, that the rings whose class of injective modules is socle fine are precisely the semiartinian rings. It is therefore natural to pose the question on the rings with socle fine class of projective modules. One first approach to the problem is given by the next theorem.

Theorem 3.1. *Let R be a ring and $\mathcal{F}$ the class of the free R-modules. Then the following assertions are equivalent:*

1) *$\mathcal{F}$ is socle fine.*

2) *R is an IBN ring with $\mathrm{Soc}(R) \neq 0$ and some homogeneous component in $\mathrm{Soc}(R)$ has finite length.*

Proof: Suppose first that $\mathcal{F}$ is socle fine. If $\mathrm{Soc}(R) = 0$ or no component has a finite length one checks immediately that $\mathrm{Soc}(R) \cong \mathrm{Soc}(R^{\mathbb{N}})$ which take us to the contradiction $R \cong R^{\mathbb{N}}$. Next we prove that R is IBN. If $R^n \cong R^m$ for $n, m \in \mathbb{N}$ then $\mathrm{Soc}(R^n) \cong \mathrm{Soc}(R^m)$. Since $\mathrm{Soc}(R) = S_{i_0}^{k_0} \oplus (\oplus_{i \neq i_0} S_i^{(I_i)})$ with $k_0 \in \mathbb{N}^*$, and S_{i_0}, S_i homogeneous components, this gives $S_{i_0}^{k_0 n} \oplus (\oplus_{i \neq i_0} S_i^{(J_i)}) \cong S_{i_0}^{k_0 m} \oplus (\oplus_{i \neq i_0} S_i^{(H_i)})$ whence $k_0 n = k_0 m$ and so $n = m$, as required. Let us prove 2) $\Rightarrow$ 1). Consider two free modules $R^{(I)}$, $R^{(J)}$ with isomorphic socles and $\mathrm{Soc}(R) = S_{i_0}^{k_0} \oplus (\oplus_{i \neq i_0} S_i^{(I_i)})$ with $k_0 \in \mathbb{N}^*$, and S_{i_0}, S_i the homogeneous components. Then:

$$\mathrm{Soc}(R^{(I)}) = S_{i_0}^{(X)} \oplus (\oplus_{i \neq i_0} S_i^{(I \times I_i)})$$

$$\mathrm{Soc}(R^{(J)}) = S_{i_0}^{(Y)} \oplus (\oplus_{i \neq i_0} S_i^{(J \times I_i)})$$

where $X = \{1, \ldots, k_0\} \times I$, $Y = \{1, \ldots, k_0\} \times J$. From the hypothesis that the socles are isomorphic one gets $|X| = |Y|$ hence $|I| = |J|$. $\qquad \square$

We recall that a ring A is left pseudo-Frobenius (a left PF ring) if A is an injective cogenerator. Left PF rings are characterized by the next theorem:

Theorem 3.2. *Let A be a ring. Then the following statements are equivalent:*

1) *A is a left PF ring.*

2) *The class of projective A-modules is socle fine and A is a left cogenerator.*

Proof: If A is a left PF ring then by definition and [**3**] and [**4**] the second assertion holds. Suppose now that A is a left cogenerator with its class of projective modules being socle fine. Then $\mathrm{Soc}(A) \neq 0$ since if $\mathrm{Soc}(A) = 0 = \mathrm{Soc}(0)$ then $A = 0$. Thus $\mathrm{Soc}(A) = \oplus_{i \in I} S_i = \oplus_{i \in I} \mathrm{Soc}(E(S_i))$. As A is a cogenerator then each $E(S_i)$ embeds in A and $E(S_i)$ is a direct factor of A. Hence $E(S_i)$ is projective. We have then that $\oplus_{i \in I} E(S_i)$ is a projective A-module. Then since $\mathrm{Soc}(A) = \mathrm{Soc}(\oplus_{i \in I} E(S_i))$ we get $A \cong \oplus_{i \in I} E(S_i)$. As A is of finite type $A \cong \oplus_{i=1}^{n} E(S_i)$ for some $n \in \mathbb{N}$, whence A is injective and as a consequence A is a left PF ring. $\qquad \square$

Theorem 3.3. *Let A be a ring, then the following properties are equivalent:*

  1) *A is a QF ring.*

  2) *The class $\mathcal{D}$ of projective or injective modules is socle fine.*

Proof: If A is QF, the class $\mathcal{D}$ is just the class of injective modules (and also the class of projective ones by [**19**, Theorem 13.6.1, p. 352]). Since A is artinian, this class is socle fine by [**16**, Theorem 2]. Next we prove that if $\mathcal{D}$ is socle fine, then A is a QF ring. Take P a projective module, then we have $\mathrm{Soc}(P) = \mathrm{Soc}(E(P))$ and $P, E(P) \in \mathcal{D}$. Consequently $P \cong E(P)$ and so P is injective. By [**19**, Theorem 13.6.1, p. 352], A is QF. $\square$

Proposition 3.1. *Let A be a cogenerator ring. Then the following assertions are equivalent:*

  1) *A is a left QF3 ring.*

  2) *The class of projective A-modules is socle fine.*

Proof: If A is a cogenerator left QF3 ring then A is a PF-ring by [**25**, p. 55]. Now by Theorem 3.2 the class of projective A-modules is socle fine. The other implication is trivial applying Theorem 3.2. $\square$

Proposition 3.2. *Let A be a left QF3 ring. Then the following assertions are equivalent:*

  1) *A is a QF-ring.*

  2) *The class of projective A-modules is socle fine.*

Proof: If A is a QF-ring then the class of projective A-modules agrees with the class of injective modules and as A is an artinian ring, this class is socle fine (see [**16**, Theorem 2]). Suppose that the class of projective A-modules is socle fine. Let P be a projective A-module and $E(P)$ its injective hull. By [**8**, Corollary II.6, p. 58], both P and $E(P)$ are projective. Since $\mathrm{Soc}(P) = \mathrm{Soc}(E(P))$ this implies $P \cong E(P)$ whence P is injective and A is a QF-ring. $\square$

4. Semiperfect rings

Let A be a ring and J its Jacobson radical. We recall that A is semiperfect if A/J is semisimple and any idempotent of A/J is of the form $e+J$ with e an idempotent of A. It is well known (see for instance [**2**, Proposition 27.10, p. 306]) that if A is semiperfect, each complete family of primitive idempotents contains a *basic family* $e_1, \ldots, e_m$ of A, and all the basic families of A have the same cardinality which is called the *capacity* of A. The A-modules $Ae_1, \ldots, Ae_m$ form a complete irredundant

family of representatives of projective indecomposable A-modules. The A-modules $Ae_1/Je_1, \ldots, Ae_m/Je_m$ form a complete irredundant family of representatives of simple A-modules. We shall use the notation $S_i = Ae_i/Je_i$ for all i. If P is a projective A-module, then P has an essentially unique decomposition of the type $P = (Ae_i)^{(I_1)} \oplus \cdots \oplus (Ae_m)^{(I_m)}$. A ring A is called *finitely embedded* if and only if it has an essential and finitely generated socle. Any artinian ring is finitely embedded but there are rings which are finitely embedded (and even local) but nonartinian (see [**24**]).

Theorem 4.1. *Let A be a finitely embedded semiperfect ring. The following assertions are equivalent:*

1) *The class of projective A-modules is socle fine.*

2) *A contains all its types of simple A-modules, and any projective indecomposable module has a homogeneous socle.*

Proof: Consider a finitely embedded semiperfect ring A with capacity m and a basic family $e_1, \ldots, e_m$ of A. Suppose that the class of projective A-modules is socle fine. If $m = 1$ then assertion 2) follows from the fact that $\text{Soc}(A) \neq 0$. Next we take $m \geq 2$, and suppose (after a suitable reordering if necessary) that $\text{Soc}(A) \cong S_1^{\alpha_1} \oplus \cdots \oplus S_r^{\alpha_r}$ with $\alpha_i \neq 0$ for $i \in \{1, \ldots, r\}$, and $r < m$. Then there is a $t \leq r$ such that $S_1 \oplus \cdots \oplus S_r$ can be embedded in $Ae_{i_1} \oplus \cdots \oplus Ae_{i_t}$. Consequently $\text{Soc}(Ae_{i_1} \oplus \cdots \oplus Ae_{i_t}) = S_1^{\beta_1} \oplus \cdots \oplus S_r^{\beta_r}$ with $\beta_i \neq 0$ for all $i \in \{1, \ldots, r\}$. The projective A-modules $A^{(\mathbb{N})}$ and $(Ae_{i_1} \oplus \cdots \oplus Ae_{i_t})^{(\mathbb{N})}$ have isomorphic socles hence we have an isomorphism

$$Ae_1^{(\mathbb{N})} \oplus \cdots \oplus Ae_m^{(\mathbb{N})} \cong Ae_{i_1}^{(\mathbb{N})} \oplus \cdots \oplus Ae_{i_t}^{(\mathbb{N})}.$$

Since $t < m$ there is a $j \in \{1, \ldots, m\}$ such that $j \notin \{i_1, \ldots, i_t\}$, but on the other hand, the theorem on the uniqueness of the decomposition of projective A-modules implies that $Ae_j \cong Ae_{i_k}$ for some $i_k \in \{i_1, \ldots, i_t\}$ which is contradictory. Next we prove that $\text{Soc}(Ae_i)$ is homogeneous for each i. Suppose that $\text{Soc}(Ae_k) = S_1^{\alpha_1} \oplus \cdots \oplus S_q^{\alpha_q}$ with $q \geq 2$ and $\alpha_i \neq 0$ for all i. Then there exist $Ae_{i_1}, \ldots, Ae_{i_n}$ pairwise different and distinct from Ae_k such that $n \leq m - q$, and $S_{q+1} \oplus \cdots \oplus S_m$ can be embedded in $Ae_{i_1} \oplus \cdots \oplus Ae_{i_n}$. Consequently $\text{Soc}(Ae_k \oplus Ae_{i_1} \oplus \cdots \oplus Ae_{i_n}) = S_1^{\beta_1} \oplus \cdots \oplus S_m^{\beta_m}$ with each $\beta_i \neq 0$. As $(Ae_k \oplus Ae_{i_1} \oplus \cdots \oplus Ae_{i_n})^{(\mathbb{N})}$ is projective and its socle is isomorphic to the socle of the projective A-module $A^{(\mathbb{N})}$ we have an isomorphism

$$Ae_k^{(\mathbb{N})} \oplus Ae_{i_1}^{(\mathbb{N})} \oplus \cdots \oplus Ae_{i_n}^{(\mathbb{N})} \cong Ae_1^{(\mathbb{N})} \oplus \cdots \oplus Ae_m^{(\mathbb{N})}$$

which implies that $n + 1 = m$ by the previously mentioned uniqueness theorem. Since we had $n \leq m - q$ and $q \geq 2$, then $n + 1 \leq m - q + 1 \leq m - 1$, a contradiction. Suppose now that 2) holds, and denote by $Ae_{\sigma(i)}$ the unique direct summand of A containing to S_i. It is clear that $i \mapsto \sigma(i)$ is a permutation of $\{1, \ldots, m\}$ and $\mathrm{Soc}(Ae_{\sigma(i)}) \cong S_i^{n_i}$ with $n_i \neq 0$. Take P and Q two projective A-modules with $P = \oplus_{j=1}^m Ae_{\sigma(j)}^{(I_j)}$ and $Q = \oplus_{j=1}^m Ae_{\sigma(j)}^{(H_j)}$. If $\mathrm{Soc}(P) \cong \mathrm{Soc}(Q)$ we have that $\oplus_j (S_j^{n_j})^{(I_j)} \cong \oplus_j (S_j^{n_j})^{(H_j)}$ and according to the uniqueness of the homogeneous components of semisimple modules we have $(S_j^{n_j})^{(I_j)} \cong (S_j^{n_j})^{(H_j)}$ for all j. The uniqueness of the decomposition of a semisimple module as a direct sum of simple ones implies the coincidence of cardinals: $|I_j| = |H_j|$ whence $P \cong Q$. $\qquad \square$

Corollary 4.1. *Let A be ring, of some of the following types:*

1) *A finitely embedded local ring.*

2) *A primary ring (that is A is artinian and $A/\mathrm{Rad}(A)$ is simple).*

3) *A commutative artinian ring.*

Then the class of projective A-modules is socle fine.

Proof: Suppose that A is as in the first possibility. Since any local ring is semiperfect and all the simple A-modules are isomorphic, from $\mathrm{Soc}(A) \subset A$ we conclude that A contains its unique type of simple A-module. As any projective A-module is free and $\mathrm{Soc}(A)$ is homogeneous we have that any projective indecomposable module has a homogeneous socle. In the second case, take into account that the simple modules over a simple artinian ring are isomorphic. It is easy to prove that the simple A-modules of the form $Ae_i/\mathrm{Rad}(A)e_i$ are also simple as $A/\mathrm{Rad}(A)$-modules. Thus they are isomorphic as $A/\mathrm{Rad}(A)$-modules and also as A-modules. In this way there is only one isomorphism class of simple A-modules and the conditions in item 2 of Theorem 4.1 are satisfied. Finally, if A is commutative and artinian, it splits into a finite direct sum of local artinian rings. The class of projective modules of any of these summands is socle fine (as proved in the first item), and from this it is easy to derive that the class of projective A-modules is socle fine.

In the noncommutative case we do not have in general this property. Take for instance a field K and A the artinian ring of triangular matrices defined by:

$$A = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}.$$

Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have $A = Ae_1 \oplus Ae_2$ and for all $i \in \{1, 2\}$ the ring $e_i Ae_i \simeq K$ is local. Ae_1 and Ae_2 are two projective A-modules with $\operatorname{Soc} Ae_1 \simeq \operatorname{Soc} Ae_2$ but Ae_1 and Ae_2 are not isomorphic.

We recall that a ring A is a left (resp. right) CEP-ring if any cyclic left (resp. right) A-module embeds essentially in a projective A-module. The QF and the uniserial rings are CEP-rings. Jain and López-Permouth [15] have proved that if A is a semiperfect CEP-ring, then the class of projective A-modules projectifs is socle fine. Any semiperfect CEP-ring is artinian. There is a finitely embedded semiperfect ring whose class of projective A-modules is socle fine and which is not a CEP-ring: let $K[x_1, x_2, \dots]$ the ring of polynomials in an infinite countable number of commuting indeterminates $x_1, x_2, \dots$, with entries in K. Let I be the ideal of $K[x_1, x_2, \dots]$ generated by $\{x_i x_j x_k, x_n x_{n+1} - x_1 x_2, x_l x_m\}$ where i, j, k, l, m and n are positive integers with $|l - m| \neq 1$. Then the ring $A = \frac{K[x_1, x_2, \dots]}{I}$ is a finitely embedded semiperfect ring which is not a CEP-ring and its class of projective A-modules is socle fine. $\square$

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