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RIESZ TRANSFORMS ON GENERALIZED HEISENBERG GROUPS AND RIESZ TRANSFORMS ASSOCIATED TO THE CCR HEAT FLOW

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Abstract

Let $1 < q < \infty$. We prove that the Riesz transforms $R_k = X_k L^{-\frac{1}{2}}$ on a generalized Heisenberg group G satisfy $\left\|\left(\sum_{k=1}^{K} |R_k(f)|^2\right)^{\frac{1}{2}}\right\|_{L^q(G)} \leq C(q,J)\,\|f\|_{L^q(G)}$ where K, J are respectively the dimensions of the first and second layer of the Lie algebra of G. We prove similar inequalities on Schatten spaces $S^q(H)$, with dimension free constants, for Riesz transforms associated to commuting inner $*$-derivations D_k and a suitable substitute of the square function. An example is given by the derivations associated to n commuting pairs of operators (P_j, Q_j) on a Hilbert space H satisfying the canonical commutation relations $[\mathrm{P}_j, Q_j] = iI_H$.

General introduction

This paper is divided in two parts: we solve similar problems in two different settings, using similar methods inspired by the first part of [**P1**], which contains a proof of the following classical inequalities [**S**]: for $1 < q < \infty$ and $f \in \mathcal{D}(\mathbb{R}^n)$,

$$D_q \|f\|_{L^q(\mathbb{R}^n)} \leq \left\|\left(\sum_{k=1}^{n} |R_k(f)|^2\right)^{\frac{1}{2}}\right\|_{L^q(\mathbb{R}^n)} \leq C_q \|f\|_{L^q(\mathbb{R}^n)}$$

where

$$R_k(f) = \frac{\partial}{\partial x_k} L^{-\frac{1}{2}}(f), \quad L = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

and the constants do not depend on n.

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The first part deals with Riesz transforms acting on $L^q(G)$, where G is a generalized Heisenberg group, and owes a lot to [**CMZ**].

The second part deals with Riesz transforms acting on the Schatten space $\mathcal{S}_q(H)$, associated to commuting $*$-inner derivations on the algebra $K(H)$ of compact operators on a Hilbert space H; an example is given by the inner derivations defined by $(P_j, Q_j)_{j=1}^n$, where P_j, Q_j satisfy the canonical commutation relation $[Q_j, P_j] = iI_H$ and the other commutators are zero.

We already used Pisier's method in other settings, see e.g. [**LP2**], [**LP3**]. However the difficulties arise at different steps in different applications.

Since the two settings we consider are very different, we present more precise introductions in each part.

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1. Riesz transforms on generalized Heisenberg groups

1.1. Introduction.

For any stratified Lie group G, we denote by $X_1, \ldots, X_K$ a basis of the top layer of the Lie algebra $\mathcal{G}$ of G, by

$$L = -\sum_{j=1}^{K} X_j^2$$

the subelliptic Kohn Laplacian on G, by

$$R_k = X_k L^{-\frac{1}{2}}, \quad 1 \leq k \leq K,$$

the Riesz transforms. The boundedness of each R_k on $L^q(G)$, $1 < q < \infty$, is known: the classical proof uses the homogeneity of its kernel and the singular integral results of [**FS**, Chapter 6]; see also Lemma 2 below. Our interest in this paper is to look for dimension free inequalities involving $\left\| \left(\sum_{k=1}^{K} |R_k(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^q(G)}$.

We first consider the case where G is a Heisenberg group $\mathbb{H}_n$ and give a simpler proof of the main result of [**CMZ**]. We extend this result to the generalized Heisenberg groups $\mathbb{H}_{K,J}$ defined by Kaplan [**K**]; they are particular step two stratified Lie groups, and K, J denote respectively the dimensions of the first and second layer in the Lie algebra of $\mathbb{H}_{K,J}$. The Heisenberg group $\mathbb{H}_n$ is the same as $\mathbb{H}_{2n,1}$.

Theorem 1. *Let $1 < q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$.*

a) [**CMZ**] *There exist constants C_q such that, for every $n \in \mathbb{N}^*$ and every $f \in \mathcal{D}(\mathbb{H}_n)$,*

$$C_{q'}^{-1} \|f\|_{L^q(\mathbb{H}_n)} \leq \left\| \left(\sum_{k=1}^{2n} |R_k(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathbb{H}_n)} \leq C_q \|f\|_{L^q(\mathbb{H}_n)} .$$

b) *The same holds for generalized Heisenberg groups $\mathbb{H}_{K,J}$, with constants $C(q,J)$ which depend on q and J but not on K.*

It is a standard fact (see e.g. [**CMZ**]) that, in the above formula, for any Lie group G, the lefthand side inequality for q is an easy consequence of the righthand side one for q', $\frac{1}{q} + \frac{1}{q'} = 1$.

Theorem 1 relies on Christ's study [**C**] of Hilbert transforms along curves for homogeneous nilpotent Lie groups, and on the use of dilations δ_t in order to get an expression of the convolution operator $e^{-\frac{1}{2}t^2 L}$ involving the (heat) kernel p of $e^{-\frac{1}{2}L}$, namely (see Lemma 2 a))

$$e^{-\frac{1}{2}t^2 L}(f)(\gamma) = \int_G f(\gamma \delta_t g^{-1}) p(g)\, dg. \tag{1}$$

Theorem 1 also uses a formula (Lemma 2 b) below)

$$\sqrt{2\pi} X L^{-\frac{1}{2}}(f)(\gamma) = \int_G F(\gamma, g)(Xp)(g)\, dg \tag{2}$$

which holds on every stratified group G, X lying in the first layer of $\mathcal{G}$, F being a Hilbert transform of f. These ingredients are already in [**CMZ**]; they are reminiscent of the method of [**P1**].

In the proof of Theorem 1 a), our improvement upon [**CMZ**] is that we do not use the explicit formula of p and avoid computations. Denoting $p = p^{(n)}$ when $G = \mathbb{H}_n$, we use only the following properties:

(i) [**FS**] p is a positive function lying in $\mathcal{S}(G)$, and $\int_G p\, dg = 1$,

(ii) $p^{(n)}(x_1, \ldots, y_n, u)$ is radial with respect to $(x_1, \ldots, y_n)$, i.e. depends on (r, u),

(iii) $p^{(n)}(x_1, \ldots, y_n, u) = p^{(1)}(x_1, y_1, u) *_u \ldots *_u p^{(1)}(x_n, y_n, u)$,

where $*_u$ denotes convolution in $\mathbb{R}$ with respect to the variable u. Property (ii) is used through the observation that

$$X_j p^{(n)} = x_j \frac{1}{r} \frac{\partial p^{(n)}}{\partial r} + 2y_j \frac{\partial p^{(n)}}{\partial u}.$$

In the proof of Theorem 1 b) we use the analogue for $\mathbb{H}_{K,J}$.

We recall that the heat kernels (i.e. the kernels of e^{-tL}) on $\mathbb{H}_n$ and the non isotropic Heisenberg groups, or rather their Fourier transform with respect to u, were first computed in [**G**] and [**H**], then in several subsequent papers, by rather complicated methods; the heat kernels on general step two stratified groups were computed in [**CY**]. A more explicit formula for generalized Heisenberg groups $\mathbb{H}_{K,J}$ is given in [**R**], using the result for $\mathbb{H}_n$. Let us mention our computation of the heat kernels for isotropic or non isotropic Heisenberg groups and the free step two stratified groups $N_{n,2}$ [**LP1**], which is simpler than the previous ones and relies on the common starting point of [**CMZ**] and the present paper, namely formula (1).

1.2. Notation.

For a background on stratified groups (which are particular homogeneous groups) we refer to [**FS**, Chapter I]. We consider stratified Lie groups G equipped with their Haar measure, denoted by dg or $d\gamma$, which is Lebesgue measure on the underlying space $\mathbb{R}^d$. $\mathcal{D}(G)$ denotes the space of $\mathcal{C}^\infty$ compactly supported functions on G, $\mathcal{S}(G)$ denotes the Schwartz class. The convolution of two functions f, h lying in $\mathcal{S}(G)$ is defined by

$$f * h(\gamma) = \int_G f(\gamma g^{-1}) h(g)\, dg.$$

The Lie algebra of left invariant vector fields on G is denoted by $\mathcal{G}$. For $X \in \mathcal{G}$,

$$X(f * p) = f * Xp. \tag{3}$$

The first layer of $\mathcal{G}$ is the linear subspace which spans $\mathcal{G}$ as a Lie algebra. We denote by σ the automorphism of G, corresponding to the automorphism σ of $\mathcal{G}$ whose action on the first layer is

$$\sigma\colon X \longrightarrow -X.$$

G is equipped with a dilation δ_t, $t > 0$, corresponding to the automorphism of $\mathcal{G}$ whose action on the first layer is

$$\delta_t X = tX. \tag{4}$$

The induced action on functions $f\colon G \to \mathbb{R}$ is denoted by $(\delta_t f)(g) = f(\delta_t g)$, and [**FS**, I C]

$$X\delta_t f = t\delta_t(Xf). \tag{5}$$

A stratified Lie group G is said to be step two if the central layer of $\mathcal{G}$ is the second one; denoting by $X_1, \dots, X_K$ a basis of the first layer, and by $U_1, \dots, U_J$ a basis of the second layer, it means that all commutators $[X_j, X_k]$ belong to the linear span of the U_j's and the other commutators are zero. Every $g \in G$ is defined in a unique way by $g = \exp\left(\sum_{k=1}^{K} x_k X_k + \sum_{j=1}^{J} u_j U_j\right)$ and we denote $g = (x, u) = (x_1, \dots, x_K, u_1, \dots, u_J)$. In this setting, the Haar measure on G, i.e. the Lebesgue measure on $\mathbb{R}^{K+J}$, is also denoted by $dx\, du$. In particular, $\sigma g = \sigma(x, u) = (-x, u)$, hence $\sigma g^{-1} = (x, -u)$, and $\delta_t(x, u) = (tx, t^2 u)$.

We will use the following easy, but crucial, result which is standard when $J = 0$ and ψ is the gaussian density on $\mathbb{R}^K$.

Lemma 2. *Let ψ be a measurable function: $\mathbb{R}^{K+J} \to \mathbb{R}$ such that $\int_{\mathbb{R}^J} |\psi(x, u)|\, du$ only depends on $|x|$, $x = (x_k)_{k=1}^K \in \mathbb{R}^K$. Then*

a) *if $1 \le q < \infty$, $a_k \in \mathbb{C}$,*

$$\left\| \sum_{k=1}^{K} a_k x_k \right\|_{L^q(|\psi|\, dx\, du)} = \left(\sum_{k=1}^{K} |a_k|^2 \right)^{\frac{1}{2}} \|x_1\|_{L^q(|\psi|\, dx\, du)} ;$$

b) *if $1 < q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$, for every $h \in L^q(|\psi|\, dx\, du)$,*

$$\left(\sum_{k=1}^{K} \left| \int_{\mathbb{R}^{K+J}} x_k h \psi\, dx\, du \right|^2 \right)^{\frac{1}{2}} \le \|x_1\|_{L^{q'}(|\psi|\, dx\, du)} \|h\|_{L^q(|\psi|\, dx\, du)} .$$

Proof: For short, we write $L^q(|\psi|)$ instead of $L^q(|\psi|\, dx\, du)$. We use polar coordinates in $\mathbb{R}^K$, namely $a = (a_1, \dots, a_K) = |a|\, w$, $x = |x|\, v = rv$, with $w, v \in \Sigma_K, v = (v_1, \dots, v_K)$, here Σ_K denotes the unit sphere of $\mathbb{R}^K$ and $d\sigma_K$ is the uniform measure on it, with total mass the area of Σ_K.

a) Since $\int_{\mathbb{R}^J} |\psi|\, du$ only depends on r,

$$\begin{aligned}
\left\| \sum_{k=1}^{K} a_k x_k \right\|_{L^q(|\psi|)}^q &= |a|^q \int_0^\infty r^q \left(\int_{\mathbb{R}^J} |\psi|\, du \right) r^{K-1}\, dr \int_{\Sigma_K} |\langle w, v\rangle|^q\, d\sigma_K(v) \\
&= |a|^q \int_0^\infty r^q \left(\int_{\mathbb{R}^J} |\psi|\, du \right) r^{K-1}\, dr \int_{\Sigma_K} |v_1|^q\, d\sigma_K(v) \\
&= |a|^q \|x_1\|_{L^q(|\psi|)}^q .
\end{aligned}$$

b) This follows from a): indeed, by Hölder inequality,

$$\begin{aligned}\left(\sum_{k=1}^{K}\left|\int x_k h\psi\,dx\,du\right|^2\right)^{\frac{1}{2}} &= \sup_{|a|=1}\left|\int\left(\sum_{k=1}^{K}a_k x_k\right)h\psi\,dx\,du\right| \\ &\leq \|h\|_{L^q(|\psi|)}\sup_{|a|=1}\left\|\sum_{k=1}^{K}a_k x_k\right\|_{L^{q'}(|\psi|)} \\ &= \|h\|_{L^q(|\psi|)}\,\|x_1\|_{L^{q'}(|\psi|)}\,. \qquad \square\end{aligned}$$

Let G be a stratified Lie group; to $g \in G$ we associate the curve: $\mathbb{R} \to G$

$$\begin{aligned} g(t) &= \delta_t g \quad \text{for} \quad t \geq 0, \\ g(t) &= \delta_{|t|}\sigma g \quad \text{for} \quad t < 0. \end{aligned}$$

In particular, if G is step two, the curve associated to $g = (x, u)$ is

$$g(t) = (tx, t^2 u), \quad t \in \mathbb{R}.$$

The Hilbert transform of f along the curve $g(t)$ is

$$\begin{aligned} F(\gamma, g) &= \mathrm{pv}\int_{-\infty}^{\infty} f(\gamma g(t)^{-1})\frac{dt}{t} \\ &= \lim_{\varepsilon\to 0^+}\int_{\varepsilon<|t|<\varepsilon^{-1}} f(\gamma g(t)^{-1})\frac{dt}{t} \\ &= \lim_{\varepsilon\to 0^+} F(\gamma, g, \varepsilon). \end{aligned}$$

The function $F(\gamma, g)$ is well defined on $G \times G$ because $f \in \mathcal{D}(G)$. $F(\gamma, g, \varepsilon)$ is called the truncated Hilbert transform.

The map $h_t^g \colon f \to f(.g(t)^{-1})$, $t \in \mathbb{R}$, $L^\infty(G) \to L^\infty(G)$, is a $*$-homomorphism of the $*$-algebra $L^\infty(G)$, but $\{h_t^g\}_{t\in\mathbb{R}}$ is not a one parameter group. This makes an important difference with the settings of [**LP1**], [**LP2**], [**LP3**] and the second part of this paper.

The next result comes from [**CMZ**, Proof of Lemmas 1 and 5]. For the sake of completeness we give a more precise proof.

Lemma 3. *Let G be a stratified Lie group and $f \in \mathcal{D}(G)$. Then*

$$e^{-\frac{1}{2}t^2L}(f)(\gamma) = \int_G f(\gamma\delta_t g^{-1})p(g)\,dg, \quad t>0, \tag{1}$$

and, for X in the first layer of $\mathcal{G}$, $d\gamma$ a.s.,

$$\sqrt{2\pi}XL^{-\frac{1}{2}}(f)(\gamma) = \int_G F(\gamma,g)(Xp)(g)\,dg, \tag{2}$$

where $F(\gamma,g)$ is the Hilbert transform of f along the curve $g(t)$, $g\in G$.
For $1<q<\infty$, the Riesz transforms satisfy

$$\left\|\sqrt{2\pi}XL^{-\frac{1}{2}}(f)\right\|_{L^q(G)} \le h_q \left\|Xp\right\|_{L^1(G)} \left\|f\right\|_{L^q(G)}.$$

Proof: Formula (1) holds true for $t=1$ by definition of p. By (4) (see also [**LP1**]),

$$e^{-\frac{1}{2}t^2L} = \delta_{t^{-1}}e^{-\frac{1}{2}L}\delta_t,$$

hence, by (5),

$$e^{-\frac{1}{2}t^2L}(f)(\gamma) = \delta_{t^{-1}}[(f\circ\delta_t)*p(\gamma)] = \int_G f(\gamma\delta_t g^{-1})p(g)\,dg. \tag{1}$$

By (5), (1) and (3),

$$Xe^{-\frac{1}{2}t^2L}(f) = t^{-1}\delta_{t^{-1}}X[(f\circ\delta_t)*p)] = t^{-1}\delta_{t^{-1}}[(f\circ\delta_t)*(Xp)].$$

The automorphism σ maps L to L, so $p = p\circ\sigma$ and

$$Xp = X(p\circ\sigma) = -(Xp)\circ\sigma.$$

For $h\in\mathcal{D}(G)$, since $\sigma^2 g = g$ and σ is measure preserving,

$$\begin{aligned} h*(Xp)(\gamma) &= \int_G h(\gamma g^{-1})(Xp)(g)\,dg \\ &= -\int_G h(\gamma g^{-1})(Xp)(\sigma g)\,dg \\ &= -\int_G h(\gamma\sigma g^{-1})(Xp)(g)\,dg. \end{aligned}$$

In particular, for $f\in\mathcal{D}(G)$,

$$2Xe^{-\frac{1}{2}t^2L}(f)(\gamma) = t^{-1}\int_G [f(\gamma\delta_t g^{-1}) - f(\gamma\delta_t\sigma g^{-1})](Xp)(g)\,dg.$$

Since

$$\sqrt{\frac{\pi}{2}}XL^{-\frac{1}{2}}(f) = \int_0^\infty Xe^{-\frac{1}{2}t^2L}(f)\,dt$$

we get

$$\sqrt{2\pi} X L^{-\frac{1}{2}}(f)(\gamma) = \int_0^\infty \left(\int_G [f(\gamma \delta_t g^{-1}) - f(\gamma \delta_t \sigma g^{-1})](Xp)(g)\, dg \right) \frac{dt}{t}.$$

Since $p \in \mathcal{S}(G)$, $Xp \in L^1(G)$, and formula (2) now comes from the subsequent Lemma 4 b).

By (2) and Hölder inequality with $\frac{1}{q} + \frac{1}{q'} = 1$, $d\gamma$ a.s.,

$$\sqrt{2\pi} \left| X L^{-\frac{1}{2}}(f)(\gamma) \right| \leq \|F(\gamma, .)\|_{L^q(|Xp|\, dg)} \|Xp\|_{L^1(G)}^{\frac{1}{q'}}.$$

By the subsequent Lemma 4 a),

$$\|F(\gamma, g)\|_{L^q(d\gamma, L^q(|Xp|\, dg))} \leq h_q \|f\|_{L^q(G)} \|Xp\|_{L^1(G)}^{\frac{1}{q}},$$

which ends the proof. □

The next lemma comes from [**C**]; it is already used in this setting in [**CMZ**], see [**CMZ**, Lemma 5].

Lemma 4. *Let G be a stratified Lie group and $f \in \mathcal{D}(G)$. For $g \in G$ let $F(\gamma, g)$ be the Hilbert transform of f along the curve $g(t)$ and let $\psi \in L^1(G)$. Then*

a) *for $1 < q < \infty$, there exists a constant h_q such that*

$$\|F\|_{L^q(d\gamma \otimes |\psi|\, dg)} \leq h_q \|f\|_{L^q(d\gamma)} \|\psi\|_1^{\frac{1}{q}}.$$

b)

$$\int_G F(\gamma, g)\psi(g)\, dg = \int_0^\infty \left[\int_G (f(\gamma \delta_t g^{-1}) - f(\gamma \delta_t \sigma g^{-1}))\psi(g)\, dg \right] \frac{dt}{t}, \quad d\gamma \ a.s.$$

Proof: a) When g runs through G, the corresponding family of curves $g(t)$ satisfies the assumptions of [**C**, pp. 579 and 594]. By the main result of [**C**], if $1 < q < \infty$, there exists a constant h_q such that, for every $g \in G$,

$$\|F(., g)\|_{L^q(G)} \leq h_q \|f\|_{L^q(G)}.$$

This proves a) by integration with respect to g and Fubini theorem.

b) For every $g \in G$, the truncated Hilbert transform $F(\gamma, g, \varepsilon)$ along the curve $g(t)$ satisfies [**C**, Lemma 6.3]

$$\|F(., g) - F(., g, \varepsilon)\|_{L^2(G)} \longrightarrow_{\varepsilon \to 0^+} 0$$

and

$$\sup_{\varepsilon>0} \|F(.,g,\varepsilon)\|_{L^2(G)} \le h_2 \|f\|_{L^2(G)}.$$

Hence, since $\psi \in L^1(G)$, the dominated convergence theorem implies

$$\lim_{\varepsilon\to 0^+} \left\| \int_G (F(\gamma,g) - F(\gamma,g,\varepsilon))\psi(g)\,dg \right\|_{L^2(G)}$$

$$\le \lim_{\varepsilon\to 0^+} \int_G \|F(.,g) - F(.,g,\varepsilon)\|_{L^2(G)} |\psi(g)|\,dg = 0.$$

This implies b), because, by Fubini theorem, $d\gamma$ a.s.,

$$\int_G F(\gamma,g,\varepsilon)\psi(g)\,dg$$

$$= \int_{\varepsilon<t<\varepsilon^{-1}} \left[\int_G (f(\gamma\delta_t g^{-1}) - f(\gamma\delta_t\sigma g^{-1}))\psi(g)\,dg \right] \frac{dt}{t}. \quad \square$$

Proof of Theorem 1: We treat Heisenberg groups first because the proof in this case is simpler and the idea is more apparent.

a) When $G = \mathbb{H}_n$, we denote $p = p^{(n)}$. The group law on $\mathbb{H}_n$ is

$$\gamma g = (x_1,y_1,\dots,x_n,y_n,u)(x_1',y_1',\dots,x_n',y_n',u')$$

$$= \left(x_1+x_1', y_1+y_1', \dots, x_n+x_n', y_n+y_n', u+u'+2\sum_{j=1}^n (y_j x_j' - x_j y_j') \right).$$

By definition, for $g = (x_1,\dots,y_n,u)$,

$$(X_j p^{(n)})(g) = \frac{\partial p^{(n)}}{\partial x_j}(g) + 2y_j \frac{\partial p^{(n)}}{\partial u}(g)$$

$$(Y_j p^{(n)})(g) = \frac{\partial p^{(n)}}{\partial y_j}(g) - 2x_j \frac{\partial p^{(n)}}{\partial u}(g).$$

The Laplacian is given by

$$-L = \sum_{j=1}^n \left[\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4(x_j^2+y_j^2)\frac{\partial^2}{\partial u^2} + 4\left(x_j \frac{\partial^2}{\partial y_j \partial u} - y_j \frac{\partial^2}{\partial x_j \partial u} \right) \right]$$

hence commutes with rotations on $(x_1,\dots,y_n)$. It follows that $p^{(n)}$ is radial with respect to $(x_1,\dots,y_n)$, hence so are $\frac{\partial p^{(n)}}{\partial u}$ and the function

$$\frac{1}{r}\frac{\partial p^{(n)}}{\partial r} = \frac{1}{x_j}\frac{\partial p^{(n)}}{\partial x_j} = \frac{1}{y_j}\frac{\partial p^{(n)}}{\partial y_j},$$

where $r = \left(\sum_{j=1}^n x_j^2 + y_j^2\right)^{\frac{1}{2}}$. Rewriting

$$(X_j p^{(n)})(g) = x_j \frac{1}{r}\frac{\partial p^{(n)}}{\partial r}(g) + 2y_j \frac{\partial p^{(n)}}{\partial u}(g)$$

we have, by (2), $d\gamma$ a.s.,

$$\sqrt{2\pi} X_j L^{-\frac{1}{2}}(f)(\gamma) = \int_G x_j F(\gamma, g)\frac{1}{r}\frac{\partial p^{(n)}}{\partial r}(g)\,dg + 2\int_G y_j F(\gamma, g)\frac{\partial p^{(n)}}{\partial u}(g)\,dg$$

and a similar formula for $Y_j L^{-\frac{1}{2}}(f)$. Denoting for $1 \le j \le n$

$$R_j = X_j L^{-\frac{1}{2}}$$
$$R_{n+j} = Y_j L^{-\frac{1}{2}}$$

by triangular inequality in l^2_{2n} and Lemma 2 applied on $\mathbb{R}^{2n+1}$, $d\gamma$ a.s.,

$$(6)\quad \sqrt{2\pi}\left(\sum_{k=1}^{2n} |R_k(f)(\gamma)|^2\right)^{\frac{1}{2}} \le \|F(\gamma, .)\|_{L^q(\frac{1}{r}|\frac{\partial p^{(n)}}{\partial r}|\,dg)} \|x_1\|_{L^{q'}(\frac{1}{r}|\frac{\partial p^{(n)}}{\partial r}|\,dg)}$$
$$+ 2\,\|F(\gamma, .)\|_{L^q(|\frac{\partial p^{(n)}}{\partial u}|\,dg)} \|x_1\|_{L^{q'}(|\frac{\partial p^{(n)}}{\partial u}|\,dg)}\,.$$

Since L is the sum of $-(X_j^2 + Y_j^2)$, $1 \le j \le n$, which act on different set of variables except for the central u [**CMZ**, p. 371], (see also the proof of b) below)

$$p^{(n)}(x_1, \dots, y_n, u) = p^{(1)}(x_1, y_1, u) *_u \dots *_u p^{(1)}(x_n, y_n, u).$$

Since $p^{(n)} \ge 0$ for every n,

$$\frac{1}{r}\left|\frac{\partial p^{(n)}}{\partial r}\right| = \left|\frac{1}{x_1}\frac{\partial p^{(n)}}{\partial x_1}\right| \le \left|\frac{1}{x_1}\frac{\partial p^{(1)}}{\partial x_1}\right| *_u p^{(n-1)}.$$

Since $\int_{\mathbb{H}_{n-1}} p^{(n-1)}(g)\,dg = 1$, for $q \ge 0$,

$$\int_{\mathbb{H}_n} |x_1|^q \left|\frac{1}{r}\frac{\partial p^{(n)}}{\partial r}\right| dg \le \int_{\mathbb{R}^3} |x_1|^q \left|\frac{1}{x_1}\frac{\partial p^{(1)}}{\partial x_1}\right| dx_1\,dy_1\,du = A_q.$$

Since $p^{(1)}$ belongs to $\mathcal{S}(\mathbb{R}^3)$, A_q is obviously finite for $q \ge 1$, and also for $q = 0$. Indeed

$$\left\|\frac{1}{x_1}\frac{\partial p^{(1)}}{\partial x_1}\right\|_{L^1(\mathbb{R}^3)} = 2\pi \int_{\mathbb{R}^+ \times \mathbb{R}} \left|\frac{\partial p^{(1)}}{\partial r}\right| dr\,du$$

and $\left|\frac{\partial p^{(1)}}{\partial r}\right| = \left(\left|\frac{\partial p^{(1)}}{\partial x_1}\right|^2 + \left|\frac{\partial p^{(1)}}{\partial y_1}\right|^2\right)^{\frac{1}{2}}$. Similarly

$$\int_{\mathbb{H}_n} |x_1|^q \left|\frac{\partial p^{(n)}}{\partial u}\right| dg \leq \int_{\mathbb{R}^3} |x_1|^q \left|\frac{\partial p^{(1)}}{\partial u}\right| dx_1 dy_1\, du = B_q$$

and B_q is finite for $q \geq 0$.

Integrating (6) with respect to γ, we get by Lemma 4 a)

$$\sqrt{2\pi} \left\| \left(\sum_{k=1}^{2n} |R_k(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathbb{H}_n)} \leq A_{q'}^{\frac{1}{q'}} \, \|F\|_{L^q(d\gamma \otimes |\frac{1}{r} \frac{\partial p^{(n)}}{\partial r}|\, dg)}$$

$$+ 2B_{q'}^{\frac{1}{q'}} \, \|F\|_{L^q(d\gamma \otimes |\frac{\partial p^{(n)}}{\partial u}|\, dg)} \leq h_q \, \|f\|_{L^q(\mathbb{H}_n)} \, (A_{q'}^{\frac{1}{q'}} A_0^{\frac{1}{q}} + 2B_{q'}^{\frac{1}{q'}} B_0^{\frac{1}{q}})$$

which gives the righthand side inequality in the statement of the theorem. Denoting

$$\mathcal{R}\colon f \longrightarrow (R_k(f))_{k=1}^K,$$

this means that $\mathcal{R}$ is bounded: $L^q(G) \to L^q(G, l^2_{2n})$, and so is

$$\mathcal{R}^*\colon (h_k)_{k=1}^{2n} \longrightarrow \sum_{k=1}^{2n} R_k^*(h_k), \quad L^{q'}(G, l^2_{2n}) \longrightarrow L^{q'}(G).$$

Since $f = \sum_{k=1}^{2n} R_k^* R_k(f) = \mathcal{R}^*\mathcal{R}(f)$ and $\|\mathcal{R}^*\|_{q \to q} = \|\mathcal{R}\|_{q' \to q'}$, we get in the standard way the lefthand side inequality:

$$\|f\|_{L^q(G)} \leq \|\mathcal{R}^*\|_{q \to q} \left\| \left(\sum_{k=1}^{2n} |R_k(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^q(G)}.$$

b) When $G = \mathbb{H}_{K,J}$, we recall [**K**], [**CDKR**] that its Lie algebra $\mathcal{G}$ is equipped with a scalar product, so that the first layer $\mathcal{V}$ is a real vector space with dimension K, orthogonal to the second layer $\mathcal{Z}$ with dimension J. We denote by $|X|$ the norm on $\mathcal{G}$ defined by the scalar product.

The specific property is that, for every X with norm 1 in $\mathcal{V}$, the operator ad_X defined on $\mathcal{V}$ by

$$\mathrm{ad}_X \colon Y \longrightarrow [X, Y]$$

is an isometry from $E = (\ker \mathrm{ad}_X)^\perp \subset \mathcal{V}$ onto $\mathcal{Z}$. In particular, E has dimension J and $|\mathrm{ad}_X(Y)| = |P_E(Y)|$ where P_E is the orthogonal projection onto E.

Let us now precise the structure of $\mathcal{V}+\mathcal{Z}$ both as a Hilbert space and a Lie algebra. For $U\in\mathcal{Z}$ let $\Phi_U\colon \mathcal{V}\to\mathcal{V}$ be the linear operator defined by

$$\langle \Phi_U(X), X'\rangle = \langle U, [X,X']\rangle .$$

In particular, $(\Phi_U)^* = -\Phi_U$, so that, if $F\subset\mathcal{V}$ is an invariant subspace for Φ_U, $F^\perp$ is also invariant. The operator $\Phi\colon \mathcal{Z}\to B(\mathcal{V})$ defined by $U\to\Phi_U$ satisfies

$$(*) \qquad \Phi_U\Phi_{U'} + \Phi_{U'}\Phi_U = -2\,\langle U, U'\rangle \,\mathrm{Id}_{\mathcal{V}} .$$

In particular, if $U_1,\dots,U_J$ is an orthonormal basis of $\mathcal{Z}$, $(\Phi_{Uj})_{j=1}^J$ are unitary skew-adjoint anticommuting operators on $\mathcal{V}$. In more sophisticated words (see [**BTV**, 3.1.2]), denoting by q the quadratic form defined on $\mathcal{Z}$ by $q(U) = -\langle U, U\rangle$, Φ induces a representation of the real Clifford algebra $\mathcal{C}_J$ built on $(\mathcal{Z}, q)$ into $B(\mathcal{V})$. In $\mathcal{C}_J$ one has $U^2 = -\,\mathrm{Id}$ for every $U\in\mathcal{Z}$ with norm 1, so that $\mathcal{C}_J$ is the linear span of Id and $U_{i_1}U_{i_2}\dots U_{i_n}$, $1\le i_1 < i_2 < \cdots < i_n \le J$, and has dimension 2^J. Φ is a direct sum of irreducible representations of $\mathcal{C}_J$. By the classification of the Clifford algebras $\mathcal{C}_J$, their irreducible representations are as follows (see e.g. [**Hu**, Chapter 11] or [**ABS**, Part I]):

If $J\ne 3 \pmod 4$, there is only one (up to equivalence) irreducible representation of $\mathcal{C}_J$. Hence $\mathcal{V}$ must be splitted as a hilbertian sum $\mathcal{V} = \mathcal{V}_1\oplus\cdots\oplus\mathcal{V}_N$ where the spaces $\mathcal{V}_l$ have the same dimension K_J and are invariant under all Φ_U, $U\in\mathcal{Z}$. In particular, the $\mathcal{V}_l$'s are commuting copies with isomorphic Lie structure. We may choose orthonormal basis $(X_{(l-1)K_J+i})_{i=1}^{K_J}$ of the $\mathcal{V}_l$'s such that $[X_{(l-1)K_J+i}, X_{(l-1)K_J+h}]$ does not depend on l, for $1\le i,\, h\le K_J$.

If $J\equiv 3 \pmod 4$, there are two non equivalent irreducible representations of $\mathcal{C}_J$, with the same dimension K_J. $\mathcal{V}$ must be splitted as a hilbertian sum $\mathcal{V}_1\oplus\cdots\oplus\mathcal{V}_k\oplus\mathcal{V}'_{k+1}\oplus\cdots\oplus\mathcal{V}'_N$ where the $\mathcal{V}_l$'s (resp. the $\mathcal{V}'_l$'s) are commuting copies with isomorphic Lie structure, and the $\mathcal{V}_l$'s commute with the the $\mathcal{V}'_l$'s. We choose orthonormal basis of the $\mathcal{V}_l$'s (resp. the $\mathcal{V}'_l$'s) as above.

Conversely, by [**K**], if $\mathcal{Z}$, $\mathcal{V}$ are real finite dimensional Hilbert spaces, from a linear isometry $\Phi\colon \mathcal{Z}\to B(\mathcal{V})$ satisfying $(*)$, one can build a structure of Lie algebra on $\mathcal{V}+\mathcal{Z}$. If $J\equiv 3 \pmod 4$, there are several non isomorphic groups $\mathbb{H}_{K,J}$ for every admissible $K > K_J$ and only one $\mathbb{H}_{K_J,J}$ [**BTV**, 3.1.2].

The value of K_J is computed as follows. By [**K**], the couple (K,J) satisfies

$$J<\rho(K)=8\alpha+2^\beta \text{ where } K=m2^{4\alpha+\beta}, \quad m \text{ odd}, \quad 0\le\beta\le 3, \quad \alpha,\beta\in\mathbb{N}.$$

In particular, K must be even. Let ρ_J be the smallest integer such that

$$\rho_J = 8\alpha_J + 2^{\beta_J} > J, \quad 0 \leq \beta_J \leq 3, \quad \alpha_J, \beta_J \in \mathbb{N}$$

and let

$$K_J = 2^{4\alpha_J + \beta_J}, \quad \text{hence} \quad \rho(K_J) = \rho_J.$$

This ensures the existence of $\mathbb{H}_{K_J,J}$ and K_J is the minimal possible dimension of $\mathcal{V}$, i.e. the dimension of each $\mathcal{V}_l$. In particular $J < K_J$.

For a given $\mathbb{H}_{K,J}$ we choose an orthonormal basis $X_1, \ldots, X_K$ of $\mathcal{V}$ as above. We denote by $g = (x, u)$ an element of $\mathbb{H}_{K,J}$, where $x = (x_k)_{k=1}^{NK_J}$ are the coordinates corresponding to $X_1, \ldots, X_K$, $u = (u_j)_{j=1}^{J}$ are the coordinates corresponding to $U_1, \ldots, U_J$, and by $y_l \in \mathbb{R}^{K_J}$ the coordinates of x corresponding to the basis of $\mathcal{V}_l$ or of $\mathcal{V}_l'$, $1 \leq l \leq N$, so that $x = (y_1, \ldots, y_N)$.

By definition [**BTV**, 3.1.5], $U_j = \frac{\partial}{\partial u_j}$, $1 \leq j \leq J$, and

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^{J}\left\langle \left[\sum_{k=1}^{K} x_k X_k, X_i\right], U_j \right\rangle \frac{\partial}{\partial u_j}, \quad 1 \leq i \leq K.$$

In particular, if X_i belongs to $\mathcal{V}_l$ or to $\mathcal{V}_l'$, it only depends on y_l.

The heat kernel $p = p^{(K,J)}$ has properties analogous to those of $p^{(n)}$. First, $p^{(K,J)}$ is radial both with respect to x and u and depends only on (K, J). This follows from the following formula established by J. Randall [**R**, Proof of Lemma 1.3.3]:

$$(\mathcal{F}_u p^{(K,J)})(x, \lambda) = (\mathcal{F}_v p^{(\frac{K}{2})})(x, |\lambda|), \quad \lambda \in \mathbb{R}^J,$$

where $\mathcal{F}_u$ denotes the Fourier transform on $\mathbb{R}^J$ with respect to u, $\mathcal{F}_v$ the Fourier transform on $\mathbb{R}$ with respect to $v \in \mathbb{R}$, and $p^{(\frac{K}{2})}$ denotes the heat kernel on the Heisenberg goup $\mathbb{H}_{\frac{K}{2}}$. In particular, there is a unique $p^{(K_J,J)}$ for any given J.

We claim that

$$p^{(K,J)}(x, u) = p^{(K_J,J)}(y_1, u) *_u \ldots *_u p^{(K_J,J)}(y_N, u)$$

where convolution is on $\mathbb{R}^J$, with respect to u. This follows from Randall's formula and the fact that

$$(\mathcal{F}_v p^{(\frac{K}{2})})(x, |\lambda|) = \Pi_{l=1}^{N}(\mathcal{F}_v p^{(\frac{K_J}{2})})(y_l, |\lambda|).$$

It can also be proved directly. Indeed, let us denote by q the convolution product on the right hand side. Since $p^{(K_J,J)}$ is a positive function with norm 1 in $L^1(\mathbb{R}^{K_J+J})$, q is a positive function with norm 1 in $L^1(\mathbb{R}^{K+J})$. As well known (see e.g. [**FS**]), the heat kernel $p_t^{(K,J)}$ is the only positive function with norm 1 in $L^1(\mathbb{R}^{K+J})$ satisfying $\frac{\partial f}{\partial t} = -\frac{1}{2}Lf$; moreover,

$p_t^{(K,J)} = t^{-(\frac{K}{2}+J)} p^{(K,J)} \circ \delta_{t^{-\frac{1}{2}}}$. Hence $p^{(K,J)}$ is the only positive function with norm 1 in $L^1(\mathbb{R}^{K+J})$ satisfying

$$(K \operatorname{Id} - L)p + \sum_{k=1}^{K} x_k X_k p = -2 \sum_{j=1}^{J} U_j(u_j p)$$

(see e.g. [**LP1**, Lemmas 2, 3]). So we only have to verify that q satisfies this equation. The left term can be splitted as $\sum_{l=1}^{N} D_l(p)$ where

$$D_l = K_J \operatorname{Id} + \sum_{i=1}^{K_J} -X_{i+(l-1)K_J}^2 + x_{i+(l-1)K_J} X_{i+(l-1)K_J}.$$

By our choice of coordinates, D_l only acts on the l^{th} factor of q. For the right term, we note that

$$\begin{aligned} &u_j q(y_1, \ldots, y_N, u) \\ &= \sum_{l=1}^{N} p^{(K_J,J)}(y_1, u) *_u \ldots *_u (u_j p^{(K_J,J)})(y_l, u) *_u p^{(K_J,J)}(y_{l+1}, u) *_u \cdots, \end{aligned}$$

which is easily verified by Fourier transform with respect to u. So it remains to verify that, for $1 \le l \le N$,

$$(D_l p^{(K_J,J)})(y_l, u) = -2 \sum_{j=1}^{J} U_j(u_j p^{(K_J,J)})(y_l, u).$$

These are the heat equations corresponding to each Lie algebra $\mathcal{V}_l + \mathcal{Z}$ or $\mathcal{V}_l' + \mathcal{Z}$. Hence they are all satisfied by $p^{(K_J,J}$ as we saw above (this is obvious when there are only $\mathcal{V}_l$'s).

For $g = (x, u) \in \mathbb{H}_{K,J}$ let $r = |x|$, $\rho = |u|$. As in a), we may rewrite for $1 \le i \le K$,

$$X_i p^{(K,J)} = \frac{x_i}{r} \frac{\partial p^{(K,J)}}{\partial r} + \frac{1}{2} \left\langle \left[\sum_{k=1}^{K} x_k X_k, X_i \right], \sum_{j=1}^{J} u_j U_j \right\rangle \frac{1}{\rho} \frac{\partial p^{(K,J)}}{\partial \rho},$$

and

$$\begin{aligned} (7) \quad \sqrt{2\pi} \left(\sum_{i=1}^{K} |R_i(f)(\gamma)|^2 \right)^{\frac{1}{2}} &\le \left(\sum_{i=1}^{K} \left| \int_G x_i F(\gamma, g) \frac{1}{r} \frac{\partial p^{(K,J)}}{\partial r}(g)\, dg \right|^2 \right)^{\frac{1}{2}} \\ &+ \frac{1}{2} \left(\sum_{i=1}^{K} \left| \int_G F(\gamma, g) \left\langle \left[\sum_{k=1}^{K} x_k X_k, X_i \right], \sum_{j=1}^{J} u_j U_j \right\rangle \frac{1}{\rho} \frac{\partial p^{(K,J)}}{\partial \rho}(g)\, dg \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The $L^q(d\gamma)$ norm of the first term in the righthand side of (7) is estimated as in a), replacing $p^{(1)}$ by $p^{(K_J,J)}$; it is less than

$$h_q \left\|x_1\right\|_{L^{q'}(\mathbb{R}^{K_J+J},|\frac{1}{x_1}\frac{\partial p^{(K_J,J)}}{\partial x_1}|\,dx\,du)} \left\|\frac{1}{x_1}\frac{\partial p^{(K_J,J)}}{\partial x_1}\right\|_{L^1(\mathbb{R}^{K_j+J})}^{\frac{1}{q}} \left\|f\right\|_{L^q(\mathbb{H}_{K,J})}$$
$$= C_1(q,J)\left\|f\right\|_{L^q(\mathbb{H}_{K,J})}$$

and the constant is finite as in a) because $p^{(K_J,J)} \in \mathcal{S}(\mathbb{H}_{K_J,J})$.

The second term is rewritten as

$$\sup_{|a|=1}\int_G F(\gamma,g)\left\langle\left[\sum_{k=1}^K x_k X_k, \sum_{i=1}^K a_i X_i\right], \sum_{j=1}^J u_j U_j\right\rangle \frac{1}{\rho}\frac{\partial p^{(K,J)}}{\partial \rho}(g)\,dg$$
$$\leq A_{q'}\left\|F(\gamma,.)\right\|_{L^q(\frac{1}{\rho}|\frac{\partial p^{(K,J)}}{\partial\rho}(g)|\,dg)}$$

where

$$A_{q'} = \sup_{|a|=1}\left\|\sum_{j=1}^J u_j\left\langle\left[\sum_{k=1}^K x_k X_k, \sum_{i=1}^K a_i X_i\right], U_j\right\rangle\right\|_{L^{q'}(\frac{1}{\rho}|\frac{\partial p^{(K,J)}}{\partial\rho}(g)|\,dg)}.$$

Hence, by Lemma 4 a), the $L^q(d\gamma)$ norm of the second term in the righthand side of (7) is estimated by

$$A_{q'}h_q\left\|\frac{1}{u_1}\frac{\partial p^{(K,J)}}{\partial u_1}\right\|_{L^1(\mathbb{R}^{K+J})}^{\frac{1}{q}}\left\|f\right\|_{L^q(\mathbb{H}_{K,J})}$$
$$\leq A_{q'}h_q\left\|\frac{1}{u_1}\frac{\partial p^{(K_J,J)}}{\partial u_1}\right\|_{L^1(\mathbb{R}^{K_J+J})}^{\frac{1}{q}}\left\|f\right\|_{L^q(\mathbb{H}_{K,J})}.$$

We now estimate $A_{q'}$. By Lemma 2 a) applied to $u_1,\ldots,u_J$ and fixed x,

$$\left\|\sum_{j=1}^J u_j\left\langle\left[\sum_{k=1}^K x_k X_k, \sum_{i=1}^K a_i X_i\right], U_j\right\rangle\right\|_{L^{q'}(\frac{1}{\rho}|\frac{\partial p^{(K,J)}}{\partial\rho}|\,du)}$$
$$= B_a(x)\left\|u_1\right\|_{L^{q'}(\frac{1}{\rho}|\frac{\partial p^{(K,J)}}{\partial\rho}|\,du)}$$

where, since $\left|\sum_{i=1}^K a_i X_i\right| = 1$,

$$B_a(x) = \left|\left[\sum_{k=1}^K x_k X_k, \sum_{i=1}^K a_i X_i\right]\right| = \left|\mathrm{ad}_{\sum_{i=1}^K a_i X_i}\left(\sum_{k=1}^K x_k X_k\right)\right|$$
$$= \left|P_{E_a}\left(\sum_{k=1}^K x_k X_k\right)\right|$$

with $E_a = (\ker \mathrm{ad}_{\sum_{i=1}^K a_i X_i})^{\perp}$. By rotation on the x variables, we may suppose that E_a is the span of $X_1, \ldots, X_J$. Since $\frac{\partial p^{(K,J)}}{\partial \rho}$ is radial with respect to x

$$A_{q'}^{q'} = \sup_{|a|=1} \int_{\mathbb{R}^{K+J}} |B_a(x)|^{q'} |u_1|^{q'} \frac{1}{\rho} \left|\frac{\partial p^{(K,J)}}{\partial \rho}(x,u)\right| dx\, du$$
$$= \int_{\mathbb{R}^{K+J}} \left(\sum_{k=1}^J |x_k|^2\right)^{\frac{q'}{2}} |u_1|^{q'-1} \left|\frac{\partial p^{(K,J)}}{\partial u_1}(x,u)\right| dx\, du$$
$$\leq \int_{\mathbb{R}^{K_J+J}} \left(\sum_{k=1}^J |x_k|^2\right)^{\frac{q'}{2}} |u_1|^{q'-1} \left|\frac{\partial p^{(K_J,J)}}{\partial u_1}(x,u)\right| dx\, du$$

where the inequality is verified as in a), replacing $p^{(1)}$ by $p^{(K_J,J)}$, since $J < K_J$. Finally

$$C_2(q,J) = h_q \left\|\left(\sum_{k=1}^J |x_k|^2\right)^{\frac{1}{2}} |u_1|\right\|_{L^{q'}(|\frac{1}{u_1}\frac{\partial p^{(K_J,J)}}{\partial u_1}|\, dx\, du)} \left\|\frac{1}{u_1}\frac{\partial p^{(K_J,J)}}{\partial u_1}\right\|_{L^1(\mathbb{R}^{K_J+J})}^{\frac{1}{q}}.$$

This constant is finite because $p^{(K_J,J)} \in \mathcal{S}(\mathbb{H}_{K_J,J})$ and we get

$$\left\|\left(\sum_{i=1}^K |R_i(f)(\gamma)|^2\right)^{\frac{1}{2}}\right\|_{L^q(\mathbb{H}_{K,J})} \leq \frac{1}{\sqrt{2\pi}}\left(C_1(q,J) + \frac{1}{2}C_2(q,J)\right)\|f\|_{L^q(\mathbb{H}_{K,J})}.$$

This implies as in a) the lefthand side inequality. $\square$

Remark. The above proof of a) does not seem to extend to the setting of non isotropic Heisenberg groups, because it uses in a crucial way the radiality of p; so does the proof in [**CMZ**]. However, it is stated in [**BDJ**, p. 59], without explanation, that the proof in [**CMZ**] can be extended to the non isotropic case.

2. Riesz transforms associated to commuting inner $*$-derivations

2.1. Introduction and notation.

We now show how the classical result on Riesz transforms on $L^q(\mathbb{R}^n, dx)$ can be extended to Riesz transforms on Schatten spaces $\mathcal{S}^q(H)$, defined by inner (bounded or unbounded) commuting $*$-derivations acting on the C* algebra $K(H)$ of compact operators on the Hilbert space H.

We denote by γ_n the standard gaussian density on $\mathbb{R}^n$ and by Π the orthogonal projection onto the linear span of the coordinates y_j in $L^2(\gamma_n)$. As well known, Π extends as a bounded operator: $L^q(\gamma_n) \to L^q(\gamma_n)$, $1 \le q < \infty$, which we still denote by Π.

The Fourier transform on $\mathbb{R}$ is defined by $\widehat{f}(u) = \int_{\mathbb{R}} e^{-iyu} f(y)\, dy$.

We recall that $\mathcal{S}^q(H)$ is the space of compact operators X on H such that $\operatorname{tr}(X^*X)^{\frac{q}{2}} < \infty$, $1 \le q < \infty$. $B(H)$ is the dual space of $\mathcal{S}^1(H)$ which is itself the dual space of $K(H)$. $\mathcal{S}^2(H)$ is the Hilbert space of Hilbert Schmidt operators on H. From now on, we assume H is separable.

We denote by $|X|_s^2 = X^*X + XX^*$ the symmetrized modulus of $X \in K(H)$.

Let h be a $*$-automorphism of $K(H)$, hence h^{**} is a $*$-automorphism of $B(H)$. By [**Pe**, Theorem 8.9.2] there exists a unitary operator U on H such that $h(X) = UXU^*$, $X \in K(H)$. Let $(h_y)_{y\in\mathbb{R}}$ be a strongly continuous one parameter group of $*$-automorphisms of $K(H)$; then there exists a one parameter group of unitaries $(U_y)_{y\in\mathbb{R}}$ such that $h_y(X) = U_y X U_y^*$, $X \in K(H)$ (see e.g. [**V**, Theorem 11.1] for an actually stronger result, or [**Par**, pp. 86–87]); by Stone's theorem $U_y = e^{yA}$, where iA is self-adjoint on H. Hence the generator D of $(h_y)_{y\in\mathbb{R}}$ is an inner $*$-derivation, in general unbounded, defined by $D(X) = [A, X]$. In particular $(h_y)_{y\in\mathbb{R}}$ is a group of isometries of $\mathcal{S}^2(H)$ (and a group of isometries of every $\mathcal{S}^q(H)$, $1 \le q < \infty$). By [**RS**, Theorem VIII.9], $(h_y)_{y\in\mathbb{R}}$ is strongly continuous on $\mathcal{S}^2(H)$. Its generator is naturally induced by D, and iD is self-adjoint on $\mathcal{S}^2(H)$ by Stone's theorem.

Let us now recall some facts about joint functional calculus for "commuting" $*$-inner derivations. We consider n strongly continuous one parameter groups of $*$-automorphisms of $K(H)$, with respective generators D_j, $1 \leq j \leq n$, which we denote respectively by $(e^{y_j D_j})_{y_j \in \mathbb{R}}$. We assume that

(i) their restrictions commute on $\mathcal{S}^2(H)$

(ii) the only $X \in \mathcal{S}^2(H)$ which is invariant under every $e^{y_j D_j}$ is $X = 0$.

In particular, by (i),

$$y \longrightarrow U(y) = e^{\sum_{j=1}^n y_j D_j}$$

is a strongly continuous map of $\mathbb{R}^n$ into the unitary operators on $\mathcal{S}^2(H)$, satisfying $U(y+z) = U(y)U(z)$, $y, z \in \mathbb{R}^n$ and $U(0) = I$. Then [**RS**, Theorem VIII.12], there is a projection valued measure E on $\mathbb{R}^n$ such that

$$\left\langle e^{\sum_{j=1}^n y_j D_j}(X), Y \right\rangle = \int_{\mathbb{R}^n} e^{i\langle y, \lambda\rangle}\, d\,\langle E_\lambda(X), Y\rangle, \qquad X, Y \in \mathcal{S}^2(H), \quad y \in \mathbb{R}^n. \tag{8}$$

By bounded functional calculus and Fubini theorem, for every $F \in \mathcal{S}(\mathbb{R}^n)$, as bounded operators on $\mathcal{S}^2(H)$ [**RS**, p. 272],

$$\widehat{F}(iD_1, \ldots, iD_n) = \int_{\mathbb{R}^n} e^{\sum_{k=1}^n y_k D_k} F(y)\, dy. \tag{9}$$

Moreover, let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function and let $X \in \mathcal{S}^2(H)$ be such that $\int_{\mathbb{R}^n} |h(\lambda)|^2 \, d\,\langle E_\lambda(X), X\rangle < \infty$; then the formula

$$\langle h(iD_1, \ldots, iD_n)(X), Y\rangle = \int_{\mathbb{R}^n} h(\lambda)\, d\,\langle E_\lambda(X), Y\rangle, \quad Y \in \mathcal{S}^2(H)$$

defines an operator $h(iD_1, \ldots, iD_n)$ which is densely defined and self-adjoint on $\mathcal{S}^2(H)$. This holds in particular for

$$L = -\sum_{k=1}^n D_k^2.$$

By assumption (ii), the projection $E_{\{0\}}$ is null. Indeed, let $Z \in \mathcal{S}^2(H)$ lying in its range. Since the measure $E - \delta_0 \otimes E_{\{0\}}$ is valued in the set of orthogonal projections P on $\mathcal{S}^2(H)$ such that $PE_{\{0\}} = 0$, hence $P(Z) = 0$, the scalar measure $1_{\{\mathbb{R}^n \setminus \{0\}\}}(\lambda)\, d\,\langle E_\lambda(Z), Y\rangle$ is zero for every $Y \in \mathcal{S}^2(H)$; then, by (8),

$$\left\langle Y, e^{\sum_{j=1}^n y_j D_j}(Z) \right\rangle = \left\langle e^{i\langle y, \lambda\rangle}, \delta_0 \right\rangle \left\langle Y, E_{\{0\}} Z \right\rangle = \langle Y, Z\rangle,$$

which, by (ii), implies $Z = 0$.

The Riesz transforms are defined by

$$R_j = D_j L^{-\frac{1}{2}}, \quad 1 \le j \le n,$$

so they are contractions on their domain in $\mathcal{S}^2(H)$. Since $E_{\{0\}} = 0$, R_j is actually defined on $\mathcal{S}^2(H)$ by

$$\langle R_j(X), Y\rangle = \int_{\mathbb{R}^n\setminus\{0\}} \frac{i\lambda_j}{|\lambda|}\, d\,\langle E_\lambda(X), Y\rangle\,. \tag{10}$$

The main result of this part is the following theorem:

Theorem 5. *Let H be a separable Hilbert space and let $(e^{y_j D_j})_{y_j\in\mathbb{R}}$, $1 \le j \le n$ be strongly continuous one parameter groups of $*$-automorphisms of $K(H)$ satisfying the above conditions (i), (ii). Let $1 < q < \infty$.*

a) *Then, for every $X \in K(H)$ and $t \in \mathbb{R}$,*

$$e^{-\frac{1}{2}t^2 L}(X) = \int_{\mathbb{R}^n} e^{t\sum_{j=1}^n y_j D_j}(X)\gamma_n(y)\, dy;$$

$e^{-\frac{1}{2}t^2 L}$ is a completely positive contraction: $K(H) \to K(H)$ and a contraction of every $\mathcal{S}^q(H)$.

b) *The operator $\mathcal{R} = \sum_{j=1}^n y_j R_j$ is a (completely) bounded operator: $\mathcal{S}^q(H) \to L^q(\gamma_n(y)\, dy, \mathcal{S}^q)$ which satisfies*

$$\sqrt{2\pi}\mathcal{R}(X) = (\Pi \otimes I_{\mathcal{S}^q})\left(pv \int_{-\infty}^{\infty} e^{t\sum_{j=1}^n y_j D_{A_j}}(X)\frac{dt}{t}\right).$$

c) *For $X \in \mathcal{S}^q$, $\|X\|_{\mathcal{S}^q}$ is respectively equivalent, with constants which depend only on q, to*

(i) $\left\|\left(\sum_{j=1}^n |R_j(X)|_s^2\right)^{\frac{1}{2}}\right\|_{\mathcal{S}^q}, \quad 2 \le q < \infty,$

and to

(ii) $\inf\left\{\left\|\left(\sum_{j=1}^n B_j B_j^*\right)^{\frac{1}{2}}\right\|_{\mathcal{S}^q} + \left\|\left(\sum_{j=1}^n C_j^* C_j\right)^{\frac{1}{2}}\right\|_{\mathcal{S}^q}\right\}, \quad 1<q<2,$
where the infimum is taken over all decompositions $R_j(X) = B_j + C_j$ in $\mathcal{S}^q(H)$.

Note that, on H, the operators $R_j(X)$, $1 \le j \le n$, do not commute in general, and $R_j(X)^* = R_j(X^*)$ does not commute in general with $R_j(X)$.

Proof of Theorem 5: The strategy is similar to Pisier's in the first part of [**P1**] and to the one we used in the first part of this paper. On one hand, things are much easier than for Heisenberg groups because we deal with a one parameter group; on the other hand, some difficulties arise from the setting of non commutative L^q's.

a) Formula (9) defines bounded operators on $K(H)$ and $\mathcal{S}^q(H)$, $1 \leq q < \infty$, with norm less than $\|F\|_{L^1(dy)}$, because $e^{\sum_{k=1}^n y_k D_k}$ is an isometry of $K(H)$ and $\mathcal{S}^q(H)$ for every y.

This formula gives a Stinespring factorization (see e.g. [**Pa**]) of $\widehat{F}(iD_1, \dots, iD_n)$ acting on $K(H)$ because

$$X \longrightarrow e^{\sum_{j=1}^n y_j D_j}(X)$$

is a $*$-homomorphism: $K(H) \to L^\infty(d\gamma_n, B(H)) \subset B(L^2(d\gamma_n, H))$.

Applying this to $F(y) = \gamma_n(y)$ and $tD_1, \dots, tD_n$ proves assertion a).

b) α) By (10) and Fubini theorem, for $X \in \mathcal{S}^2(H)$,

$$\begin{aligned}
\sqrt{\frac{\pi}{2}} \langle R_j(X), X\rangle &= \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \frac{i\lambda_j}{|\lambda|} \, d\langle E_\lambda(X), X\rangle \\
&= \int_{\mathbb{R}^n} i\lambda_j \left(\int_0^\infty e^{-\frac{1}{2}t^2|\lambda|^2} \, dt \right) d\langle E_\lambda(X), X\rangle \\
&= \int_0^\infty \left(\int_{\mathbb{R}^n} i\lambda_j e^{-\frac{1}{2}t^2|\lambda|^2} \, d\langle E_\lambda(X), X\rangle \right) dt \\
&= \int_0^\infty \left\langle D_j e^{-\frac{1}{2}t^2 L}(X), X \right\rangle dt.
\end{aligned}$$

β) Let

$$F(y) = y_j \gamma_n(y), \quad \text{hence} \quad \widehat{F}(u) = -iu_j e^{-\frac{1}{2}|u|^2}.$$

By (9) applied to F and $tD_1, \dots, tD_n$, by symmetry with respect to t, for $X \in \mathcal{S}^q(H)$,

$$\begin{aligned}
tD_j e^{-\frac{1}{2}t^2 L}(X) &= \int_{\mathbb{R}^n} e^{t\sum_{k=1}^n y_k D_k}(X) y_j \gamma_n(y) \, dy \\
&= \frac{1}{2} \int_{\mathbb{R}^n} (e^{t\sum_{k=1}^n y_k D_k}(X)) - e^{-t\sum_{k=1}^n y_k D_k}(X)) y_j \gamma_n(y) \, dy.
\end{aligned}$$

Hence, by α), for $X \in \mathcal{S}^2(H)$,

$$\begin{aligned}\sqrt{2\pi}R_j(X) &= \int_0^\infty \left[\int_{\mathbb{R}^n} (e^{t\sum_{k=1}^n y_k D_k}(X) \right.\\ &\qquad \left. - e^{-t\sum_{k=1}^n y_k D_k}(X))y_j\gamma_n(y)\,dy\right]\frac{dt}{t} \qquad (11)\\ &= \int_{-\infty}^\infty \left[\int_{\mathbb{R}^n} e^{t\sum_{k=1}^n y_k D_k}(X)y_j\gamma_n(y)\,dy\right]\frac{dt}{t}.\end{aligned}$$

γ) We claim that, for every $X \in \mathcal{S}^q(H)$, $y \in \mathbb{R}^n$, $1 < q < \infty$,

$$F(y,X) = \mathrm{pv}\int_{-\infty}^\infty e^{t\sum_{k=1}^n y_k D_k}(X)\frac{dt}{t}$$

is well defined as the norm limit in $\mathcal{S}^q(H)$ of

$$F(y,X,\varepsilon) = \int_{\varepsilon<|t|<\varepsilon^{-1}} e^{t\sum_{k=1}^n y_k D_k}(X)\frac{dt}{t}$$

and satisfies

$$\|F(y,X)\|_{\mathcal{S}^q} \le h_q \|X\|_{\mathcal{S}^q}. \qquad (12)$$

Indeed, $\mathcal{S}^q(H)$ is UMD for $1 < q < \infty$ [**BGM**, Theorem 6.1] and, for fixed y, $(e^{t\sum_{k=1}^n y_k D_k})_{t\in\mathbb{R}}$ is a strongly continuous one parameter group of isometries of $\mathcal{S}^q(H)$. Hence [**BGM**, Theorems 5.12 and 5.16] prove the claim.

δ) It follows that, for the $\mathcal{S}^2(H)$ norm, by Fubini theorem and (11),

$$\begin{aligned}\int_{\mathbb{R}^n} F(y,X)y_j\gamma_n(y)\,dy &= \lim_\varepsilon \int_{\mathbb{R}^n} F(y,X,\varepsilon)y_j\gamma_n(y)\,dy\\ &= \lim_\varepsilon \int_{\varepsilon<|t|<\varepsilon^{-1}}\left(\int_{\mathbb{R}^n} e^{t\sum_{k=1}^n y_k D_k}(X)y_j\gamma_n(y)\,dy\right)\frac{dt}{t}\\ &= \sqrt{2\pi}R_j(X).\end{aligned}$$

Hence, for $X \in \mathcal{S}^q(H) \cap \mathcal{S}^2(H)$,

$$\sqrt{2\pi}\mathcal{R}(X) = \sum_{j=1}^n y_j \int_{\mathbb{R}^n} F(y,X)y_j\gamma_n(y)\,dy = (\Pi \otimes I_{\mathcal{S}^q})(F(y,X)),$$

and, by (12),

$$\|F(y,X)\|_{L^q(\gamma_n\,dy,\mathcal{S}^q)} \le h_q \|X\|_{\mathcal{S}^q}.$$

ε) By [**P2**, Remark 8.4.6], $\Pi \otimes I_{\mathcal{S}^q}$ is bounded on $L^q(\gamma_n\, dy, \mathcal{S}^q)$, $1 < q < \infty$, and its norm depends only on q. By δ) this proves the boundedness of $\mathcal{R}$ on $\mathcal{S}^q(H)$ and ends the proof of b).

Actually, Π is completely bounded, hence so is $\Pi \otimes I_{\mathcal{S}^q}$, and the same argument as above, applied to $(e^{t\sum_{j=1}^n y_j D_{A_j}} \otimes I_{\mathcal{S}^q})_{t\in\mathbb{R}}$ shows that, for fixed y, $\operatorname{pv}\int_{-\infty}^{\infty} e^{t\sum_{j=1}^n y_j D_{A_j}} \frac{dt}{t}$ is completely bounded: $\mathcal{S}^q(H) \to \mathcal{S}^q(H)$.

c) By [**P2**, Theorem 8.4.1], the norms in the statement are equivalent, with constants which depend only on q, to

$$\left\| \sum_{j=1}^n y_j R_j(X) \right\|_{L^q(d\gamma_n, \mathcal{S}^q)} = \|\mathcal{R}(X)\|_{L^q(d\gamma_n, \mathcal{S}^q)}.$$

Hence, we have to show that $\|X\|_{\mathcal{S}^q}$ is equivalent to $\|\mathcal{R}(X)\|_{L^q(d\gamma_n, \mathcal{S}^q)}$. One inequality has been proved in b). For the other one, we notice that $\mathcal{R}^*\mathcal{R} = \mathrm{Id}$ on $\mathcal{S}^2(H)$ because

$$\begin{aligned}\langle \mathcal{R}(X), \mathcal{R}(Y)\rangle &= \int_{\mathbb{R}^n} \left\langle \sum_{j=1}^n y_j R_j(X), \sum_{k=1}^n y_k R_k(Y) \right\rangle \gamma_n(y)\, dy \\ &= \sum_{j=1}^n \langle R_j(X), R_j(Y)\rangle = \sum_{j=1}^n \left\langle R_j^* R_j(X), Y\right\rangle = \langle X, Y\rangle .\end{aligned}$$

Hence, for $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\|X\|_{\mathcal{S}^q} \le \|\mathcal{R}^*\|_{q\to q} \|\mathcal{R}(X)\|_{L^q(\gamma_n\, dy, \mathcal{S}^q)} = \|\mathcal{R}\|_{q'\to q'} \|\mathcal{R}(X)\|_{L^q(\gamma_n dy, \mathcal{S}^q)}. \qquad \square$$

Example 6. The CCR heat flow on $K(L^2(\mathbb{R}^n))$.

The assumptions of Theorem 5 are satisfied by the following example, taken from [**A**], where assertion a) is proved in this special case.

The operator $P = -i\frac{d}{dx}$ and the operator Q of multiplication by x are well defined: $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and satisfy the Canonical Commutation Relation

$$[Q, P] = iI.$$

P, Q are formally selfadjoint unbounded operators on $L^2(\mathbb{R})$. They generate two one parameter unitary groups of operators on $L^2(\mathbb{R})$, respectively e^{isP} (translation by $-s$) and e^{itQ} (multiplication by e^{itx}), satisfying

$$e^{isP} e^{itQ} = e^{ist} e^{itQ} e^{isP}, \quad s, t \in \mathbb{R}. \tag{13}$$

We may define the inner $*$-derivations

$$D_P(X) = i[P, X],$$
$$D_Q(X) = i[Q, X]$$

for operators X on $L^2(\mathbb{R})$ which are defined by kernels $k(x, y) \in \mathcal{S}(\mathbb{R}^2)$. Note that the kernels of $D_P(X)$, $D_Q(X)$ are respectively $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) k$ and $(x - y)k$. D_P, D_Q are generators of groups of $*$-automorphisms of $K(L^2(\mathbb{R}))$, namely

$$e^{sD_P}(X) = e^{isP} X e^{-isP},$$
$$e^{tD_Q}(X) = e^{itQ} X e^{-itQ}$$

and, owing to (13), e^{sD_P} and e^{tD_Q} commute on $K(L^2(\mathbb{R}))$. Obviously, only $X = 0$ is stable under these groups.

On $\mathcal{S}(\mathbb{R}^n)$ we consider in the same way $P_j = -i\frac{\partial}{\partial x_j}$, Q_j the operator of multiplication by x_j. The $*$-automorphisms $e^{s_j D_{P_j}}$ and $e^{t_k D_{Q_k}}$, $1 \leq j, k \leq n$ all commute on $K(L^2(\mathbb{R}^n))$. Denoting

$$L = -\sum_{j=1}^{n} D_{P_j}^2 + D_{Q_j}^2,$$

the semigroup e^{-tL}, $t \geq 0$, acting on $K(L^2(\mathbb{R}^n))$, is called the CCR heat flow. The $2n$ Riesz transforms $D_{P_j} L^{-\frac{1}{2}}$, $D_{Q_k} L^{-\frac{1}{2}}$, $1 \leq j, k \leq n$ satisfy the conclusion of Theorem 5.

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