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A NOTE ON THE CONTINUOUS EXTENSIONS OF INJECTIVE MORPHISMS BETWEEN FREE GROUPS TO RELATIVELY FREE PROFINITE GROUPS

Thierry Coulbois, Mark Sapir and Pascal Weil

Abstract

Let $\mathbf{V}$ be a pseudovariety of finite groups such that free groups are residually $\mathbf{V}$, and let $\varphi\colon F(A) \to F(B)$ be an injective morphism between finitely generated free groups. We characterize the situations where the continuous extension $\hat{\varphi}$ of φ between the pro-$\mathbf{V}$ completions of $F(A)$ and $F(B)$ is also injective. In particular, if $\mathbf{V}$ is extension-closed, this is the case if and only if $\varphi(F(A))$ and its pro-$\mathbf{V}$ closure in $F(B)$ have the same rank. We examine a number of situations where the injectivity of $\hat{\varphi}$ can be asserted, or at least decided, and we draw a few corollaries.

In this paper, we are interested in the pro-$\mathbf{V}$ topologies on finitely generated free groups, where $\mathbf{V}$ is a *pseudovariety of groups* (a class of finite groups closed under taking subgroups, quotients and finite direct products). These topologies were introduced in the 1950s by Hall. When $\mathbf{V}$ is the class of all finite groups, the finite index subgroups are exactly the open subgroups, and Hall proved [**6**] that every finitely generated subgroup is closed. More recent papers (Ribes and Zalesskiĭ [**9**], Margolis, Sapir and Weil [**7**], Weil [**12**]) focused on the problem of effectively computing the pro-$\mathbf{V}$ closure $\mathrm{Cl}_{\mathbf{V}}(H)$ of a given finitely generated subgroup H of a free group. It is known for instance that if $\mathbf{V}$ is extension-closed, then $\mathrm{Cl}_{\mathbf{V}}(H)$ has finite rank, at most equal to the rank of H [**9**]. In general, a finite rank subgroup may have an infinite rank closure (e.g. if $\mathbf{V}$ is the pseudovariety of finite abelian groups), or it may be the case that if H is a finite rank subgroup, then its closure

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always has finite rank, possibly greater than the rank of H (e.g. if $\mathbf{V}$ is the pseudovariety of finite nilpotent groups [**7**]). It is interesting to note that deciding whether a given subgroup is pro-$\mathbf{V}$-closed is equivalent to deciding an extension property for a certain set of partial isomorphisms of a finite set [**7**]. If $\mathbf{V}$ consists of all finite p-groups, for some fixed prime p, the closure of a given finite rank subgroup can be effectively computed [**9**], in polynomial time [**7**]. On the other hand, it is not known whether the pro-solvable closure of a finite rank subgroup is effectively computable; a positive solution for this difficult open question would have interesting consequences in finite monoid theory [**7**] and in computational complexity (Straubing and Thérien [**10**]).

It is also known that finite rank closed subgroups are free factors of clopen subgroups, and that the converse holds if $\mathbf{V}$ is extension-closed [**9**]. Moreover, again in the extension-closed case, if H is a finite rank pro-$\mathbf{V}$-closed subgroup of a free group F, then the pro-$\mathbf{V}$ topology of H coincides with the topology it inherits from F. The central result in this paper (Theorem 1.1) characterizes the situations where this property of coincidence of topologies holds: it is equivalent to another extension property, namely to the fact that a certain injective morphism between two free groups F and F' admits an injective continuous extension between the pro-$\mathbf{V}$ completions of F and F'. It turns out that in the extension-closed case, this is equivalent to the fact that H and its pro-$\mathbf{V}$ closure have equal rank.

After the proof of the main result, we list a number of immediate consequences: for instance, it follows from our result that if $\mathbf{V}$ is extension-closed, the continuous extension of an injective endomorphism of the free group of rank 2 is always injective. In the last section, we illustrate our result by considering a simple example of an injective morphism $\varphi\colon F \to F'$ between finitely generated free groups whose continuous extension $\hat{\varphi}$ to the pro-p completions is not injective, and we exhibit a sequence $(t_n)_n$ of elements of F whose limit points are non-trivial elements of $\ker\hat{\varphi}$.

1. Injective extendability

If A is an alphabet (that is, a finite non-empty set), then $F(A)$ denotes the free group on A. Let $\mathbf{V}$ be a pseudovariety of finite groups: the pro-$\mathbf{V}$ topology on a group G is the least topology which makes every morphism from G into an element of $\mathbf{V}$ continuous. A basis of neighborhoods of 1 in this topology is given by the finite-index normal subgroups K of G such that $G/K \in \mathbf{V}$.

The pro-**V** topology on G is Hausdorff if and only if G is residually **V**. In that case, the pro-**V** topology on G can be defined by an ultrametric distance function. This situation arises in particular if G is a free group and **V** is a non-trivial extension-closed pseudovariety. In the sequel, we consider only pseudovarieties **V** such that free groups are residually **V**.

If G is a group, we denote by $\hat{G}$ the pro-**V** completion of G: it is compact and totally disconnected. If $G = F(A)$, we write $\hat{F}_{\mathbf{V}}(A)$ for $\hat{G}$; this is also the free pro-**V** group on A. If $H \subseteq G$, we write $\mathrm{Cl}_{\mathbf{V}}(H)$ (or simply $\mathrm{Cl}(H)$) for the closure of H in G. If $H \subseteq \hat{G}$, we write $\overline{H}$ for the closure of H in $\hat{G}$. In particular, if $H \subseteq G$, $\overline{H} = \overline{\mathrm{Cl}(H)}$ and, if G is residually **V**, then $\mathrm{Cl}(H) = \overline{H} \cap G$.

We note that every morphism $\varphi \colon F \to F'$ between free groups is uniformly continuous when both groups are equipped with their respective pro-**V** topologies. In particular, φ admits a (uniquely defined) continuous extension between the pro-**V** completions, written $\hat{\varphi} \colon \hat{F} \to \hat{F}'$.

For a justification of these assertions, we refer the readers, for instance, to [**7**]. We now consider injective morphisms, and we state our main result.

Theorem 1.1. *Let $\varphi \colon F(A) \to F(B)$ be an injective morphism and let $H = \varphi(F(A))$. Let* **V** *be a pseudovariety of groups such that free groups are residually* **V**. *The following conditions are equivalent:*

- *The continuous extension of φ, $\hat{\varphi} \colon \hat{F}_{\mathbf{V}}(A) \to \hat{F}_{\mathbf{V}}(B)$ is one-to-one.*
- *The pro-***V** *topology on H coincides with the topology on H induced by the pro-***V** *topology on $F(B)$.*

If, in addition, **V** *is extension-closed, these properties are equivalent to:*

- *H and $\mathrm{Cl}(H)$ have the same rank.*

The proof of Theorem 1.1 follows directly from Propositions 1.4, 1.6 and 1.8 below.

1.1. Comparing the pro-V topologies on a subgroup.

We will use the following elementary remark.

Lemma 1.2. *Let $\varphi \colon F(A) \to F(B)$ be a morphism between finitely generated free groups, let $H = \varphi(F(A))$ be the range of φ and let $\hat{\varphi} \colon \hat{F}_{\mathbf{V}}(A) \to \hat{F}_{\mathbf{V}}(B)$ be the continuous extension of φ between the pro-***V** *completions of $F(A)$ and $F(B)$. Then the range of $\hat{\varphi}$ is $\overline{H}$.*

Proof: By continuity, we have $\hat{\varphi}(\hat{F}_{\mathbf{V}}(A)) = \hat{\varphi}(\overline{F(A)}) \subseteq \overline{\varphi(F(A))} = \overline{H}$. Of course, we also have $H \subseteq \hat{\varphi}(\hat{F}_{\mathbf{V}}(A))$. Finally, as $\hat{F}_{\mathbf{V}}(A)$ is compact and $\hat{\varphi}$ is continuous, the group $\hat{\varphi}(\hat{F}_{\mathbf{V}}(A))$ is closed, so $\hat{\varphi}(\hat{F}_{\mathbf{V}}(A)) = \overline{H}$. □

Next we consider the following property of a finitely generated subgroup H of a free group $F(B)$:

Property Coinc(V). The pro-$\mathbf{V}$ topology on H coincides with the topology on H induced by the pro-$\mathbf{V}$ topology on $F(B)$.

This property translates as follows.

Lemma 1.3. *Let H be a finitely generated subgroup of a free group $F(B)$, let $\imath\colon H \to F(B)$ be the natural injection of H into $F(B)$, and let $\hat{\imath}\colon \hat{H} \to \hat{F}_{\mathbf{V}}(B)$ be the continuous extension of $\imath$ between the pro-$\mathbf{V}$ completions of H and $F(B)$. Then H has Property* Coinc($\mathbf{V}$) *if and only if $\hat{\imath}$ is injective. In particular, $\overline{H}$ is homeomorphic to $\hat{H}$.*

Proof: This is immediate once we observe that $\hat{\imath}$ has range $\overline{H}$ (by Lemma 1.2). ☐

We are now ready to prove the first equivalence in Theorem 1.1.

Proposition 1.4. *Let $\mathbf{V}$ be a pseudovariety of groups such that free groups are residually $\mathbf{V}$. Let φ, $\hat{\varphi}$ and H be as in the statement of Theorem 1.1. Then $\hat{\varphi}$ is injective if and only if H has Property* Coinc($\mathbf{V}$).

Proof: Let $\psi\colon F(A) \to H$ be the restriction of φ to an isomorphism between $F(A)$ and H, and let $\hat{\psi}\colon \hat{F}_{\mathbf{V}}(A) \to \hat{H}$ be the continuous extension of ψ. As ψ is an isomorphism, $\hat{\psi}$ is a homeomorphism.

Let $\imath\colon H \to F(B)$ and $\hat{\imath}\colon \hat{H} \to \hat{F}_{\mathbf{V}}(B)$ be as in Lemma 1.3. We observe that $\varphi = \imath \circ \psi$, so that $\hat{\varphi} = \hat{\imath} \circ \hat{\psi}$. As $\hat{\psi}$ is a homeomorphism, $\hat{\varphi}$ is one-to-one if and only if $\hat{\imath}$ is. By Lemma 1.3, this is equivalent to H having Property Coinc($\mathbf{V}$), as we wanted to prove. ☐

1.2. The extension-closed case.

The following sufficient condition for a finitely generated subgroup to have Property Coinc($\mathbf{V}$) was proved in [**7**, Proposition 2.17]:

Proposition 1.5. *If $\mathbf{V}$ is extension-closed, then every finitely generated, closed subgroup of $F(A)$ has Property* Coinc($\mathbf{V}$).

We will see that this sufficient condition is not necessary (Proposition 1.7 below). However, we can immediately use this property to prove one half of the remaining equivalence.

Proposition 1.6. *Let $\mathbf{V}$ be a non-trivial extension-closed pseudovariety of groups. Let $\varphi\colon F(A) \to F(B)$ be a morphism between free groups, let $H = \varphi(F(A))$, and let $\hat{\varphi}$ be the continuous extension of φ between the pro-$\mathbf{V}$ completions of $F(A)$ and $F(B)$. If φ and $\hat{\varphi}$ are injective, then H and* Cl(H) *have equal ranks.*

Proof: By Lemma 1.2, the range of $\hat{\varphi}$ is $\overline{H}$. Since $\hat{\varphi}$ is injective between two compact spaces, it is a homeomorphism onto its image, so $\overline{H}$ is homeomorphic to the free pro-**V** group of rank $|A| = \mathrm{rank}(H)$.

On the other hand, we know from Proposition 1.5 that $\mathrm{Cl}(H)$ has Property Coinc(**V**). Applying Lemma 1.3 to $\mathrm{Cl}(H)$, we find that $\overline{\mathrm{Cl}(H)}$ is homeomorphic to the free pro-**V** group of rank $\mathrm{rank}(\mathrm{Cl}(H))$.

But $\overline{H} = \overline{\mathrm{Cl}(H)}$, so we have proved that $\mathrm{rank}(H) = \mathrm{rank}(\mathrm{Cl}(H))$. □

Before we prove the reverse implication, we show the following result, which does not require the hypothesis that **V** is extension-closed.

Proposition 1.7. *Let H be a finitely generated subgroup of the free group $F(A)$. If H is dense in the pro-**V** topology of $F(A)$ and if* $\mathrm{rank}(H) = \mathrm{rank}(F(A))$*, then H has Property* Coinc(**V**).

Proof: As H and $F(A)$ have the same rank, we may consider an injective endomorphism ψ of $F(A)$ with range H. Let $\hat{\psi}\colon \hat{F}_{\mathbf{V}}(A) \to \hat{F}_{\mathbf{V}}(A)$ be the continuous extension of ψ. By Lemma 1.2, $\hat{\psi}$ is an onto endomorphism of $\hat{F}_{\mathbf{V}}(A)$. But every onto continuous endomorphism of a finitely generated profinite group is injective [**5**, Proposition 15.3]. So $\hat{\varphi}$ is injective: by Proposition 1.4, this implies that H has Property Coinc(**V**). □

We can now give the last element in the proof of Theorem 1.1.

Proposition 1.8. *Let H be a finitely generated subgroup of the free group $F(A)$. If **V** is extension-closed and* $\mathrm{rank}(H) = \mathrm{rank}(\mathrm{Cl}(H))$*, then H has Property* Coinc(**V**).

Proof: By Proposition 1.5, the pro-**V** topology on $\mathrm{Cl}(H)$ coincides with the topology on $\mathrm{Cl}(H)$ induced by the pro-**V** topology on $F(A)$. Therefore H is dense in the pro-**V** topology on $\mathrm{Cl}(H)$. Now Proposition 1.7 implies that the pro-**V** topology on H coincides with the topology on H induced by the pro-**V** topology on $\mathrm{Cl}(H)$, and this concludes the proof. □

2. Corollaries

The following collection of remarks is immediately deduced from Theorem 1.1. Throughout this section, **V** denotes a pseudovariety of groups such that free groups are residually **V**, $\varphi\colon F(A) \to F(B)$ is an injective morphism between free groups, $\hat{\varphi}\colon \hat{F}_{\mathbf{V}}(A) \to \hat{F}_{\mathbf{V}}(B)$ is the continuous extension of φ between the pro-**V** completions of $F(A)$ and $F(B)$, and $H = \varphi(F(A))$.

Corollary 2.1. *Whether $\hat{\varphi}$ is injective depends only on H, not on φ.*

If $\mathbf{V} = \mathbf{G}$, the pseudovariety of all finite groups, the pro-$\mathbf{V}$ completion of a group is called its profinite completion. It is well-known that for the pro-$\mathbf{G}$ topology, every finitely generated subgroup of the free group is closed [**6**]. As a result, we have:

Corollary 2.2. *Every injective morphism between free groups of finite rank admits an injective continuous extension to the profinite completions of these groups.*

Let p be a prime number and let $\mathbf{G}_p$ be the pseudovariety of finite p-groups. The pro-$\mathbf{G}_p$ completion of a group is called its pro-p completion. It is shown in [**9**] that if p is a prime number, one can effectively compute the pro-p closure of a finitely generated subgroup of the free group (see [**7**] for a polynomial time algorithm). It follows that:

Corollary 2.3. *Given a prime number p, one can decide whether the continuous extension of φ to the pro-p completions is injective.*

Let $\mathbf{G}_{\mathrm{sol}}$ be the pseudovariety of finite solvable groups; the pro-$\mathbf{G}_{\mathrm{sol}}$ completion of a group is called its pro-solvable completion. It is also shown in [**12**] that one can compute the rank of the pro-solvable closure of a finite index subgroup:

Corollary 2.4. *If H has finite index, one can decide whether the continuous extension of φ to the pro-solvable completions is injective.*

For the general case however, we do not know whether one can effectively compute the rank of the pro-solvable closure of a given finitely generated subgroup (see the conclusion of [**7**] or [**12**] for a discussion). In particular, we do not know whether the injectivity of the continuous extension of φ to the pro-solvable completions is decidable.

In [**7**], the pro-$\mathbf{V}$ topology is considered also when $\mathbf{V}$ is the pseudovariety $\mathbf{G}_{\mathrm{nil}}$ of finite nilpotent groups, a pseudovariety which is not extension-closed. An example is given of a finitely generated subgroup which is closed in that topology yet does not have Property Coinc($\mathbf{V}$) [**7**, Example 1.10]. This shows that the extension-closed assumption in Proposition 1.8 cannot be dispensed with. However, we also know [**7**] that the pro-nilpotent completion of the free group (its pro-$\mathbf{G}_{\mathrm{nil}}$ completion) is a subdirect product of its pro-p completions (p prime). Therefore we have:

Corollary 2.5. *The continuous extension of φ to the pro-nilpotent completions is injective if and only if, for each prime p, the continuous extension of φ to the pro-p completions is injective.*

In view of Theorem 1.1, deciding the injectivity of the pro-nilpotent extension of φ is equivalent to deciding whether every p-closure of H has the same rank as H. But there are only finitely many subgroups of $F(A)$ of the form $\mathrm{Cl}_p(H)$, and they are effectively computable [**7**], [**12**]. It follows that:

Corollary 2.6. *It is decidable whether the extension of φ to the pro-nilpotent completions is injective.*

Returning to extension-closed pseudovarieties, it is shown in [**9**] that $\mathrm{rank}(\mathrm{Cl}(H)) \leq \mathrm{rank}(H)$. As free groups and free pro-$\mathbf{V}$ groups of rank 1 are commutative, it follows that if H has rank 1 or 2, then $\mathrm{rank}(H) = \mathrm{rank}(\mathrm{Cl}(H))$. This translates into the following result.

Corollary 2.7. *If $\mathbf{V}$ is extension-closed and φ is defined on the free group of rank 1 or 2, then $\hat{\varphi}$ is injective.*

This last result leads to the following consequences. Let $\mathbf{G}$ be the pseudovariety of all finite groups, B be a finite alphabet, and $u \in \hat{F}_{\mathbf{G}}(B)$. We say that a finite group G *satisfies the pseudo-identity* $u = 1$ if, for every continuous morphism $\psi\colon \hat{F}_{\mathbf{G}}(B) \to G$, we have $\psi(u) = 1$. The class of finite groups which satisfy a given set of pseudo-identities is a pseudovariety, and every pseudovariety can be defined in this fashion (Reiterman's theorem, see [**1**]).

Let $A = \{a, b\}$, B be a finite alphabet, $\varphi\colon F(A) \to F(B)$ be an injective morphism (that is, $\varphi(a)\varphi(b) \neq \varphi(b)\varphi(a)$) and $\hat{\varphi}\colon \hat{F}_{\mathbf{G}}(A) \to \hat{F}_{\mathbf{G}}(B)$ be the continuous extension of φ to the free profinite groups over A and B.

Corollary 2.8. *Let $\mathbf{V}$ be a non-trivial extension-closed pseudovariety and let $(u_i)_{i\in I}$ be a collection of elements of $\hat{F}_{\mathbf{G}}(A)$. If $\mathbf{V}$ satisfies the pseudo-identities $\hat{\varphi}(u_i) = 1$, then $\mathbf{V}$ satisfies the pseudo-identities $u_i = 1$.*

To build from this result, let us observe that, if p is a prime number, it is immediate that a finite group is a p-group if and only if every one of its cyclic subgroup is a p-group. That is equivalent to saying that $\mathbf{G}_p$ is defined by a set of one-variable pseudo-identities. In fact, it is even the case that there exists a single element $u_p \in \hat{F}_{\mathbf{G}}(a)$ such that a finite group is a p-group if and only if it satisfies the pseudo-identity $u_p = 1$ (u_p is the limit in the profinite topology of the sequence $a^{p^{n!}}$, denoted $u_p = a^{p^{\omega}}$ [**2**, Example 2.6(1)]).

It is also known that a finite group is nilpotent (resp. solvable) if and only if each of its 2-generated subgroups is nilpotent (resp. solvable), so that the pseudovarieties $\mathbf{G}_{\mathrm{nil}}$ and $\mathbf{G}_{\mathrm{sol}}$ are both defined by a set of

2-variable pseudo-identities. In the nilpotent case, this is a result of Neumann and Taylor [**8**] and in the solvable case, it was proved by Thompson [**11**], see also Flavell [**4**]. In fact, it is known that there exists an element $u_{\rm nil}(a,b)$ (resp. $u_{\rm sol}(a,b)$) of $\hat{F}_{\mathbf{G}}(A)$ such that the single 2-variable pseudo-identity $u_{\rm nil}(a,b)=1$ defines exactly $\mathbf{G}_{\rm nil}$, Almeida [**2**, Example 2.7(1)] (resp. $u_{\rm sol}(a,b)=1$ defines exactly $\mathbf{G}_{\rm sol}$, Bandman *et al.* [**3**]).

Thus, Corollary 2.8 implies the following.

Corollary 2.9. *Let B be a finite alphabet, and let $x,y\in F(B)$ such that $xy\neq yx$. Let $\mathbf{V}$ be an extension-closed pseudovariety of groups.*

- *If $\mathbf{V}$ satisfies the pseudo-identity $x^{p^\omega}=1$, then $\mathbf{V}=\mathbf{G}_p$.*
- *If $\mathbf{V}$ satisfies the pseudo-identity $u_{\rm nil}(x,y)=1$, then $\mathbf{V}=\mathbf{G}_p$ for some prime p.*
- *If $\mathbf{V}$ satisfies the pseudo-identity $u_{\rm sol}(x,y)=1$, then $\mathbf{V}=\mathbf{G}_{\rm sol}$.*

This can be rewritten in the, perhaps more readable, following form (without actually using the subtle results of the existence of a *single* pseudo-identity defining $\mathbf{G}_p$, $\mathbf{G}_{\rm nil}$ or $\mathbf{G}_{\rm sol}$).

Corollary 2.10. *Let B be a finite alphabet, and let $x,y\in F(B)$ such that $xy\neq yx$. Let $\mathbf{V}$ be an extension-closed pseudovariety of groups.*

- *If for every morphism $\psi\colon F(B)\to G$ into an element $G\in\mathbf{V}$, $\psi(x)$ has exponent a power of p (for some fixed prime p), then $\mathbf{V}=\mathbf{G}_p$.*
- *If for every morphism $\psi\colon F(B)\to G$ into an element $G\in\mathbf{V}$, $\psi(x)$ and $\psi(y)$ generate a nilpotent subgroup of G, then $\mathbf{V}=\mathbf{G}_p$ for some prime p.*
- *If for every morphism $\psi\colon F(B)\to G$ into an element $G\in\mathbf{V}$, $\psi(x)$ and $\psi(y)$ generate a solvable subgroup of G, then $\mathbf{V}=\mathbf{G}_{\rm sol}$.*

3. An example

We now give an explicit example of an injective morphism φ between finitely generated free groups whose continuous extension $\hat{\varphi}$ between the corresponding free pro-p groups is not injective, and we exhibit a sequence of words $(t_n)_n$ such that $\lim\varphi(t_n)=1$, yet 1 is not a limit point of $(t_n)_n$: thus the limit points of $(t_n)_n$ are non-trivial elements of $\ker\hat{\varphi}$. Put differently, this means that p-groups ultimately satisfy $\varphi(t_n)=1$, yet there exists a p-group that does not satisfy any of the identities $t_n=1$.

Let q be a fixed odd prime, let $B = \{x, y\}$, and let H be the kernel of the morphism from $F(B)$ into the additive group $\mathbb{Z}/q\mathbb{Z}$ which maps letters x and y to 1. Then H has rank $q+1$ and if $A = \{a_0, a_1, \ldots, a_q\}$, then H is the range of the injective morphism $\varphi\colon F(A) \to F(B)$ given by

$$\varphi(a_i) = x^i y x^{-(i+1)}, \quad i = 0, \ldots, q-2$$

$$\varphi(a_{q-1}) = x^{q-1} y$$

$$\varphi(a_q) = x^q.$$

As $F(B)/H$ is a q-group, H is closed in the pro-q topology: by Theorem 1.1, the continuous extension of φ to the pro-q completions of $F(A)$ and $F(B)$ is injective. On the other hand, one can show that for every other prime number p, H is dense in the pro-p topology (see [**7**, Section 3.1]), and by Theorem 1.1 again, the continuous extension of φ to the pro-p completions is not injective.

Let $u_0 = x$, $v_0 = y$, and for each $n \geq 0$, $u_{n+1} = u_n v_n$ and $v_{n+1} = v_n u_n$. We now fix a prime number $p \neq q$. It is well-known that the identities $u_n = v_n$ are ultimately verified by every finite p-group (Engel identities, [**8**]). This means that the sequence $(u_n v_n^{-1})_n$ converges to 1 in the pro-p topology. It is also easily verified that, for each $n \geq 0$, the word $u_n v_n^{-1}$ is reduced and lies in H. Thus there exists a (unique) word $t_n \in F(A)$ such that $\varphi(t_n) = u_n v_n^{-1}$ for each $n \geq 0$.

Let S be the q-dimensional vector space over the p-element field $\mathbb{F}_p$ with basis $e_0, \ldots, e_{q-1}$. Let π be the projection of $F(A)$ onto S defined by $\pi(a_i) = e_i$ for $i = 0, \ldots, q-1$ and $\pi(a_q) = 0$. We prove that $\pi(t_n)$ is never 0 in S, so that the additive group S (an abelian p-group) does not satisfy $t_n = 1$.

We consider the morphism $\pi \circ \varphi^{-1}$ from H to S. For each $n \geq 0$, let $s_n = \pi(t_n) = \pi \circ \varphi^{-1}(u_n v_n^{-1})$. Then

$$s_{n+1} = \pi \circ \varphi^{-1}(u_{n+1} v_{n+1}^{-1}) = \pi \circ \varphi^{-1}(u_n v_n u_n^{-1} v_n^{-1}).$$

Since H is normal, $u_n v_n u_n^{-1} u_n^{-1} \in H$ and we have

$$s_{n+1} = \pi\circ\varphi^{-1}(u_n v_n u_n^{-1} u_n^{-1}) + \pi\circ\varphi^{-1}(u_n v_n^{-1}) = s_n - \pi\circ\varphi^{-1}(u_n u_n v_n^{-1} u_n^{-1}).$$

Let σ be the linear isomorphism of S given by $\sigma(e_i) = e_{i+1}$ where the indices i and $i+1$ are taken modulo q. We leave it to the reader to verify that, for each $i = 0, \ldots, q-1$, we have

$$\pi \circ \varphi^{-1}(x\varphi(a_i)x^{-1}) = \pi \circ \varphi^{-1}(y\varphi(a_i)y^{-1}) = \sigma \circ \pi(a_i).$$

It follows that for each $g \in F(A)$, we have

$$\pi \circ \varphi^{-1}(x\varphi(g)x^{-1}) = \pi \circ \varphi^{-1}(y\varphi(g)y^{-1}) = \sigma \circ \pi(g).$$

Now u_n is a positive word in x and y of length 2^n, so we have

$$s_{n+1} = s_n - \sigma^{2^n}(s_n) = (\mathrm{id} - \sigma^{2^n})(s_n).$$

We observe that the linear transformation σ of V has order q and a one-dimensional eigenspace associated to the eigenvalue 1 (generated by $w = e_0 + \cdots + e_{q-1}$) and the supplementary hyperplane W of equation $x_0 + \cdots + x_{q-1} = 0$ is stable under σ. Now we use the fact that $\sigma^q = \mathrm{id}$ and q is odd: for each integer $n \geq 0$, σ is equal to a power of σ^{2^n}. As 1 is not an eigenvalue of the restriction $\sigma_{|W}$, it cannot be an eigenvalue of any 2^n-th power of $\sigma_{|W}$. It follows that the linear transformation $\mathrm{id} - \sigma^{2^n}$ of W is invertible for all $n \geq 0$. As $s_0 \neq 0$, we find that $s_n = \pi(t_n) \neq 0$ for each n, as announced.

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Thierry Coulbois:
Equipe Logique
Université Paris-7
2 place Jussieu
75251 Paris Cedex 05
France

and

Al-Quds University
Occupied Palestinian Territories
E-mail address: coulbois@logique.jussieu.fr

Mark Sapir:
Department of Mathematics
Vanderbilt University
Nashville, TN 37240-0001
U.S.A.
E-mail address: msapir@math.vanderbilt.edu
http://www.math.vanderbilt.edu/~msapir

Pascal Weil:
LaBRI, CNRS
351 cours de la Libération
33405 Talence Cedex
France
E-mail address: pascal.weil@labri.fr
http://www.labri.fr/Perso/~weil

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