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# CENTRE-BY-METABELIAN GROUPS WITH A CONDITION ON INFINITE SUBSETS

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*Abstract*

In this note, we consider some combinatorial conditions on infinite subsets of groups and we obtain in terms of these conditions some characterizations of the classes $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$ and $\mathcal{F}\mathcal{L}(\mathcal{N}_k)$ for the finitely generated centre-by-metabelian groups, where $\mathcal{L}(\mathcal{N}_k)$ (respectively, $\mathcal{F}$) denotes the class of groups in which the normal closure of each element is nilpotent of class at most $k$ (respectively, finite groups).

## 1. Introduction and results

Following a question of Erdös, B. H. Neumann proved in [**13**] that a group is centre-by-finite if, and only if, every infinite subset contains a commuting pair of distinct elements. Since this result, problems of similar nature have been the object of many papers (for example [**1**], [**2**], [**3**], [**4**], [**5**], [**9**], [**11**], [**10**], [**16**], [**17**]). We present here some further results of the same type.

Let $k$ be a fixed positive integer. Denote by $E_k^*$ the class of groups such that for every infinite subset $X$ there exist two distinct elements $x$, $y$ in $X$, and integers $t_0, t_1, \dots, t_k$ depending on $x$, $y$, and satisfying $\left[z_0^{t_0}, z_1^{t_1}, \dots, z_k^{t_k}\right] = 1$, where $z_i \in \{x, y\}$ for every $i \in \{0, 1, \dots, k\}$ and $z_0 \neq z_1$. Denote also by $E_k^{\#}$ the class of groups $G \in E_k^*$ for which the integers $t_0, \dots, t_k$ belong to $\{-1, 1\}$. In [**3**], it is proved that if $G$ is a finitely generated soluble group in the class $E_k^*$ (respectively $E_k^{\#}$), then there is an integer $c$, depending only on $k$, such that $G$ is in $\mathcal{N}_c\mathcal{F}$ (respectively $\mathcal{F}\mathcal{N}_c$); where $\mathcal{N}_c$ and $\mathcal{F}$ denote respectively the class of nilpotent groups of class at most $c$ and the class of finite groups. In [**3**], it is also proved that a finitely generated metabelian group $G$ is in $E_k^*$ (respectively $E_k^{\#}$) if, and only if, $G$ belongs to $\mathcal{N}_k\mathcal{F}$ (respectively $\mathcal{F}\mathcal{N}_k$); and it is

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observed that these results are not true if the derived length of $G$ is $\geq 3$. Among the examples cited, which are due to Newman [**14**] (see also [**2**]), there is a finitely generated torsion-free nilpotent group $G$ of class 4, of derived length 3, and whose 2-generated subgroups are nilpotent of class at most 3. So $G$ is a finitely generated centre-by-metabelian group which belongs to $E_3^*$ (respectively $E_3^\#$) and such that $G \notin \mathcal{N}_3\mathcal{F}$ (respectively $G \notin \mathcal{F}\mathcal{N}_3$). Note that if a group belongs to $\mathcal{N}_k$, then it is in $\mathcal{L}(\mathcal{N}_{k-1})$, where $\mathcal{L}(\mathcal{N}_{k-1})$ denotes the class of groups in which the normal closure of each element is nilpotent of class at most $k-1$. Considering this weaker condition we are able to prove the following results:

**Theorem 1.1.** *A finitely generated centre-by-metabelian group $G$ is in $E_{k+1}^*$ if, and only if, $G$ belongs to $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$.*

**Theorem 1.2.** *A finitely generated centre-by-metabelian group $G$ is in $E_{k+1}^\#$ if, and only if, $G$ belongs to $\mathcal{F}\mathcal{L}(\mathcal{N}_k)$. In particular, a torsion-free centre-by-metabelian group $G$ is in $E_{k+1}^\#$ if, and only if, $G$ belongs to $\mathcal{L}(\mathcal{N}_k)$.*

In [**7**], it is proved that a metabelian group $G$ is $(k+1)$-Engel if, and only if, $G$ belongs to $\mathcal{L}(\mathcal{N}_k)$. Morse [**12**] extended this result to a certain class of soluble groups of derived length $\leq 5$ which contains the centre-by-metabelian groups. So our theorems improve Morse's result for the centre-by-metabelian groups.

Denote by $\mathcal{B}_k^*$ the class of groups such that every infinite subset contains an element $x$ such that $\langle x \rangle$ is subnormal of defect $k$. It is proved in [**8**, Corollary 2.5] that a metabelian non-torsion group is a $k$-Baer group (that is every cyclic subgroup of $G$ is subnormal of defect $k$) if, and only if, $G$ is a $k$-Engel group. Here, using Theorem 1.2, we shall improve this result with the following:

**Theorem 1.3.** *Let $G$ be a finitely generated centre-by-metabelian group. If $G$ is in $\mathcal{B}_k^*$, then $G$ is finite-by-($k$-Engel). In particular, a torsion-free centre-by-metabelian group $G$ belongs to $\mathcal{B}_k^*$ if, and only if, $G$ is $k$-Engel.*

## 2. Proof of the results

**Lemma 2.1.** *Let $G$ be a finitely generated torsion-free nilpotent group of class at most $k+1$. If $G$ belongs to $E_k^*$, then $G$ is a $k$-Engel group.*

*Proof:* Let $G$ be a group in $E_k^*$ and assume that $G$ is not $k$-Engel. Therefore there exist $x$, $y$ in $G$ such that $[x,{}_k y] \neq 1$. The group $G$, being a finitely generated torsion-free nilpotent group, is a residually finite $p$-group for every prime $p$. So $G$ has a normal subgroup $N$ such that $[x,{}_k y] \notin N$ and $|G/N| = p^r$ for some positive integer $r$. Considering the infinite subset $\left\{x^{p^{r+i}} y : i \text{ integer}\right\}$, there are integers $n, m, t_0, t_1, \ldots, t_k$ such that $\left[z_0^{t_0}, z_1^{t_1}, \ldots, z_k^{t_k}\right] = 1$, where $z_i \in \left\{x^{p^{r+n}} y, x^{p^{r+m}} y\right\}$, $n \neq m$ and $z_0 \neq z_1$. Since $G$ is nilpotent of class at most $k+1$, the commutator $\left[z_0^{t_0}, z_1^{t_1}, \ldots, z_k^{t_k}\right]$ is linear in each argument [**1**, Lemma 1], so we get that $[z_0, z_1, \ldots, z_k]^{t_0 t_1 \ldots t_k} = 1$, and therefore $[z_0, z_1, \ldots, z_k] = 1$ since $G$ is torsion-free. Put $z_0 = x^{p^{r+s_0}} y$ and $z_1 = x^{p^{r+s_1}} y$, where $s_0 \neq s_1 \in \{m, n\}$. So

$$1 = [z_0, z_1, \ldots, z_k] = \left[\left[x^{p^{r+s_0}} y, x^{p^{r+s_1}} y\right], z_2, \ldots, z_k\right]$$
$$= \left[\left[x^{(p^{r+s_0} - p^{r+s_1})}, y\right]^{z_1}, z_2, \ldots, z_k\right] = \left[x^{(p^{r+s_0} - p^{r+s_1})}, y, z_2, \ldots, z_k\right]^{z_1}.$$

Hence

$$1 = \left[x^{(p^{r+s_0} - p^{r+s_1})}, y, z_2, \ldots, z_k\right] = [x, y, z_2, \ldots, z_k]^{(p^{r+s_0} - p^{r+s_1})}.$$

Thus $[x, y, z_2, \ldots, z_k] = 1$ as $G$ is torsion-free and $s_0 \neq s_1$. Consequently $[x, y, z_2, \ldots, z_k]\, N = N$. Now $x^{p^{r+n}}, x^{p^{r+m}} \in N$, so $z_i N = yN$. It follows that $[x,{}_k y]\, N = N$; this means that $[x,{}_k y] \in N$, a contradiction which completes the proof. □

It is proved in [**12**, Theorem 1] that if $G$ is nilpotent of class at most $k+2$, then $G$ is $(k+1)$-Engel if and only if $G \in \mathcal{L}(\mathcal{N}_k)$. So combining this result and Lemma 2.1, we have the following consequence:

**Lemma 2.2.** *Let $G$ be a finitely generated nilpotent group of class at most $k+2$. If $G$ is in $E_{k+1}^*$, then $G$ belongs to $\mathcal{FL}(\mathcal{N}_k)$. In particular, a torsion-free nilpotent group $G$ of class at most $k+2$ is in $E_{k+1}^*$ if, and only if, $G$ belongs to $\mathcal{L}(\mathcal{N}_k)$.*

*Proof:* Let $G$ be a finitely generated nilpotent group of class at most $k+2$ and suppose that $G$ is in $E_{k+1}^*$. Then $T$, the torsion subgroup of $G$, is finite and $G/T$ is a finitely generated torsion-free group of nilpotency class at most $k+2$ which belongs to $E_{k+1}^*$. It follows, from Lemma 2.1, that $G/T$ is a $(k+1)$-Engel group, and by [**12**, Theorem 1], $G/T$ belongs to $\mathcal{L}(\mathcal{N}_k)$. Hence, $G$ is in $\mathcal{FL}(\mathcal{N}_k)$; as claimed.

Now, we suppose that $G$ is a torsion-free group of nilpotency class at most $k+2$ which belongs to $E^*_{k+1}$ and let $x, y_1, \ldots, y_{k+1} \in G$. Then $H = \langle x, y_1, \ldots, y_{k+1}\rangle$ is a finitely generated group of nilpotency class at most $k+2$ which belongs to $E^*_{k+1}$. It follows, from the first part of the proof, that $H$ is in $\mathcal{FL}(\mathcal{N}_k)$. So $H$ is in $\mathcal{L}(\mathcal{N}_k)$ since it is torsion-free. Hence, $[x^{y_1}, \ldots, x^{y_{k+1}}] = 1$, and this means that $G$ belongs to $\mathcal{L}(\mathcal{N}_k)$.

Clearly, any group in $\mathcal{L}(\mathcal{N}_k)$ is $(k+1)$-Engel, so it belongs to $E^*_{k+1}$. $\square$

*Proof of Theorem 1.1:* Let $G$ be a finitely generated centre-by-metabelian group in $E^*_{k+1}$. So $G/Z(G)$ is a finitely generated metabelian group in $E^*_{k+1}$. Therefore, by [**3**, Theorem 1.3], $G/Z(G)$ is in $\mathcal{N}_{k+1}\mathcal{F}$. Hence, $G$ belongs to $\mathcal{N}_{k+2}\mathcal{F}$. Since finitely generated nilpotent groups are (torsion-free)-by-finite [**15**, 5.4.15(i)], $G$ has a normal subgroup $H$, of finite index such that $H$ is a torsion-free nilpotent group of class at most $k+2$ which belongs to $E^*_{k+1}$. It follows, by Lemma 2.2, that $H$ is in $\mathcal{L}(\mathcal{N}_k)$; so $G$ belongs to $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$.

Conversely, suppose that $G$ is in $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$. Therefore there is a positive integer $n$ and a normal subgroup $H$ such that $H \in \mathcal{L}(\mathcal{N}_k)$ and $|G/H| = n$. So $H$ is a $(k+1)$-Engel group and $x^n, y^n \in H$ for any $x$, $y$ in $G$. Hence, $[x^n, {}_{k+1}y^n] = 1$ and consequently $G$ belongs to $E^*_{k+1}$. $\square$

*Proof of Theorem 1.2:* Let $G$ be a finitely generated centre-by-metabelian group in $E^{\#}_{k+1}$. So $G/Z(G)$ is a finitely generated metabelian group which belongs to $E^{\#}_{k+1}$. Therefore, by [**3**, Theorem 1.6], $\frac{G/Z(G)}{Z_{k+1}(G/Z(G))}$ is finite; so $G/Z_{k+2}(G)$ is finite. It follows, by [**6**, Theorem 1], that $G$ is in the class $\mathcal{FN}_{k+2}$. Let $H$ be a finite normal subgroup such that $G/H$ is nilpotent of class at most $k+2$. If $T/H$ is the torsion subgroup of $G/H$, then $T/H$ is finite; so $T$ is finite and $G/T$ is a torsion-free finitely generated nilpotent group of class at most $k+2$ which belongs to $E^{\#}_{k+1}$. It follows, by Lemma 2.2, that $G/T$ is in $\mathcal{L}(\mathcal{N}_k)$; so $G$ belongs to $\mathcal{FL}(\mathcal{N}_k)$; as required.

Conversely, suppose that $G$ is in the class $\mathcal{FL}(\mathcal{N}_k)$. Therefore there is a finite normal subgroup $H$ such that $G/H$ is $(k+1)$-Engel. Since $G$ is a finitely generated soluble group, $G/H$ is therefore nilpotent. It follows that $G$ is finite-by-nilpotent, so $G$ is residually finite. Consequently, there is a normal subgroup $N$ of finite index such that $H \cap N = 1$. Since $G/N$ is finite, if $X$ is an infinite subset of $G$, then there are $x, y \in X$ such that $x \neq y$ and $xN = yN$. We have $[x, {}_{k+1}y] \in H$ and $\frac{\langle x,y\rangle N}{N}$ is cyclic, since $G/H$ is $(k+1)$-Engel and $xN = yN$. Thus, $[x, {}_{k+1}y] \in H \cap N$. It follows that $[x, {}_{k+1}y] = 1$ and, therefore, $G$ belongs to $E^{\#}_{k+1}$.

Now we suppose that $G$ is a torsion-free centre-by-metabelian group in the class $E^{\#}_{k+1}$ and let $x, y_1, \dots, y_{k+1} \in G$. Then $H = \langle x, y_1, \dots, y_{k+1}\rangle$ is a torsion-free finitely generated centre-by-metabelian group. It follows, from the first part of the proof, that $H$ belongs to $\mathcal{FL}(\mathcal{N}_k)$, and consequently $H \in \mathcal{L}(\mathcal{N}_k)$ since it is torsion-free. Hence, $[x^{y_1}, \dots, x^{y_{k+1}}] = 1$ and, therefore, $G$ belongs to $\mathcal{L}(\mathcal{N}_k)$. □

For the proof of Theorem 1.3, we need further lemmas. Note that it is proved in [**8**, Theorem 2.3] that every non-torsion $k$-Baer group is a $k$-Engel group. But the converse is shown only in the metabelian case. As a consequence of Morse's result [**12**], we will extend this result with the following lemma:

**Lemma 2.3.** *Let $G$ be a non-torsion centre-by-metabelian group. Then, $G$ is a $k$-Baer group if, and only if, $G$ is a $k$-Engel group.*

*Proof:* Let $G$ be a non-torsion centre-by-metabelian group, and suppose that $G$ is a $k$-Engel group. From [**12**, Theorem 2], $G$ is in $\mathcal{L}(\mathcal{N}_{k-1})$. Let $x$ in $G$; then $x^G$, the normal closure of $x$ in $G$, is in $\mathcal{N}_{k-1}$. Now, it is well known that subgroups of a group of nilpotency class at most $k-1$ are subnormal of defect $k-1$. Thus, $\langle x\rangle$ is $(k-1)$-subnormal in $x^G$, so $\langle x\rangle$ is $k$-subnormal in $G$. It follows that $G$ is a $k$-Baer group. □

**Lemma 2.4.** *Let $G$ be a torsion-free group in $\mathcal{L}(\mathcal{N}_k)$. If $G$ belongs to $\mathcal{B}^*_k$, then $G$ is a $k$-Engel group.*

*Proof:* Let $x$, $y$ in $G$; since $G$ is torsion-free, the subset $\{x^i : i \text{ positive integer}\}$ is infinite. Therefore there is a positive integer $i$ such that $\langle x^i\rangle$ is $k$-subnormal in $G$. Thus, $[x^i, [y, {}_{k-1}x^i]] \in \langle x^i\rangle$, so $[x^i, [y, {}_{k-1}x^i]] = x^r$ for some integer $r$. Since $G$ belongs to $\mathcal{L}(\mathcal{N}_k)$, we have that $G$ is a $(k+1)$-Engel group. Hence, $1 = [x^i, {}_{k+1}[y, {}_{k-1}x^i]] = x^{r^{k+1}}$; and this gives that $r = 0$ as $G$ is torsion-free. It follows that $[x^i, [y, {}_{k-1}x^i]] = 1$, so $[y, {}_k x^i] = 1$. Now, because $x^G$ is in $\mathcal{N}_k$, we have that every commutator in $x^G$ of length $k$ is multilinear. Thus $1 = [y, {}_k x^i] = [[y, x^i], {}_{k-1}x^i] = [y, {}_k x]^{i^k}$. Once again, as $G$ is torsion-free, we obtain that $[y, {}_k x] = 1$; this means that $G$ is a $k$-Engel group. □

*Proof of Theorem 1.3:* Let $G$ be a finitely generated centre-by-metabelian group in the class $\mathcal{B}^*_k$. So every infinite subset of $G$ contains an element $x$ such that $\langle x\rangle$ is $k$-subnormal in $G$. Hence, for any $y$ in $G$ we have $[y, {}_{k+1}x] = 1$. Thus, $G$ belongs to $E^{\#}_{k+1}$. It follows, from [**11**, Theorem 1], that $G$ is finite-by-nilpotent. Therefore there is a finite normal subgroup $T$ such that $G/T$ is a torsion-free centre-by-metabelian group which belongs to $E^{\#}_{k+1}$. It follows from Theorem 1.2 that $G/T$ is in $\mathcal{L}(\mathcal{N}_k)$, and by Lemma 2.4, we obtain that $G/T$ is a $k$-Engel group. Therefore, $G$ is finite-by-($k$-Engel); as claimed.

Now, assume that $G$ is a torsion-free centre-by-metabelian group in $\mathcal{B}^*_k$ and let $x$, $y$ in $G$. Then, from the first part of the proof, $H = \langle x, y\rangle$ is finite-by-($k$-Engel). Since $G$ is torsion-free we deduce that $H$ is $k$-Engel. Hence, $[y, {}_k x] = 1$, so $G$ is a $k$-Engel group.

Conversely, suppose that $G$ is a torsion-free centre-by-metabelian and a $k$-Engel group. From Lemma 2.3 we get that $G$ is a $k$-Baer group, so $G$ is in $\mathcal{B}^*_k$. □

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