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MEAN DIRECTIONALLY CURVED LINES ON SURFACES IMMERSED IN \mathbb{R}^4

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Abstract

The notion of principal configuration of immersions of surfaces into \mathbb{R}^3 , due to Sotomayor and Gutierrez [16] for lines of curvature and umbilics, is extended to that of *mean directional configuration* for immersed surfaces in \mathbb{R}^4 . This configuration consists on the families of *mean directionally curved lines*, along which the second fundamental form points in the direction of the *mean curvature vector*, and their singularities, called here *H-singularities*.

The concepts of *H-singularities* and periodic mean directionally curved lines are studied here in detail. Also the notion of principal structural stability of immersions of surfaces into \mathbb{R}^3 is extended to that of mean directional structural stability, for the case of surfaces in \mathbb{R}^4 . Sufficient conditions for immersions to be *mean directionally structurally stable* are provided in terms of *H-singularities*, periodic mean directionally curved lines and the asymptotic behavior of all the other mean directionally curved lines.

1. Introduction

Principal curvatures, principal direction fields, their integral foliations and umbilic singularities are classical topics of the theory of surfaces immersed in \mathbb{R}^3 . Nevertheless the global behavior for a large class of these geometric objects has been understood only recently with the introduction, by Gutierrez and Sotomayor, of the notion of structural stability and genericity, originated from differential equations and dynamical systems. Their original works [16] and [10] are also presented in [11].

The global behavior of other geometric structures of the theory of surfaces immersed in \mathbb{R}^3 , such as asymptotic lines [4] and lines of mean curvature [7], is still the subject of current works.

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General aspects of the curvature theory for surfaces in \mathbb{R}^4 are presented in the works of Forsyth [3], Wong [18], Little [12] and Asperti [1].

The ideas of Gutierrez and Sotomayor have also been applied to some aspects of this theory. Such as the description of generic singularities of asymptotic lines [5], the study of principal lines on minimal surfaces [8], and the analysis of generic singularities of lines of curvature, [9] and [15]. Recently the global behavior of lines of axial curvature on surfaces immersed in \mathbb{R}^4 was studied by Garcia and Sotomayor [6].

The main feature of this paper is the study of the global generic structure of mean directionally curved lines on surfaces immersed in \mathbb{R}^4 . Along these lines the second fundamental form points in the direction of the mean curvature vector. A review of properties of the first and second fundamental forms, the ellipse of curvature and related geometric objects is presented below. Afterwards, the main conclusions and the structure of this paper are formulated.

In this paper immersions are assumed to be C^∞ . Nevertheless the results can be adapted for C^r immersions, $r \geq 4$. Let $\alpha: M \rightarrow \mathbb{R}^4$ be an immersion of a smooth and oriented surface into \mathbb{R}^4 , which is endowed with the euclidean inner product $\langle \cdot, \cdot \rangle$ and oriented by a once for all fixed orientation. Denote respectively by TM and NM the tangent and the normal bundles of α and by T_pM and N_pM the respective fibers, i.e., the tangent and the normal planes at $p \in M$. Let $\{N_1, N_2\}$ be a frame of vector fields orthonormal to α . Assume that (u, v) is a positive chart of M and that $\{\alpha_u, \alpha_v, N_1, N_2\}$ is a positive frame of \mathbb{R}^4 . In such a chart (u, v) the first fundamental form of α , I_α , is given by

$$I = I_\alpha = \langle d\alpha, d\alpha \rangle = E du^2 + 2F du dv + G dv^2,$$

where $E = \langle \alpha_u, \alpha_u \rangle$, $F = \langle \alpha_u, \alpha_v \rangle$ and $G = \langle \alpha_v, \alpha_v \rangle$. The second fundamental form of α , II_α , is defined in terms of the NM -valued quadratic form

$$II = II_\alpha = \langle d^2\alpha, N_1 \rangle N_1 + \langle d^2\alpha, N_2 \rangle N_2 = II_{1,\alpha} N_1 + II_{2,\alpha} N_2,$$

where

$$II_i = II_{i,\alpha} = e_i du^2 + 2f_i du dv + g_i dv^2,$$

$e_i = \langle \alpha_{uu}, N_i \rangle$, $f_i = \langle \alpha_{uv}, N_i \rangle$, and $g_i = \langle \alpha_{vv}, N_i \rangle$, for $i = 1, 2$.

The following functions are associated to α (see [12]):

1. The *mean curvature vector* of α

$$H = H_\alpha = H_1 N_1 + H_2 N_2,$$

where

$$H_i = H_{i,\alpha} = \frac{Eg_i - 2Ff_i + Ge_i}{2(EG - F^2)}, \quad \text{for } i = 1, 2.$$

2. The *normal curvature* of α

$$k_N = k_{N,\alpha} = \frac{E(f_1g_2 - f_2g_1) - F(e_1g_2 - e_2g_1) + G(e_1f_2 - e_2f_1)}{2(EG - F^2)}.$$

3. The *resultant* Δ of $II_{1,\alpha}$ and $II_{2,\alpha}$

$$\Delta = \Delta_\alpha = \frac{1}{4(EG - F^2)} \begin{vmatrix} e_1 & 2f_1 & g_1 & 0 \\ e_2 & 2f_2 & g_2 & 0 \\ 0 & e_1 & 2f_1 & g_1 \\ 0 & e_2 & 2f_2 & g_2 \end{vmatrix}.$$

4. The *normal curvature vector* of α defined by $\eta: TM \rightarrow NM$, where $\eta(p, v) = \frac{II(p,v)}{I(p,v)}$.

The image of the unitary circle S^1 of T_pM by $\eta(p): T_pM \rightarrow N_pM$ describes an ellipse in N_pM called *ellipse of curvature* of α at p and denoted by $\varepsilon_\alpha(p)$. This ellipse may degenerate into a line segment, a circle or a point. The center of the ellipse of curvature is the mean curvature vector H and their area is given by $\frac{\pi}{2} |k_N(p)|$. The map $\eta(p)$ restricted to S^1 , being quadratic, is a double covering of the ellipse of curvature. Thus every point on the ellipse corresponds to two diametrically opposed points on the unitary tangent circle. The ellipse of curvature is invariant by rotations in the tangent and normal planes.

A point $p \in M$ is called a *minimal point* of α if $H(p) = 0$ and it is called an *inflection point* of α if $\Delta(p) = k_N(p) = 0$. It follows that $p \in M$ is an inflection point if and only if its ellipse of curvature is a radial line segment.

Based in the above results we have: from any well-defined continuous choice of points on the ellipse of curvature, continuous *tangent direction fields* may be constructed on M . If the construction fails for special points we say that they are *singular points* of the direction field. Consider the following construction.

The line through the mean curvature vector $H(p)$ meets the ellipse of curvature $\varepsilon_\alpha(p)$ at two points. This construction induces two orthogonal directions on T_pM . Making this construction for all $p \in M$ we define two direction fields on M , called *H-direction fields*. The singularities of these fields, called *H-singularities*, are the points where either $H = 0$ (minimal

points) or at which the ellipse of curvature becomes a radial line segment (inflection points). The set of H -singularities will be denoted by $S(\alpha)$.

A *mean directionally curved line* is a regular curve $\varphi: (a, b) \rightarrow M$ which at each of its points is tangent to an H -direction and it contains any regular curve with this property which intersects it. The mean directionally curved lines and their singularities are assembled into the *mean directional configuration*.

In this work, the notion of principal structural stability of immersions of surfaces into \mathbb{R}^3 is extended for the mean directional configurations of the case \mathbb{R}^4 . Sufficient conditions are provided to extend to the present setting the Theorem on Structural Stability for Principal Configurations due to Gutierrez and Sotomayor [11]. Two local cases, treated in detail here, are essential for this extension: H -singularities with their separatrix structure and closed (i.e., the cycles or periodic) mean directionally curved lines.

This paper is organized as follows:

In Section 2 we analyse the differential equation of mean directionally curved lines in an arbitrary chart. It is shown that this differential equation fits into the class of quadratic or binary differential equations.

Section 3 is devoted to the analysis of H -singularities. For this purpose the differential equation of mean directionally curved lines is written in a Monge chart. The H -singularity condition is explicitly stated in terms of the coefficients of second order jet of the two functions which represent the immersion in a Monge chart. The condition of stability at H -singularity is expressed in an invariant form involving the third order jets.

In Section 4 the derivative of first return map along a mean directionally curved cycle is established. It consists of an integral involving geometric functions along the cycle.

In Section 5 the results presented in Sections 3 and 4 are put together to provide sufficient conditions for mean directional structural stability.

In Section 6 we analyse the case where the surface M is immersed in S^3 . In this case H -singularities only appear at inflection points. A correspondence between mean directionally curved lines on M and lines of mean curvature on $\phi^{-1}(M)$ is established, where $\phi: \mathbb{R}^3 \rightarrow S^3$ is the stereographic projection. Furthermore it is shown that, for surfaces in S^3 , the quartic differential equation of lines of axial curvature factors into the quadratic differential equation of mean directionally curved lines and the quadratic differential equation of asymptotic lines.

2. Differential equation for mean directionally curved lines

The differential equation of mean directionally curved lines is given by

$$(2.1) \quad \eta = \mu H,$$

where $\mu \in \mathbb{R}$. Eliminating μ in (2.1) we have a quadratic differential equation of the form

$$(2.2) \quad A(u, v) du^2 + 2B(u, v) du dv + C(u, v) dv^2 = 0,$$

where

$$(2.3) \quad A = A(u, v) = (e_1g_2 - e_2g_1)E + 2(e_2f_1 - e_1f_2)F,$$

$$(2.4) \quad B = B(u, v) = (f_1g_2 - f_2g_1)E + (e_2f_1 - e_1f_2)G,$$

$$(2.5) \quad C = C(u, v) = 2(f_1g_2 - f_2g_1)F + (e_2g_1 - e_1g_2)G.$$

The H -singularities are determined by $A = B = C = 0$ in (2.2). But it is immediate that the equation $EC = 2FB - GA$ holds.

We have established the following proposition.

Proposition 2.1. *Let $\alpha: M \rightarrow \mathbb{R}^4$ be an immersion of a smooth and oriented surface into \mathbb{R}^4 . With the above notations we have:*

1. *The differential equation of mean directionally curved lines is given by (2.2).*
2. *The H -singularities of α are given by $A = B = 0$, where A and B are defined in (2.3) and (2.4) respectively.*

Proposition 2.2. *$A(p) = B(p) = 0$ if and only if p is either a minimal point or an inflection point of M .*

Proof: It is enough to prove the proposition for the isothermic coordinates where $E = G = \lambda^2$ and $F = 0$. In this case equation (2.2) has the form

$$A_1(u, v) du^2 + 2B_1(u, v) du dv - A_1(u, v) dv^2 = 0,$$

where

$$A_1 = e_1g_2 - e_2g_1,$$

and

$$B_1 = f_1(e_2 + g_2) - f_2(e_1 + g_1).$$

Thus

$$0 = A_1 f_1 + B_1 g_1 = (e_1 + g_1)(f_1 g_2 - f_2 g_1)$$

and

$$0 = A_1 f_2 + B_1 g_2 = (e_2 + g_2)(f_1 g_2 - f_2 g_1).$$

If $f_1 g_2 - f_2 g_1 \neq 0$ then $e_1 + g_1 = e_2 + g_2 = 0$ and this implies that $H(p) = 0$, i.e., p is a minimal point. If $f_1 g_2 - f_2 g_1 = 0$ then $e_1 f_2 - e_2 f_1 = e_1 g_2 - e_2 g_1 = f_1 g_2 - f_2 g_1 = 0$ and this implies that p is an inflection point. The reciprocal is immediate. \square

From the Proposition 2.2, $S(\alpha) = S_1(\alpha) \cup S_2(\alpha)$, where $S_1(\alpha)$ is the set of minimal points of M and $S_2(\alpha)$ is the set of inflection points of M .

The differential equation (2.2) is also obtained equivalently by

$$(2.6) \quad \text{Jac}\{\text{Jac}(II_1, II_2), I\} = 0,$$

where

$$\text{Jac}(\cdot, \cdot) = \frac{\partial(\cdot, \cdot)}{\partial(du, dv)}.$$

Equation (2.6) suggests the definition of the quadratic form

$$J = J_\alpha = \text{Jac}(II_1, II_2)$$

$$= (e_1 f_2 - e_2 f_1) du^2 + (e_1 g_2 - e_2 g_1) du dv + (f_1 g_2 - f_2 g_1) dv^2.$$

Denote by $\lambda_1(p)$ and $\lambda_2(p)$, with $\lambda_1(p) \leq \lambda_2(p)$, the extreme values of J as w ranges on the unitary circle of $T_p M$. These extreme values are the roots of

$$\lambda^2 - 2k_N \lambda + \Delta = 0.$$

In fact, for $w = a \frac{\partial}{\partial u} + b \frac{\partial}{\partial v}$, $J(w, w)$ restricted to $I(w, w) = 1$ is stationary if and only if, for some (Lagrange multiplier) λ ,

$$(2.7) \quad \frac{\partial}{\partial a}(J(w, w)) = \lambda \frac{\partial}{\partial a}(I(w, w)) \quad \text{and} \quad \frac{\partial}{\partial b}(J(w, w)) = \lambda \frac{\partial}{\partial b}(I(w, w)).$$

Performing the differentiation and eliminating a and b , λ will satisfy the above equation. Thus the unitary vectors $\pm e_1(p)$ and $\pm e_2(p)$, at which the extreme values $\lambda_1(p)$ and $\lambda_2(p)$ are attained, are well defined. They are mutually orthogonal for p outside the set $S(\alpha)$, at which $\lambda_1(p) = \lambda_2(p)$. Now, performing the differentiation of (2.7) and eliminating λ gives that, in the chart (u, v) , the components a and b of $\pm e_1(p)$ and $\pm e_2(p)$ satisfies

$$A(u, v)a^2 + 2B(u, v)ab + C(u, v)b^2 = 0,$$

i.e., the differential equation of mean directionally curved lines (2.2). We define $l_H(\alpha) = \mathbb{R}(\pm e_1)$ and $L_H(\alpha) = \mathbb{R}(\pm e_2)$.

If $p \notin S(\alpha)$ then $(B^2 - AC)(p) > 0$. This implies the existence of two orthogonal solutions of the differential equation of mean directionally curved lines (2.2). Thus in a neighborhood of this point there exist two families of orthogonal curves. Under the orientability hypothesis it is possible to extend these lines to the whole M . Each family defines a foliation, denoted by $F_H(\alpha)$ and $f_H(\alpha)$ respectively, on the surface without the H -singularities. The foliation $F_H(\alpha)$ ($f_H(\alpha)$) is associated to $L_H(\alpha)$ ($l_H(\alpha)$). Each isolated H -singularity defines an isolated singularity of both foliations.

The *mean directional configuration* of an immersion $\alpha: M \rightarrow \mathbb{R}^4$ is the triple

$$\mathcal{H}(\alpha) = \{S(\alpha), F_H(\alpha), f_H(\alpha)\}.$$

It synthesizes the qualitative properties of the foliations $F_H(\alpha)$ and $f_H(\alpha)$ and represents the way their lines approach the H -singularity set. It is a natural analog of the principal configuration of an immersion $\alpha: M \rightarrow \mathbb{R}^3$.

3. Mean directional configurations near H -singularities

Let $p \in M$ be an H -singularity. We say that p is a *transversal H -singularity* if

$$J(A, B)(p) = \frac{\partial(A, B)}{\partial(u, v)}(p) \neq 0,$$

where A and B are defined in (2.3) and (2.4) respectively. This condition means that the curves $A = 0$ and $B = 0$, whose intersection defines the H -singularities, are regular and meet transversally at p . It follows that transversal H -singularities are isolated.

Consider the surface M in a Monge chart, i.e., the surface M is the graph of the map $\alpha(u, v) = (u, v, S(u, v), R(u, v))$, where S and R are C^∞ functions defined on a neighborhood $U \subset \mathbb{R}^2$ of $(0, 0)$ with the conditions

$$S(0, 0) = R(0, 0) = S_u(0, 0) = R_u(0, 0) = S_v(0, 0) = R_v(0, 0) = 0.$$

For each point $\alpha(u, v) \in M$ the tangent plane $T_{\alpha(u, v)}M$ is generated by $\{\alpha_u(u, v) = (1, 0, S_u(u, v), R_u(u, v)), \alpha_v(u, v) = (0, 1, S_v(u, v), R_v(u, v))\}$ and the normal plane $N_{\alpha(u, v)}M$ is generated by $\{\bar{N}_1, \bar{N}_2\}$, where

$$\bar{N}_1 = (-S_u, -S_v, 1, 0)$$

and

$$\begin{aligned} \bar{N}_2 = & (-R_u(1 + S_v^2) + S_u S_v R_v, \\ & -R_v(1 + S_u^2) + S_u S_v R_v, -S_u R_u - S_v R_v, 1 + S_u^2 + S_v^2). \end{aligned}$$

Therefore

$$\begin{aligned} E &= \langle \alpha_u, \alpha_u \rangle, & F &= \langle \alpha_u, \alpha_v \rangle, & G &= \langle \alpha_v, \alpha_v \rangle, \\ e_i &= \langle \alpha_{uu}, N_i \rangle, & f_i &= \langle \alpha_{uv}, N_i \rangle, & g_i &= \langle \alpha_{vv}, N_i \rangle, \end{aligned}$$

where

$$N_i = \frac{\bar{N}_i}{\|\bar{N}_i\|},$$

for $i = 1, 2$.

Write the Taylor's expansion of the functions S and R near $(0, 0)$

$$(3.1) \quad S(u, v) = \frac{s_{20}}{2}u^2 + s_{11}uv + \frac{s_{02}}{2}v^2 + \frac{a}{6}u^3 + \frac{d}{2}u^2v + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + O(4),$$

$$(3.2) \quad R(u, v) = \frac{r_{20}}{2}u^2 + r_{11}uv + \frac{r_{02}}{2}v^2 + \frac{\bar{a}}{6}u^3 + \frac{\bar{d}}{2}u^2v + \frac{\bar{b}}{2}uv^2 + \frac{\bar{c}}{6}v^3 + O(4).$$

Thus the coefficients of the first and the second fundamental forms in a Monge chart are given by

$$E = 1 + O(2), \quad F = O(2), \quad G = 1 + O(2),$$

$$e_1 = s_{20} + au + dv + O(2), \quad f_1 = s_{11} + du + bv + O(2), \quad g_1 = s_{02} + bu + cv + O(2),$$

$$e_2 = r_{20} + \bar{a}u + \bar{d}v + O(2), \quad f_2 = r_{11} + \bar{d}u + \bar{b}v + O(2), \quad g_2 = r_{02} + \bar{b}u + \bar{c}v + O(2).$$

Define

$$J = s_{02}r_{20} - s_{20}r_{02},$$

$$K = s_{02}\bar{a} + r_{20}b - s_{20}\bar{b} - r_{02}a,$$

$$L = s_{02}\bar{d} + r_{20}c - s_{20}\bar{c} - r_{02}d,$$

$$M = s_{11}(r_{20} + r_{02}) - r_{11}(s_{20} + s_{02}),$$

$$N = s_{11}(\bar{a} + \bar{b}) - r_{11}(a + b) + (r_{20} + r_{02})d - (s_{20} + s_{02})\bar{d},$$

$$P = s_{11}(\bar{c} + \bar{d}) - r_{11}(c + d) + (r_{20} + r_{02})b - (s_{20} + s_{02})\bar{b}.$$

The differential equation of mean directionally curved lines (2.2) in a Monge chart has the form

$$(3.3) \quad C(u, v) dv^2 + 2B(u, v) du dv + A(u, v) du^2 = 0,$$

where

$$C(u, v) = J + Ku + Lv + Q_1(u, v),$$

$$B(u, v) = M + Nu + Pv + Q_2(u, v),$$

$$A(u, v) = -J - Ku - Lv + Q_3(u, v),$$

and Q_1, Q_2 and Q_3 are of order $O(2)$.

From (3.3) the condition for $(0, 0)$ to be an H -singularity is that

$$J = s_{02}r_{20} - s_{20}r_{02} = 0$$

and

$$M = s_{11}(r_{20} + r_{02}) - r_{11}(s_{20} + s_{02}) = 0.$$

Consider the tangent projective bundle $PM = \{TM - \{0\}/\{v = rv, r \neq 0\}\}$ of M . The natural projection is given by $\pi: PM \rightarrow M$. We put $p = dv/du$ and $q = du/dv$. Thus PM can be parametrized by charts $(u, v; p)$ and $(u, v; q)$. On PM consider the surface W defined by the differential equation of mean directionally curved lines. In coordinates $(u, v; p)$ this surface is defined by $T^{-1}(0)$ where, from (3.3),

$$(3.4) \quad T(u, v; p) = (J + Ku + Lv + Q_1)p^2 + 2(M + Nu + Pv + Q_2)p - (J + Ku + Lv + Q_3).$$

If $(0, 0)$ is an H -singularity then $(0, 0; p) \in W$, for all p . Furthermore the surface W is smooth in a neighborhood of the p -axis if and only if $(0, 0)$ is a transversal H -singularity, according to [11] and [2].

Suppose that $(0, 0)$ is a transversal H -singularity. In this case consider the following vector field on W

$$(3.5) \quad X(u, v; p) = (T_p(u, v; p), pT_p(u, v; p), -(T_u(u, v; p) + pT_v(u, v; p))).$$

The vector field X has generically either one or three singularities on p -axis, which are of type saddle or node, according to [11].

Proposition 3.1. *Suppose the case where $K = 0$. With the above construction $p \in M$ is a transversal H -singularity if and only if*

$$(3.6) \quad NL \neq 0.$$

Proof: The condition of transversality is given by $2(KP - NL) \neq 0$, which is equivalent to (3.6). \square

Proposition 3.2. *Suppose the case where $K = 0$. Consider a transversal H -singularity as above. Then we have:*

1. *If*

$$(3.7) \quad \frac{N}{L} > \frac{1}{2} \left[\left(\frac{P}{L} \right)^2 + 1 \right],$$

then the H -singularity is of type S_1 (see Figure 1).

2. *If*

$$(3.8) \quad \frac{1}{2} \left[\left(\frac{P}{L} \right)^2 + 1 \right] > \frac{N}{L} > 0, \quad 2N \neq L,$$

then the H -singularity is of type S_2 (see Figure 1).

3. *If*

$$(3.9) \quad \frac{N}{L} < 0,$$

then the H -singularity is of type S_3 (see Figure 1).

Proof: The singularities of the vector field X (3.5) on the p -axis are given by

$$\varphi(p) = T_u(0, 0; p) + pT_v(0, 0; p) = 0.$$

But

$$0 = \varphi(p) = p \left[Lp^2 + 2Pp + 2N - L \right],$$

whose roots are

$$p_0 = 0,$$

$$p_1 = \frac{-P}{L} + \sqrt{\left(\frac{P}{L} \right)^2 - \frac{2N}{L} + 1}$$

and

$$p_2 = \frac{-P}{L} - \sqrt{\left(\frac{P}{L}\right)^2 - \frac{2N}{L} + 1}.$$

The Jacobian matrix of the vector field X at $(0, 0; p)$ is given by

$$\begin{pmatrix} 2N & 2(Lp + P) & 0 \\ 2Np & 2p(Lp + P) & 0 \\ * & * & -(3Lp^2 + 4Pp + 2N - L) \end{pmatrix}.$$

Therefore the Jacobian determinant of X at $p_0 = 0$ is $JX(0) = 2N(L - 2N)$. Thus, we have

$$\frac{JX(0)}{L^2} = \frac{2N}{L} \left(1 - \frac{2N}{L}\right).$$

The vector field X has:

1. One singularity which is a saddle if (3.7) holds. In fact, in this case we have

$$\frac{2N}{L} > \left(\frac{P}{L}\right)^2 + 1,$$

and this implies that $JX(0)$ is negative and the only singular point of X is a saddle.

2. Three singularities which are one node and two saddles if (3.8) holds. In fact, in this case there are two possibilities at $p_0 = 0$:

(a) If

$$1 > \frac{2N}{L} > 0,$$

then $JX(0)$ is positive and the origin is a hyperbolic node.

(b) If

$$\frac{2N}{L} > 1,$$

then $JX(0)$ is negative and the origin is a saddle.

As any singular point of X can be taken to the origin by appropriate change of coordinates, the analysis above imply that one of the singularities of X , that in the middle, is a node and the other two are saddles.

3. Three singularities which are three saddles if (3.9) holds. In fact, in this case we have

$$\frac{2N}{L} < 0,$$

and this implies that $JX(0)$ is negative and the origin is a saddle. The singularities p_1 and p_2 are also saddle points.

The local configurations of the H -singularities S_1 , S_2 and S_3 illustrated in the Figure 1 are obtained by projection onto the uv -plane of the local configurations of the above singularities, according to [11].

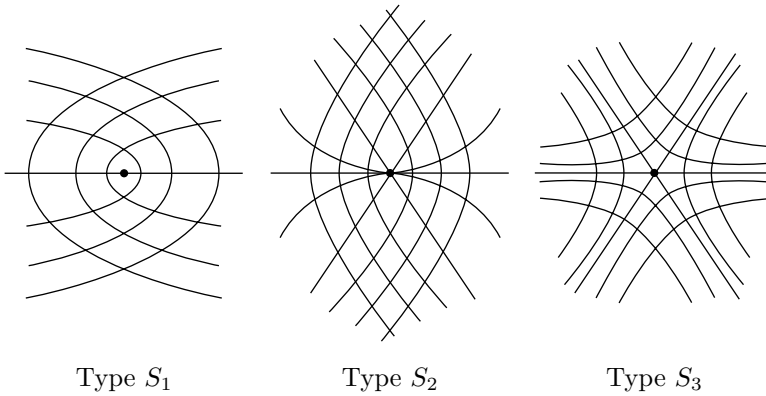


FIGURE 1. H -singularities of S -type.

□

Remark 3.3. The index $i = 1, 2, 3$ of S_i denotes the number of *separatrices* of the H -singularity. These are mean directionally curved lines which approach of the H -singularity and which separate regions of different patterns of approach to it. Figure 1 shows the local mean directional configuration of three different types of H -singularity denoted by S_1 , S_2 and S_3 and called *H -singularity of S -type*.

Proposition 3.4. *Suppose that $p \in M$ is an H -singularity with $H(p) = 0$ and $k_N(p) \neq 0$. Then (3.1) and (3.2) can be written as*

$$(3.10) \quad S(u, v) = \frac{A}{2}(u^2 - v^2) + \frac{a}{6}u^3 + \frac{d}{2}u^2v + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + O(4),$$

$$(3.11) \quad R(u, v) = Cuv + \frac{\bar{a}}{6}u^3 + \frac{\bar{d}}{2}u^2v - \frac{\bar{a}}{2}uv^2 + \frac{\bar{c}}{6}v^3 + O(4).$$

Proof: Through an appropriate rotation in the normal plane it is possible to write $r_{20} = r_{02} = 0$. This implies that $s_{11} = 0$. Through an appropriate rotation in the uv -plane it is possible to write $\bar{b} = -\bar{a}$. Substituting these expressions in (3.1) and (3.2) the proposition is proved. □

From (3.10) and (3.11) we have

$$\begin{aligned} J &= 0, \\ K &= 0, \\ L &= -A(\bar{c} + \bar{d}), \\ M &= 0, \\ N &= -C(a + b), \\ P &= -C(c + d). \end{aligned}$$

Proposition 3.5. *Suppose that $p \in M$ is an H -singularity which is an inflection point and that $H(p) \neq 0$. Then (3.1) and (3.2) can be written as*

$$(3.12) \quad S(u, v) = \frac{A}{2}(u^2 + v^2) + Buv + O(3),$$

$$(3.13) \quad R(u, v) = \frac{\bar{a}}{6}u^3 + \frac{\bar{d}}{2}u^2v + \frac{\bar{a}}{2}uv^2 + \frac{\bar{c}}{6}v^3 + O(4).$$

Proof: Through an appropriate rotation in the normal plane it is possible to write $r_{20} = r_{02} = r_{11} = 0$. Through an appropriate rotation in the uv -plane it is possible to write $\bar{b} = \bar{a}$. Substituting these expressions in (3.1) and (3.2) the proposition is proved. \square

From (3.12) and (3.13) we have

$$\begin{aligned} J &= 0, \\ K &= 0, \\ L &= A(\bar{d} - \bar{c}), \\ M &= 0, \\ N &= 2(\bar{a}B - A\bar{d}), \\ P &= B(\bar{c} + \bar{d}) - 2A\bar{a}. \end{aligned}$$

Remark 3.6. As a consequence of Propositions 3.1, 3.2, 3.4 and 3.5 we have the description of H -singularities of S -type in terms of the coefficients of third order jets.

4. Mean directionally curved cycles

In terms of geometric invariants, here is established the formula of the first derivative of the first return map of a periodic mean directionally curved line, called here *mean directionally curved cycle*. This first return map, denoted by π , is called holonomy in foliation theory and Poincaré map in dynamical systems.

A mean directionally curved cycle is called *hyperbolic* if the first derivative of the first return map at the fixed point is different from one.

Let $\gamma: I \rightarrow M$ be a mean directionally curved cycle of the foliation $f_H(\alpha)$, parametrized by arc length s and of length L . We take $\{T_1(s) = \gamma'(s), T_2(s)\}$ an orthonormal frame of $T_{\gamma(s)}M$, $\{N_1(s), N_2(s)\}$ an orthonormal frame of $N_{\gamma(s)}M$ such that $\{T_1, T_2, N_1, N_2\}$ be a positive frame of \mathbb{R}^4 . We choose the frame $\{N_1(s), N_2(s)\}$ with the following properties

$$\eta(\gamma(s), T_1(s)) = a(s)H_1(s)N_1(s)$$

and

$$\eta(\gamma(s), T_2(s)) = b(s)H_1(s)N_1(s).$$

This means that the functions a and b are linked by a relation of the form $a + b \equiv 2$.

The tangent projection of the vector field $\gamma''(s)$ along γ is given by $k_g(s)T_2(s)$. Define the *geodesic curvature* of γ by $k_g(s) = \langle \gamma''(s), T_2(s) \rangle$. Now the normal component of $\gamma''(s)$ is given by $\eta(\gamma(s), T_1(s))$. Define the *geodesic torsion vector* by $\tau_g = \tau_{g,1}N_1 + \tau_{g,2}N_2$ and the *normal torsion* of the frame $\{N_1, N_2\}$ by $\tau_n = \langle N'_1, N_2 \rangle$.

In a similar way as in the case of surfaces in \mathbb{R}^3 (see [17, p. 131]), we will obtain a system of equations of the *Darboux frame* $\{T_1(s), T_2(s), N_1(s), N_2(s)\}$ associated to γ :

$$T'_1(s) = k_g(s)T_2(s) + \eta(\gamma(s), T_1(s)) = k_g(s)T_2(s) + a(s)H_1(s)N_1(s);$$

$$T'_2(s) = -k_g(s)T_1(s) + \tau_g(s) = -k_g(s)T_1(s) + \tau_{g,1}(s)N_1(s) + \tau_{g,2}(s)N_2(s);$$

$$N'_1(s) = -a(s)H_1(s)T_1(s) - \tau_{g,1}(s)T_2(s) + \tau_n(s)N_2(s);$$

$$N'_2(s) = -\tau_{g,2}(s)T_2(s) - \tau_n(s)N_1(s).$$

Here $H = H_1N_1$ is the mean curvature vector.

At the point $\gamma(s)$, the intersection of the surface M with the hyperplane generated by $\{T_2(s), N_1(s), N_2(s)\}$ is a curve Γ_s , tangent to $T_2(s)$

at $\gamma(s)$. This curve can be parametrized by

$$\Gamma_s(t) = \gamma(s) + tT_2(s) + V_1(s, t)N_1(s) + V_2(s, t)N_2(s)$$

and its curvature is given by $\eta(\gamma(s), T_2(s))$. Thus we have the following expressions

$$V_1(s, t) = \frac{b(s)H_1(s)}{2}t^2 + \bar{A}(s, t)\frac{t^3}{6} + O(4)$$

and

$$V_2(s, t) = \bar{B}(s, t)\frac{t^3}{6} + O(4),$$

with $\bar{A}(s) = \bar{A}(s, 0)$ and $\bar{B}(s) = \bar{B}(s, 0)$.

Lemma 4.1. *Let γ be a mean directionally curved cycle of the foliation $f_H(\alpha)$, parametrized by arc length s and of length L . We take the orthonormal positive Darboux frame along γ . Then the expression*

$$(4.1) \quad \alpha(s, t) = \gamma(s) + tT_2(s) + \left[\frac{b(s)H_1(s)}{2}t^2 + \bar{A}(s, t)\frac{t^3}{6} + O(4) \right] N_1(s) + \left[\bar{B}(s, t)\frac{t^3}{6} + O(4) \right] N_2(s),$$

defines a local chart of M , L periodic in s , in a neighborhood of γ .

Proof: The lemma follows from the Inverse Function Theorem applied to the map

$$\alpha(s, t, v_1, v_2) = \gamma(s) + tT_2(s) + v_1N_1(s) + v_2N_2(s),$$

which defines a tubular neighborhood of γ . □

Differentiation of the equation (4.1) gives that

$$E(s, 0) = G(s, 0) = 1, \quad F(s, 0) = 0,$$

$$(4.2) \quad e_1(s, 0) = a(s)H_1(s), \quad f_1(s, 0) = \tau_{g,1}(s), \quad g_1(s, 0) = b(s)H_1(s), \\ e_2(s, 0) = g_2(s, 0) = 0, \quad f_2(s, 0) = \tau_{g,2}(s).$$

We note that

$$k_N(s) = k_N(s, 0) = \tau_{g,2}(s)(a(s) - b(s))H_1(s)$$

and

$$\Delta(s) = \Delta(s, 0) = -(\tau_{g,2}(s))^2 a(s)b(s)(H_1(s))^2.$$

Since $H_1(s) \neq 0$ and $(a(s))^2 + (b(s))^2 \neq 0$, for all $s \in [0, L]$, we have that $\tau_{g,2}(s) \neq 0$, for all $s \in [0, L]$.

Lemma 4.2. *Let γ be a mean directionally curved line and consider the chart (s, t) given in (4.1). Then the orthonormal frame $\{N_1, N_2\}$ of the normal bundle satisfies the following equations*

$$(4.3) \quad (N_1)_t(s, 0) = -\tau_{g,1}(s)T_1(s) - b(s)H_1(s)T_2(s) + a_{12}^3(s)N_2(s)$$

and

$$(4.4) \quad (N_2)_t(s, 0) = -\tau_{g,2}(s)T_1(s) - a_{12}^3(s)N_1(s),$$

where $a_{12}^3(s) = a_{12}^3(s, 0) = \langle (N_1)_t(s, 0), N_2(s, 0) \rangle$ is the normal torsion of the frame $\{N_1, N_2\}$ associated to the mean directionally curved line orthogonal to γ at the point $\gamma(s)$.

Proof: In a chart (s, t) the following equations hold

$$(N_1)_t = \frac{g_1F - f_1G}{EG - F^2}\alpha_s + \frac{f_1F - g_1E}{EG - F^2}\alpha_t + a_{12}^3N_2$$

and

$$(N_2)_t = \frac{g_2F - f_2G}{EG - F^2}\alpha_s + \frac{f_2F - g_2E}{EG - F^2}\alpha_t - a_{12}^3N_1.$$

Using (4.2) the lemma is proved. □

Direct calculation shows that the following equations hold

$$E_t(s, 0) = -2k_g, \quad F_t(s, 0) = G_t(s, 0) = 0,$$

$$(e_1)_t(s, 0) = \tau'_{g,1} - \tau_{g,2}\tau_n - k_gH_1(a + b),$$

$$(f_1)_t(s, 0) = (bH_1)' + k_g\tau_{g,1} + \tau_{g,2}a_{12}^3,$$

$$(g_1)_t(s, 0) = \bar{A},$$

$$(e_2)_t(s, 0) = \tau_{g,1}\tau_n + \tau'_{g,2} - aH_1a_{12}^3,$$

$$(f_2)_t(s, 0) = k_g\tau_{g,2} - \tau_{g,1}a_{12}^3,$$

$$(g_2)_t(s, 0) = \bar{B} - bH_1a_{12}^3.$$

Lemma 4.3. *The function \bar{B} introduced in (4.1) is given by*

$$(4.5) \quad \bar{B} = 2(H_2)_t + H_1a_{12}^3(a + b) - \tau_{g,1}\tau_n - \tau'_{g,2}.$$

Proof: In the coordinates (s, t) we have

$$H_2 = \frac{Eg_2 - 2Ff_2 + Ge_2}{2(EG - F^2)}.$$

Differentiating H_2 and using the above equations, the lemma is proved. \square

Theorem 4.4. *Let γ be a mean directionally curved cycle of the foliation $f_H(\alpha)$, parametrized by arc length s and of length L . Then the derivative of the first return map is given by*

(4.6)

$$\pi'(0) = \exp \left[\frac{1}{2} \int_0^L \left[\frac{a(s) \left((H_2)_t(s) + H_1(s) a_{12}^3(s) \right) - \tau_{g,1}(s) \tau_n(s)}{\tau_{g,2}(s)} \right] ds \right].$$

Proof: The derivative of the first return map satisfies the following linear differential equation

$$\frac{d}{ds} \left(\frac{dt}{dt_0} \right) = - \frac{1}{2B} \frac{\partial A}{\partial t} \frac{dt}{dt_0}.$$

Therefore

(4.7)

$$\pi'(0) = \exp \left[\int_0^L \frac{-A_t(s, 0)}{2B(s, 0)} ds \right],$$

where the functions A and B are given in (2.3) and (2.4). Using (4.2), (4.3) and (4.4) we have

(4.8)

$$2B(s, 0) = -2H_1(s) \tau_{g,2}(s) (a(s) + b(s)),$$

and

(4.9)

$$A_t(s, 0) = H_1(s) \left[a(s) \bar{B}(s) - b(s) \left(\tau_{g,1}(s) \tau_n(s) + \tau'_{g,2}(s) \right) \right].$$

Substituting (4.5), (4.8) and (4.9) in (4.7) the theorem follows. \square

Remark 4.5. The corresponding formula for the first derivative of the first return map when γ is a mean directionally curved cycle of the foliation $F_H(\alpha)$ is given by

(4.10)

$$\pi'(0) = \exp \left[\frac{1}{2} \int_0^L \left[\frac{b(s) \left((H_2)_t(s) + H_1(s) a_{12}^3(s) \right) - \tau_{g,1}(s) \tau_n(s)}{\tau_{g,2}(s)} \right] ds \right].$$

Proposition 4.6. *Let $\alpha: M \rightarrow \mathbb{R}^4$ be an immersion of a smooth and oriented surface into \mathbb{R}^4 , and let γ be a mean directionally curved cycle of the foliation $f_H(\alpha)$, parametrized by arc length s and of length L . Consider a chart (s, t) as in the Lemma 4.1 and consider the deformation*

$$(4.11) \quad \beta_\epsilon(s, t) = \beta(\epsilon, s, t) = \alpha(s, t) + \epsilon \left[\frac{a(s)\tau_{g,2}(s)}{6} t^3 \right] \delta(t) N_2(s),$$

where $\delta = 1$ in neighborhood of $t = 0$, with small support and $a \not\equiv 0$. Then γ is a mean directionally curved cycle of β_ϵ , for all $\epsilon \geq 0$ small, and γ is a hyperbolic mean directionally curved cycle for β_ϵ , $\epsilon \neq 0$.

Proof: From direct calculation with the deformation β_ϵ it follows that γ is a mean directionally curved cycle for all β_ϵ , and at $t = 0$ we have

$$2(H_2)_t(s, 0) = \tau_{g,1}\tau_n + \tau'_{g,2} - 2H_1a^3_{12} + \bar{B} + \epsilon a\tau_{g,2}.$$

Therefore, assuming $a \not\equiv 0$, it results that

$$\frac{d}{d\epsilon} (\ln \pi'(0)) \Big|_{\epsilon=0} = \frac{1}{2} \int_0^L \frac{(a(s))^2 \tau_{g,2}(s)}{2\tau_{g,2}(s)} ds = \frac{1}{4} \int_0^L (a(s))^2 ds \neq 0. \quad \square$$

5. Mean directional structural stability

Let $I(M, \mathbb{R}^4)$ be the space of the immersions of M into \mathbb{R}^4 , where M is a compact, smooth and oriented surface, endowed with the Whitney topology.

Lemma 5.1. *Let M be a surface which is the graph of the map $\alpha(u, v) = (u, v, S(u, v), R(u, v))$, as in the Section 3. Suppose that $(0, 0)$ is an H -singularity. We call β_{ab} the immersion*

$$\beta(\alpha; u, v; a, b) = \left(u, v, S(u, v) + \frac{a}{2}u^2 + \frac{b}{2}v^2, R(u, v) + auv + \frac{b}{2}u^2 \right),$$

with $(a, b) \in V$, where V is a neighborhood of $(0, 0)$ and $(u, v) \in U$. Then there exists $D \subset U$ a compact disc on which

$$a_0(u, v) = \frac{\partial(C, B)}{\partial(a, b)}(u, v; 0, 0) \neq 0,$$

where C and B are obtained from the differential equation of mean directionally curved lines of β_{ab} .

Proof: The differential equation of mean directionally curved lines of β_{ab} is given by

$$C(u, v; a, b)(dv)^2 + 2B(u, v; a, b) du dv + A(u, v; a, b)(du)^2 = 0,$$

where

$$C = C(u, v; a, b) = \left[2(f_1g_2 - f_2g_1)F + (e_2g_1 - e_1g_2)G \right] (u, v; a, b)$$

and

$$B = B(u, v; a, b) = \left[(f_1g_2 - f_2g_1)E + (e_2f_1 - e_1f_2)G \right] (u, v; a, b).$$

The form of $A(u, v; a, b)$ is not important here.

Without loss of generality we have:

1. If the H -singularity is a minimal point then through an appropriate rotation in the normal plane it is possible to consider the frame $\{N_1, N_2\}$ and the principal axes of the ellipse of curvature as being parallels. This implies that $e_2 = g_2 = f_1 = 0$ and $f_2g_1 \neq 0$.
2. If the H -singularity is an inflection point then through an appropriate rotation in the normal plane it is possible to consider the ellipse of curvature on N_1 -direction. This implies that $e_2 = f_2 = g_2 = 0$, $g_1 \neq 0$ and $e_1 + g_1 \neq 0$.

From extensive calculation we have

$$a_0(0, 0) = \frac{\partial(C, B)}{\partial(a, b)}(0, 0; 0, 0) = (e_2 + g_1)(e_1 + g_1 + f_2) \neq 0.$$

Therefore there exists $D \subset U$ a compact disc on which

$$a_0(u, v) = \frac{\partial(C, B)}{\partial(a, b)}(u, v; 0, 0) \neq 0.$$

This ends the proof. □

Lemma 5.2. *Let M be a surface which is the graph of the map $\alpha(u, v) = (u, v, S(u, v), R(u, v))$. Suppose that $(0, 0)$ is an H -singularity and let β_{ab} be the immersion*

$$\beta(\alpha; u, v; a, b) = \left(u, v, S(u, v) + \frac{a}{2}u^2 + \frac{b}{2}v^2, R(u, v) + auv + \frac{b}{2}u^2 \right),$$

with $(a, b) \in V$, where V is a neighborhood of $(0, 0)$ and $(u, v) \in U$. Let $D \subset U$ be a compact disc on which

$$a_0(u, v) \neq 0,$$

as in the Lemma 5.1.

Call $V_{ab} = V_{ab}(D) \subset V$ the set of the pairs (a, b) for which all H -singularities of the immersion β_{ab} , with $(u, v) \in D$, satisfy the transversality condition. There is a small $\rho > 0$ such that the intersection of V_{ab} with the disc of radius ρ has full Lebesgue measure.

Proof: The set

$$S_{ab} = S(\beta_{ab}) = \left\{ (u, v; a, b) \in \mathbb{R}^2 \times \mathbb{R}^2 / C(u, v; a, b) = 0 = B(u, v; a, b) \right\}$$

is the locus of points where the immersion β_{ab} has H -singularities. Here C and B are the coefficients of the differential equation of mean directionally curved lines of β_{ab} .

At transversal H -singularities S_{ab} is a smooth surface, since at these points

$$\frac{\partial(C, B)}{\partial(u, v)} \neq 0.$$

Now we prove that if (a, b) is close to $(0, 0)$, the surface S_{ab} is regular even at non transversal H -singularities.

From the Lemma 5.1, there exists $D \subset U$ a compact disc on which

$$a_0(u, v) = \frac{\partial(C, B)}{\partial(a, b)}(u, v; 0, 0) \neq 0.$$

This shows that there exists a neighborhood $V_{ab} = \{|(a, b)| < \rho\}$ such that if $(u, v) \in D$ and $(a, b) \in V_{ab}$ then

$$\frac{\partial(C, B)}{\partial(a, b)}(u, v; a, b) \neq 0.$$

This concludes the proof of the smoothness of S_{ab} . Now the lemma is a consequence of Sard's Theorem, by the identification of V_{ab} with the regular values of the orthogonal projection of S_{ab} to the ab -plane. \square

Theorem 5.3. *The set $\alpha \in I(M, \mathbb{R}^4)$ of immersions such that all H -singularities are of S -type is open and dense in $I(M, \mathbb{R}^4)$.*

Proof: The conditions imposed to the immersion α near an H -singularity of S -type depend on the derivatives of order three and are open, so it implies that this set is open.

If the transversality condition holds for an H -singularity, a small perturbation on the parameters of the coordinate chart defines an H -singularity of S -type. Thus it is enough to consider H -singularities for which the transversality condition holds.

The set of H -singularities of an immersion α is compact, therefore it can be covered by a finite number of Monge charts. Using the local Lemma 5.2, which can be globalized by a standard argument, the theorem is proved. \square

An immersion α is said to be *mean directional stable* if it has a neighborhood $V(\alpha)$, such that for any $\beta \in V(\alpha)$ there exists a homeomorphism $h: M \rightarrow M$ mapping $S(\alpha)$ onto $S(\beta)$, mapping $F_H(\alpha)$ onto $F_H(\beta)$ and mapping $f_H(\alpha)$ onto $f_H(\beta)$.

Consider the subset $\Sigma \subset I(M, \mathbb{R}^4)$ of immersions α defined by the following conditions:

1. All H -singularities are S -type (Section 3).
2. All mean directionally curved cycles are hyperbolic (Section 4).
3. The limit set of every mean directionally curved line is contained in the set of H -singularities and mean directionally curved cycles.
4. There are no connections or self connections of H -singularity separatrices (Section 3).

Theorem 5.4. *The set Σ is open in $I(M, \mathbb{R}^4)$ and every $\alpha \in \Sigma$ is mean directional stable.*

Proof: We take on the projective bundle PM , as in Section 3, the surface W_α defined by the differential equation of mean directionally curved lines (2.2). In coordinates $(u, v; p)$ this surface is defined by $W_\alpha = T_\alpha^{-1}(0)$, where $T_\alpha(u, v; p) = C(u, v)p^2 + 2B(u, v)p + A(u, v)$. This surface is regular under the H -singularity hypothesis. The restriction of the natural projection $\pi: PM \rightarrow M$ to W_α is a double covering outside the preimage of the set $S(\alpha)$.

On PM we define the involution $I(u, v; [du : dv]) = (u, v; [dv : -du])$ which amounts to a rotation of lines by an angle $\pi/2$. The surface W_α is invariant under I . On $W_\alpha - \pi^{-1}(S(\alpha))$ we define the vector field X_α , which in the coordinates $(u, v; p)$ has the form $X_\alpha = ((T_\alpha)_p, p(T_\alpha)_p, -((T_\alpha)_u + p(T_\alpha)_v))$. This vector field has an unique regular extension to $\pi^{-1}(S(\alpha))$.

We consider the induced line field I_*X_α . Thus it is obtained a transversal pair $\{X_\alpha, I_*X_\alpha\}$ on $W_\alpha - \pi^{-1}(S(\alpha))$. Therefore we have defined a net outside $\pi^{-1}(S(\alpha))$, with the following properties: this net is invariant under I and by π projects to the net $(F_H(\alpha), f_H(\alpha))$.

With the above constructions this situation is connected with the case of the principal line fields and their canonical regions [11]. Thus the construction and continuation to a small neighborhood $V(\alpha)$ of α of canonical regions follow also from the openness and unique continuation, for β near α , of singularities and of cycles due to the hyperbolicity of these elements in the fields of the pair $\{X_\alpha, I_*X_\alpha\}$. This leads of the openness of Σ and gives uniquely a correspondence between H -singularities, separatrices, cycles and their intersections for $\{X_\alpha, I_*X_\alpha\}$ and

$\{X_\beta, I_*X_\beta\}$. The extension of this correspondence to a topological equivalence $H: W_\alpha \rightarrow W_\beta$ which, by projection, gives the topological equivalence $h: M \rightarrow M$ between $\mathcal{H}(\alpha)$ and $\mathcal{H}(\beta)$ is carried out as in the case of nets of asymptotic lines on surfaces immersed in \mathbb{R}^3 , according to [4]. \square

6. A special case

Let $\alpha: M \rightarrow S^3$ be an immersion of a smooth and oriented surface into S^3 . Consider the natural inclusion $i: S^3 \rightarrow \mathbb{R}^4$ and the composition $\alpha = i \circ \alpha$ still denoted by α . Assume that (u, v) is a positive chart of M and that $\{\alpha_u, \alpha_v, N_1, N_2\}$ is a positive frame of \mathbb{R}^4 , $\{N_1, N_2\}$ being a frame of vector fields orthonormal to α , where $N_1(p) \in T_p S^3$ and $N_2(p)$ is the inward normal to S^3 , for all $p \in M$. Thus $N_2 \equiv -\alpha$.

In such a chart (u, v)

$$e_2 = E, \quad f_2 = F \quad \text{and} \quad g_2 = G,$$

where E, F and G are the coefficients of the first fundamental form of α . It follows that $II_2 = I$. Now

$$\eta = \frac{II}{I} = \frac{II_1}{I}N_1 + \frac{II_2}{I}N_2 = \frac{II_1}{I}N_1 + N_2.$$

This implies that the ellipse of curvature is degenerate as a line segment on $N_2 = 1$, for all $p \in M$. In classic literature, this type of points are called semiumbilics and this result has been already obtained in [14]. We have

$$H_2 = \frac{Eg_2 - 2Ff_2 + Ge_2}{2(EG - F^2)} = 1,$$

for all $p \in M$. It follows that $H(p) \neq 0$, for all $p \in M$. So, if p is an H -singularity of M then p is an inflection point of M . In this point the ellipse of curvature becomes a point.

As an example consider the following construction. Let $\phi: \mathbb{R}^3 \rightarrow S^3 \subset \mathbb{R}^4$ be the stereographic projection given by

$$\phi(x, y, z) = \frac{1}{1+w}(x, y, z, w),$$

where $w = \frac{1}{2}(x^2 + y^2 + z^2 - 1)$. We recall that ϕ is conformal.

Let $\alpha: M \rightarrow \mathbb{R}^3$ be an immersion of a smooth and oriented surface M into \mathbb{R}^3 . Assume that (u, v) is a positive chart of M and that $\{\alpha_u, \alpha_v, N\}$ is a positive frame of \mathbb{R}^3 , where

$$N = \frac{\alpha_u \wedge \alpha_v}{\|\alpha_u \wedge \alpha_v\|}$$

is the normal vector field to α . Let $\bar{\alpha} = \phi \circ \alpha$ be the stereographic projection of M in S^3 and let $\tilde{\alpha} = i \circ \bar{\alpha}$ be the immersion of M into \mathbb{R}^4 , where $\{\bar{\alpha}_u, \bar{\alpha}_v, N_1, N_2\}$ is a positive frame of \mathbb{R}^4 , being

$$N_1 = \frac{d\phi(N)}{\|d\phi(N)\|}$$

and N_2 the inward unitary normal to S^3 .

From extensive calculation

$$\bar{E} = (1 + w)^{-2}E,$$

$$\bar{F} = (1 + w)^{-2}F,$$

$$\bar{G} = (1 + w)^{-2}G,$$

$$\bar{e}_1 = (1 + w)^{-2}[(1 + w)e + Et],$$

$$\bar{f}_1 = (1 + w)^{-2}[(1 + w)f + Ft],$$

$$\bar{g}_1 = (1 + w)^{-2}[(1 + w)g + Gt],$$

$$\bar{e}_2 = \bar{E},$$

$$\bar{f}_2 = \bar{F}, \quad \text{and}$$

$$\bar{g}_2 = \bar{G},$$

where the expressions without (with respectively) bar are associated to α ($\bar{\alpha}$ respectively), and $t = \langle \alpha, N \rangle$ is the *support function* of α .

Lines of axial curvature on surfaces immersed in \mathbb{R}^4 are lines along which the second fundamental form points in the direction of principal axes of the ellipse of curvature. The differential equation of lines of axial curvature is given by [6],

$$(6.1) \quad \text{Jac}(\|\eta - H\|^2, I) = 0,$$

which is a quartic differential equation.

Asymptotic lines on surfaces immersed in \mathbb{R}^4 are lines along which the second fundamental form points in the direction of the tangent lines to the ellipse of curvature. The differential equation of asymptotic lines is given by

$$(6.2) \quad \text{Jac}(II_1, II_2) = 0.$$

Remark 6.1. Through the above construction lines of principal curvature of α are carried over into asymptotic lines of $\bar{\alpha}$, lines of mean curvature of α are carried over into mean directionally curved lines of $\bar{\alpha}$ and umbilic points of α are carried over into inflection points of $\bar{\alpha}$. These results are also presented in [12] and [13].

Examples of immersions $\beta \in \Sigma$ can be obtained from the Remark 6.1. Consider the set Ψ of immersions of surfaces into \mathbb{R}^3 where every $\alpha \in \Psi$ is mean curvature structurally stable, according to [7]. Thus if $\alpha \in \Psi$ then $\bar{\alpha} \in \Sigma$, where $\bar{\alpha} = i \circ \phi \circ \alpha$ is as above.

Let $\alpha: M \rightarrow \mathbb{R}^4$ be an immersion of a smooth and oriented surface into \mathbb{R}^4 . The quartic differential equation (6.1) can be written as the product of two quadratic differential equations if the image of the surface M by α is contained into \mathbb{R}^3 , according to [6]. We have the following theorem.

Theorem 6.2. *Let $\alpha: M \rightarrow S^3$ be an immersion of a smooth and oriented surface into S^3 . Consider the natural inclusion $i: S^3 \rightarrow \mathbb{R}^4$ and the composition $\alpha = i \circ \alpha$ still denoted by α . Then the quartic differential equation (6.1) can be written as*

$$(6.3) \quad \text{Jac}\{\text{Jac}(II_1, I), I\} \text{Jac}(II_1, I) = 0,$$

where the first expression in (6.3) is the quadratic differential equation of mean directionally curved lines (2.6) and the second one is the quadratic differential equation of asymptotic lines (6.2).

Proof: From the coefficients of the first and the second fundamental forms listed above write the product $\text{Jac}\{\text{Jac}(II_1, I), I\} \text{Jac}(II_1, I)$ and compare the result with $\text{Jac}(\|\eta - H\|^2, I)$. The theorem is proved. \square

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