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WHEN IS EACH PROPER OVERRING OF R AN S(EIDENBERG)-DOMAIN?

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Abstract

A domain R is called a maximal "non-S" subring of a field L if $R \subset L$, R is not an S-domain and each domain T such that $R \subset T \subseteq L$ is an S-domain. We show that maximal "non-S" subrings R of a field L are the integrally closed pseudo-valuation domains satisfying $\dim(R) = 1$, $\dim_v(R) = 2$ and $L = \mathrm{qf}(R)$.

1. Introduction

Throughout this paper, $R \hookrightarrow S$ denotes an extension of commutative integral domains, $\mathrm{qf}(R)$ the quotient field of an integral domain R and $\mathrm{tr.deg}[S : R]$ the transcendence degree of $\mathrm{qf}(S)$ over $\mathrm{qf}(R)$. If $\mathrm{tr.deg}[S : R] = 0$, we say that S is algebraic over R. We recall that a ring R of finite *Krull dimension* n is a *Jaffard ring* if its *valuative dimension* (the limit of the sequence $(\dim(R[X_1, \dots, X_n]) - n,\ n \in \mathbb{N})$) $\dim_v(R)$, is also n. Prüfer domains and Noetherian domains are Jaffard domains. Recall that a domain R is an S-*domain* [**12**] if for each height 1 prime ideal p of R, the extended prime $p[X]$ in one indeterminate is also height 1 in $R[X]$. We assume familiarity with these concepts as in [**1**] and [**12**].

In [**3**], the author and M. Ben Nasr considered maximal non-Jaffard subrings of a field L, that is, the domains R where R is a non Jaffard domain and each ring T, $R \subset T \subseteq L$ is Jaffard. They characterized these domains in terms of pseudo-valuation domains. On the other hand the author and I. Yengui in [**11**] studied the domains R such that each domain contained between R and its quotient field is an S-domain. They are said to be absolutely S-domains. To complete this circle of ideas and to honor Seidenberg we deal with *maximal "non-*S*" subring(s) of a field*; that is, the domains R, where R is not an S-domain and each ring T, $R \subset T \subseteq L$ is an S-domain. First we show that if R is a maximal

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"non-S" subring of a field L, then $L = \mathrm{qf}(R)$. Hence, we may restrict ourselves to the case where $L = \mathrm{qf}(R)$. Let us recall some terminology: Let T be a ring, I an ideal of T, D be a subring of T/I and let R be the subring of T defined by the following *pullback construction*:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Following [**4**], we say that R is the *ring of the* (T, I, D) *construction* and we set $R := (T, I, D)$. Note that $R := (T, I, D)$ if and only it is contained in T and *shares the ideal* I with the ring T. The (T, I, D) constructions were considered for the first time in [**7**], in the contest of general pullback construction. Particularly the last construction to be noted here concerns the notion of a pseudo-valuation domain (for short, a PVD), which was introduced by J. R. Hedstrom and E. G. Houston [**9**] and has been studied subsequently in [**2**], [**5**], [**6**] and [**10**]. A domain R is said to be a PVD in case each prime ideal p of R is strongly prime, in the sense that whenever $x, y \in \mathrm{qf}(R)$ satisfy $xy \in p$, then either $x \in p$ or $y \in p$, equivalently, in case R has a (uniquely determined) valuation overring V such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ as sets, equivalently (by [**2**, Proposition 2.6]) in case R is a pullback of the form $V \times_K k$, where V is a valuation domain with residue field K and k is a subfield of K. As the terminology suggests, any valuation domain is a PVD [**9**, Proposition 1.1]. Although the converse is false [**9**, Example 2.1], any PVD must, at least, be local [**9**, Corollary 1.3]. The main result of this paper is Theorem 2.2, which states that R is a maximal "non-S" subring of $\mathrm{qf}(R)$ if and only if R is an integrally closed pseudo-valuation domain with $\dim(R) = 1$ and $\dim_v(R) = 2$. As an application of Theorem 2.2, we give necessary and sufficient conditions for certain pullbacks to be maximal "non-S" subrings of their quotient fields.

2. Main results

Let R be a domain contained in a field L. We say that R is *a maximal "non-*S*"subring of* L if R is not an S-domain and each ring T such that $R \subset T \subseteq L$ is an S-domain.

First of all, we establish the following:

Proposition 2.1. *Let R be a domain and L a field containing R. If R is a maximal "non-*S*"subring of L, then $L = \mathrm{qf}(R)$.*

Proof: First notice that L is algebraic over R. Indeed, if not then there exists an element t of L transcendental over R. Hence each overring of $R[t]$ should be an S-domain that is $R[t]$ is an absolutely S-domain. Hence by [**11**, Proposition 1.14] R is a field which contradicts the fact that R is not an S-domain. Now our task is to show that $L = \mathrm{qf}(R)$. Assume that $\mathrm{qf}(R) \subset L$, and let $\alpha \in L \setminus \mathrm{qf}(R)$. Then α is algebraic over R. Thus there exists an element $r \in R$ such that $r\alpha$ is integral over R. Thus $R \subset R[r\alpha]$ is an integral extension. But $R[r\alpha]$ is an S-domain. Hence R is an S-domain, the desired contradiction to complete the proof. □

As a direct consequence of Proposition 2.1, the study of maximal "non-S" subring(s) of a field L can be reduced to the case where $L = \mathrm{qf}(R)$. Now notice that if R is a maximal "non-S"subring of $\mathrm{qf}(R)$, then R is integrally closed. Indeed, if $R \neq R'$, then R' is an S-domain, and hence so is R (since $R \subset R'$ is an integral extension), which is impossible.

Our main result is the following:

Theorem 2.2. *Let R be a domain. Then the following statements are equivalent:*

(i) *R is a maximal "non-S" subring of $\mathrm{qf}(R)$;*

(ii) *R is an integrally closed PVD with $\dim(R) = 1$ and $\dim_v(R) = 2$.*

Proof: (i) $\Rightarrow$ (ii). We have already noticed that R is integrally closed. On the other hand since R is not an S-domain, then there is a height 1 prime ideal p of R such that $ht(p[X]) = 2$. Then there is a nonzero prime ideal P of $R[X]$ contained in $p[X]$ such that $P \cap R = (0)$. Thus R is a subring of $R_1 = R[X]/P$ which is isomorphic to $R[u]$, where u is an algebraic element over R. By [**8**, Corollary 19.7], there is a valuation overring W of R_1 containing a prime ideal P' of height 1 such that $P' \cap R_1 = p[X]/P$. Denoting $V = W \cap \mathrm{qf}(R)$, V is a valuation overring of R containing a height 1 prime ideal $q = P' \cap \mathrm{qf}(R)$ [**8**, Theorem 19.16] such that $q \cap R = p$. Now, $\mathrm{tr.deg}[W/P' : V/q] = 0$ [**8**, Theorem 19.16]. Hence

$$\begin{aligned}
\mathrm{tr.deg}[V/q : R/p] &= \mathrm{tr.deg}[W/P' : R/p] \\
&\geq \mathrm{tr.deg}[R_1/(p[X]/P) : R/p] \\
&= \mathrm{tr.deg}[(R[X]/P)/(p[X]/P) : R/p] \\
&= \mathrm{tr.deg}[(R[X]/p[X]) : R/p] = 1.
\end{aligned}$$

Assume that $R \neq (V_q, qV_q, R_p/pR_p)$, then the domain $(V_q, qV_q, R_p/pR_p)$ is a proper overring of R and it should be an S-domain and by [**11**, Proposition 1.4], we get $\operatorname{tr.deg}[V_q/qV_q : R_p/pR_p] = 0$ which is impossible. Therefore $R := (V_q, qV_q, R_p/pR_p)$. Hence R is a PVD (cf. [**2**]). Our task now is to show that $\operatorname{tr.deg}[V_q/qV_q : R_p/pR_p] = 1$. The extension $R_p/pR_p \subset V_q/qV_q$ can not be algebraic since R is not an S-domain [**11**, Proposition 1.4]. Assume that $\operatorname{tr.deg}[V_q/qV_q : R_p/pR_p] \geq 2$, and let X, Y be two transcendental algebraically independent elements of V_q/qV_q over R_p/pR_p. Then the domain $T := (V_q, qV_q, (R_p/pR_p)[X])$ is a proper overring of R, thus T is an S-domain. Hence by [**11**, Proposition 1.4], we get $\operatorname{tr.deg}[V_q/qV_q : (R_p/pR_p)[X]] = 0$, which is impossible. Hence $\operatorname{tr.deg}[V_q/qV_q : R_p/pR_p] = 1$. Therefore by [**1**, Proposition 2.5], $\dim(R) = 1$ and $\dim_v(R) = 2$.

(ii) $\Rightarrow$ (i). Since R is a PVD, then $R := (V, M, k)$, where V is a valuation domain with maximal ideal M and k is a field. It is clear that R is not an S-domain because $\operatorname{tr.deg}[V/M : R/M] = 1$. Now, let T be a domain such that $R \subset T \subseteq \operatorname{qf}(R)$. Then by [**3**, Lemma 1.3], either T is an overring of V, so it is an S-domain, or T is an intermediate domain between R and V, so $T := (V, M, D)$, where $R/M \subset D \subseteq V/M$. Since R is integrally closed, then $\operatorname{tr.deg}[V/M : D] = 0$. Thus T is an S-domain. Hence R is a maximal "non-S" subring of $\operatorname{qf}(R)$. $\square$

Now we determine when a pullback R is a maximal "non-S" subring of its quotient field. We recall some notation for conductors. If R is a domain and I, J are R-submodules of $\operatorname{qf}(R)$, then $(I : J) = \{x \in \operatorname{qf}(R) \mid xJ \subset I\}$. If R is a PVD with associated valuation domain V and maximal ideal M, assume that $R \neq V$, then M is not a principal ideal of R and $V = (M : M)$ [**2**, Proposition 2.3], and by [**2**, Lemma 2.4], we get $V = (R : M) = (M : M)$.

We establish the following theorem.

Theorem 2.3. *Let T be a domain, M a maximal ideal of T and D a subring of the field $K = T/M$. Let $R := (T, M, D)$. Then the following statements are equivalent:*

(i) *R is a maximal "non-S" subring of* $\operatorname{qf}(R)$*;*

(ii) *D is a field algebraically closed in $(M : M)/M$, with* $\operatorname{tr.deg}[K : D] = 1$ *and T is a one-dimensional Jaffard PVD.*

Proof: (i) $\Rightarrow$ (ii). By Theorem 2.2, R is a PVD. Hence there exists a valuation domain V with m as a maximal ideal such that $R := (V, m, k)$, where k is a field. Since T is an overring of R, then by [**3**, Lemma 1.3], either $R \subset T \subseteq V$ or $V \subseteq T$.

Case 1: If $R \subset T \subseteq V$, then T shares the ideal m with R and V, so $T := (V, m, T/m)$. But we have $M \subseteq m$ (since R is local with maximal ideal m). Thus $M = m$ because M is a maximal ideal of T. Hence $T := (V, M, K)$, $D = R/M = R/m = k$, so D is a field. On the other hand R is integrally closed (Theorem 2.2), thus D is algebraically closed in $V/M = (M : M)/M$. We have $\dim(T) = \dim(V) = \dim(R) = 1$, and since T is an S-domain, then $\dim(T) = \dim_v(T) = 1$. Now $\operatorname{tr.deg}[K : D] = \dim_v(R) - \dim_v(T) = 1$.

Case 2: If T is an overring of V, then $T = V$ since V is a one-dimensional valuation domain. Thus $m = M$. This yields $D = R/M = R/m = k$ and it is obvious that D is algebraically closed in $V/M = (M : M)/M$. On the other hand $\operatorname{tr.deg}[K : D] = \dim_v(R) - \dim_v(T) = 1$.

(ii) $\Rightarrow$ (i). Since $D \subset K$ is not an algebraic extension, then R is not an S-domain [**11**, Proposition 1.4]. The ring T is a PVD, so there is a valuation domain W with maximal ideal M such that $T := (W, M, K)$. But $R := (T, M, D)$. Hence R is a PVD with associated valuation domain $W = (M : M)$. Furthermore, $\dim(R) = \dim(T) = 1$ and $\dim_v(R) = \dim_v(T) + \dim_v(D) + \operatorname{tr.deg}[K : D] = 2$. Since D is algebraically closed in W/M, then R is integrally closed. Thus by Theorem 2.2, R is a maximal "non-S" subring of $\mathrm{qf}(R)$. $\square$

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