

Publ. Mat. **46** (2002), 257–325

CARLESON MEASURES, TREES, EXTRAPOLATION, AND $T(b)$ THEOREMS

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Abstract

The theory of Carleson measures, stopping time arguments, and atomic decompositions has been well-established in harmonic analysis. More recent is the theory of phase space analysis from the point of view of wave packets on tiles, tree selection algorithms, and tree size estimates. The purpose of this paper is to demonstrate that the two theories are in fact closely related, by taking existing results and reproving them in a unified setting. In particular we give a dyadic version of extrapolation for Carleson measures, as well as a two-sided local dyadic $T(b)$ theorem which generalizes earlier $T(b)$ theorems of David, Journé, Semmes, and Christ.

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2000 *Mathematics Subject Classification.* 42B20, 42B25.

Key words. Haar wavelets, BMO, accretivity, $T(b)$ theorems, extrapolation, Calderón-Zygmund theory.

1. Introduction

The purpose of this article is to demonstrate the close connection between two sets of techniques in harmonic analysis: the theory of Carleson measures and related objects, and the theory of trees and related objects.

A Carleson measure is a positive measure μ on the upper half space such that $\mu(I \times (0, \ell(I))) \lesssim |I|$ for every cube $I \subseteq \mathbb{R}^n$ with side length $\ell(I)$. There is also an analogous notion for domains more general than the half-space, as well as a discrete version: if μ is a mapping from dyadic cubes into the non-negative reals, then μ satisfies a (discrete) Carleson measure condition $\sum_{I \subseteq J} \mu_I \lesssim |J|$ for every dyadic cube J , where the sum runs over all dyadic sub-cubes of J . Carleson measures are intimately connected with many aspects of harmonic analysis, including non-tangential behavior of functions in the half-space (or in a domain) (see e.g. [51]), H^p theory and BMO [27], boundedness of singular integrals, square functions and maximal functions (e.g. [12], [22], [35], [50], [37], [38]), geometric measure theory (e.g. [24], [35], [36]), and PDE (e.g. [3], [28], [32], [42]). Moreover, via their connection with the theory of trees, Carleson measures have played a significant role in recent work on Bilinear Singular Integrals [39], [40], [45], [46], [53], and (rather appropriately!) Carleson's theorem on a.e. convergence of Fourier Series [25], [41]. In these latter connections it is more convenient to work in the phase plane than in the Carleson half-space, and we have deliberately chosen our notation to reflect this fact.

This article is mainly expository. Apart from one main new result (a local $T(b)$ theorem), we shall mostly take existing results (atomic decompositions, paraproduct estimates, Carleson embedding) and reprove them in a framework which unifies both the Carleson measure theory and the theory of trees and tiles. (As such there is some overlap with the recent lecture notes in [49].)

Since this is an expository article, we shall simplify matters and only work in one dimension \mathbb{R} . Also, we shall mostly work in the dyadic setting instead of the continuous one, to avoid issues such as rapidly decreasing tails or use of the Vitali covering lemma. Thus, our results will be phrased using dyadic intervals and the Haar basis instead of arbitrary intervals and Gaussians (or similar smooth kernels). However most of our results have continuous analogues (see e.g. [29] for a comparison between dyadic and continuous harmonic analysis). We also will truncate all our spaces to be finite-dimensional to avoid technicalities.

The paper is organized as follows. After setting up the notation of dyadic Carleson measures and BMO, Haar wavelets, and tiles and trees, we will give a quick review of the standard “ L^∞ ” theory of BMO (i.e. measuring the ways in which BMO is close to L^∞), but from the perspective of trees and tiles. As part of this L^∞ theory, we give a trees-based proof of the (dyadic analogue of the) extrapolation lemma for Carleson measures developed recently in [3], [32], [42]. We also give an alternate proof of the extrapolation lemma due to John B. Garnett.

We then show how BMO is also useful in “ L^p ” contexts, mainly through a BMO version of the Calderón-Zygmund decomposition. This type of lemma is used often in the recent work on Carleson’s theorem and the bilinear Hilbert transform, and is implicit in earlier work on Carleson measures and similar objects; we illustrate this by using the BMO Chebyshev inequality to re-prove the standard atomic decomposition of H^p .

Next, we prove the Carleson embedding theorem and give its usual applications to paraproduct estimates and the $T(1)$ theorem. We also give a short proof of the boundedness of paraproducts below L^1 ; the proof is more direct than earlier proofs in that one does not go explicitly through the $T(1)$ theorem.

Finally, we consider Calderón-Zygmund operators. We prove a two-sided local $T(b)$ theorem which generalizes the existing local and global $T(b)$ theorems [23], [4], [11], [50]; for instance, we can prove the standard global $T(b)$ theorem assuming that b is only in BMO rather than L^∞ .

The $T(b)$ Theorem, in its various guises, has its roots in a question posed by Yves Meyer, who asked whether the $T(1)$ Theorem of David and Journé [22] (see also Section 6 below) remains true if the constant function 1 is replaced by some function $b \in L^\infty$ with $\operatorname{Re} b \geq \delta$ (such b are said to be “accretive”). The question was motivated by its applicability to the L^2 boundedness of the Cauchy integral operator on a Lipschitz graph. Indeed, if Γ denotes the graph, in the plane, of a real-valued Lipschitz function A , then by Cauchy’s theorem, we have that in the sense of BMO (that is, modulo constants),

$$0 = \text{p.v.} \int_{\Gamma} \frac{1}{z-w} dw,$$

for $z \in \Gamma$. But in graph co-ordinates, this amounts to saying that (again in the sense of BMO),

$$0 = \text{p.v.} \int_{-\infty}^{\infty} \frac{1 + iA'(y)}{x-y + i(A(x) - A(y))} dy \equiv T(b)(x),$$

where b is the accretive function $1 + iA'$, and T is the singular integral operator naturally associated to the antisymmetric Calderón-Zygmund kernel $K(x, y) = (x - y + i(A(x) - A(y)))^{-1}$.

The L^2 boundedness of T , and hence also that of the Cauchy integral operator

$$C_{\Gamma}f(x) \equiv \text{p.v.} \int_{\Gamma} \frac{f(w)}{z - w} dw,$$

thus follows from an analogue of the $T(1)$ Theorem in which the condition $T(1), T^*(1) \in \text{BMO}$ is replaced by the condition $T(b) = 0 = T^*(b)$, for some accretive function b . Just such a result was proved by McIntosh and Meyer [43], who consequently obtained an alternative proof of their earlier joint result with Coifman [13] concerning the L^2 boundedness of the operator C_{Γ} .

The “ $T(b)$ Theorem” of [43] was generalized by David, Journé and Semmes [23] to allow $T(b), T^*(b) \in \text{BMO}$ (indeed, they allowed other generalizations as well, for example that there could be two different accretive functions b_1, b_2 such that $T(b_1), T^*(b_2) \in \text{BMO}$, and moreover that the pointwise accretivity condition could be relaxed to a condition holding on various sorts of averages —see, e.g., the notion of “pseudo-accretivity” defined in Section 6.1 below).

This led to a proof of the $T(b)$ Theorem by constructing Haar wavelets adapted to the function b [20] (we shall base our proof on a variation of these adapted Haar wavelets).

A very simple proof of a “one-sided version” of the $T(b)$ Theorem was obtained by Semmes [50], who observed that in the special case $T(b) \in \text{BMO}, T^*(1) = 0$, one can readily show that $T(1) \in \text{BMO}$, thus reducing matters to the $T(1)$ Theorem. It is worth noting that a suitable adaptation of Semmes’s argument is applicable to the solution of the square root problem of Kato. Indeed, one of the present authors (Auscher), along with Tchamitchian [5], formulated a version of the $T(b)$ Theorem whose proof was based upon the argument of [50], and which was subsequently used to solve the Kato problem in higher dimensions [1], [31], [2]. We further note that there are local versions of the $T(b)$ Theorem, due to M. Christ [11] (cf. Theorem 6.8 below), which also have interesting applications, namely to questions of analytic capacity; see for instance [54] for further discussion.

This work was conducted at University of Missouri, University of California at Los Angeles (UCLA), and the Centre for Mathematics and its Applications (CMA) at the Australian National University (ANU). The

authors are particularly grateful to CMA for their warm hospitality during the visit of three of us (P. Auscher, C. Thiele, T. Tao). S. Hofmann is supported by NSF grant DMS 0088920. C. Muscalu is supported by NSF grant DMS 0100796. T. Tao is a Clay Prize Fellow and is supported by a grant from the Packard Foundation. C. Thiele is supported by a Sloan fellowship and NSF grants DMS 9985572 and DMS 9970469. The authors are indebted to Stephanie Molnar, John B. Garnett, Joan Verdera and the referee for many helpful corrections and comments.

2. Notation

We use $A \lesssim B$ to denote the estimate $A \leq CB$ for some absolute constant C which may vary from line to line.

If E is a set, we use $|E|$ to denote the Lebesgue measure of E . We will always be ignoring sets of measure zero, thus we only consider two sets E, F to be intersecting if $|E \cap F| > 0$.

Although our functions may be complex valued, we shall use the real inner product

$$\langle f, g \rangle := \int f(x)g(x) dx$$

throughout.

2.1. Tiles and trees. We shall be working with dyadic intervals throughout the paper. The number of dyadic intervals is infinite, but to simplify the arguments we shall restrict ourselves to a finite set on the half-line; in applications, this restriction can always be removed by a standard translation and limiting argument. Specifically, we fix a large integer $M > 0$; none of our estimates will depend on M . We define *dyadic interval* to be an interval¹ of the form $I = [j2^k, (j+1)2^k]$, where j, k are integers such that $-M \leq k \leq M$ and $I \subseteq [0, 2^M]$. Let \mathbf{I} denote the set of all dyadic intervals; observe that \mathbf{I} is finite. All sums and unions involving I or J will be assumed to be over \mathbf{I} unless otherwise specified. If f is a function on \mathbb{R} , we define $[f]_I := \frac{1}{|I|} \int_I f$ to denote the mean of f on I . We use $2I$ to denote the parent² of I , and I_l, I_r to denote the left and right children of I (these are undefined if $|I| = 2^M$ or $|I| = 2^{-M}$ respectively). We refer to the intervals I_l and I_r as *siblings*.

¹We will be careless about whether our intervals are closed, half-open, or open because of our convention of ignoring sets of measure zero.

²On the non-dyadic theory $2I$ is often used to denote the interval with the same center as I but twice the length; this can be thought of as a non-dyadic version of the parent of I . However, in this paper we use $2I$ to exclusively refer to the dyadic parent of I , i.e. the unique dyadic interval of twice the length which contains I .

Since our dyadic intervals have been restricted to a finite set, all norms will automatically be finite and all stopping time processes will automatically terminate. This allows us to avoid some minor technicalities in our arguments, although it also means that we occasionally have to treat the smallest scale $|I| = 2^{-M}$ or the largest scale $|I| = 2^M$ a little differently from all the other scales.

A major advantage of the dyadic setting is the *nesting property*: if I, J are dyadic intervals which intersect each other, then either $I \subseteq J$ or $J \subseteq I$. In particular, for any collection of dyadic intervals, the maximal intervals in this collection will always be disjoint.

The theory of Carleson measures is usually set in the *upper half-space*

$$\mathbb{R}_+^2 := \{(x, t) : x \in \mathbb{R}, t \in \mathbb{R}^+\}.$$

Actually, because of our truncation parameter M we will work in the compact subset

$$(\mathbb{R}_+^2)_M := \{(x, t) : x \in [0, 2^M], t \in [2^{-M}, 2^{M-1}]\}.$$

The variable x represents spatial position, while the t variable represents time, wavelength, or spatial scale. For every dyadic interval $I \in \mathbf{I}$, we let $l(I) = |I|$ denote the side-length of I , and define the *Carleson box* $Q(I) \subset (\mathbb{R}_+^2)_M$ by

$$Q(I) := I \times [2^{-M}, l(I)]$$

and the *Whitney box* $Q^+(I) \subset Q(I)$ by

$$Q^+(I) := I \times \left[\frac{l(I)}{2}, l(I) \right].$$

We remark that we have the partition $Q(I) = \bigcup_{J: J \subseteq I} Q^+(J)$.

Meanwhile, the theory of trees and tiles is usually set in *phase space*

$$\mathbb{R}^2 := \{(x, \xi) : x \in \mathbb{R}, \xi \in \mathbb{R}\}.$$

Because of our truncation, and because we are in the dyadic setting, we will instead work in the region³

$$(\mathbb{R}^2)_M := \{(x, \xi) : x \in [0, 2^M], \xi \in [0, 2^M]\}.$$

The variable x represents spatial position, while ξ represents frequency. A *Heisenberg tile* or simply *tile* is a rectangle in \mathbb{R}^2 of the form $P := I_P \times \omega_P$, where I_P and ω_P are dyadic intervals such that $|P| = |I_P| |\omega_P| = 1$.

If P and Q are tiles, we say that $P \leq Q$ if P intersects Q and $I_P \subseteq I_Q$. This is a partial order on tiles.

³In truth, we are working not with the Euclidean field \mathbb{R} , but with the Walsh field $\mathbb{R}^+ \equiv (\mathbf{Z}_2)^{\mathbf{Z}}$. See e.g. [52].

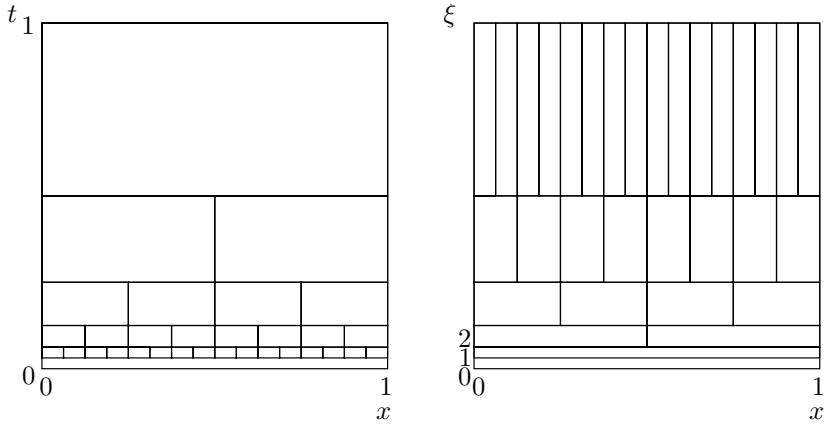


FIGURE 1. The geometry of the Carleson half-plane (partitioned into Whitney boxes) and phase space (partitioned into non-lacunary tiles). The heuristic $t = 1/\xi$ provides a one-to-one correspondence between the two partitions.

If I is a dyadic interval, we define the *lacunary tile* $P^+(I)$ by

$$P^+(I) := I \times \left[\frac{1}{l(I)}, \frac{2}{l(I)} \right]$$

and the *non-lacunary tile* $P^0(I)$ by

$$P^0(I) := I \times \left[0, \frac{1}{l(I)} \right].$$

Let \mathbf{P}^+ denote the set of all lacunary tiles, and \mathbf{P}^0 the set of all non-lacunary tiles. We define $P^+(I) \leq' P^+(J)$ if and only if $P^0(I) \leq P^0(J)$, or equivalently if $I \subseteq J$. Thus \leq' is a partial ordering on \mathbf{P}^+ . Of course, there are many tiles which are not in either of these two sets, and many results in this paper can be extended to general tiles. However, for simplicity we shall mostly restrict ourselves to the lacunary and non-lacunary tiles. We write $[f]_P$ as shorthand for the averages $[f]_{I_P}$.

If $P^+(I)$ is a lacunary tile, we define the *parent* $2P^+(I)$ of $P^+(I)$ by $2P^+(I) := P^+(2I)$. Similarly define $2P^0(I) := P^0(2I)$.

A *lacunary*⁴ tree (henceforth abbreviated as *tree*) is a collection $T \subseteq \mathbf{P}^+$ of lacunary tiles with a *top tile* $P_T \in T$, such that $P \leq' P_T$ for all $P \in T$. We use I_T as short-hand for I_{P_T} . If $P \in \mathbf{P}^+$, we define the *complete tree* $\text{Tree}(P)$ to be the tree

$$\text{Tree}(P) := \{Q \in \mathbf{P}^+ : Q \leq' P\}$$

with top P . We sometimes write $\text{Tree}(I)$ for $\text{Tree}(P^+(I))$. Note that every tree T lies inside a complete tree $\text{Tree}(P_T)$. If T is a tree inside a collection \mathbf{P} of tiles, we say that T is *complete with respect to \mathbf{P}* if $T = \text{Tree}(P_T) \cap \mathbf{P}$.

Let $\alpha > 0$ and T be a tree. We define an α -*packing* of T to be a set $\mathbf{P} \subset T$ of tiles such that

$$\sum_{P \in \mathbf{P}} |I_P| \leq \alpha |I_T|.$$

We say that \mathbf{P} is a *uniform α -packing*⁵ of T if

$$\sum_{P \in \mathbf{P} : I_P \subset J} |I_P| \leq \alpha |J|$$

for all dyadic intervals J .

If $\alpha < 1/2$ and \mathbf{P} is an α -packing of T , observe that the parent tiles $2\mathbf{P} := \{2P : P \in \mathbf{P}\}$ form a 2α -packing of T . Similarly if \mathbf{P} is a uniform α -packing of T , then $2\mathbf{P}$ is a uniform 2α -packing of T .

We say that a collection of lacunary tiles \mathbf{P} is *convex* if for every pair of tiles $P_1 \leq' P_2$ in \mathbf{P} , the set $\{P \in \mathbf{P}^+ : P_1 \leq' P \leq' P_2\}$ is also contained in \mathbf{P} . We will usually be dealing with convex trees in this paper.

The correspondence between the upper half-space and phase space is given by the heuristic formula⁶

$$(1) \quad t = 1/\xi;$$

in other words, frequency is the reciprocal of wavelength. This correspondence identifies Whitney boxes $Q^+(I)$ with lacunary tiles $P^+(I)$, and identifies a Carleson box $Q(I)$ with the complete tree $\text{Tree}(I)$.

⁴Non-lacunary trees $T \subseteq \mathbf{P}^0$ are also useful in the study of the bilinear Hilbert transform and Carleson’s operator; more precisely, when treating the bilinear Hilbert transform $\langle B(f, g), h \rangle$ one uses a triple of trees associated to f, g, h respectively, with two of the trees lacunary and the third non-lacunary (but possibly with a non-zero frequency origin). Similarly when treating the Carleson operator $\langle \mathcal{C}_{N(x)}f, \chi_E \rangle$ one uses a pair of trees associated to f and χ_E respectively, with one lacunary and one non-lacunary. See [40], [41]. However we will not use non-lacunary trees explicitly in this paper, although they appear implicitly in Lemma 4.2 and in the paraproduct theory.

⁵This is roughly equivalent to $\sum_{P \in \mathbf{P}} \chi_{I_P}$ having a BMO norm bounded by α .

⁶To be completely precise, one would have to adjust this formula when $\xi \leq 2^{1-M}$, but as this is only a heuristic anyway we will not bother to do this.

(Incomplete trees T are identified with the portion of a Carleson box above a “dyadic Lipschitz graph”, cf. [3].) Note how this correspondence clearly gives a privileged position to the frequency origin $\xi = 0$.

The thesis of this paper is that the theory of Carleson measures can be equated with the theory of lacunary tiles. The theory of general tiles—which is needed for applications such as Carleson’s theorem and the bilinear Hilbert transform, in which the frequency origin plays no distinguished role—can then be thought of as a generalization of Carleson measure theory⁷.

| Tiles and trees | Carleson measures |
|---------------------------------|--------------------------------------|
| Phase space \mathbb{R}^2 | Upper half-plane \mathbb{R}_+^2 |
| Lacunary tile $P^+(I)$ | Whitney box $Q^+(I)$ |
| Non-lacunary tile $P^0(I)$ | “Tower” $I \times [l(I), \infty)$ |
| Complete tree $\text{Tree}(I)$ | Carleson box $Q(I)$ |
| Convex tree | Carleson box above a Lipschitz graph |
| Size $\ \mu\ _{\text{size}(T)}$ | Normalized mass $\mu(Q(I))/ I $ |
| Bounded maximal size | Carleson measure (or BMO function) |
| ξ | $1/t$ |

TABLE 1. A partial dictionary between tree terminology, and Carleson measure terminology. In our paper the two viewpoints are essentially equivalent, however the phase space viewpoint is better adapted to handle more general situations where one needs to modulate in frequency. Conversely, Carleson measures are better adapted to complex analysis applications.

⁷In our paper, we will only need tiles which are centered at or near the frequency origin, in which case it does not particularly matter whether we use the Carleson half-plane or the phase plane. However, we have chosen to use phase space notation (using frequency ξ instead of wavelength t) as this is more compatible with the more general theory of multilinear operators such as the bilinear Hilbert transform (or the Carleson maximal operator), which are invariant under translations of the frequency variable. Note that the modulation operation $f \mapsto e^{2\pi i \xi_0 x} f$ can be represented easily in the phase plane as a translation by ξ_0 in the ξ variable, but is not so elegantly representable in the Carleson half-plane. Nevertheless, we will not need to modulate in frequency in this paper, so the Carleson viewpoint and the phase space viewpoint are essentially equivalent here.

2.2. Size and Carleson measures. Let T be a convex lacunary tree, and suppose that we have a function $a: T \mapsto \mathbb{R}^+$ assigning non-negative numbers to each tile in T . We define the *size* of a on T by

$$(2) \quad \|a\|_{\text{size}(T)} := \frac{1}{|I_T|} \sum_{P \in T} a(P).$$

Now let \mathbf{P} be any collection of lacunary tiles, and let $a: \mathbf{P} \mapsto \mathbb{R}^+$. We define the *maximal size* of a on T by

$$(3) \quad \|a\|_{\text{size}^*(\mathbf{P})} := \sup_{T \subset \mathbf{P}} \|a\|_{\text{size}(T)}$$

where T ranges over all convex lacunary trees in \mathbf{P} ; we adopt the convention that $\|a\|_{\text{size}^*(\mathbf{P})} = 0$ if \mathbf{P} is empty. The notion of size and maximal size is analogous to α -packings and uniform α -packings. For instance, the following lemma is immediate from the definitions:

Lemma 2.1. *If \mathbf{P} is a uniform α -packing of a tree T , and $a(P)$ obeys the weak Carleson condition*

$$|a(P)| \leq A|I_P| \text{ for all } P \in \mathbf{P}$$

for some $A > 0$, then $\|a\|_{\text{size}^(\mathbf{P})} \leq A\alpha$.*

If μ is a non-negative measure on the truncated upper half-space $(\mathbb{R}_+^2)_M$, then it assigns a non-negative number to each Whitney box. By the correspondence (1), we can thus assign to each lacunary tile $P = P^+(I)$ a number $\mu(P)$ by the formula $\mu(P^+(I)) := \mu(Q^+(I))$. We say that μ is a *Carleson measure* if

$$\|\mu\|_{\text{size}^*(\mathbf{P}^+)} < \infty.$$

The reader may easily verify that this is equivalent to the usual formulation of a Carleson measure, namely that $\mu(Q(I)) \leq C|I|$ for some constant C .

2.3. Wavelets, phase space projections, and BMO. Let P be a lacunary tile. We define the (*mother*) *Haar wavelet* ϕ_P to be the L^2 -normalized function

$$\phi_P := |I_P|^{-1/2}(\chi_{I_P^l} - \chi_{I_P^r})$$

where I_P^l and I_P^{right} are the left and right halves of I_P respectively. Similarly, if P is a non-lacunary tile, we define the (*father*) *Haar wavelet* ϕ_P by

$$\phi_P := |I_P|^{-1/2}\chi_{I_P}.$$

Observe that these functions are normalized in L^2 , and that ϕ_P and $\phi_{P'}$ are orthogonal whenever P and P' are disjoint⁸.

It is in fact possible to assign a function ϕ_P to every Heisenberg tile; these functions are known as *Walsh wave packets*, see e.g. [52] for a discussion. These Walsh packets can then be used to efficiently decompose such operators as the (Walsh) bilinear Hilbert transform or the (Walsh) Carleson operator, just as the Haar wavelets can be used to decompose (dyadic) paraproducts or (dyadic) Calderón-Zygmund operators; see in particular the remarks after (48). However, we shall not make any use of the Walsh wave packets for the results in this paper.

We define a (*dyadic*) *test function* to be any finite linear combination of mother and father Haar wavelets ϕ_P . We use S to denote the space of all test functions, and S_0 to denote the test functions with mean zero. For any dyadic interval I , we define $S(I)$ to be the elements of S which are supported in I , and similarly define $S_0(I)$. Note that $S(I)$ is only one dimension larger than $S_0(I)$, and is in fact spanned by $S_0(I)$ and any function $b_I \in S(I)$ with non-zero mean. This fact will be used much later on when we discuss local $T(b)$ theorems.

Since the mother Haar wavelets are orthonormal, we have the representation formula⁹

$$f = \sum_{P \in \mathbf{P}^+} \langle f, \phi_P \rangle \phi_P$$

for all $f \in S_0$.

If $f \in S$ and T is any collection of disjoint tiles in $\mathbf{P}^+ \cup \mathbf{P}^0$ (i.e. the tiles in T can be lacunary or non-lacunary), we define the phase space projection Π_T by

$$\Pi_T f := \sum_{P \in T} \langle f, \phi_P \rangle \phi_P.$$

This is an orthogonal projection from $L^2(\mathbb{R})$ to the space of functions spanned by $\{\phi_P : P \in T\}$. For instance, we have $\Pi_{P^0(I)} f = [f]_I \chi_I$ and

$$(4) \quad \Pi_{\text{Tree}(I)} f = (f - [f]_I) \chi_I.$$

More generally, for any convex tree $T \subset \mathbf{P}^+$ and any $f \in S$ we have

$$(5) \quad \Pi_T f(x) = [f]_{J(x,T)} - [f]_{I(x,T)}$$

for some intervals $I(x, T), J(x, T)$ containing x ; the exact choice of these intervals depends on T . The formula (5) can be derived by writing T

⁸For a pair of lacunary tiles, this means that $I_P \neq I_{P'}$; for a pair of non-lacunary tiles, this means that I_P and $I_{P'}$ are disjoint. For lacunary P and non-lacunary P' , this means that $I_{P'}$ is not a proper subset of I_P .

⁹The continuous version of this would be a Calderón reproducing formula such as $f(x) = \int 2t^2 \Delta e^{t^2 \Delta} f(x) \frac{dt}{t}$.

as a complete tree $\text{Tree}(P_T)$ with some smaller complete trees removed, and then using (4).

If $f \in S$, we define the *wavelet transform* Wf of f to be the function

$$Wf(P) := \langle f, \phi_P \rangle$$

defined on \mathbf{P}^+ ; this is an isometry¹⁰ between S_0 (endowed with the L^2 norm) and $l^2(\mathbf{P}^+)$. In particular, the function $|Wf|^2$ maps \mathbf{P}^+ to \mathbb{R}^+ , and so one can compute the size of $|Wf|^2$ on various collections of trees. We observe in particular that

$$(6) \quad \| |Wf|^2 \|_{\text{size}(T)} = \frac{1}{|I_T|} \| \Pi_T f \|_2^2 \leq \frac{1}{|I_T|} \int_{I_T} |f - [f]_{I_T}|^2 \leq \frac{1}{|I_T|} \int_{I_T} |f|^2$$

for all trees T .

If $f \in S$, we define the *dyadic BMO norm* of f by

$$(7) \quad \|f\|_{\text{BMO}} := \| |Wf|^2 \|_{\text{size}^*(\mathbf{P}^+)}^{1/2}.$$

The reader may easily verify that this definition corresponds to the usual (L^2 -based) definition of dyadic BMO. Note that the projections Π_T defined earlier are bounded on L^2 and BMO (in fact they are contractions).

Thus the concepts of Carleson measure, BMO, and maximal size are essentially the same concept. However, the concept of maximal size extends more easily to general families of tiles (not necessarily lacunary) than the other two notions. (In particular, the notion of maximal size on a tree centered at an arbitrary frequency ξ_0 is central to the boundedness of the bilinear Hilbert transform, see [39].)

2.4. Mean. We shall need a notion of *mean* (or “normalized mass”), which can be thought of as a “non-lacunary” variant of size. Given any function f on \mathbb{R} and a tile $P \in \mathbf{P}^+$, we define

$$\|f\|_{\text{mean}(P)} := [f]_{I_P} = \frac{1}{|I_P|} \int_{I_P} |f|$$

and for any collection $\mathbf{P} \subset \mathbf{P}^+$ of lacunary tiles we define

$$\|f\|_{\text{mean}^*(\mathbf{P})} := \sup_{P \in \mathbf{P}} \|f\|_{\text{mean}(P)}.$$

Like the notion of BMO, the notion of mean has the scaling of L^∞ . One can extend the notion of mean to arbitrary tiles; the function f should then be replaced by a measure on phase space. For instance, in applications to Carleson’s theorem [10] the notion of mean is applied to a measure of the form $\chi_E(x)\delta(\xi - N(x))$, where N is an arbitrary function. See [25], [41].

¹⁰The continuous analogue of this in the upper half-plane with measure $\frac{dx dt}{t}$ would be the function $Q_t f(x)$, where Q_t is a suitable cancellative averaging operator with wavelength t , e.g. $Q_t := t^2 \Delta e^{t^2 \Delta}$.

3. The “ L^∞ theory”

It is well known that the notion of maximal size or BMO can be thought of as a stable substitute for the L^∞ norm, which is often ill-suited for applications. In this section we develop the standard theory for this norm.

We begin with a simple but very useful principle: to bound the maximal size of a collection of tiles, it suffices to do so outside of an $(1 - \eta)$ -packing of each tree in the collection.

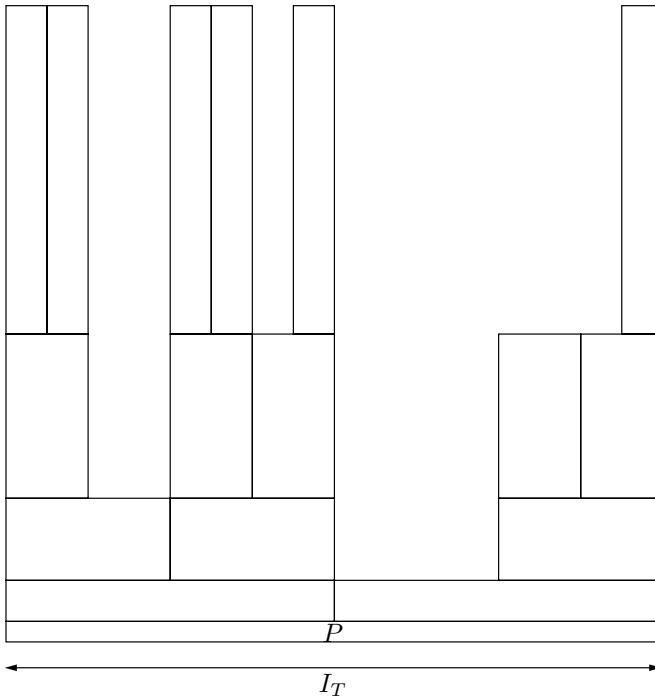


FIGURE 2. A convex tree T with top $P_T = P$. Note how this tree can be thought of as a complete tree $\text{Tree}(P)$ with some smaller complete trees removed. In this particular tree, the tops of the trees removed form a $\frac{5}{8}$ -packing of $\text{Tree}(P)$, so that $\frac{3}{8}$ of the tree T “makes it all the way to the top”. On the Carleson half-plane, this region resembles the portion of a Carleson box above a Lipschitz graph.

Lemma 3.1. *Suppose \mathbf{P} is a collection of lacunary tiles, and let $a: \mathbf{P} \rightarrow \mathbb{R}^+$, $A > 0$, and $0 < \eta < 1$ be such that for every tree T which is complete with respect to \mathbf{P} , one has*

$$\|a\|_{\text{size}(T \setminus \bigcup_{T' \in \mathbf{T}} T')} \leq A$$

for some collection \mathbf{T} of trees in T whose tops $\{P_{T'} : T' \in \mathbf{T}\}$ are a $(1 - \eta)$ -packing of T . Then we have

$$\|a\|_{\text{size}^*(\mathbf{P})} \leq A/\eta.$$

Proof: Let T be a tree in \mathbf{P} . From hypothesis we have

$$\begin{aligned} \sum_{P \in T} a(P) &= \sum_{P \in T \setminus \bigcup_{T' \in \mathbf{T}} T'} a(P) + \sum_{T' \in \mathbf{T}} \sum_{P \in T'} a(P) \\ &\leq A|I_T| + \sum_{T' \in \mathbf{T}} \|a\|_{\text{size}^*(\mathbf{P})} |I_{T'}| \\ &\leq |I_T| (A + (1 - \eta) \|a\|_{\text{size}^*(\mathbf{P})}). \end{aligned}$$

Dividing by $|I_T|$ and taking suprema of both sides we obtain

$$\|a\|_{\text{size}^*(\mathbf{P})} \leq A + (1 - \eta) \|a\|_{\text{size}^*(\mathbf{P})}$$

and the claim follows. □

An alternate (and perhaps more intuitive) proof of Lemma 3.1 is to start with a tree T , estimate the “good” part $T \setminus \bigcup_{T' \in \mathbf{T}} T'$ of the tree, and then pass to the “bad” trees $T' \in \mathbf{T}$ and iterate this process until the tree is completely exhausted. Since the geometric series $\sum_{n=0}^{\infty} (1 - \eta)^n$ converges to $1/\eta$, the claim follows.

Corollary 3.2 (Good-lambda characterization of maximal size). *Let \mathbf{P} be a collection of lacunary tiles, and let $a: \mathbf{P} \rightarrow \mathbb{R}^+$ be such that*

$$\left| \left\{ x \in I_T : \sum_{P \in T} a(P) \frac{\chi_{I_P}(x)}{|I_P|} \geq A \right\} \right| \leq (1 - \eta) |I_T|$$

for some $A > 0$, $0 < \eta < 1$ and all trees $T \subseteq \mathbf{P}$. Then we have $\|a\|_{\text{size}^*(\mathbf{P})} \leq \frac{A}{\eta}$.

This lemma goes back at least to Fritz John [34]. A partial converse can be obtained from Markov’s inequality:

$$\begin{aligned} \left| \left\{ x \in I_T : \sum_{P \in T} a(P) \frac{\chi_{I_P}(x)}{|I_P|} \geq A \right\} \right| &\leq \frac{1}{A} \left\| \sum_{P \in T} a(P) \frac{\chi_{I_P}}{|I_P|} \right\|_1 \\ &\leq \frac{1}{A} |I_T| \|a\|_{\text{size}(T)} \\ &\leq \frac{1}{A} |I_T| \|a\|_{\text{size}^*(\mathbf{P})}. \end{aligned}$$

Proof: Let T be any tree in \mathbf{P} . Consider the set \mathbf{Q} of all tiles $Q \in T$ such that

$$\sum_{P \in T: Q \leq' P} a(P)/|I_P| \geq A$$

and such that Q is maximal with respect to the ordering \leq' . By assumption \mathbf{Q} is a $(1 - \eta)$ -packing of T .

By Lemma 3.1 it suffices to show that $\|a\|_{\text{size}(T \setminus \bigcup_{Q \in \mathbf{Q}} \text{Tree}(Q))} \leq A$, or equivalently that

$$\int_{I_T} \sum_{P \in T \setminus \bigcup_{Q \in \mathbf{Q}} \text{Tree}(Q)} a(P) \frac{\chi_{I_P}(x)}{|I_P|} dx \leq A |I_T|.$$

But by construction of \mathbf{Q} , the integrand is bounded by A for all $x \in I_T$, and the claim follows. \square

A similar argument gives the well-known L^p characterization of BMO:

Corollary 3.3. *Let $0 < p < \infty$, and let $f \in S$ be such that $\frac{1}{|I|} \int_I |f - [f]_I|^p \lesssim 1$, or equivalently that $\int_I |\Pi_{\text{Tree}(I)} f|^p \lesssim |I|$ for all dyadic intervals I . Then*

$$\|f\|_{\text{BMO}} \lesssim 1.$$

(The implicit constants depend on p .)

Applying this corollary with $p = 1$ we obtain in particular that

$$\|f\|_{\text{BMO}} \sim \sup_I \frac{1}{|I|} \int_I |f - [f]_I|;$$

in fact, this is sometimes taken as the definition of (dyadic) BMO.

Proof: We need to show that $|Wf|^2$ has bounded maximal size. Let $T = \text{Tree}(I)$ be any complete tree for some interval I . By Lemma 3.1,

it suffices to find a collection \mathbf{T} of trees in T whose tops are a $\frac{1}{2}$ -packing of T such that

$$(8) \quad \| |Wf|^2 \|_{\text{size}(T \setminus \bigcup_{T' \in \mathbf{T}} T')} \lesssim 1.$$

First observe from (4) that if $J \subset I$ and $x \in J$ then

$$\Pi_{\text{Tree}(I)} f(x) - [\Pi_{\text{Tree}(I)} f]_J = \Pi_{\text{Tree}(J)} f(x)$$

so from hypothesis we have

$$(9) \quad \int_J |\Pi_{\text{Tree}(I)} f - [\Pi_{\text{Tree}(I)} f]_J|^p \lesssim |J|.$$

Now let C_p be a large constant to be chosen later. Let \mathbf{Q} denote the tiles $Q \in \text{Tree}(I)$ such that $|\Pi_{\text{Tree}(I)} f|_Q| \geq C_p$ and that Q is maximal with respect to \leq' . If C_p is sufficiently large, we see from (9)

$$\int_{I_Q} |\Pi_{\text{Tree}(I)} f|^p \gtrsim C_p^p |I_Q|$$

and hence that \mathbf{Q} is a $\frac{1}{4}$ -packing of T . In particular the collection $2\mathbf{Q} = \{2Q : Q \in \mathbf{Q}\}$ of parents of tiles in \mathbf{Q} is a $\frac{1}{2}$ -packing of T .

We set $\mathbf{T} := \{\text{Tree}(2Q) : 2Q \in 2\mathbf{Q}\}$, and define

$$F := \Pi_{T \setminus \bigcup_{T \in \mathbf{T}} T} f = \Pi_{\text{Tree}(I)} f - \sum_{Q \in \mathbf{Q}} \Pi_{\text{Tree}(2Q)} f.$$

Then we can rewrite the left-hand side of (8) as $\frac{1}{|I|} \|F\|_2^2$.

By construction we see that F is supported on I . Since $[\Pi_{\text{Tree}(2Q)} f]_{2Q} = 0$, we see that F is constant on each I_{2Q} and that

$$\|F\|_{L^\infty(2Q)} = |[F]_{2Q}| = |[\Pi_{\text{Tree}(I)} f]_{2Q}| \lesssim C_p.$$

If $x \in I$ is not in any of the I_{2Q} , then

$$|F(x)| = |\Pi_{\text{Tree}(I)} f(x)| \lesssim C_p.$$

Thus we have $\|F\|_\infty \lesssim C_p$. Combining this with the previous we obtain (8) as desired. \square

We now give the well-known converse to the above lemma:

Lemma 3.4 (John-Nirenberg inequality). *Let I be a dyadic interval, and let $f \in S_0(I)$ be real-valued. Then we have*

$$\|f\|_p \lesssim (1+p)|I|^{1/p} \|f\|_{\text{BMO}}$$

for all $0 < p < \infty$ and

$$|\{x \in I : f(x) > 2n\|f\|_{\text{BMO}}\}| \leq 2^{-n+1}|I| \text{ for all } n \in \mathbf{Z}^+.$$

Proof: It suffices to prove the latter inequality, as the former easily follows.

We prove the claim by induction on n . The claim is clear for $n = 1$. Now suppose that $n > 1$ and the claim has already been proven for $n - 1$.

Fix I, f . Let \mathbf{P} denote those tiles P in $\text{Tree}(I)$ such that $[f]_P > 2\|f\|_{\text{BMO}}$, and such that \mathbf{P} is maximal with respect to \leq' . For each P we have

$$\int_{I_P} |f|^2 \geq |I_P|[f]_P^2 \geq 4|I_P|\|f\|_{\text{BMO}}^2.$$

On the other hand, from (6), (7) we have

$$\int_I |f|^2 \leq |I|\|Wf\|_{\text{size}(\text{Tree}(I))}^2 \leq |I|\|f\|_{\text{BMO}}^2.$$

Thus \mathbf{P} is a $\frac{1}{4}$ -packing of $\text{Tree}(I)$, so that the collection $2\mathbf{P} = \{2P : P \in \mathbf{P}\}$ of parents of tiles in \mathbf{P} form a $\frac{1}{2}$ -packing of $\text{Tree}(I)$.

By construction we have $[f]_{2P} \leq 2\|f\|_{\text{BMO}}$ for all $P \in \mathbf{P}$, and $f(x) \leq 2\|f\|_{\text{BMO}}$ for all $x \notin \bigcup_{P \in \mathbf{P}} I_{2P}$. Thus

$$\{x \in I : f(x) > 2n\|f\|_{\text{BMO}}\} \subseteq \bigcup_{2P \in 2\mathbf{P}} \{x \in I_{2P} : f - [f]_{2P}(x) > 2(n-1)\|f\|_{\text{BMO}}\}.$$

The claim then follows from the inductive hypothesis. □

3.1. Chopping big trees into little trees, and extrapolation of Carleson measures. Let $a : \mathbf{P}^+ \rightarrow \mathbb{R}^+$ be a function. Suppose we have a convex lacunary tree T_0 with a large size, let's say

$$(10) \quad \|a\|_{\text{size}^*(T_0)} \leq C_0.$$

Let $0 < \delta \leq C_0$ be a small number. An obvious question to ask is whether one can decompose the large tree T_0 into small trees, each of which has size less than or equal to δ . This is clearly impossible, as the example of a singleton tree T_0 with large size demonstrates. However, one can do the next best thing:

Theorem 3.5. *With the above assumptions, we have the disjoint partition*

$$(11) \quad T_0 = \bigcup_{T \in \mathbf{T}} T \cup \mathbf{P}$$

where the trees T in \mathbf{T} are convex and satisfy

$$(12) \quad \|a\|_{\text{size}^*(T)} \leq \delta$$

while the tiles $P \in \mathbf{P}$ obey the estimate

$$(13) \quad a(P) \leq C_0|I_P|.$$

Furthermore, the tiles \mathbf{P} and the tree tops $\{P_T : T \in \mathbf{T}\}$ are both uniform $C(C_0, \delta)$ -packings of T_0 .

Note that Lemma 2.1 gives an easy converse to the above theorem: if T_0 can be partitioned by (11) with the above properties then $\|a\|_{\text{size}^*(T_0)}$ is bounded (but by a much large constant than C_0). Thus, if one is willing to ignore losses in constants, the above theorem gives a complete characterization of trees of large size in terms of trees of small size. As we shall see in this section, this theorem can be applied to give extrapolation lemma for Carleson measures, and seems likely to be useful in other contexts also.

A continuous parameter version of Theorem 3.5 is at least implicit in [3], where, as here, it is used to prove the ‘‘Extrapolation Lemma for Carleson Measures’’ (see Corollary 3.9 below). The latter, in its continuous parameter form, was then used to establish the ‘‘restricted version’’ of the Kato square root conjecture, for L^∞ perturbations of real, symmetric, elliptic coefficient matrices. The essential idea of the extrapolation method had previously been introduced by J. L. Lewis in his work with M. A. M. Murray [42] on the heat equation in non-cylindrical domains, and refined further by Lewis and one of the present authors [32] in their work on parabolic and elliptic equations. Similar ideas had also appeared previously in the work of David and Semmes on uniform rectifiability: indeed Theorem 4.5 is very closely related to the ‘‘Corona Decomposition’’ of [24].

Roughly speaking, in applications of the extrapolation method, the idea is first to show that some ‘‘scale-invariant estimate on cubes’’ (like a Carleson measure estimate, a BMO estimate, or a reverse Holder or A_∞ estimate for a weight) holds when some controlling Carleson measure is suitably small in a certain sense, which will be made precise in the sequel. The term ‘‘extrapolation’’ refers to the removal of the smallness restriction. In that sense it is analogous to G. David’s technique for bootstrapping the Lipschitz constant (see, e.g., [21]), although it is not clear whether there exists an explicit connection between the two methods.

In [42], [32], for example, the controlling Carleson measure was a condition on either the boundary of the domain, or on the coefficients of the elliptic or parabolic operator, and one proved reverse Holder inequalities for the associated elliptic-harmonic or parabolic measures. In particular, in [32], the authors give an alternative proof, via extrapolation, of the main theorem of R. A. Fefferman, C. E. Kenig and J. Pipher [28], in which the controlling Carleson measure is a condition on the disagreement between the coefficients of two elliptic (or parabolic) operators, in the case that reverse Holder estimates are known to hold for the elliptic-harmonic measures associated to the first operator, and one wishes to prove such estimates for the second.

In [3], the authors exploit the fact that proving Kato’s square root estimate is equivalent (by “ $\mathbf{T}(1)$ ” type reasoning) to proving that a certain positive measure in the upper half space is Carleson. Here, the controlling Carleson measure was the one associated to the original, self-adjoint operator, and the extrapolation technique was used to prove that the analogous measure, related to the square root estimate for the perturbed operator, was also Carleson. It is in this setting that Corollary 3.9, or rather its continuous parameter analogue, is directly applicable.

The proof we give here follows the approach in [3]. At the end of this section we give an alternate proof, due to John B. Garnett, which gives better dependence on constants.

Before we give the rigorous proof, we first informally describe the idea of the argument. Suppose the original tree T_0 has size $\|a\|_{\text{size}(T_0)} = c$. Then $0 \leq c \leq C_0$ by (10). To create a tree of maximal size less than δ , we start with T_0 and remove from it some sub-trees of size between $c + \delta/2$ and $c - \delta/2$, which we select by a straightforward stopping time argument. It then remains to control the sub-trees that were removed. By shrinking the trees slightly (putting the error into \mathbf{P}) we can assume that the trees have size either greater than $c + \delta/2$ or less than $c - \delta/2$ (so that the tree that remains must have size at most δ). We call the first type of tree “heavy” and the second type “light”. Because the original tree had size c , it cannot be the case that I_{T_0} is covered by heavy sub-trees, and so a positive proportion of I_{T_0} must be covered by light trees or by nothing. We then pass to the light sub-trees and iterate this process, finding a positive proportion I_{T_0} occupied by increasingly lighter sub-trees. After about $O(C_0/\delta)$ steps, we must terminate, finding a positive proportion of I_{T_0} which are not covered by any further sub-trees. We then pass to the remaining portion of I_{T_0} and all the heavy trees which have until now been neglected, and iterate once again; since we have replaced I_{T_0} with a strictly smaller fraction of I_{T_0} , this procedure will converge geometrically to obtain the desired estimates.

We now prove Theorem 3.5. We shall drop (13) since it follows from (10). In the spirit of Lemma 3.1, it will suffice to prove the apparently weaker

Theorem 3.6. *With the above assumptions, we can find a (possibly empty) collection $\mathbf{T}_{\text{iterate}}$ of disjoint convex trees in T_0 whose tops have disjoint spatial intervals and form a $(1 - \eta)$ -packing of T_0 for some $\eta = \eta(C_0, \delta) > 0$, such that we have the disjoint partition*

$$(14) \quad T_0 = \bigcup_{T \in \mathbf{T}_{\text{iterate}}} T \cup \bigcup_{T \in \mathbf{T}} T \cup \mathbf{P}$$

where the trees $T \in \mathbf{T}$ obey (12), and \mathbf{P} and the tree tops of \mathbf{T} are both uniform $C(C_0, \delta)$ -packings of T_0 .

Indeed, if Theorem 3.6 holds, then we can construct the collection in Theorem 3.5 by starting with the partition (14), and then taking each of the trees in $\mathbf{T}_{\text{iterate}}$ and breaking them up by a further application of Theorem 3.6. We continue on in this way until the original tree \mathbf{T}_0 is completely broken up into trees T obeying (12) and tiles P obeying (13). The fact that \mathbf{P} and the tree tops of \mathbf{T} are $C(C_0, \delta)$ -packings of T_0 then follows from Theorem 3.6 and the fact that the geometric series $\sum_n (1 - \eta)^n$ converges. A similar argument can then be used to improve “ $C(C_0, \delta)$ -packing” to “uniform $C(C_0, \delta)$ -packing”. We omit the details.

Proof of Theorem 3.6: Define the quantity c by

$$(15) \quad c := \|a\|_{\text{size}(T_0)},$$

thus $0 \leq c \leq C_0$. We shall prove the theorem by induction on c . Specifically, we fix $0 \leq c \leq C_0$ and assume that the theorem has already been proven in the case $\|a\|_{\text{size}(T_0)} \leq c - \delta/2$. Note that we only have to apply this induction a finite number of times (about $O(C_0/\delta)$) so we will be allowed to let the constants get worse with each induction step.

The main lemma used in the proof of the theorem will be

Lemma 3.7. *We can partition*

$$(16) \quad T_0 = \bigcup_{T \in \mathbf{T}_{\text{small}}} T \cup \mathbf{P}_{\text{buffer}} \cup \bigcup_{T \in \mathbf{T}_{\text{heavy}}} T \cup \bigcup_{T \in \mathbf{T}_{\text{light}}} T$$

where $\mathbf{T}_{\text{small}}$ is a collection of convex trees which all obey (12) and whose tree tops are a uniform 4-packing of T_0 , $\mathbf{P}_{\text{buffer}}$ is uniform 3-packing of T_0 , and $\mathbf{T}_{\text{heavy}}$, $\mathbf{T}_{\text{light}}$ are collections of disjoint convex sub-trees of T_0 which are complete with respect to T_0 , and are such that we have the tree counting estimates

$$(17) \quad \sum_{T \in \mathbf{T}_{\text{heavy}}} |I_T| + \sum_{T \in \mathbf{T}_{\text{light}}} |I_T| \leq |I_{T_0}|$$

$$(18) \quad \sum_{T \in \mathbf{T}_{\text{heavy}}} |I_T| \leq \frac{c}{c + \delta/2} |I_{T_0}|$$

and the size bounds

$$(19) \quad \|a\|_{\text{size}(T)} \leq c - \delta/2 \text{ for all } T \in \mathbf{T}_{\text{light}}.$$

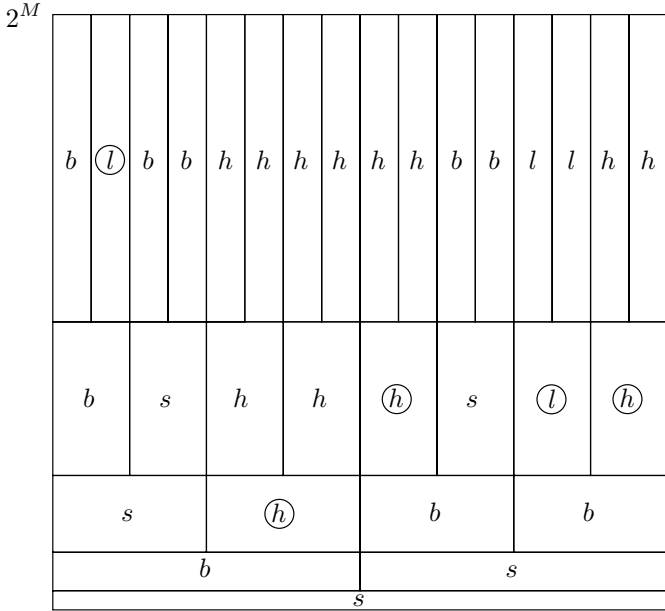


FIGURE 3. A convex tree T_0 and its decomposition from Lemma 3.7. The circled h and l tiles are the tops of maximal sub-trees of T_0 for which the size of a fluctuates by at least $\delta/2$ from c ; the uncircled h and l tiles are the remaining tiles in those maximal sub-trees. $\mathbf{T}_{\text{heavy}}$ thus consists of the (three) h trees while $\mathbf{T}_{\text{light}}$ consists of the (two) l trees. $\mathbf{P}_{\text{buffer}}$ consists of those remaining tiles (labeled b) which lie just below a heavy or light tile, or are at the very top of the phase plane. The remaining tiles (labeled s) form the (three) small trees $\mathbf{T}_{\text{small}}$.

Proof: Define $\mathbf{T}_{\text{fluctuate}}$ to be those sub-trees T of T_0 which are complete with respect to T_0 , such that $|||a||_{\text{size}(T)} - c| \geq \delta/2$, and such that T is maximal with respect to set inclusion and the above two properties. Note that such trees are automatically convex.

By construction, none of the trees in $\mathbf{T}_{\text{fluctuate}}$ contain the top tile P_{T_0} . We may subdivide¹¹

$$\mathbf{T}_{\text{fluctuate}} = \mathbf{T}_{\text{heavy}} \cup \mathbf{T}_{\text{light}}$$

¹¹With reference to Figure 3, $\mathbf{T}_{\text{fluctuate}}$ consists of the h and l trees, T_1 consists of the s and b tiles, and T_2 consists of just the s tiles.

where $\mathbf{T}_{\text{heavy}}$ consists of those trees $T \in \mathbf{T}_{\text{fluctuate}}$ with

$$(20) \quad \|a\|_{\text{size}(T)} \geq c + \delta/2$$

and $\mathbf{T}_{\text{light}}$ consists of those trees $T \in \mathbf{T}_{\text{fluctuate}}$ with

$$\|a\|_{\text{size}(T)} \leq c - \delta/2.$$

The trees in $\mathbf{T}_{\text{fluctuate}}$ are disjoint, convex, and have disjoint spatial supports, so (17) holds. On the other hand if one multiplies (20) by $|I_T|$ and sums over all $T \in \mathbf{T}_{\text{heavy}}$ one obtains

$$|I_{T_0}|c = \sum_{P \in T_0} a(P) \geq \sum_{P \in \bigcup_{T \in \mathbf{T}_{\text{heavy}}} T} a(P) \geq (c + \delta/2) \sum_{T \in \mathbf{T}_{\text{heavy}}} |I_T|.$$

Dividing by $c + \delta/2$ we obtain (18).

Let T_1 denote the convex tree $T_1 := T_0 \setminus \bigcup_{T \in \mathbf{T}_{\text{fluctuate}}} T$ with top P_{T_0} . Informally, T_1 represents the portion of T_0 below the fluctuating tiles. The tree T_1 contains P_{T_0} and is hence non-empty. Let $\mathbf{P}_{\text{buffer}}$ denote the tiles¹²

$$\mathbf{P}_{\text{buffer}} := \{P \in T_1 : P = 2Q \text{ for some } Q \notin T_1\} \cup \{P \in T_1 : |I_P| = 2^{-M}\}.$$

In other words, $\mathbf{P}_{\text{buffer}}$ consists of those tiles in T_1 which touch the upper boundary of T_1 (which in particular may include the tiles of minimal width $|I_P| = 2^{-M}$). Since the tiles Q in the definition of $\mathbf{P}_{\text{buffer}}$ have disjoint spatial supports and $|I_P| = 2|I_Q|$ we see that $\{P \in T_1 : P = 2Q \text{ for some } Q \notin T_1\}$ is a uniform 2-packing of T_1 . Since $\{P \in T_1 : |I_P| = 2^{-M}\}$ is clearly a uniform 1-packing of T_1 , we thus see that $\mathbf{P}_{\text{buffer}}$ is a uniform 3-packing of T_1 .

Let T_2 denote the (possibly empty) tree $T_2 := T_1 \setminus \mathbf{P}_{\text{buffer}}$ with top P_{T_0} . This tree is not necessarily convex, however we shall invoke the following lemma to split it into convex trees.

Lemma 3.8. *Let T be a convex tree, and let $\mathbf{P} \subset T$ be a uniform α -packing of T for some $\alpha > 0$. Then $T \setminus \mathbf{P}$ can be partitioned into*

$$T \setminus \mathbf{P} = \bigcup_{T' \in \mathbf{T}} T'$$

where \mathbf{T} is a collection of convex trees T' whose tops $\{P_T : T \in \mathbf{T}\}$ form a uniform $(\alpha + 1)$ -packing of T .

¹²Here we are taking advantage of our decision to work in a finite model, where the tiles have a minimal width 2^{-M} . One can replicate this argument in the infinite setting but one has to treat the portion of T_1 which “goes all the way to infinity” separately. See [3].

Proof: Let \mathbf{Q} denote those dyadic intervals $Q \subset I_T$ such that $Q \in T \setminus \mathbf{P}$ and $2Q \notin T \setminus \mathbf{P}$. For any $Q \in \mathbf{Q}$, we see that either $Q = P_T$ or $2Q \in \mathbf{P}$. Since \mathbf{P} is a uniform α -packing, this implies that \mathbf{Q} is a uniform $(\alpha + 1)$ -packing.

For each $Q \in \mathbf{Q}$, define the convex tree T_Q with top Q by

$$T_Q := \{P \in T \setminus \mathbf{P} : P \leq' Q\},$$

and there does not exist $Q' \in \mathbf{Q}$ such that $P <' Q' <' Q$.

If we then set $\mathbf{T} := \{T_Q : Q \in \mathbf{Q}\}$ we see that the lemma follows. \square

By Lemma 3.8 we may write $T_2 = \bigcup_{T \in \mathbf{T}_{\text{small}}} T$ where the trees in $\mathbf{T}_{\text{small}}$ are distinct and the tree tops of $\mathbf{T}_{\text{small}}$ are a uniform 4-packing of \mathbf{T} .

We now verify that each tree $T \in \mathbf{T}_{\text{small}}$ obeys (12). It suffices to show that

$$(21) \quad \sum_{P \in \text{Tree}(I) \cap T} a(P) \leq \delta |I|$$

for all $I \subseteq I_{T_0}$.

Fix I . The idea is to write $\text{Tree}(I) \cap T$ as the difference of trees, each of which has size $c + O(\delta)$.

We may assume that $P^+(I) \in T$ since the claim is trivial otherwise. We observe that

$$\text{Tree}(I) \cap T = (\text{Tree}(I) \cap T_0) \setminus \bigcup_{J \in \mathbf{J}} (\text{Tree}(J) \cap T_0)$$

where \mathbf{J} consists of those intervals $J \subseteq I$ such that $P^+(J) \notin T$, and which are maximal with respect to this property.

The tile $P^+(I)$ is in T and hence in T_1 . By construction of T_1 , we thus have

$$\sum_{P \in \text{Tree}(I) \cap T_0} a(P) = |I| \|a\|_{\text{size}(\text{Tree}(I) \cap T_0)} \leq |I|(c + \delta/2)$$

(since otherwise $\text{Tree}(I) \cap T_0$ would belong to $\mathbf{T}_{\text{heavy}}$, a contradiction). Similarly, for every $J \in \mathbf{J}$, the tile $\mathbf{P}^+(J)$ is contained in T_1 (otherwise $P^+(2J)$ would be both in T and in $\mathbf{P}_{\text{buffer}}$, a contradiction), so

$$\sum_{P \in \text{Tree}(J) \cap T_0} a(P) \geq |J|(c - \delta/2).$$

By the construction of \mathbf{J} , the intervals J in \mathbf{J} partition I , thus

$$\sum_{J \in \mathbf{J}} \sum_{P \in \text{Tree}(J) \cap T_0} a(P) \geq |I|(c - \delta/2).$$

Subtracting this from the previous we obtain (21) as desired. \square

We apply the above lemma and place $\mathbf{P}_{\text{buffer}}$ into \mathbf{P} , $\mathbf{T}_{\text{small}}$ into \mathbf{T} , and $\mathbf{T}_{\text{heavy}}$ into $\mathbf{T}_{\text{iterate}}$. For the remaining trees $\mathbf{T}_{\text{light}}$ we use the induction hypothesis, which splits each of the trees in $\mathbf{T}_{\text{light}}$ into $\mathbf{T}_{\text{iterate}}$, \mathbf{T} , and \mathbf{P} . All the desired conclusions of Theorem 3.6 are easily verified except perhaps for the claim that the tops of $\mathbf{T}_{\text{iterate}}$ form a $(1 - \eta)$ -packing of T , or in other words

$$\sum_{T \in \mathbf{T}_{\text{iterate}}} |I_T| \leq (1 - \eta) |I_{T_0}|.$$

To prove this inequality, note that the trees in $\mathbf{T}_{\text{heavy}}$ contribute $\sum_{T \in \mathbf{T}_{\text{heavy}}} |I_T|$ to the left-hand side, while from the induction hypothesis the trees in $\mathbf{T}_{\text{light}}$ contribute at most $(1 - \eta) \sum_{T \in \mathbf{T}_{\text{light}}} |I_T|$ for some $\eta > 0$. There are no other contributions. The claim then follows from (17), (18) (reducing the value of η as necessary). \square

The constants $C(C_0, \delta)$ given by this argument are about $(C_0/\delta)^{C C_0/\delta}$. This bound can be improved substantially; see below.

The following corollary allows one to use one Carleson measure μ to prove the Carleson measure property of a related measure μ' . It is the dyadic version of an extrapolation lemma in [3], which in turn is based on ideas in [42], [32].

Corollary 3.9 (Extrapolation of Carleson measures). *Let $\mu: \mathbf{P}^+ \rightarrow \mathbb{R}^+$ have bounded maximal size and let $\delta > 0$. Let μ' be a non-negative measure on \mathbb{R}_+^2 obeying the “weak Carleson condition”*

$$\mu'(P) \leq C_1 |I_P| \text{ for all } P \in \mathbf{P}^+$$

and such that $\|\mu'\|_{\text{size}(T)} \leq C_2$ for all convex trees T such that $\|\mu\|_{\text{size}^(T)} \leq \delta$. Then μ' also has bounded maximal size:*

$$\|\mu'\|_{\text{size}^*(\mathbf{P}^+)} \leq C(\|\mu\|_{\text{size}^*(\mathbf{P}^+)}, \delta)(C_1 + C_2).$$

Proof: Let T_0 be any convex tree. We need to show that

$$\|\mu'\|_{\text{size}(T_0)} \leq C(\|\mu\|_{\text{size}^*(\mathbf{P}^+)}, \delta)(C_1 + C_2).$$

By Theorem 3.5, we can partition $T_0 = \bigcup_{T \in \mathbf{T}} T \cup \mathbf{P}$ where $\|\mu\|_{\text{size}^*(T)} \leq \delta$ for all $T \in \mathbf{T}$, and

$$\sum_{T \in \mathbf{T}} |I_T| + \sum_{P \in \mathbf{P}} |I_P| \leq C(\|\mu\|_{\text{size}^*(\mathbf{P}^+)}, \delta) |I_{T_0}|.$$

From this and assumptions on μ' we see that

$$\begin{aligned} \sum_{P \in T_0} \mu'(P) &= \sum_{T \in \mathbf{T}} \sum_{P \in T} \mu'(P) + \sum_{P \in \mathbf{P}} \mu'(P) \\ &\leq C(\|\mu\|_{\text{size}^*(\mathbf{P}^+)}, \delta)C_2|I_{T_0}| + C(\|\mu\|_{\text{size}^*(\mathbf{P}^+)}, \delta)C_1|I_{T_0}| \end{aligned}$$

and the claim follows. □

As mentioned earlier, this lemma has applications to the Kato problem. In [3], this lemma was used to establish a restricted version of Kato’s conjecture, for perturbations of real, symmetric coefficient matrices. In that case, μ was a Carleson measure which controlled the original operator, and μ' was the analogous measure controlling the perturbed operator. The point was to establish that μ' was also a Carleson measure. We remark that the fact that the final bound on μ' was linear in C_2 was crucial to this application.

It is possible to eliminate the weak Carleson condition by allowing the tree measured by μ' to be a little larger than the tree measured by μ , but we will not pursue this type of generalization here.

3.2. An alternate argument. In this section we give an alternate proof of Theorem 3.5, due to John B. Garnett (personal communication). The idea of this argument is similar to some arguments in [8].

Fix T_0, a . We first observe that it suffices to prove the theorem under the additional “weak Carleson” assumption

$$(22) \quad a(P) \leq \frac{\delta}{2}|I_P| \text{ for all } P \in T_0.$$

To see this, suppose that we are in the general case when (22) need not hold. We set \mathbf{P} to be the set of tiles where (22) fails:

$$(23) \quad \mathbf{P} := \left\{ P \in T_0 : a(P) \geq \frac{\delta}{2}|I_P| \right\}.$$

From (10) we see that \mathbf{P} is a uniform $2C_0/\delta$ -packing of T_0 (cf. Lemma 2.1). By Lemma 3.8 we thus see that we can split $T_0 \setminus \mathbf{P}$ into a collection of disjoint convex subtrees of T_0 , whose tops form a uniform $2C_0/\delta + 1$ -packing of T_0 . On each such subtree (22) holds. Thus if we apply Theorem 3.5 to each sub-tree and then combine all the decompositions, we obtain the desired decomposition for the original tree T_0 (with the constants $C(C_0, \delta)$ worsened by a factor of $2C_0/\delta + 1$).

Henceforth we assume (22). Under this assumption we will not need \mathbf{P} any more, and will set it equal to the empty set.

We can assume without loss of generality that T_0 is a complete tree, since if T_0 is incomplete then one can replace T_0 by its completion, and extend a by zero; note that the intersection of two convex trees is convex, so one does not lose convexity when one restricts back to T_0 .

We can now make the technical assumption that the minimal tiles have large coefficient:

$$(24) \quad a(P) = \frac{\delta}{2}|I_P| \text{ whenever } P \in T_0 \text{ and } |I_P| = 2^{-M}.$$

This is because in the general case one can simply increase $a(P)$ for these tiles to equal $\frac{\delta}{2}|I_P|$; observe that this only increases $\|a\|_{\text{size}^*(T)}$ by at most $\delta/2$, so the claim follows by redefining C_0 as necessary. This technical assumption is needed to make sure that a certain stopping argument always halts before it reaches the smallest scale.

We now use the greedy algorithm to select a subtree T of T_0 of size roughly comparable to δ :

Lemma 3.10. *Let T_0 be a complete tree, and let $a: T_0 \rightarrow \mathbb{R}^+$ obey (22) and (24). Then there exists a convex subtree $T \subseteq T_0$ of T_0 with $P_T = P_{T_0}$ such that*

$$\frac{\delta}{2} \leq \|a\|_{\text{size}(T)} \leq \|a\|_{\text{size}^*(T)} \leq \delta.$$

Proof: Consider the top tile P_{T_0} . If $a(P_{T_0}) = \frac{\delta}{2}|I_{T_0}|$ then we can set T to be the singleton tree $\{P_{T_0}\}$, so we can assume that $a(P_{T_0})$ is strictly less than $\frac{\delta}{2}|I_{T_0}|$.

Let \mathbf{T} be the class of all convex subtrees $T' \subseteq T_0$ with top $P_{T'} = P_{T_0}$ such that

$$(25) \quad \|a\|_{\text{size}^*(T')} < \delta/2.$$

By the previous paragraph, this class \mathbf{T} is non-empty; also, by (24), this class cannot contain any tiles with the minimal width 2^{-M} .

Let T_* be a tree in \mathbf{T} which is maximal with respect to set inclusion. Let \mathbf{P} denote the set of tiles P in $T_0 \setminus T_*$ such that $2P \in T_*$; these are the tiles which lie just above T_* . Since T_0 is complete and T_* does not contain any tiles of minimal width, we see that the spatial intervals $\{I_P : P \in \mathbf{P}\}$ partition I_{T_0} . In particular the \mathbf{P} are a uniform 1-packing of T_0 .

Set $T := T_* \cup \mathbf{P}$; this is clearly a convex sub-tree of T_0 with top $P_T = P_{T_0}$. From (25), (22), and the uniform 1-packing property of \mathbf{P} we see that $\|a\|_{\text{size}^*(T)} \leq \delta$ (cf. Lemma 2.1). It thus remains to show the lower

bound on size, i.e.

$$(26) \quad \sum_{P \in T} a(P) \geq \frac{\delta}{2} |I_{T_0}|.$$

Call a tile $Q \in T$ *heavy* if $\|a\|_{\text{size}(\text{Tree}(Q) \cap T)} \geq \frac{\delta}{2}$, or in other words

$$(27) \quad \sum_{P \in T: P \leq' Q} a(P) \geq \frac{\delta}{2} |I_Q|.$$

Observe that for every tile $P \in \mathbf{P}$, there must exist a heavy tile $Q \in T$ such that $P \leq' Q$, since otherwise one could add P to the tree T while retaining the property (25), contradicting the maximality of T .

Let \mathbf{Q} denote the set of heavy tiles Q in T which are maximal with respect to the ordering \leq' . By the previous paragraph we see that the spatial intervals of \mathbf{Q} partition T_0 . If one adds up (27) for all such tiles one obtains (26). The proof of the lemma is now complete. \square

When one removes T from the complete tree T_0 we obtain a union of disjoint complete tree, which are of course smaller than T_0 but still obey (22) and (24). Thus we can iterate the above lemma to obtain

Corollary 3.11. *Let T_0 be a complete tree, and let $a: T_0 \rightarrow \mathbb{R}^+$ obey (22) and (24). Then we can partition $T_0 = \bigcup_{T \in \mathbf{T}} T$, where \mathbf{T} is a collection of disjoint convex trees T such that*

$$\frac{\delta}{2} \leq \|a\|_{\text{size}(T)} \leq \|a\|_{\text{size}^*(T)} \leq \delta$$

for all $T \in \mathbf{T}$.

We apply the above corollary to the tree T_0 in the theorem, obtaining the collection \mathbf{T} of trees. We now claim that the tops $\{P_T : T \in \mathbf{T}\}$ of these trees form a uniform $2C_0/\delta$ -packing of T_0 . Indeed, for any dyadic interval $J \subseteq I_{T_0}$ we see that

$$\begin{aligned} \sum_{T \in \mathbf{T}: I_T \subseteq J} \frac{\delta}{2} |I_T| &\leq \sum_{T \in \mathbf{T}: I_T \subseteq J} \sum_{P \in T} a(P) \\ &\leq \sum_{P \in T_0: I_P \subseteq J} a(P) \\ &\leq \|a\|_{\text{size}^*(T_0)} |J| \\ &\leq C_0 |J|. \end{aligned}$$

This completes the proof of Theorem 3.5 (with \mathbf{P} empty). \square

Observe that the constants obtained in this manner are significantly superior to the previous argument, being polynomial in C_0/δ instead of exponential.

4. The “ L^p theory”

In the previous section we established some estimates in the case when the maximal size was bounded; this can be thought of as the “ L^∞ theory” of maximal size. Now we study what happens when our collection of tiles does not have a good maximal size bound. In this case we can subdivide the collection into disjoint trees, such that the size of each of the trees is under control:

Lemma 4.1 (“Calderón-Zygmund decomposition for size”). *Let $n \in \mathbb{Z}$, \mathbf{P}_n be a convex collection of lacunary tiles, and let $a: \mathbf{P}_n \rightarrow \mathbb{R}^+$ be a function such that $\|a\|_{\text{size}^*(\mathbf{P}_n)} \leq 2^n$. Then there exists a disjoint partition*

$$(28) \quad \mathbf{P}_n = \bigcup_{T \in \mathbf{T}_n} T \cup \mathbf{P}_{n-1}$$

where \mathbf{P}_{n-1} is a convex collection of tiles such that

$$(29) \quad \|a\|_{\text{size}^*(\mathbf{P}_{n-1})} \leq 2^{n-1}$$

and \mathbf{T}_n is a collection of convex trees T with disjoint spatial intervals I_T such that

$$(30) \quad \|a\|_{\text{size}(T)} \sim \|a\|_{\text{size}^*(T)} \sim 2^n$$

for all $T \in \mathbf{T}_n$.

In the particular case that $a = |Wf|^2$ for some $f \in S_0$, we then have

$$(31) \quad \|\Pi_T f\|_{\text{BMO}} \sim 2^{n/2}$$

and

$$(32) \quad \|\Pi_T f\|_p \sim |I_T|^{1/p} 2^{n/2}$$

for all $0 < p < \infty$ (with the implicit constant depending on p).

Proof: We set \mathbf{T}_n to be the collection of all trees $T \subset \mathbf{P}_n$ such that $\|a\|_{\text{size}(T)} \geq 2^{n-1}$, and are maximal with respect to set inclusion. Clearly these trees are disjoint (otherwise the union of the two trees would also qualify, and contradict maximality). They are also complete with respect to \mathbf{P}_n (i.e. $T = \text{Tree}(I_T) \cap \mathbf{P}_n$) and thus convex. One then sets

$$(33) \quad \mathbf{P}_{n-1} := \mathbf{P}_n \setminus \bigcup_{T \in \mathbf{T}_n} T.$$

The properties (28), (29), (30) are easily verified. To prove the last two properties, observe from (30) that

$$\|\Pi_T f\|_{\text{BMO}} = \| |Wf|^2 \|_{\text{size}^*(T)}^{1/2} \sim 2^{n/2}$$

and

$$\|\Pi_T f\|_2 = |I_T|^{1/2} \| |Wf|^2 \|_{\text{size}(T)}^{1/2} \sim 2^{n/2} |I_T|^{1/2}.$$

The claim then follows from the John-Nirenberg inequality (Lemma 3.4) and Hölder’s inequality. \square

The above lemma should be compared with the standard Calderón-Zygmund decomposition, which if given a function f with $\|f\|_\infty \leq 2^n$, will subdivide $f = g + \sum_I b_I$ where $\|g\|_\infty \leq 2^{n-1}$, the b_I are supported on disjoint intervals I and have mean zero and $\|b_I\|_p \sim 2^n |I|^{1/p}$ for all $0 < p < \infty$. The trees in \mathbf{T}_n are the analogues of the intervals I , and can be thought of as the region of phase space where a (or f) “has size $\sim 2^n$ ”. Lemma 4.1 can also be thought of as a sort of BMO version of the Chebyshev’s inequality

$$(34) \quad |\{x : |f(x)| \gtrsim 2^{n/2}\}| \lesssim 2^{-np/2} \|f\|_p^p.$$

Indeed, if f , n , \mathbf{T}_n is as in the above lemma; then by (32) and the disjointness of the I_T we have

$$\|f\|_p^p \gtrsim \sum_{T \in \mathbf{T}_n} \|\Pi_T f\|_p^p \sim \sum_{T \in \mathbf{T}_n} |I_T| 2^{np/2}$$

and hence

$$(35) \quad \left| \bigcup_{T \in \mathbf{T}_n} I_T \right| \lesssim 2^{-np/2} \|f\|_p^p$$

(compare with (34), (31), (29)).

In practice one iterates the above lemma, starting with a large n and decrementing n repeatedly, thus decomposing \mathbf{P}^+ into trees of various sizes (plus a remainder of size 0).

We have a similar selection lemma for mean, which can be thought of as the analogue of the previous lemma for the non-lacunary tiles \mathbf{P}^0 .

Lemma 4.2 (“Calderón-Zygmund decomposition for mean”). *Let $n \in \mathbb{Z}$, \mathbf{P}_n be a convex collection of lacunary tiles, and let $f \in S$ such that $\|f\|_{\text{mean}^*(\mathbf{P}_n)} \leq 2^n$. Then there exists a disjoint partition*

$$(36) \quad \mathbf{P}_n = \bigcup_{T \in \mathbf{T}_n} T \cup \mathbf{P}_{n-1}$$

where \mathbf{P}_{n-1} is a convex collection of tiles such that

$$(37) \quad \|f\|_{\text{mean}^*(\mathbf{P}_{n-1})} \leq 2^{n-1}$$

and \mathbf{T}_n is a collection of convex trees T with disjoint spatial intervals I_T such that

$$(38) \quad \|f\|_{\text{mean}(P_T)} \sim \|f\|_{\text{mean}^*(T)} \sim 2^n$$

for all $T \in \mathbf{T}$. In particular, we have

$$(39) \quad \sum_{T \in \mathbf{T}_n} |I_T| \lesssim 2^{-n} \int_{|f| \gtrsim 2^n} |f| \lesssim 2^{-np} \|f\|_p^p$$

for any $1 \leq p < \infty$ (with the implicit constant depending on p).

Proof: We define \mathbf{T}_n to be the set of all trees T such that $\|f\|_{\text{mean}^*(T)} \geq 2^{n-1}$, and are maximal with respect to set inclusion, and then define \mathbf{P}_{n-1} by (33). As before the trees T are convex and complete with respect to \mathbf{P}_n . The properties (36), (37), (38) are easily verified. By (38) we have $\int_{x \in I_T: |f(x)| \gtrsim 2^n} |f| \gtrsim 2^n |I_T|$, and (39) follows from the disjointness of the I_T . \square

In later sections we give some applications of the above machinery. These applications will all have a similar flavor, in that they follow the following broad strategy:

- Begin with a sum over a collection of lacunary tiles.
- Use Lemma 4.1 and/or Lemma 4.2 to extract disjoint trees in this collection of a certain size (plus a remainder of size 0, which is usually trivial to handle).
- Estimate the contribution of each tree in terms of the width $|I_T|$ of the tree and the size and/or mean of the tree.
- Estimate the total width $\sum_T |I_T|$ of the trees (using such estimates as (31), (32), (35), (39)).
- Remove these trees from the collection, and repeat the above steps until the collection has been exhausted.
- Sum up.

This type of argument is fundamental to the general theory of tiles, as can be seen in the work on the bilinear Hilbert transform and Carleson’s theorem (see e.g. [25], [39], [41], [46]). Apart from several technical details, the main differences between the arguments in those papers and the ones here are to eliminate the word “lacunary” from the above strategy, and replace the above lemmata by more sophisticated tree selection algorithms.

4.1. Example: Atomic decomposition of dyadic H^p . Let $0 < p \leq 1$; implicit constants will be allowed to depend on p . In this section we reprove the standard atomic decomposition of H^p , but we first need some notation.

For any $f \in S$, we define the *dyadic Littlewood-Paley square function* Sf to be the vector-valued function

$Sf(x) := (\Pi_P f(x))_{P \in \mathbf{P}^+} = (\langle f, \phi_P \rangle \phi_P(x))_{P \in \mathbf{P}^+} = (Wf(P)\phi_P(x))_{P \in \mathbf{P}^+}$ taking values in $l^2(\mathbf{P}^+)$. Note that

$$(40) \quad |Sf(x)| = \left(\sum_{P \in \mathbf{P}^+} |Wf(P)|^2 \frac{\chi_{I_P}(x)}{|I_P|} \right)^{1/2}.$$

The adjoint operator S^* takes $l^2(\mathbf{P}^+)$ -valued functions to scalar-valued functions, and is given by the formula

$$S^*(f_P)_{P \in \mathbf{P}^+} = \sum_{P \in \mathbf{P}^+} \Pi_P f_P.$$

In particular S^*S is the identity on S_0 . Also observe¹³ that $|WSf(P)| = |Wf(P)|$ for all $P \in \mathbf{P}^+$, so in particular we have

$$(41) \quad \|Sf\|_2 = \|f\|_2; \quad \|Sf\|_{\text{BMO}} = \|f\|_{\text{BMO}}$$

on S_0 and S respectively.

We also define the *cancellative dyadic maximal function* $\tilde{M}f$ by

$$\tilde{M}f(x) := \sup_{P \in \mathbf{P}^0} |\Pi_P f(x)| = \sup_{I: x \in I} \frac{1}{|I|} \left| \int_I f \right| = \sup_{I: x \in I} |[f]_I|;$$

this operator should not be confused with the dyadic Hardy-Littlewood maximal function $Mf := \tilde{M}|f|$. From (5) we have

$$(42) \quad |\Pi_T f(x)| \leq 2\tilde{M}f(x)$$

whenever T is a convex tree in \mathbf{P}^+ .

¹³Here we are allowing W to act on vector-valued functions in the obvious manner, i.e. if $f = (f_1, \dots, f_n)$ is a vector, then $Wf(P) := (Wf_1(P), \dots, Wf_n(P))$. Similarly we can define vector-valued BMO, etc.

If $\text{Tree}(I)$ is a complete tree, we define a (dyadic) H^p atom on $\text{Tree}(I)$ to be a function $a \in S_0(I)$ such that $\|a\|_2 \leq |I|^{1/2-1/p}$. Equivalently, $a \in S_0$ is an H^p atom on $\text{Tree}(I)$ if and only if the wavelet transform Wa of a is supported on $\text{Tree}(I)$ and $\| |Wa|^2 \|_{\text{size}(\text{Tree}(I))} \leq |I|^{-2/p}$; this is because of (6).

In this section we show

Theorem 4.3 (Equivalent definitions of H^p). *Let $f \in S_0$ and $0 < p \leq 1$. Then the following statements are equivalent.*

- (i) $\|Sf\|_p \lesssim 1$.
- (ii) $\|\tilde{M}f\|_p \lesssim 1$.
- (iii) *There exists a collection \mathbf{I} of dyadic intervals, and to each $I \in \mathbf{I}$ there exists a non-negative number c_I and an H^p atom a_I on $\text{Tree}(I)$ such that $f = \sum_I c_I a_I$ and $\sum_I c_I^p \lesssim 1$.*

Proof: We first show that (iii) implies (i) and (ii). From the quasi-triangle inequality

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$$

we see that it suffices to verify this on atoms, i.e. to show that $\|Sa_I\|_p, \|\tilde{M}a_I\|_p \lesssim 1$ whenever a_I is a H^p atom on $\text{Tree}(I)$.

Fix I, a . By construction $\tilde{M}a_I$ and Sa_I are supported on I , so by Hölder it suffices to show $\|Sa_I\|_2, \|\tilde{M}a_I\|_2 \lesssim |I|^{1/2-1/p}$. But this follows from the L^2 normalization of a_I and the fact that S, \tilde{M} are bounded on L^2 .

It remains to show that either one of (i) or (ii) are enough to imply (iii). Let f be any element of S_0 , thus $f = \sum_{P \in \mathbf{P}^+} Wf(P)\phi_P$.

Set $a := |Wf|^2$. We apply Lemma 4.1 repeatedly, starting with a sufficiently large n and setting $\mathbf{P}_n := \mathbf{P}^+$, and then decrementing n indefinitely. Eventually one obtains a partition

$$\mathbf{P}^+ = \bigcup_{n \in \mathbf{Z}} \bigcup_{T \in \mathbf{T}_n} T \cup \mathbf{P}_{-\infty}$$

where the \mathbf{T}_n are as in Lemma 4.1, and $\| |Wf|^2 \|_{\text{size}^*(\mathbf{P}_{-\infty})} = 0$. Thus Wf vanishes on $\mathbf{P}_{-\infty}$, and only a finite number of \mathbf{T}_n are non-empty. We thus have $f = \sum_{n \in \mathbf{Z}} \sum_{T \in \mathbf{T}_n} \Pi_T f$. If we then set $c_{I_T} := 2^{n/2} |I_T|^{1/p}$ and $a_{I_T} := \Pi_T f / c_{I_T}$ then we have

$$f = \sum_{n \in \mathbf{Z}} \sum_{T \in \mathbf{T}_n} c_{I_T} a_{I_T}.$$

By (32) (with $p = 2$) we see that each a_{I_T} is an H^p atom. To show (iii) it thus remains to show that

$$\sum_n \sum_{T \in \mathbf{T}_n} c_{I_T}^p = \sum_n 2^{np/2} \sum_{T \in \mathbf{T}_n} |I_T| \lesssim 1.$$

First suppose that (i) holds. For each n and each $T \in \mathbf{T}_n$, we see that

$$\int_{I_T} |S\Pi_T f|^q \lesssim |I_T| \|S\Pi_T f\|_{\text{BMO}}^q = |I_T| \|\Pi_T f\|_{\text{BMO}}^q \sim 2^{nq/2} |I_T|$$

for all $2 < q < \infty$ by the John-Nirenberg inequality (Lemma 3.4), (41), and (31). Also, we have

$$\int_{I_T} |S\Pi_T f|^2 = \int_{I_T} |\Pi_T f|^2 \sim 2^n |I_T|.$$

By Hölder we thus have

$$\int_{I_T} |Sf|^r \geq \int_{I_T} |S\Pi_T f|^r \gtrsim 2^{nr/2} |I_T|,$$

for any $0 < r < p$, with the implicit constant depending on r . This clearly implies

$$\int_{x \in I_T: |Sf(x)| \gtrsim 2^{n/2}} |Sf|^r \gtrsim 2^{nr/2} |I_T|.$$

Summing over all $T \in \mathbf{T}_n$ and using the disjointness of the I_T we obtain

$$\int_{|Sf(x)| \gtrsim 2^{n/2}} |Sf|^r \gtrsim 2^{nr/2} \sum_{T \in I_T} |I_T|.$$

Multiplying by $2^{n(p-r)/2}$ and summing over n we obtain

$$\int \sum_{n: |Sf(x)| \gtrsim 2^{n/2}} 2^{n(p-r)/2} |Sf|^r \gtrsim \sum_n 2^{np/2} \sum_{T \in I_T} |I_T|.$$

Since the left-hand side is comparable to $\|Sf\|_p^p$, the claim follows.

Now suppose instead that (ii) holds. By (32), (42) we have $\int_{I_T} |\tilde{M}f|^r \gtrsim 2^{nr/2} |I_T|$ for all $0 < r < p$. Now we argue as with Sf to obtain (iii) from (ii). \square

From the above proof we see that the atoms in fact obey the BMO bound $\|a_{I_T}\|_{\text{BMO}} \lesssim |I|^{-1/p}$. One can improve this BMO control to L^∞ control by repeating the argument in the John-Nirenberg inequality (Lemma 3.4). Namely, one locates the maximal sub-intervals where the averages of a_{I_T} are large and separates off those trees, leaving behind a

bounded atom. One then repeats the process until only bounded atoms remain, in the spirit of Lemma 3.1. We omit the details.

Let f be in BMO. From (7), (6), we have

$$\|f\|_{\text{BMO}} = \sup_T \frac{1}{|I_T|^{1/2}} \|\Pi_T f\|_2.$$

Clearly it suffices to take suprema over complete trees T , thus by (4)

$$\|f\|_{\text{BMO}} = \sup_I \frac{1}{|I|^{1/2}} \left(\int_I |f - [f]_I|^2 \right)^{1/2}.$$

By duality we thus have

$$(43) \quad \|f\|_{\text{BMO}} = \sup_I \sup_{a \in S_0(I): \|a\|_2=1} |I|^{-1/2} |\langle f, a \rangle|$$

or equivalently

$$\|f\|_{\text{BMO}} = \sup\{|\langle f, a \rangle| : a \text{ is a } H^1 \text{ atom}\}.$$

Thus, as is well known, BMO is the dual of H^1 .

5. The Carleson embedding theorem and paraproducts

We now give a slight variant of the above method, in which one selects trees using the averages $[f]_I$ instead of the sizes. This type of argument is of course very old, and the arguments here are by no means new. On the other hand, this type of tree selection method is a special case of the “mean selection” algorithm used (together with a size selection algorithm) in the proof of Carleson’s theorem in [41].

We begin with

Lemma 5.1 (Carleson embedding theorem). *Let \mathbf{P} be a collection of lacunary tiles, $a: \mathbf{P} \rightarrow \mathbb{R}^+$ be a function, and $1 < p < \infty$. Then we have*

$$\sum_{P \in \mathbf{P}} a(P) |[f]_{I_P}|^p \lesssim \|a\|_{\text{size}^*(\mathbf{P})} \|f\|_p^p$$

for all locally integrable functions f , with the implicit constants depending on p .

Proof: We apply Lemma 4.2 repeatedly, starting with a sufficiently large n and decrementing n repeatedly. This gives us a partition $\mathbf{P} = \bigcup_{n \in \mathbb{Z}} \bigcup_{T \in \mathbf{T}_n} T \cup \mathbf{P}_{-\infty}$ where the \mathbf{T}_n are as in Lemma 4.2, and $\|f\|_{\text{mean}^*(\mathbf{P}_{-\infty})} = 0$. The contribution of $\mathbf{P}_{-\infty}$ is zero, so it suffices to control

$$\sum_n \sum_{T \in \mathbf{T}_n} \sum_{P \in T} a(P) | [f]_{I_P} |^p.$$

If $P \in T \in \mathbf{T}_n$, then $| [f]_{I_P} | \leq \|f\|_{\text{mean}(P)} \leq \|f\|_{\text{mean}^*(T)} \lesssim 2^n$. From this and (2), (3), (39) we may estimate the previous by

$$\begin{aligned} &\lesssim \sum_n \sum_{T \in \mathbf{T}_n} \sum_{P \in T} a(P) 2^{np} \\ &\lesssim \sum_n \sum_{T \in \mathbf{T}_n} \|a\|_{\text{size}^*(\mathbf{P})} |I_T| 2^{np} \\ &\lesssim \sum_n \|a\|_{\text{size}^*(\mathbf{P})} 2^{np} 2^{-n} \int_{|f| \gtrsim 2^n} |f| \\ &\sim \|a\|_{\text{size}^*(\mathbf{P})} \int |f|^p \end{aligned}$$

as desired. □

We can apply this theorem to various linear and bilinear operators. To do this we shall need some notation.

For any sequence $(a_P)_{P \in \mathbf{P}^+}$ of real numbers, we define the *wavelet multiplier* $W^{-1}a_P W$ from S_0 to S_0 by

$$W^{-1}a_P W f := \sum_{P \in \mathbf{P}^+} a_P W f(P) \phi_P.$$

Wavelet multipliers are the discrete analogue of pseudo-differential operators, with a_P being the discrete analogue of a symbol $a(x, \xi)$. Observe that if a_P is bounded, then $W^{-1}a_P W$ is bounded on L^2 and also bounded on BMO. One can of course extend the domain $W^{-1}a_P W$ from S_0 to S , although some of the algebra properties are lost in doing so (since W is not injective on S).

Let f, g be elements of S . We define the “high-low”, “low-high”, and “high-high” paraproducts¹⁴

$$\begin{aligned}\pi_{hl}(f, g) &:= \sum_{P \in \mathbf{P}^+} Wf(P)[g]_P \phi_P \\ \pi_{lh}(f, g) &:= \sum_{P \in \mathbf{P}^+} [f]_P Wg(P) \phi_P \\ \pi_{hh}(f, g) &:= \sum_{P \in \mathbf{P}^+} Wf(P)Wg(P) \frac{\chi_{I_P}}{|I_P|}.\end{aligned}$$

These paraproducts have the symmetries

$$\begin{aligned}(44) \quad & \int \pi_{hh}(f, g)h = \int \pi_{hl}(g, h)f = \int \pi_{lh}(h, f)g \\ & = \int \pi_{hh}(g, f)h = \int \pi_{hl}(f, h)g = \int \pi_{lh}(h, g)f \\ & = \sum_{P \in \mathbf{P}^+} Wf(P)Wg(P)[h]_P\end{aligned}$$

and can be expressed in terms of the Littlewood-Paley square function S :

$$\pi_{hl}(f, g) = S^*(gSf); \quad \pi_{lh}(f, g) = S^*(fSg); \quad \pi_{hh}(f, g) = Sf \cdot Sg.$$

When f, g have mean zero (i.e. $f, g \in S_0$), then the paraproducts decompose the pointwise product operator:

$$(45) \quad fg = \pi_{hl}(f, g) + \pi_{lh}(f, g) + \pi_{hh}(f, g).$$

To see this, it suffices by bilinearity to reduce to the case when $f = \phi_P$ and $g = \phi_Q$ for some $P, Q \in \mathbf{P}^+$. If I_P and I_Q are disjoint then both sides are zero. Thus there are only three cases: $P >' Q$, $P <' Q$, and $P = Q$. In these three cases the reader may easily verify that fg is equal to the high-low, low-high, or high-high paraproduct of f and g respectively, and that the other two paraproducts vanish.

We observe that the high-low and low-high paraproducts can be written as wavelet multipliers:

$$(46) \quad \pi_{hl}(f, g) = W^{-1}[g]_P Wf; \quad \pi_{lh}(f, g) = W^{-1}[f]_P Wg.$$

¹⁴The continuous counterparts would be something like $\int (Q_t f)(P_t g) \frac{dt}{t}$, $\int (P_t f)(Q_t g) \frac{dt}{t}$, and $\int (Q_t f)(Q_t g) \frac{dt}{t}$, where Q_t is as before and P_t is a suitable approximation to the identity at width $1/t$, e.g. $P_t := e^{t^2 \Delta}$. The precise definition of a paraproduct is not standardized, for instance π_{hh} is not considered a paraproduct in some texts.

The high-high paraproduct cannot be written in this way, but we have the useful relationship

$$(47) \quad \pi_{hh}(W^{-1}a_P Wf, g) = \pi_{hh}(f, W^{-1}a_P Wg).$$

One can also write paraproducts using both lacunary and non-lacunary tiles, for instance

$$(48) \quad \text{int } \pi_{hh}(fg)h = \sum_{I \text{ dyadic}} |I|^{-1/2} \langle f, \phi_{P+(I)} \rangle \langle g, \phi_{P+(I)} \rangle \langle h, \phi_{P^0(I)} \rangle.$$

The bilinear Hilbert transform turns out to have a similar expansion, but with the sum ranging over a larger collection of triples of tiles than the ones for paraproducts (specifically, the tiles need not be lacunary or non-lacunary, and range over a three-parameter family rather than a two-parameter one). See e.g. [39], [40], [52], [45], [53], [46].

From the Carleson embedding theorem we have paraproduct estimates:

Corollary 5.2 ($L^2 \times \text{BMO} \rightarrow L^2$ paraproduct estimates). *We have*

$$\|\pi_{hl}(f, g)\|_2 \lesssim \|f\|_2 \|g\|_\infty$$

and

$$\|\pi_{hh}(f, g)\|_2, \|\pi_{th}(f, g)\|_2 \lesssim \|f\|_2 \|g\|_{\text{BMO}}$$

for all $f, g \in S$.

In other words, paraproducts map $L^2 \times L^\infty$ to L^2 (just as the pointwise product does), and the L^∞ factor can be relaxed to BMO as long as one only considers high frequencies of a BMO function. Note that the low frequency portion of a BMO function is somewhat ill-defined since a BMO function might only be determined up to a constant.

Proof: The first bound follows from (46) since $[g]_P$ is bounded by $\|g\|_\infty$. To prove the second bound, it suffices by (44) to consider π_{th} . By orthogonality we have

$$\|\pi_{th}(f, g)\|_2 = \left(\sum_{P \in \mathbb{P}^+} |Wg(P)|^2 |[f]_{I_P}|^2 \right)^{1/2}.$$

The claim now follows from Carleson embedding (Lemma 5.1). □

Lemma 5.3 ($\text{BMO} \times \text{BMO} \rightarrow \text{BMO}$ paraproduct estimate). *We have*

$$\|\pi_{hh}(f, g)\|_{\text{BMO}} \lesssim \|f\|_{\text{BMO}} \|g\|_{\text{BMO}}$$

for all $f, g \in S$.

For the other paraproducts π_{hl}, π_{lh} one must place the “low” factor in L^∞ rather than BMO, as in Lemma 5.2; this is again an easy consequence of (46).

Proof: By Lemma 3.3 with $p=1$ it suffices to show $\int |\Pi_{\text{Tree}(I)}\pi_{hh}(f, g)| \lesssim |I|$ for all dyadic intervals I .

Fix I , and expand the left-hand side as

$$\int \left| \sum_{P \in \mathbf{P}^+} Wf(P)Wg(P)\Pi_{\text{Tree}(I)} \left(\frac{\chi_{I_P}}{|I_P|} \right) \right|.$$

The summand vanishes unless $P \in \text{Tree}(I)$. Thus we can write the above as

$$\int \left| \Pi_{\text{Tree}(I)} \sum_{P \in \text{Tree}(I)} Wf(P)Wg(P) \frac{\chi_{I_P}}{|I_P|} \right|.$$

Since $\Pi_{\text{Tree}(I)}$ is bounded on L^1 , we can bound this by

$$\lesssim \int \left| \sum_{P \in \text{Tree}(I)} Wf(P)Wg(P) \frac{\chi_{I_P}}{|I_P|} \right|.$$

Putting the absolute values inside and performing the integration, we can bound this by

$$\lesssim \sum_{P \in \text{Tree}(I)} |Wf(P)||Wg(P)|.$$

The claim then follows from Cauchy-Schwarz and (7). □

5.1. Weak-type estimates. We now show how to use the above machinery to prove $L^{p,\infty}$ paraproduct estimates, where $L^{p,\infty}$ is the weak L^p (quasi-)norm

$$\|f\|_{L^{p,\infty}} := \sup_{\lambda>0} \lambda |\{x : |f(x)| \geq \lambda\}|^{1/p}.$$

We need the following basic characterization of weak L^p for $0 < p < \infty$:

Lemma 5.4. *Let $0 < p < \infty$ and $A > 0$. Then the following statements are equivalent up to constants:*

- (i) $\|f\|_{p,\infty} \lesssim A$.
- (ii) *For every set E with $0 < |E| < \infty$, there exists a subset $E' \subset E$ with $|E'| \sim |E|$ and $|\langle f, \chi_{E'} \rangle| \lesssim A|E|^{1/p'}$.*

Here p' is defined by $1/p' + 1/p = 1$ (note that p' can be negative!).

Proof: To see that (i) implies (ii), set

$$E' := E \setminus \{x : |f(x)| \geq CA|E|^{-1/p}\}.$$

If C is a sufficiently large constant, then (i) implies $|E'| \sim |E|$, and the claim follows.

To see that (ii) implies (i), let $\lambda > 0$ be arbitrary and set $E := \{x : \operatorname{Re}(f(x)) > \lambda\}$. Then by (ii) we have

$$\lambda|E| \sim \lambda|E'| \lesssim A|E|^{1/p'},$$

and (i) easily follows (replacing Re by $-\operatorname{Re}$, Im , $-\operatorname{Im}$ as necessary). \square

When $p > 1$ we can always set $E' = E$, and the above lemma then reflects the duality between $L^{p,\infty}$ and $L^{p',1}$. However for $p \leq 1$ the freedom to set E' to be smaller than E is necessary (since f need not be locally integrable).

A typical application of Lemma 5.4 is

Proposition 5.5 ($L^p \times L^q \rightarrow L^{r,\infty}$ paraproduct estimates). *We have*

$$\|\pi_{hl}(f, g)\|_{r,\infty} \lesssim \|f\|_p \|g\|_q$$

whenever $1 < p, q < \infty$ and $1/p + 1/q = 1/r$. Similarly for π_{lh}, π_{hh} .

Note that r can be less than 1. One can strengthen the weak L^r to strong L^r by multilinear interpolation (see e.g. [6], [45], [33]). The continuous version of these dyadic paraproduct estimates can be found in, e.g. [14]–[19]; the version for $r < 1$ was first proven in [30] (with some special cases in [9], [14]). It is possible to obtain the continuous estimates from the dyadic ones via averaging arguments, but we shall not do so here.

Proof: We first consider π_{hl} . We may normalize $\|f\|_p = \|g\|_q = 1$; we may assume that f and g are dyadic test functions. Let E be a measurable set with $0 < |E| < \infty$. We need to find a set $E' \subset E$ with $|E'| \sim |E|$ such that

$$(49) \quad |\langle \pi_{hl}(f, g), \chi_{E'} \rangle| \lesssim |E|^{1/r'}.$$

By rescaling (using the hypothesis $1/p + 1/q = 1/r$) we may take $|E| \sim 1$.

We choose E' as

$$E' := E \setminus \{x : M|f|^p(x) + M|g|^q(x) \geq C\}.$$

If C is large enough, then $|E| \sim |E'|$ by the Hardy-Littlewood maximal inequality (see e.g. [51]).

We wish to show (49) with $|E| \sim 1$. By (44) it suffices to show that $|\sum_{P \in \mathbf{P}} Wf(P)[g]_{I_P} W\chi_{E'}(P)| \lesssim 1$ for all convex collections \mathbf{P} of tiles.

We may remove all tiles in \mathbf{P} for which $I_P \cap E' = \emptyset$, since $W\chi_{E'}$ vanishes on these tiles. For any remaining tile P we then have

$$(50) \quad \int_{I_P} |f|^p \lesssim |I_P|$$

by construction of E' . Similarly, we have

$$\|g\|_{\text{mean}^*(\mathbf{P})} = \sup_{P \in \mathbf{P}} \|[g]\|_{I_P} \lesssim \|[g]^q\|_{I_P}^{1/q} \lesssim 1.$$

Thus we reduce to showing

$$(51) \quad \sum_{P \in \mathbf{P}} |Wf(P)| |W\chi_{E'}(P)| \lesssim 1.$$

From (50) and the L^p boundedness of the Littlewood-Paley square function (see e.g. [51]) we have

$$\int_{I_P} |S\Pi_{\text{Tree}(I_P)}f|^p \lesssim \int_{I_P} |\Pi_{\text{Tree}(I_P)}f|^p \lesssim \int_{I_P} |f|^p \lesssim |I_P|,$$

for all $P \in \mathbf{P}$, thus

$$\int_{I_P} \left(\sum_{P \in \text{Tree}(I_P)} |Wf(P)|^2 \frac{\chi_{I_P}(x)}{|I_P|} \right)^{p/2} \lesssim |I_P|.$$

Applying Chebyshev's inequality and Corollary 3.2 we thus see that $\| |Wf|^2 \|_{\text{size}^*(\mathbf{P})} \lesssim 1$. Also, we have

$$\| |W\chi_{E'}|^2 \|_{\text{size}^*(\mathbf{P})} \leq \| \chi_{E'} \|_{\text{BMO}}^2 \leq \| \chi_{E'} \|_{\infty}^2 \leq 1.$$

Thus we can find an $n = O(1)$ such that

$$\| |Wf|^2 \|_{\text{size}^*(\mathbf{P}_n)} \leq 2^{2n} \text{ and } \| |W\chi_{E'}|^2 \|_{\text{size}^*(\mathbf{P}_n)} \leq 2^{2np/s},$$

where we have set $\mathbf{P}_n := \mathbf{P}$, and $s > 1$ is an exponent close to 1 to be chosen later.

By a finite number of applications of Lemma 4.1 with $a := |Wf|^2$ or $a := |W\chi_{E'}|^2$ we may partition $\mathbf{P}_n = \bigcup_{T \in \mathbf{T}_n} T \cup \mathbf{P}_{n-1}$ where \mathbf{P}_{n-1} is a convex collection of tiles such that

$$\| |Wf|^2 \|_{\text{size}^*(\mathbf{P}_{n-1})} \leq 2^{2(n-1)} \text{ and } \| |W\chi_{E'}|^2 \|_{\text{size}^*(\mathbf{P}_{n-1})} \leq 2^{2(n-1)p/s}$$

and \mathbf{T}_n is a collection of convex trees with disjoint spatial supports such that either

$$\int_{I_T} |\Pi_T f|^p \gtrsim 2^{np} |I_T|$$

or

$$\int_{I_T} |\Pi_T \chi_{E'}|^s \gtrsim 2^{np} |I_T|$$

for each $T \in \mathbf{T}_n$. From (42) we thus have

$$\int (\tilde{M}f)^p + \int (\tilde{M}\chi_{E'})^s \gtrsim 2^{np} \sum_{T \in \mathbf{T}_n} |I_T|;$$

from our assumptions on f , $\chi_{E'}$ and the Hardy-Littlewood maximal inequality (see e.g. [51]) we thus have $\sum_{T \in \mathbf{T}_n} |I_T| \lesssim 2^{-np}$. We now return to (51), and estimate the contribution of the trees in \mathbf{T}_n by

$$\sum_{T \in \mathbf{T}_n} \sum_{P \in T} |Wf(P)| |W\chi_{E'}(P)|.$$

We apply Cauchy-Schwarz followed by the size control on Wf and $W\chi_{E'}$ we may bound this by

$$\begin{aligned} \sum_{T \in \mathbf{T}_n} |I_T| \left(\|Wf\|_{\text{size}(T)}^{1/2} \right) \left(\|W\chi_{E'}\|_{\text{size}(T)}^{1/2} \right) \\ \lesssim \sum_{T \in \mathbf{T}_n} |I_T| 2^n 2^{np/s} \lesssim 2^{-np} 2^n 2^{np/s}. \end{aligned}$$

We now turn to the contribution of the tiles in \mathbf{P}_{n-1} . We may iterate the above procedure, decomposing \mathbf{P}_{n-1} into \mathbf{T}_{n-1} and \mathbf{P}_{n-2} , and continue in this fashion until we are left with a collection of tiles $\mathbf{P}_{-\infty}$ with size zero, which we can discard. Summing up, we can thus control the left-hand side of (51) by $\lesssim \sum_{n \leq O(1)} 2^{-np} 2^n 2^{np/s}$. If one chooses s sufficiently close to 1, then this sum converges, and we are done.

A similar argument handles π_{hl} . The remaining paraproduct π_{hh} then follows from (45) and Hölder's inequality (which is still valid for $r < 1$). □

One can modify the above argument to obtain the corresponding estimate for the bilinear Hilbert transform (in the Walsh model, at least); see e.g. [39], [53], [46]. A difficulty in that case is that the tiles are no longer lacunary, and one cannot guarantee the spatial disjointness of the trees T in \mathbf{T}_n . However one can still make the trees essentially disjoint in phase space, but then one can only use L^2 estimates to control $\sum_{T \in \mathbf{T}_n} |I_T|$ instead of L^p estimates. Because of this, the above strategy only seems to work for the bilinear Hilbert transform when $r > 2/3$; it appears that one needs very different techniques to handle the remaining case $1/2 < r \leq 2/3$.

6. Calderón-Zygmund operators, and $T(b)$ theorems

We now consider the theory of Calderón-Zygmund operators, and specifically the aspects of the theory related to $T(1)$ and $T(b)$ theorems.

For simplicity we shall restrict ourselves to model operators of the form

$$Tf(x) := \int K(x, y)f(y) dy$$

where K is a locally integrable function on $(x, y) \in [0, 2^M] \times [0, 2^M]$ which obeys the kernel condition

$$(52) \quad |K(x, y)| \lesssim \frac{1}{|x - y|}$$

and the *perfect dyadic Calderón-Zygmund conditions*

$$(53) \quad |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| = 0$$

whenever $x, x' \in I$ and $y \in J$ for some disjoint dyadic intervals I and J . Equivalently, K is constant on all rectangles $\{I \times J : I, J \text{ are siblings}\}$. We also impose the technical truncation conditions¹⁵ that K vanishes on all diagonal squares $I \times I$ where $|I| = 2^{-M}$, and also vanishes on the squares $[0, 2^{M-1}] \times [2^{M-1}, 2^M]$ and $[2^{M-1}, 2^M] \times [0, 2^{M-1}]$.

We refer to operators T of the above form as *perfect dyadic Calderón-Zygmund operators*. These operators can be thought of as the dyadic analogue of truncated Calderón-Zygmund operators, where the cancellation conditions are perfect. (For ordinary Calderón-Zygmund operators one can bound the left-hand side of (53) by something like $O(|x - x'|/|x - y|^2)$.)

Let T be a perfect dyadic Calderón-Zygmund operator. From (53) we observe that T maps S to S , and furthermore if I is any dyadic interval and $f \in S_0(I)$, then Tf is supported in I . Similarly for T^* . We shall use this cancellation heavily in the sequel.

Examples of perfect dyadic Calderón-Zygmund operators include multiplier operators $W^{-1}a_P W$ and paraproducts $f \mapsto \pi(a, f)$. It turns out that these are essentially the only such operators. Indeed, we have the well-known splitting (see [7], [44]) of a perfect dyadic Calderón-Zygmund operator into three parts: the diagonal part, the $T(1)$ paraproduct, and the $T^*(1)$ paraproduct:

¹⁵The continuous analogue of these conditions would be that $K(x - y)$ vanishes when $|x - y| \leq 2^{-M}$ or $|x - y| \geq 2^M$. It is possible to formulate $T(b)$ theorems which do not require truncated operators, but this introduces some additional technicalities which are not relevant to the present discussion, and so we have chosen to ignore the issue for expository reasons.

Lemma 6.1. *If T is a perfect Calderón-Zygmund operator, then we have*

$$(54) \quad Tf \equiv W^{-1} \langle T\phi_P, \phi_P \rangle Wf + \pi_{hl}(T(1), f) + \pi_{hh}(T^*(1), f)$$

for all $f \in S$, where $f \equiv g$ denotes the statement¹⁶ that $Wf = Wg$.

Proof: We need to show that

$$(55) \quad \begin{aligned} \langle Tf, \phi_Q \rangle &= \langle T\phi_Q, \phi_Q \rangle Wf(Q) \\ &\quad + W(T(1))(Q)[f]_Q \\ &\quad + \sum_{P \in \mathbf{P}^+} W(T^*(1))(P)Wf(P)[\phi_Q]_P \end{aligned}$$

for all $f \in S$ and $Q \in \mathbf{P}^+$.

When $f = 1$ the identity is clear, so by subtracting off a multiple of 1 if necessary we may assume $f \in S_0$, thus $f = \sum_P Wf(P)\phi_P$. We can then decompose

$$\langle Tf, \phi_Q \rangle = \sum_{P \in \mathbf{P}^+} Wf(P) \langle T\phi_P, \phi_Q \rangle.$$

If I_P and I_Q are disjoint then $\langle T\phi_P, \phi_Q \rangle = 0$ since T is a perfect dyadic Calderón-Zygmund operator. Thus we may partition the sum into the portions $P = Q$, $Q <' P$, or $P <' Q$.

The diagonal term $P = Q$ is the first term in (55). Now consider the $Q <' P$ portion. We write this as

$$\left\langle \sum_{P \in \mathbf{P}^+: Q <' P} Wf(P)\phi_P, T^*\phi_Q \right\rangle.$$

The function $T^*\phi_Q$ is supported on I_Q , while

$$\sum_{P \in \mathbf{P}^+: Q <' P} Wf(P)\phi_P = f - \sum_{P \in \mathbf{P}^+: Q \not<' P} Wf(P)\phi_P$$

is constant on I_Q and has the same mean as f on I_Q , we thus have

$$\left\langle \sum_{P \in \mathbf{P}^+: Q <' P} Wf(P)\phi_P, T^*\phi_Q \right\rangle = \langle [f]_Q, T^*\phi_Q \rangle = \langle T(1), \phi_Q \rangle [f]_Q$$

which is the second term in (55).

¹⁶This is equivalent to $f - g$ being constant on $[0, 2^M]$.

Finally, we consider the $P <' Q$ term. Observe that the P summation in (55) vanishes unless $P <' Q$. It thus suffices to show that

$$Wf(P)\langle T\phi_P, \phi_Q \rangle = W(T^*(1))(P)Wf(P)[\phi_Q]_P.$$

But this follows since $T\phi_P$ is supported on I_P and ϕ_Q is constant on I_P . \square

Corollary 6.2 (Dyadic global $T(1)$ theorem). *Let T be a perfect dyadic Calderón-Zygmund operator such that*

$$\|T(1)\|_{\text{BMO}}, \|T^*(1)\|_{\text{BMO}} \lesssim 1$$

and we have the weak boundedness property

$$(56) \quad |\langle T\phi_P, \phi_P \rangle| \lesssim 1 \text{ for all } P \in \mathbf{P}^+.$$

Then T is bounded on L^2 .

Note that the converse of this theorem is easy: if T is bounded on L^2 , then we certainly have the weak boundedness property, and by (43)

$$\begin{aligned} \|T(1)\|_{\text{BMO}} &= \sup_I \sup_{a \in S_0(I): \|a\|_2=1} |I|^{-1/2} |\langle T(1), a \rangle| \\ &= \sup_I \sup_{a \in S_0(I): \|a\|_2=1} |I|^{-1/2} |\langle T(\chi_I), a \rangle| \lesssim 1 \end{aligned}$$

and similarly for $T^*(1)$. Indeed we observe that T and T^* must map L^∞ to BMO.

Proof: From the conditions on the kernel K and duality it suffices to show that $|\langle Tf, g \rangle| \lesssim \|f\|_2 \|g\|_2$ for all $f, g \in S$. By splitting $[0, 2^M]$ into two intervals it suffices to show this when $f \in S([0, 2^{M-1}])$ or $f \in S([2^{M-1}, 2^M])$.

Without loss of generality we may assume that $f \in S([0, 2^{M-1}])$. From (54), Corollary 5.2, and the hypotheses on T we see that this estimate holds for all $g \in S_0$. Also, from the truncation hypothesis the claim trivially holds for $g = \chi_{[2^{M-1}, 2^M]}$. Since S is spanned by S_0 and $\chi_{[2^{M-1}, 2^M]}$, the claim follows. \square

We can rephrase Corollary 6.2 as an equivalent “local” version:

Corollary 6.3 (Dyadic local $T(1)$ theorem). *Let T be a perfect dyadic Calderón-Zygmund operator such that*

$$(57) \quad \|T(\chi_{I_P})\|_{L^1(I_P)}, \|T^*(\chi_{I_P})\|_{L^1(I_P)} \lesssim |I_P| \text{ for all } P \in \mathbf{P}^+.$$

Then T is bounded on L^2 .

Proof: From (57) we see that $|\int_{I_P \times I_P} K(x, y) dx dy| \lesssim |I_P|$. From (52) we thus have that

$$\left| \int_{I_P \times I_P} \phi_P(x) K(x, y) \phi_P(y) dx dy \right| \lesssim 1,$$

or in other words that (56) holds. By Corollary 6.2 and symmetry it thus suffices to show that $T(1) \in \text{BMO}$, or equivalently (by Corollary 3.3 and duality, cf. (43)) that

$$|\langle T(1), h_I \rangle| \lesssim |I| \|h_I\|_\infty$$

whenever I is a dyadic interval and $h_I \in S_0(I)$. But since $T^*(h_I)$ is supported on I , we have $\langle T(1), h_I \rangle = \langle T\chi_I, h_I \rangle$, and the claim follows from (57). \square

Note that the converse of the above corollary is immediate from Hölder’s inequality. One can also deduce Corollary 6.2 from Corollary 6.3, but we leave this to the reader.

We shall consider generalizations of the global and local $T(1)$ theorem next, after some preliminaries on accretivity.

6.1. Accretivity and one-sided $T(b)$ theorems. Let $b \in S$ be a complex-valued function, and $\mathbf{P} \subseteq \mathbf{P}^+$ be a collection of tiles. We say that b is *pseudo-accretive on \mathbf{P}* if

$$(58) \quad |[b]_P| \gtrsim 1 \text{ for all } P \in \mathbf{P}.$$

If we in addition have the property

$$(59) \quad |[b]_{P_l}|, |[b]_{P_r}| \gtrsim 1$$

for the two children¹⁷ P_l, P_r of tiles $P \in \mathbf{P}$, we say that b is *strongly pseudo-accretive on \mathbf{P}* . Note that we are *not* assuming any L^∞ control on b in the above definition.

If b is pseudo-accretive on the entire tile set \mathbf{P}^+ , we simply say that b is *pseudo-accretive*. Examples of pseudo-accretive functions include the *accretive functions*, for which $\text{Re } b(x) \gtrsim 1$ for all $x \in [0, 2^M]$.

The $T(1)$ theorem can now be generalized to “one-sided $T(b)$ theorems” in which we control $T(b)$ and $T^*(1)$. We give the dyadic version of an argument of Semmes [50] (who considered the case $T(b) \in \text{BMO}$, $T^*(1) = 0$):

¹⁷For this to be well defined, \mathbf{P} cannot contain any tiles with minimal length $|I_P| = 2^{-M}$.

Theorem 6.4 (One-sided global $T(b)$ theorem). *Let b be a pseudo-accretive function on \mathbf{P}^+ with $\|b\|_{\text{BMO}} \lesssim 1$. Let T be a perfect dyadic Calderón-Zygmund operator obeying the weak boundedness property (56) such that*

$$\|T(b)\|_{\text{BMO}}, \|T^*(1)\|_{\text{BMO}} \lesssim 1.$$

Then T is bounded on L^2 .

Proof: From the dyadic $T(1)$ theorem (Corollary 6.2) it suffices to show that $\|T(1)\|_{\text{BMO}} \lesssim 1$, or in other words that $|W(T(1))|^2$ has bounded maximal size (i.e. is a Carleson measure).

From (54) and (46)

$$(60) \quad T(b) \equiv W^{-1} \langle T\phi_P, \phi_P \rangle Wb + W^{-1}[b]_P WT(1) + \pi_{hh}(T^*(1), b).$$

From (58) we may thus solve for $T(1)$:

$$(61) \quad T(1) \equiv W^{-1}[b]_P^{-1} W[T(b) - W^{-1} \langle T\phi_P, \phi_P \rangle Wb - \pi_{hh}(T^*(1), b)]$$

and the claim follows from (58), (56), the hypotheses $\|T(b)\|_{\text{BMO}}, \|b\|_{\text{BMO}} \lesssim 1$, and Lemma 5.2. \square

Note that one only needs b in BMO in the above argument instead of the more usual L^∞ .

One drawback to the above theorem is that it requires the function b to be pseudo-accretive. Fortunately, it is not too difficult to construct pseudo-accretive functions. The following basic lemma says that if a function has large mean, then it is pseudo-accretive on a non-trivial set of tiles. (Equivalently, if a function has small mean on too many small tiles, then it must have small mean globally.)

Lemma 6.5. *Let $T_0 \subseteq \mathbf{P}^+$ be a convex tree, and let b be a function such that*

$$(62) \quad \|\Pi_{T_0} b\|_2 \leq C_0 |T_0|^{1/2}$$

and $\|[b]_{I_{T_0}}\| \geq \delta$ for some $C_0, \delta > 0$. Then there exists $0 < \varepsilon \ll 1$ depending only on C_0 and δ and a family \mathbf{T} of disjoint convex sub-trees of T_0 whose tops form a $(1 - \varepsilon)$ -packing of T_0 , and such that $\|[b]_P\| > \varepsilon$ for all $P \in T_0 \setminus \bigcup_{T \in \mathbf{T}} T$. Furthermore we have $\|[b]_{P_T}\| \leq \varepsilon$ for all $T \in \mathbf{T}$.

Proof: Let \mathbf{P} denote those tiles in T_0 for which $\|[b]_P\| \leq \varepsilon$, and which are maximal with respect to the ordering $<'$. Clearly the tiles in \mathbf{P} have disjoint spatial intervals and obey

$$\left| \int_{I_P} b \right| < \varepsilon |I_P|.$$

To prove the lemma it will suffice to show the $(1 - \varepsilon)$ -packing property. Suppose for contradiction that

$$(63) \quad \left| \bigcup_{P \in \mathbf{P}} I_P \right| > (1 - \varepsilon) |I_{T_0}|.$$

Using the identity $\int_{I_P} b = \int_{I_P} ([b]_{I_{T_0}} + \Pi_{T_0} b)$ and summing in P we obtain

$$\left| \int_{\bigcup_{P \in \mathbf{P}} I_P} [b]_{I_{T_0}} + \Pi_{T_0} b \right| \leq \varepsilon \left| \bigcup_{P \in \mathbf{P}} I_P \right| \leq \varepsilon |I_{T_0}|;$$

if ε is sufficiently small with respect to δ , we thus see that

$$\left| \int_{\bigcup_{P \in \mathbf{P}} I_P} \Pi_{T_0} b \right| \geq \frac{\delta}{2} |I_{T_0}|.$$

Since $\Pi_{T_0} b$ has mean zero on I_{T_0} , we thus have

$$\left| \int_{I_{T_0} \setminus \bigcup_{P \in \mathbf{P}} I_P} \Pi_{T_0} b \right| \geq \frac{\delta}{2} |I_{T_0}|.$$

But this will contradict (63) and (62) by Cauchy-Schwarz, if ε is sufficiently small. \square

This lemma combines nicely with Lemma 3.1. Together, these lemmata heuristically allow us to treat “large mean” as being equivalent to “pseudo-accretive”, at least for the purposes of placing something in BMO. As an application we now use this lemma to give a localized version of Theorem 6.4. Similar results¹⁸ have been used to solve the Kato problem in higher dimensions (see e.g. [2]):

Theorem 6.6 (Local one-sided $\mathbb{T}(b)$ theorem). *Let \mathbb{T} be a perfect Calderón-Zygmund operator obeying $\|\mathbb{T}^*(1)\|_{\text{BMO}} \lesssim 1$ and (56). Suppose also that for every $P \in \mathbf{P}^+$ there exists a function $b_P \in S(I_P)$ with the normalization condition*

$$[b_P]_P = 1$$

and the L^2 bounds

$$\int_{I_P} |b_P|^2 + |\mathbb{T}b_P|^2 \lesssim |I_P|$$

for all $P \in \mathbf{P}^+$. Then \mathbb{T} is bounded on L^2 .

¹⁸More precisely, the solution to the Kato problem requires a matrix-valued analogue of this theorem in which the b_P are matrix valued, and \mathbb{T} maps matrices to vectors. The argument then requires an additional subtlety, namely a preliminary partition of the tile set \mathbf{P}^+ into $O(1)$ pieces, where on each piece the vector-valued coefficients $W(\mathbb{T}(\mathbf{1}))(P)$ lie in a narrow conical region. See [2].

The weak boundedness condition (56) can actually be removed; see the remarks after Theorem 6.8. Informally, this theorem asserts that to prove the L^2 boundedness of an operator T it actually suffices to establish boundedness for a single function b_P for each interval I_P , provided that b_P is not degenerate (in the sense that its mean is large) and provided that $T^*(1)$ is under control. In the next section we shall remove the condition on $T^*(1)$, obtaining a “two-sided” version of this theorem.

Proof: Again it suffices to show that $T(1)$ is in BMO. By Lemma 3.1 it suffices to show that for every complete tree T we have

$$(64) \quad \sum_{P \in T \setminus \bigcup_{T' \in \mathbf{T}} T'} |W(T(1))(P)|^2 \lesssim |I_T|$$

for some collection \mathbf{T} of disjoint convex trees in T whose tops form a $(1 - \varepsilon)$ -packing of T for some $\varepsilon > 0$.

Fix T . By Lemma 6.5 we can indeed find such a collection \mathbf{T} with the additional property that b is pseudo-accretive on $T \setminus \bigcup_{T' \in \mathbf{T}} T'$. The claim (64) then follows from the argument used to prove Corollary 6.4. \square

The above arguments do not extend well to two-sided situations in which one controls $T(b_1)$ and $T^*(b_2)$ (unless one of b_1, b_2 is close to a constant, e.g. in BMO norm). In order to handle the general case we need adapted Haar bases, to which we now turn.

6.2. Adapted Haar bases, and two-sided $T(b)$ theorems. Let \mathbf{P} be a collection of tiles, and let b be a function which is strongly pseudo-accretive on \mathbf{P} . For each $P \in \mathbf{P}$, we define the adapted Haar wavelet ϕ_P^b (introduced in [20]; see also [4]) by

$$(65) \quad \phi_P^b := |I_P|^{-1/2} \frac{[b]_{P_r}}{[b]_P} \chi_{I_{P_r}} - |I_P|^{-1/2} \frac{[b]_{P_l}}{[b]_P} \chi_{I_{P_l}}.$$

Observe that this collapses to ϕ_P if b is constant on I_P . For non-constant b , ϕ_P^b is no longer mean zero, but one can easily verify that ϕ_P^b still obeys the weighted mean zero condition

$$(66) \quad \int b \phi_P^b = 0.$$

As a consequence we have the orthogonality property

$$(67) \quad \int \phi_P^b b \phi_Q^b = 0 \text{ for all distinct } P, Q \in \mathbf{P}.$$

From (65) we see that

$$(68) \quad \int \phi_P^b b \phi_P^b = \frac{[b]_{P_r} [b]_{P_l}}{[b]_P} = \frac{2}{[b]_{P_l}^{-1} + [b]_{P_r}^{-1}}.$$

In particular, from the strong pseudo-accretivity condition (59) we have the bound

$$(69) \quad \left| \int \phi_P^b b \phi_P^b \right| \gtrsim 1.$$

It is interesting that this bound uses only the strong pseudo-accretivity of b , and in particular does not require L^∞ control on b .

Define the dual adapted Haar wavelet ψ_P^b by

$$\psi_P^b := \frac{\phi_P^b b}{\int \phi_P^b b \phi_P^b}.$$

By (67), (69) we thus have that $\langle \psi_P^b, \phi_Q^b \rangle = \delta_{PQ}$ where δ is the Kronecker delta. In particular we have the representation formula

$$(70) \quad f = \sum_{P \in \mathbf{P}} W_b f(P) \psi_P^b$$

whenever g is in the span of $\{\psi_P^b : P \in \mathbf{P}\}$, where the adapted wavelet coefficients $W_b f(P)$ are defined by

$$W_b f := \langle f, \phi_P^b \rangle.$$

We have the following basic orthogonality property:

Lemma 6.7. *Let T be a convex tree, and let b be a function which is pseudo-accretive on T and obeys the mean bound*

$$(71) \quad \| |b|^2 \|_{\text{mean}^*(T)} \lesssim 1.$$

Then for any function¹⁹ $f \in S$ we have

$$(72) \quad \left(\sum_{P \in T} |W_b f(P)|^2 \right)^{1/2} \lesssim \|f\|_2.$$

In fact, the more general estimate

$$(73) \quad \left(\sum_{P \in T} |W_b(b'f)(P)|^2 \right)^{1/2} \lesssim \|f\|_2 \| |b'|^2 \|_{\text{mean}^*(T)}^{1/2}$$

holds for any $f, b' \in S$.

¹⁹It can easily be seen, by aid of (70), that the estimate (72) can be reversed for all f in the span of the ϕ_P^b , but we will not use this.

Proof: From (71) and (6) we observe that

$$(74) \quad \| |Wb|^2 \|_{\text{size}^*(T)} \lesssim 1.$$

We first prove (72). From (65) and the identity $|I_P|^{-1/2}Wb(P) = [b]_{P_l} - [b]_P = [b]_P - [b]_{P_r}$, we obtain the identity

$$W^b f(P) = Wf(P) - \frac{Wb(P)}{[b]_P} [f]_P.$$

If we replace W_b by W_b then (72) follows from Bessel’s inequality and the orthonormality of the Haar wavelets ϕ_P . By the previous identity and the triangle inequality it thus suffices to show

$$\left(\sum_{P \in T} \left| \frac{Wb(P)}{[b]_P} [f]_P \right|^2 \right)^{1/2} \lesssim \|f\|_2.$$

We may discard $[b]_P$ by pseudo-accretivity (58). The claim then follows from Carleson embedding (Lemma 5.1) and (74).

Now we prove (73). Let \mathbf{Q} denote the collection of tiles in \mathbf{P}^+ which are children of tiles in T , but are not in T itself. In order to ensure that the intervals $\{I_Q : Q \in \mathbf{Q}\}$ partition I_T we will allow the tiles Q to have spatial intervals $|I_Q| = 2^{-M-1}$; the partition property then follows from the convexity of T .

The function $b'f - \sum_{Q \in \mathbf{Q}} [b'f]_Q \chi_{I_Q}$ has mean zero on every interval I_Q , and is thus orthogonal to ϕ_P^b for every $P \in T$. We may thus freely replace $b'f$ by the averaged function $\sum_{Q \in \mathbf{Q}} [b'f]_Q \chi_{I_Q}$ in (73). By (72) it thus suffices to show that

$$\left\| \sum_{Q \in \mathbf{Q}} [b'f]_Q \chi_{I_Q} \right\|_2^2 \lesssim \|f\|_2^2 \| |b'|^2 \|_{\text{mean}^*(T)}.$$

But from Cauchy-Schwarz we have

$$|[b'f]_Q|^2 |I_Q| \lesssim \|f\|_{L^2(I_Q)}^2 \| |b'|^2 \|_{\text{mean}(2Q)} \leq \|f\|_{L^2(I_Q)}^2 \| |b'|^2 \|_{\text{mean}^*(T)},$$

and the claim follows by summing in Q . □

We can now give our main result, namely a dyadic local $T(b)$ theorem.

Theorem 6.8 (Dyadic local $T(b)$ theorem). *Let T be a perfect Calderón-Zygmund operator, and suppose that for each $P \in \mathbf{P}^+$ we can find functions b_P^1, b_P^2 in $S(I_P)$ obeying the normalization*

$$(75) \quad [b_P^1]_P = [b_P^2]_P = 1$$

and the bounds

$$(76) \quad \int_{I_P} |b_P^1|^2 + |Tb_P^1|^2 + |b_P^2|^2 + |T^*b_P^2|^2 \lesssim |I_P|.$$

Then T is bounded on L^2 .

This theorem is a stronger version of the local $T(b)$ theorem in [11] (but for the dyadic setting with perfect cancellation), which required L^∞ control in (76) instead of L^2 control. (This was generalized to BMO control and to non-doubling situations in [47].) Also it required the global $T(b)$ theorem of David, Journé and Semmes [23] (which we instead deduce as a corollary of Theorem 6.8). We make some further remarks after the proof of the theorem.

Proof: This proof is somewhat lengthy and so we split the argument into several stages.

Step 0. Preliminary estimates: We begin with a basic lemma which already shows the importance of the normalization (75).

Lemma 6.9 (b_P^1 spans $S(I_P)/S_0(I_P)$). *For any tile $P \in \mathbf{P}^+$ and any $f \in S(I_P)$, we have*

$$\|f\|_{L^2(I_P)} \lesssim \|f - [f]_P\|_{L^2(I_P)} + |I_P|^{-1/2} |\langle f, b_P^1 \rangle|.$$

Similarly for b_P^2 .

Proof: Let h be an arbitrary element of $S(I_P)$ with $\|h\|_2 = 1$. Then

$$\langle f, h \rangle = \langle f, h - [h]_P b_P^1 \rangle + [h]_P \langle f, b_P^1 \rangle = \langle f - [f]_P, h - [h]_P b_P^1 \rangle + [h]_P \langle f, b_P^1 \rangle.$$

By Cauchy-Schwarz and (76) we thus have

$$|\langle f, h \rangle| \lesssim \|f - [f]_P\|_{L^2(I_P)} + |I_P|^{-1/2} |\langle f, b_P^1 \rangle|.$$

Taking suprema over all h , the claim follows. □

A useful application of the above lemma is the following convenient truncation property of the b_P^1 and b_P^2 (already observed in [11]):

Corollary 6.10. *Let P, Q be lacunary tiles with $Q \leq' P$. If we have the estimate*

$$(77) \quad \int_{I_Q} |\mathbb{T}b_P^1|^2 + |b_P^1|^2 \lesssim K|I_Q|$$

for some $K \gtrsim 1$, then we have

$$\int_{2I_Q} |\mathbb{T}(b_P^1 \chi_{I_Q})|^2 \lesssim K|I_Q|.$$

Similarly for b_P^2 (but with \mathbb{T} replaced by \mathbb{T}^*).

Proof: By (52) and (77) the portion of the integral on $2I_Q \setminus I_Q$ is acceptable, so it suffices to bound the integral on I_Q . From Cauchy-Schwarz, (76) and (77) we have

$$|\langle \mathbb{T}(b_P^1 \chi_{I_Q}), b_Q^2 \rangle| = |\langle b_P^1 \chi_{I_Q}, \mathbb{T}^* b_Q^2 \rangle| \leq \|b_P^1\|_{L^2(I_Q)} \|\mathbb{T}^* b_Q^2\|_{L^2(I_Q)} \lesssim K^{1/2} |I_Q|.$$

By Lemma 6.9 it thus suffices to show that

$$\|\mathbb{T}(b_P^1 \chi_{I_Q}) - [\mathbb{T}(b_P^1 \chi_{I_Q})]_Q\|_{L^2(I_Q)} \lesssim K^{1/2} |I_Q|^{1/2}.$$

Now observe that for every $h \in S_0(I_Q)$ we have

$$\langle \mathbb{T}(b_P^1 \chi_{I_Q}), h \rangle = \langle b_P^1 \chi_{I_Q}, \mathbb{T}^* h \rangle = \langle b_P^1, \mathbb{T}^* h \rangle = \langle \mathbb{T}b_P^1, h \rangle.$$

By duality this implies that

$$\mathbb{T}(b_P^1 \chi_{I_Q}) - [\mathbb{T}(b_P^1 \chi_{I_Q})]_Q = \mathbb{T}(b_P^1) - [\mathbb{T}(b_P^1)]_Q$$

on I_Q . The claim then follows from (77). □

This corollary will be useful in estimating the operator \mathbb{T} when acting on objects such as $\psi_Q^{b_P^1}$ which can be expressed as linear combinations of truncated versions of b_P^1 . Similarly when estimating \mathbb{T}^* on objects such as $\psi_Q^{b_P^2}$.

Step 1. Overview of main argument: We now begin the main argument. Let A be the best constant such that

$$\|\mathbb{T}^* \chi_{I_P}\|_{L^1(I_P)} \leq A|I_P|$$

for all tiles $P \in \mathbf{P}^+$. We claim that $A = O(1)$; from this and the corresponding claim for $\mathbb{T} \chi_{I_P}$ (which is of course symmetric) the theorem will follow from the local $\mathbb{T}(1)$ theorem (Corollary 6.3).

In fact we will show

$$(78) \quad \left| \int_{I_P} \mathbb{T}f \right| \leq ((1 - \varepsilon)A + O(1))|I_P| \|f\|_\infty$$

for all tiles $P \in \mathbf{P}^+$ and $f \in S(I_P)$, and some $0 < \varepsilon \ll 1$ depending only on the implicit constant in (76). By duality this implies that $A \leq (1 - \varepsilon)A + O(1)$, which will prove the desired bound on A .

Fix P, f . We shall prove the estimate (78) in three steps. Firstly (in Step 2), we decompose f and reduce matters to proving a Carleson measure type estimate on the wavelet coefficients $|\langle \mathbf{T}^* \chi_{I_P}, \psi_Q^{b_P^1} \rangle|^2$; this argument shall use stopping-time arguments (which we encapsulate as Lemma 6.11) based on b_P^1 but not on b_P^2 . Then (in Step 3), we decompose χ_{I_P} and use stopping time arguments (again using Lemma 6.11) based on b_P^2 but not on b_P^1 . It will be important not to try to handle b_P^1 and b_P^2 at the same time as we will lose the crucial $(1 - \varepsilon)$ packing property of the trees left out by the stopping time algorithm if we do so.

The purpose of these stopping arguments is to impose some pseudo-accretivity and other regularity properties on the b_P^1 and b_P^2 . Once we have enough regularity properties, we can then (in Step 4) do an elementary computation to estimate the wavelet coefficients $|\langle \mathbf{T}^* \chi_{I_P}, \psi_Q^{b_P^1} \rangle|$ pointwise by the quantities which we know to be controlled by hypothesis (see (76) below).

Step 2. Pruning the bad tiles of b_P^1 : We now begin the first of the three steps outlined above. We would like to break up f into linear combinations of the wavelets $\psi_Q^{b_P^1}$, but we cannot do this for all Q because we do not control the strong pseudo-accretivity of b_P^1 . However, by using Lemma 6.5 and some other selection algorithms we can find a large subtree of $\text{Tree}(P)$ for which we can decompose f as desired, modulo acceptable errors:

Lemma 6.11. *Let $P \in \mathbf{P}^+$ be a tile. Then we can partition*

$$\text{Tree}(P) = T_1 \cup \mathbf{P}_{\text{buffer}} \cup \bigcup_{T' \in \mathbf{T}} T'$$

where

- \mathbf{T} is a collection of disjoint complete trees in $\text{Tree}(P)$ whose tops form a $(1 - \varepsilon)$ -packing of $\text{Tree}(P)$ for some $0 < \varepsilon \ll 1$ (depending only on the implicit constant in (76));
- T_1 is a tree with top P such that $b_{T_1}^1$ is strongly pseudo-accretive on T_1 (with constants perhaps depending on ε);
- $\mathbf{P}_{\text{buffer}}$ is a 2-packing of $\text{Tree}(P)$, $T_1 \cup \mathbf{P}_{\text{buffer}}$ is convex, $b_{T_1}^1$ is pseudo-accretive on $T_1 \cup \mathbf{P}_{\text{buffer}}$ and we have the mean bounds

$$(79) \quad \left\| |b_{T_1}^1|^2 + |\mathbf{T}b_{T_1}^1|^2 \right\|_{\text{mean}^*(T_1 \cup \mathbf{P}_{\text{buffer}})} \lesssim 1$$

(with the implicit constant depending on ε).

- We have the decomposition

$$(80) \quad f = [f]_P b_P^1 + \sum_{Q \in T_1} W_{b_P^1} f(Q) \psi_Q^{b_P^1} + \sum_{T' \in \mathbf{T}} (f \chi_{I_{T'}} - [f]_{P_{T'}} b_{P_{T'}}^1) + \sum_{Q \in \mathbf{P}_{\text{buffer}}} \varphi_Q$$

whenever $f \in S(I_P)$, where the “buffer functions” φ_Q are supported on I_Q , have mean zero, and take the form

$$\varphi_Q = a_Q b_P^1 \chi_{I_{Q_l}} + a'_Q b_P^1 \chi_{I_{Q_r}} + a''_Q b_{Q_l}^1 + a'''_Q b_{Q_r}^1$$

where the coefficients a_Q, a'_Q, a''_Q, a'''_Q depend on f and the b_P^1 and (when $|I_Q| \neq 2^{-M}$) obey the bounds

$$(81) \quad |a_Q| + |a'_Q| + |a''_Q| + |a'''_Q| \lesssim \|f\|_\infty.$$

A similar statement holds with b_P^1 and $T b_P^1$ replaced by b_P^2 and $T^* b_P^2$ (but the sets $T_1, \mathbf{P}_{\text{buffer}}$ and \mathbf{T} are different then).

The tree T_1 represents the “good” portion of the tree $\text{Tree}(P)$, in which b_P^1 is neither too large nor too small (so in particular the $W^{b_P^1}$ wavelet system is well-behaved on T_1). The buffer tiles P_{buffer} are those tiles immediately above T_1 (and are thus slightly less “good”), while the remaining trees \mathbf{T} have no good properties at all, except that they only occupy at most $(1 - \varepsilon)$ of the tree $\text{Tree}(P)$. This decomposition shares many features in common with Lemma 3.7 (for instance, the trees \mathbf{T} are formed from those intervals where b is too “heavy” or too “light”).

In terms of the phase plane (but adapted to the $W_{b_P^1}$ wavelet system instead of the Haar wavelet system), one can interpret the right-hand side of (80) as follows. The first term corresponds to the region of phase space below the tree T_1 . The second term corresponds to T_1 itself. The third term corresponds to the region above $T_1 \cup \mathbf{P}_{\text{buffer}}$, while the last term is an error term corresponding to the region $\mathbf{P}_{\text{buffer}}$. In the model case $b_P^1 = \chi_{I_P}$, (80) simplifies to

$$f = [f]_P \chi_{I_P} + \Pi_{T_1} f + \sum_{T' \in \mathbf{T}} \Pi_{T'} f + \sum_{Q \in \mathbf{P}_{\text{buffer}}} W f(Q) \phi_Q.$$

Proof: We begin by applying Lemma 6.5 to $\text{Tree}(P)$ to find a preliminary collection \mathbf{T}_0 of disjoint convex trees in $\text{Tree}(P)$ such that the tops of \mathbf{T}_0 are a $(1 - 2\varepsilon)$ -packing, and such that b_P^1 is pseudo-accretive (with constants depending on ε) on the tree

$$T_2 := \text{Tree}(P) \setminus \bigcup_{T' \in \mathbf{T}_0} T'.$$

However we do not yet have (79). To obtain these bounds we let \mathbf{Q} denote the set of all tiles $Q \in T_2$ for which

$$\| |b_P^1|^2 + |Tb_P^1|^2 \|_{\text{mean}(Q)} \geq C/\varepsilon$$

and which are maximal with respect to \leq' . If the constant C is chosen large enough, then \mathbf{Q} is a ε -packing of $\text{Tree}(P)$. Thus if we define

$$\mathbf{T} := \mathbf{T}_0 \cup \bigcup_{Q \in \mathbf{Q}} (\text{Tree}(Q) \cap T_2)$$

then we see that (79) holds on the tree

$$T_3 := \text{Tree}(P) \setminus \bigcup_{T' \in \mathbf{T}} T',$$

while the tops of \mathbf{T} are still a $(1 - \varepsilon)$ -packing of $\text{Tree}(P)$.

We now perform one minor modification to \mathbf{T} to make \mathbf{T} sibling-free. If \mathbf{T} contains two trees whose tops $P_{T'}$, $P_{T''}$ are siblings, we can concatenate these trees and add a new tile $2P_{T'} = 2P_{T''}$ to join these trees to a larger tree without affecting the $(1 - \varepsilon)$ -packing nature of the tree tops. Repeating this process as often as necessary (it must terminate since $\text{Tree}(P)$ only has a finite number of tiles) we can make \mathbf{T} sibling-free.

For similar reasons we may assume that the trees²⁰ in \mathbf{T} are complete, since we can always replace an incomplete tree by the completion of that tree, absorbing any sub-trees that were also in \mathbf{T} if necessary.

We now define²¹ $\mathbf{P}_{\text{buffer}}$ to be the set of tiles Q in T_3 such that one or both²² of the children Q_l , Q_r of Q are not in T_3 . Since T_3 is a convex tree, the children of Q who are not in T_3 must have disjoint spatial supports as Q varies in $\mathbf{P}_{\text{buffer}}$. This implies that $\mathbf{P}_{\text{buffer}}$ is a 2-packing. We now set $T_1 := T_3 \setminus \mathbf{P}_{\text{buffer}}$. Note that all children of tiles in T_1 lie in $T_1 \cup \mathbf{P}_{\text{buffer}}$ so that b_P^1 is strongly pseudo-accretive on T_1 , but is merely pseudo-accretive on $T_1 \cup \mathbf{P}_{\text{buffer}}$.

²⁰Alternatively, we could avoid these modifications by combining the stopping time argument here with the one in Lemma 6.5.

²¹The algorithm here is extremely similar to the one used to prove Theorem 3.5. Indeed, one can even re-use Figure 3. The trees \mathbf{T}_0 are the “light” trees where b_P^1 has too small a mean; the tiles \mathbf{Q} correspond to the circled “heavy” tiles, where b_P^1 or Tb_P^1 has too large an L^2 norm. The tiles $\mathbf{P}_{\text{buffer}}$ are thus the buffer tiles, which are the ones just below the heavy or light tiles, as well as the tiles at the very finest scale.

²²Of course, because we made \mathbf{T} sibling-free, the only way both the children of Q fail to be in T_3 is if Q is at the finest scale, i.e. if $|I_Q| = 2^{-M}$.

The only property left to verify is the decomposition (80). If f is a constant multiple of b_P^1 then only the first term is non-zero (thanks to (66)) and the claim is easily verified. By subtracting off a constant multiple we may thus assume that f has mean zero on I_P .

It will suffice to prove the identity assuming that

$$[b_P^1]_Q, [b_P^1]_{Q_l}, [b_P^1]_{Q_r} \neq 0 \text{ for all } Q \in \text{Tree}(P),$$

since the general case then follows by an obvious limiting argument (the bounds (81) will not depend quantitatively on the above condition). In this case (70) applies²³. Comparing this with (80) and using the mean zero condition, we reduce to showing that

$$\begin{aligned} \sum_{Q \in \mathbf{P}_{\text{buffer}}} W_{b_P^1} f(Q) \psi_Q^{b_P^1} + \sum_{T' \in \mathbf{T}} \sum_{Q' \in T'} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1} \\ = \sum_{T' \in \mathbf{T}} (f \chi_{I_{T'}} - [f]_{P_{T'}} b_{P_{T'}}^1) + \sum_{Q \in \mathbf{P}_{\text{buffer}}} \varphi_Q \end{aligned}$$

for suitable φ_Q .

Let $Q \in \mathbf{P}_{\text{buffer}}$. First suppose that neither child of Q is a top of a tree in \mathbf{T} (since $\text{Tree}(P)$ is complete, this can only happen when $|I_Q| = 2^{-M}$). In this case we simply set $\varphi_Q := W_{b_P^1} f(Q) \psi_Q^{b_P^1}$.

Now suppose that one child of Q is a top of a tree T' in \mathbf{T} ; without loss of generality we assume Q_l is such a top. Since \mathbf{T} is sibling-free, Q_r is not a top and must therefore lie in $T_1 \cup \mathbf{P}_{\text{buffer}}$. In particular we have the lower bounds

$$(82) \quad |[b_P^1]_{Q_r}|, |[b_P^1]_Q| \gtrsim 1.$$

We do not have good lower bounds on $|[b_P^1]_{Q_l}|$, but fortunately we can take advantage of some “wobble room” in the buffer, and avoid using this in our computations by exploiting the identity

$$\begin{aligned} W_{b_P^1} f(Q) \psi_Q^{b_P^1} + \sum_{Q' \in T'} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1} \\ = \sum_{Q' \in \text{Tree}(Q)} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1} - \sum_{Q' \in \text{Tree}(Q_r)} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1}. \end{aligned}$$

²³One can easily verify (either by a dimension counting argument, or by inductively working from the finest scale upwards) that the wavelets $\psi_Q^{b_P^1}$ for $Q \in \text{Tree}(P)$ span $S_0(P)$.

The function $\sum_{Q' \in \text{Tree}(Q)} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1}$ clearly is supported on I_Q and has mean zero, while

$$f - \sum_{Q' \in \text{Tree}(Q)} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1} = \sum_{Q' \in \text{Tree}(P) \setminus \text{Tree}(Q)} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1}$$

is a constant multiple of b_P^1 on $I_{T'}$. Thus we have

$$\sum_{Q' \in \text{Tree}(Q)} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1} = f \chi_{I_Q} - \frac{[f]_Q}{[b_P^1]_Q} b_P^1 \chi_{I_Q}.$$

Similarly we have

$$\sum_{Q' \in \text{Tree}(Q_r)} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1} = f \chi_{I_{Q_r}} - \frac{[f]_{Q_r}}{[b_P^1]_{Q_r}} b_P^1 \chi_{I_{Q_r}}.$$

Subtracting the two we thus see that

$$W_{b_P^1} f(Q) \psi_Q^{b_P^1} + \sum_{Q' \in T'} W_{b_P^1} f(Q') \psi_{Q'}^{b_P^1} = f \chi_{I_{T'}} - \frac{[f]_Q}{[b_P^1]_Q} b_P^1 \chi_{I_Q} + \frac{[f]_{Q_r}}{[b_P^1]_{Q_r}} b_P^1 \chi_{I_{Q_r}}.$$

If we thus define

$$\varphi_Q := [f]_{P_{T'}} b_{P_{T'}}^1 - \frac{[f]_Q}{[b_P^1]_Q} b_P^1 \chi_{I_Q} + \frac{[f]_{Q_r}}{[b_P^1]_{Q_r}} b_P^1 \chi_{I_{Q_r}}$$

we see that (81) follows; the mean zero condition can be seen by (75) and inspection. This completes the proof of Lemma 6.11. \square

Roughly speaking, the above lemma states that we can find a large tree T_1 on which b_P^1 is pseudo-accretive, on which b_P^1 and $\mathbb{T}b_P^1$ are effectively bounded, and for which we have a representation of the form (70). We now run an argument in the spirit of Lemma 3.1 to localize matters exclusively to this tree T_1 .

We apply the above lemma first with the b_P^1 . We decompose f using (80), thus estimating the left-hand side of (78) by the sum of the term below T_1

$$(83) \quad \left| \int_{I_P} [f]_P \mathbb{T}b_P^1 \right|,$$

the terms coming from T_1

$$(84) \quad \left| \sum_{Q \in T_1} W_{b_P^1} f(Q) \int_{I_P} \mathbb{T}\psi_Q^{b_P^1} \right|,$$

the terms above $T_1 \cup \mathbf{P}_{\text{buffer}}$

$$(85) \quad \sum_{T' \in \mathbf{T}} \left| \int_{I_P} \mathbf{T}(f\chi_{I_{T'}} - [f]_{P_{T'}} b_{P_{T'}}^1) \right|,$$

and the term from $\mathbf{P}_{\text{buffer}}$

$$(86) \quad \sum_{Q \in \mathbf{P}_{\text{buffer}}} \left| \int_{I_P} \mathbf{T}\varphi_Q \right|.$$

The contribution of (83) is $O(|I_P| \|f\|_\infty)$ by (76) and Hölder. For the contribution of (84), we observe from (72) that

$$\left(\sum_{Q \in T_1} |W_{b_P^1} f(Q)|^2 \right)^{1/2} \lesssim \|f\|_2 \leq \|f\|_\infty |I_P|^{1/2}.$$

By Cauchy-Schwarz it will thus suffice to show the bound

$$\sum_{Q \in T_1} \left| \int_{I_P} \mathbf{T}\psi_Q^{b_P^1} \right|^2 \lesssim |I_P|$$

or equivalently that

$$(87) \quad \|\langle \mathbf{T}^* \chi_{I_P}, \psi_Q^{b_P^1} \rangle\|_{\text{size}(T_1)}^2 \lesssim 1.$$

One can think of (87) as a localized, b_P^1 -adapted version of the statement $\mathbf{T}^*(1) \in \text{BMO}$.

We will defer the proof of (87) to Step 3 of the argument. Assuming the bound (87) for now, we move on to (85). Observe that the expression inside the $\mathbf{T}()$ in (85) is supported on $I_{T'}$ and has mean zero, so we may reduce the integral from I_P to $I_{T'}$. From the definition of A we have

$$\left| \int_{I_{T'}} \mathbf{T}(f\chi_{I_{T'}}) \right| \leq A|I_{T'}| \|f\|_\infty$$

while from (76) and Cauchy-Schwarz we have

$$\left| \int_{I_{T'}} \mathbf{T}([f]_{P_{T'}} b_{P_{T'}}^1) \right| \lesssim |I_{T'}| \|f\|_\infty.$$

Adding all this up and using the fact that the tops of \mathbf{T} are a $(1 - \varepsilon)$ packing we can bound (85) by $((1 - \varepsilon)A + O(1))|I_P|$ as desired.

Finally we consider (86). Since $\varphi_Q \in S_0(I_Q)$ we can estimate this term by

$$\sum_{Q \in \mathbf{P}_{\text{buffer}}} \left| \int_{I_Q} \mathbf{T}\varphi_Q \right|.$$

If $|I_Q| = 2^{-M}$ then this vanishes because we truncated the operator \mathbf{T} , so we assume that $|I_Q| > 2^{-M}$. Using Cauchy-Schwarz, (76), (81), and Corollary 6.10 we can bound this by

$$O\left(\sum_{Q \in \mathbf{P}_{\text{buffer}}} |I_Q| \|f\|_\infty\right)$$

which is acceptable since $\mathbf{P}_{\text{buffer}}$ is a 2-packing.

Step 3. Pruning the bad tiles of b_P^2 : In Step 2 we reduced the proof of (78) (and thus of Theorem 6.8 to that of proving the Carleson measure estimate (87)). Along the way we managed to prune all the tiles for which b_P^1 was “bad” (in that the mean of b_P^1 was too small, or the L^2 norm of b_P^1 or $\mathbf{T}b_P^1$ was too large). However, b_P^2 is still not under control. Thus the next step shall be to prune b_P^2 .

We define

$$B := \|\langle \mathbf{T}^* \chi_{I_P}, \psi_Q^{b_P^1} \rangle\|^2_{\text{size}^*(T_1)};$$

it thus suffices to prove that $B = O(1)$. In fact will suffice to show that

$$\sum_{Q \in T_1 \cap \text{Tree}(P')} |\langle \mathbf{T}^* \chi_{I_P}, \psi_Q^{b_P^1} \rangle|^2 \leq ((1 - \varepsilon)B + O(1))|I_{P'}|$$

for all tiles $P' \leq P$, since the claim then follows by taking suprema over P' and solving for B (cf. Lemma 3.1).

Fix P' . We apply Lemma 6.11 again but with the $b_{P'}^2$, partitioning

$$\text{Tree}(P') = T_2 \cup \mathbf{P}'_{\text{buffer}} \cup \bigcup_{T' \in \mathbf{T}'} T'.$$

From the definition of B and the fact that \mathbf{T}' is a $(1 - \varepsilon)$ -packing we have (cf. Lemma 3.1)

$$\sum_{Q \in T_1 \cap \bigcup_{T' \in \mathbf{T}'} T'} |\langle \mathbf{T}^* \chi_{I_P}, \psi_Q^{b_P^1} \rangle|^2 \leq (1 - \varepsilon)B|I_{P'}|,$$

so it suffices to show that

$$\sum_{Q \in T_1 \cap (T_2 \cup \mathbf{P}'_{\text{buffer}})} |\langle \mathbf{T}^* \chi_{I_P}, \psi_Q^{b_P^1} \rangle|^2 \lesssim |I_{P'}|.$$

From (69) it will suffice to show

$$\sum_{Q \in T_1 \cap (T_2 \cup \mathbf{P}'_{\text{buffer}})} |\langle \mathbf{T}^* \chi_{I_P}, b_P^1 \phi_Q^{b_P^1} \rangle|^2 \lesssim |I_{P'}|.$$

We first observe that

$$\begin{aligned} |\langle T^* \chi_{I_P}, b_P^1 \phi_Q^{b_P^1} \rangle| &= \left| \int_{I_P} T(b_P^1 \phi_Q^{b_P^1}) \right| \\ &= \left| \int_{I_Q} T(b_P^1 \phi_Q^{b_P^1}) \right| \leq |I_Q|^{1/2} \|T(b_P^1 \phi_Q^{b_P^1})\|_2; \end{aligned}$$

from Corollary 6.10 and (79) we thus have the weak Carleson bound

$$|\langle T^* \chi_{I_P}, b_P^1 \phi_Q^{b_P^1} \rangle|^2 \lesssim |I_Q|$$

for all $Q \in T_1$. In particular we see that the contribution of $\mathbf{P}'_{\text{buffer}}$ will be acceptable since $\mathbf{P}'_{\text{buffer}}$ is a 2-packing. We are thus left with showing

$$\sum_{Q \in T_1 \cap T_2} |\langle T^* \chi_{I_P}, b_P^1 \phi_Q^{b_P^1} \rangle|^2 \lesssim |I_{P'}|.$$

In Step 4 we shall show the pointwise bound

$$\begin{aligned} (88) \quad |\langle T^* \chi_{I_P}, b_P^1 \phi_Q^{b_P^1} \rangle| &\lesssim |Wb_{P'}^2(Q)| + |W_{b_P^1}(b_P^1 T^*(b_{P'}^2))(Q)| \\ &\quad + |W_{b_P^1}(b_{P'}^2 T(b_P^1))(Q)| + |W_{b_P^1}(T(b_P^1))(Q)| \end{aligned}$$

for all $Q \in T_1 \cap T_2$. The claim will then follow from (72), (73)²⁴.

Step 4. Pointwise estimates on wavelet coefficients: In the previous step we reduced matters to proving the pointwise estimate (88). The proof of this estimate is really the core of our argument, although it was necessary to do all the above prunings to get to a point where this estimate became both provable and useful²⁵.

We now prove (88). Fix $Q \in T_1 \cap T_2$. On I_Q , we can decompose $\chi_{I_P} = \frac{b_{P'}^2}{[b_{P'}^2]_Q} + F$ where F is the mean zero function $F := \chi_{I_Q} - \frac{b_{P'}^2 \chi_{I_Q}}{[b_{P'}^2]_Q}$.

Since $T(b_P^1 \phi_Q^{b_P^1})$ is supported on I_Q , we can thus estimate the left-hand side of (88) by

$$\left| \left\langle T^* \frac{b_{P'}^2}{[b_{P'}^2]_Q}, b_P^1 \phi_Q^{b_P^1} \right\rangle \right| + |\langle T^* F, b_P^1 \phi_Q^{b_P^1} \rangle|.$$

²⁴While the tree $T_1 \cap T_2$ is not necessarily convex, the larger tree $(T_1 \cup \mathbf{P}'_{\text{buffer}}) \cap (T_2 \cup \mathbf{P}'_{\text{buffer}})$ is, and on this larger tree we have pseudo-accretivity and (79) for both b_P^1 and $b_{P'}^2$.

²⁵If one wished to prove a global $T(b)$ theorem, with globally para-accretive L^∞ functions $b_P^1, b_{P'}^2$, one could dispense with the selection algorithms and go directly to (a suitable analogue of) (88). We omit the details.

The first term is $|O(W_{b_P^1}(b_P^1 T^*(b_{P'}^2))(Q))|$ by the pseudo-accretivity of $b_{P'}^2$. For the second term, observe that if the $\phi_Q^{b_P^1}$ could be moved inside the T^* , thus

$$|\langle T^*(\phi_Q^{b_P^1} F), b_P^1 \rangle|$$

then by moving the T^* to the other side, we could bound this by

$$O(|W_{b_P^1}(FT(b_P^1))|) = O(|W_{b_P^1}(b_{P'}^2 T(b_P^1))(Q)|) + O(|W_{b_P^1}(T(b_P^1))(Q)|)$$

again using the pseudo-accretivity of $b_{P'}^2$. Thus it suffices to control the commutator

$$|\langle T^* F, b_P^1 \phi_Q^{b_P^1} \rangle - \langle T^*(\phi_Q^{b_P^1} F), b_P^1 \rangle|.$$

If F had mean zero on both I_{Q_l} and I_{Q_r} then this commutator would be zero from (53) since $\phi_Q^{b_P^1}$ is constant on I_{Q_l} and I_{Q_r} . Thus we may freely replace F by $[F]_{Q_l}(\chi_{I_{Q_l}} - \chi_{I_{Q_r}})$ since the difference has mean zero on both I_{Q_l} and I_{Q_r} . Throwing the T^* on to the other side and using Cauchy-Schwarz and Corollary 6.10, we can thus bound this commutator by

$$O(|I_Q|^{1/2} |[F]_{Q_l}|).$$

However a computation shows that

$$[F]_{Q_l} = \frac{1}{2}([F]_{Q_l} - [F]_{Q_r}) = \frac{[b_{P'}^2]_{Q_r} - [b_{P'}^2]_{Q_l}}{2[b_{P'}^2]_Q} = -|I_Q|^{-1/2} \frac{W b_{P'}^2(Q)}{[b_{P'}^2]_Q}$$

and the claim then follows by the pseudo-accretivity of $b_{P'}^2$. The proof of Theorem 6.8 is now complete. \square

One can generalize (76) to

$$\int_{I_P} |b_P^1|^p + |T b_P^1|^{q'} + |b_P^2|^q + |T^* b_P^2|^{p'} \lesssim |I_P|$$

for any $1 \leq p, q \leq \infty$, with $1/p + 1/p' = 1/q + 1/q' = 1$ (this was already suspected in [11] when $p = q = \infty$); the dual exponents are necessary to control such expressions as $\langle T b_P^1, b_P^2 \rangle$. Most of the argument proceeds *mutatis mutandis* except for Lemma 6.7. Firstly in (71) the $|b|^2$ mean has to be replaced by some other $|b|^p$ mean, but because of Corollary 3.3 we still recover (74). However we still must modify (72), (73) to

$$\left(\sum_{P \in T} |W_b f(P)|^2 \right)^{1/2} \lesssim |I_T|^{1/2} \| |f|^{q'} \|_{\text{mean}^*(T)}^{1/q'}$$

and

$$\left(\sum_{P \in T} |W_b(b'f)(P)|^2 \right)^{1/2} \lesssim |I_T|^{1/2} \| |f|^{q'} \|_{\text{mean}^*(T)}^{1/q'} \| |b'|^q \|_{\text{mean}^*(T)}^{1/q}.$$

This proceeds by replacing f and $b'f$ with averaged variants as in the proof of (73); the averages will then be controlled in L^∞ and hence in L^2 . We omit the details.

It is also straightforward to generalize Theorem 6.8 to Calderón-Zygmund operators which do not obey the perfect dyadic cancellation condition (53), and instead obey a more classical cancellation condition such as $|\nabla_x K(x, y)| + |\nabla_y K(x, y)| \lesssim 1/|x - y|^2$. However it is still convenient to impose a truncation condition on the kernel when $|x - y|$ is extremely small or extremely large (as in e.g. [11]).

The main new difference with these kernels is that when $f_I \in S_0(I)$, the function $\mathbb{T}f_I$ is no longer supported in I but has a tail at infinity. However the cancellation conditions ensure that this tail is quite rapidly decaying (like $1/|x|^2$ if we assume the above gradient bounds). This causes many of the identities used in the arguments above to pick up some error terms, for instance if g is constant on I it is no longer true that $\langle \mathbb{T}f_I, g \rangle = [g]_I \langle \mathbb{T}f_I, 1 \rangle$, however the error term incurred is quite manageable due to the good decay (especially if g is in fact constant on a much wider interval than I). We will not pursue the details further here as they are rather standard (see e.g. [20], [11]). A perhaps more interesting generalization would be to non-doubling situations as in [47], as this may have applications to analytic capacity problems, but this seems to require much more technical arguments.

As a corollary of the local $\mathbb{T}(b)$ theorem we can conclude a global $\mathbb{T}(b)$ theorem²⁶. We recall that a function b is *para-accretive* if for every tile P there exists a tile $Q \leq' P$ with $|I_Q| \sim |I_P|$ and $||[b]_Q| \gtrsim 1$. Every pseudo-accretive function is para-accretive (just take $Q := P$), but not conversely.

²⁶Of course, the global $\mathbb{T}(b)$ theorem could be proven directly in a much simpler manner, but one advantage of doing things this way is that we can relax the hypotheses of the global $\mathbb{T}(b)$ theorem slightly. If b_1, b_2 were bounded and strongly pseudo-accretive, one could obtain a direct proof of the global $\mathbb{T}(b)$ theorem by using the modified wavelet transforms W_{b_1}, W_{b_2} to define paraproducts by adapting (44), and then finding an analogue of (54), and repeating the proof of the global $\mathbb{T}(1)$ theorem. See e.g. [4] for details. An alternate approach based on (88) is also possible.

Corollary 6.12 (Dyadic global $T(b)$ theorem). *Let b_1, b_2 be para-accretive functions with*

$$\|b_1\|_{\text{BMO}}, \|b_2\|_{\text{BMO}} \lesssim 1.$$

Let T be a perfect dyadic Calderón-Zygmund operator obeying the modified weak boundedness property

$$(89) \quad |\langle Tb_1\chi_I, b_2\chi_J \rangle| \lesssim |K|$$

for all intervals I, J, K with $|I| \sim |J| \sim |K|$ and $I, J \subseteq K$

and such that

$$\|T(b_1)\|_{\text{BMO}}, \|T^*(b_2)\|_{\text{BMO}} \lesssim 1.$$

Then T is bounded on L^2 .

Proof: We apply Theorem 6.8 with

$$b_P^1 := \frac{b_1\chi_{I_Q}}{[b_1]_Q}$$

where for each P , we choose the tile $Q \leq' P$ so that $|I_Q| \sim |I_P|$ and $[b_1]_Q \gtrsim 1$. We define b_P^2 similarly.

The normalization (75) is clear. To prove (76), we observe that

$$b_P^1 = \frac{(b_1 - [b_1]_Q)\chi_{I_Q}}{[b_1]_Q} + 1$$

and so the L^2 bound on b_P^1 follows from the BMO control on b_1 and the lower bound on $[b_1]_Q$. To control Tb_P^1 , it suffices from (52) and the lower bound on $[b_1]_Q$ to show that

$$\|T(b_1\chi_{I_Q})\|_{L^2(I_Q)} \lesssim |I_Q|^{1/2},$$

or in other words that

$$|\langle T(b_1\chi_{I_Q}), h \rangle| \lesssim |I_Q|^{1/2} \|h\|_2$$

for all $h \in S(I_Q)$.

Select a tile $R \leq' Q$ such that $[b_2]_R \gtrsim 1$ and $|I_R| \sim |I_Q|$. If h is a scalar multiple of $b_2\chi_R$, the claim follows from (89). Thus we may subtract multiples of $b_2\chi_R$ and reduce to the case when h has mean zero (cf. Lemma 6.9). But then $\langle T(b_1\chi_{I_Q}), h \rangle = \langle Tb_1, h \rangle$, and the claim follows since $Tb_1 \in \text{BMO}$.

One can control b_P^2 and $T^*b_P^2$ by identical arguments. □

Notice that b_1, b_2 are only assumed to be in BMO^{27} rather than L^∞ . This generalization of the standard $T(b)$ theorem appears to be new. Also observe that the above argument also works in the special case $T(b_1) = T^*(b_2) = 0$ if we drop the para-accretivity and BMO hypotheses on b_1, b_2 and instead impose the reverse Hölder conditions

$$\frac{(\frac{1}{|I_P|} \int_{I_P} |b_1|^2)^{1/2}}{|[b_1]_P|}, \frac{(\frac{1}{|I_P|} \int_{I_P} |b_2|^2)^{1/2}}{|[b_2]_P|} \lesssim 1$$

on b_1, b_2 . It is in fact likely that we can obtain a $T(b)$ -type theorem for arbitrary (complex) dyadic A_∞ weights b_1, b_2 (see [28]), but we will not attempt to give the most general statements here.

References

- [1] P. AUSCHER, S. HOFMANN, A. MCINTOSH AND P. TCHAMITCHIAN, The Kato square root problem for higher order elliptic operators and systems on \mathbb{R}^n , dedicated to the memory of Tosio Kato, *J. Evol. Equ.* **1(4)** (2001), 361–385.
- [2] P. AUSCHER, S. HOFMANN, M. LACEY, A. MCINTOSH AND P. TCHAMITCHIAN, The solution of the Kato square root problem for second order elliptic operators on R^n , *Ann. of Math. (2)* (to appear).
- [3] P. AUSCHER, S. HOFMANN, J. L. LEWIS AND P. TCHAMITCHIAN, Extrapolation of Carleson measures and the analyticity of Kato's square-root operators, *Acta Math.* **187(2)** (2001), 161–190.
- [4] P. AUSCHER AND P. TCHAMITCHIAN, Bases d'ondelettes sur les courbes corde-arc, noyau de Cauchy et espaces de Hardy associés, *Rev. Mat. Iberoamericana* **5(3–4)** (1989), 139–170.
- [5] P. AUSCHER AND P. TCHAMITCHIAN, Square root problem for divergence operators and related topics, *Astérisque* **249** (1998), 172 pp.
- [6] J. BERGH AND J. LÖFSTRÖM, “*Interpolation spaces. An introduction*”, Grundlehren der Mathematischen Wissenschaften **223**, Springer-Verlag, Berlin-New York, 1976.
- [7] G. BEYLKIN, R. R. COIFMAN AND V. ROKHLIN, Fast wavelet transforms and numerical algorithms. I, *Comm. Pure Appl. Math.* **44(2)** (1991), 141–183.

²⁷For untruncated Calderón-Zygmund operators in the limit $M \rightarrow \infty$ it may well be necessary to require b_1, b_2 to be in VMO rather than BMO.

- [8] C. J. BISHOP, L. CARLESON, J. B. GARNETT AND P. W. JONES, Harmonic measures supported on curves, *Pacific J. Math.* **138(2)** (1989), 233–236.
- [9] C. P. CALDERÓN, On commutators of singular integrals, *Studia Math.* **53(2)** (1975), 139–174.
- [10] L. CARLESON, Lennart On convergence and growth of partial sumas of Fourier series, *Acta Math.* **116** (1966), 135–157.
- [11] M. CHRIST, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.* **60/61(2)** (1990), 601–628.
- [12] M. CHRIST AND J.-L. JOURNÉ, Polynomial growth estimates for multilinear singular integral operators, *Acta Math.* **159(1–2)** (1987), 51–80.
- [13] R. R. COIFMAN, A. MCINTOSH AND Y. MEYER, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, *Ann. of Math. (2)* **116(2)** (1982), 361–387.
- [14] R. R. COIFMAN AND Y. MEYER, On commutators of singular integrals and bilinear singular integrals, *Trans. Amer. Math. Soc.* **212** (1975), 315–331.
- [15] R. R. COIFMAN AND Y. MEYER, Commutateurs d'intégrales singulières et opérateurs multilinéaires, *Ann. Inst. Fourier (Grenoble)* **28(3)** (1978), 177–202.
- [16] R. R. COIFMAN AND Y. MEYER, Fourier analysis of multilinear convolutions, Calderón's theorem, and analysis of Lipschitz curves, in: “*Euclidean harmonic analysis*” (Proc. Sem., Univ. Maryland, College Park, Md., 1979), *Lecture Notes in Math.* **779**, Springer, Berlin, 1980, pp. 104–122.
- [17] R. R. COIFMAN AND Y. MEYER, Au delà des opérateurs pseudo-différentiels, *Astérisque* **57** (1978), 185 pp.
- [18] R. R. COIFMAN AND Y. MEYER, Nonlinear harmonic analysis, operator theory and P.D.E., in: “*Beijing lectures in harmonic analysis*” (Beijing, 1984), *Ann. of Math. Stud.* **112**, Princeton Univ. Press, Princeton, NJ, 1986, pp. 3–45.
- [19] R. R. COIFMAN AND Y. MEYER, “*Ondelettes et opérateurs. III. Opérateurs multilinéaires*”, *Actualités Mathématiques*, Hermann, Paris, 1991.
- [20] R. R. COIFMAN, P. W. JONES AND S. SEMMES, Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves, *J. Amer. Math. Soc.* **2(3)** (1989), 553–564.
- [21] G. DAVID, “*Wavelets and singular integrals on curves and surfaces*”, *Lecture Notes in Mathematics* **1465**, Springer-Verlag, Berlin, 1991.

- [22] G. DAVID AND J.-L. JOURNÉ, A boundedness criterion for generalized Calderón-Zygmund operators, *Ann. of Math. (2)* **120(2)** (1984), 371–397.
- [23] G. DAVID, J.-L. JOURNÉ AND S. SEMMES, Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation, *Rev. Mat. Iberoamericana* **1(4)** (1985), 1–56.
- [24] G. DAVID AND S. SEMMES, “*Analysis of and on uniformly rectifiable sets*”, Mathematical Surveys and Monographs **38**, American Mathematical Society, Providence, RI, 1993.
- [25] C. FEFFERMAN, Pointwise convergence of Fourier series, *Ann. of Math. (2)* **98** (1973), 551–571.
- [26] C. FEFFERMAN AND E. M. STEIN, Some maximal inequalities, *Amer. J. Math.* **93** (1971), 107–115.
- [27] C. FEFFERMAN AND E. M. STEIN, H^p spaces of several variables, *Acta Math.* **129(3–4)** (1972), 137–193.
- [28] R. A. FEFFERMAN, C. E. KENIG AND J. PIPHER, The theory of weights and the Dirichlet problem for elliptic equations, *Ann. of Math. (2)* **134(1)** (1991), 65–124.
- [29] J. B. GARNETT AND P. W. JONES, BMO from dyadic BMO, *Pacific J. Math.* **99(2)** (1982), 351–371.
- [30] L. GRAFAKOS AND N. J. KALTON, The Marcinkiewicz multiplier condition for bilinear operators, *Studia Math.* **146(2)** (2001), 115–156.
- [31] S. HOFMANN, M. LACEY AND A. MCINTOSH, The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds, *Ann. of Math. (2)* (to appear).
- [32] S. HOFMANN AND J. L. LEWIS, The Dirichlet problem for parabolic operators with singular drift terms, *Mem. Amer. Math. Soc.* **151(719)** (2001), 113 pp.
- [33] S. JANSON, On interpolation of multilinear operators, in: “*Function spaces and applications*” (Lund, 1986), Lecture Notes in Math. **1302**, Springer, Berlin, 1988, pp. 290–302.
- [34] F. JOHN, Quasi-isometric mappings, in: “*Seminari 1962/63 Anal. Alg. Geom. e Topol.*”, vol. 2, Ist. Naz. Alta Mat., Ediz. Cremonese, Rome, 1965, pp. 462–473.
- [35] P. W. JONES, Square functions, Cauchy integrals, analytic capacity, and harmonic measure, in: “*Harmonic analysis and partial differential equations*” (El Escorial, 1987), Lecture Notes in Math. **1384**, Springer, Berlin, 1989, pp. 24–68.
- [36] P. W. JONES, Rectifiable sets and the traveling salesman problem, *Invent. Math.* **102(1)** (1990), 1–15.

- [37] N. H. KATZ, Maximal operators over arbitrary sets of directions, *Duke Math. J.* **97(1)** (1999), 67–79.
- [38] N. H. KATZ, Remarks on maximal operators over arbitrary sets of directions, *Bull. London Math. Soc.* **31(6)** (1999), 700–710.
- [39] M. LACEY AND C. THIELE, L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, *Ann. of Math. (2)* **146(3)** (1997), 693–724.
- [40] M. LACEY AND C. THIELE, On Calderón’s conjecture, *Ann. of Math. (2)* **149(2)** (1999), 475–496.
- [41] M. LACEY AND C. THIELE, A proof of boundedness of the Carleson operator, *Math. Res. Lett.* **7(4)** (2000), 361–370.
- [42] J. L. LEWIS AND M. A. M. MURRAY, The method of layer potentials for the heat equation in time-varying domains, *Mem. Amer. Math. Soc.* **114(545)** (1995), 157 pp.
- [43] A. MCINTOSH AND Y. MEYER, Algèbres d’opérateurs définis par des intégrales singulières, *C. R. Acad. Sci. Paris Sér. I Math.* **301(8)** (1985), 395–397.
- [44] Y. MEYER, “*Ondelettes et opérateurs. I. Ondelettes*”, Actualités Mathématiques, Hermann, Paris, 1990.
- [45] C. MUSCALU, T. TAO AND C. THIELE, Multi-linear operators given by singular multipliers, *J. Amer. Math. Soc.* **15(2)** (2002), 469–496 (electronic).
- [46] C. MUSCALU, T. TAO AND C. THIELE, L^p estimates for the “bi-est”, submitted to *Math. Ann.*
- [47] F. NAZAROV, S. TREIL AND A. VOLBERG, Accretive system T_b theorems on nonhomogeneous spaces, *Duke Math. J.* (to appear).
- [48] F. NAZAROV, S. TREIL AND A. VOLBERG, T_b theorems on non-homogeneous spaces, *Acta Math.* (to appear).
- [49] M. C. PEREYRA, Lecture notes on dyadic harmonic analysis, in: “*Second Summer School in Analysis and Mathematical Physics*” (Cuernavaca, 2000), Contemp. Math. **289**, Amer. Math. Soc., Providence, RI, 2001, pp. 1–60.
- [50] S. SEMMES, Square function estimates and the $T(b)$ theorem, *Proc. Amer. Math. Soc.* **110(3)** (1990), 721–726.
- [51] E. M. STEIN, “*Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*”, Princeton Mathematical Series **43**, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.

- [52] C. THIELE, Time-frequency analysis in the discrete phase plane, in: *“Topics in analysis and its applications”*, World Sci. Publishing, River Edge, NJ, 2000, pp. 99–152.
- [53] C. THIELE, On the Bilinear Hilbert transform, Universität Kiel, Habilitationsschrift (1998).
- [54] J. VERDERA, L^2 boundedness of the Cauchy integral and Menger curvature, in: *“Harmonic analysis and boundary value problems”* (Fayetteville, AR, 2000), Contemp. Math. **277**, Amer. Math. Soc., Providence, RI, 2001, pp. 139–158.

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Primera versió rebuda el 20 de setembre de 2001,
darrera versió rebuda el 25 de juny de 2002.