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PERFECT RINGS FOR WHICH THE CONVERSE OF SCHUR'S LEMMA HOLDS

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Abstract

If M is a simple module over a ring R then, by the Schur's lemma, the endomorphism ring of M is a division ring. However, the converse of this result does not hold in general, even when R is artinian. In this short note, we consider perfect rings for which the converse assertion is true, and we show that these rings are exactly the primary decomposable ones.

1. Introduction

Let M be a module over a ring R. If M is simple, then the Schur's lemma states that $\mathrm{End}_R(M)$ is a division ring (a skew field). The converse of this statement is false. For example, if R is an integral (commutative) domain which is not a field, then its quotient field Q, considered as an R-module, is not simple, although $\mathrm{End}_R(Q) \cong Q$ is a division ring.

For an example in the artinian case, one can take: $R = \left(\begin{smallmatrix} K & K \\ 0 & K \end{smallmatrix}\right)$, the ring of upper triangular 2×2 matrices over a field K. Then for the R-module $M = Re$, where $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have $\mathrm{End}_R(M) \cong K$, but M is not simple.

Definition 1.1. We shall say that a ring R has the CSL property (abreviation of: **C**onverse of the **S**chur's **L**emma), or that R is a CSL-ring, if every module is simple whenever its endomorphism ring is a division ring.

The CSL property, has been studied by some authors. In [**4**], Ware and Zelmanowitz, considered modules with simple endomorphism ring over a commutative ring. From their results, it can be shown that a commutative ring R is a CSL-ring iff every prime ideal of R is maximal. In [**3**] some classes of noncommutative von Neumann regular rings with the CSL property has been studied.

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The full class of CSL-rings seems to be very hard to characterize, the present note deals with perfect CSL-rings. Our main result is:

Theorem 1.2. *For a perfect ring R, the following assertions are equivalent:*

(i) *Every R-module with semiprime endomorphism ring is semisimple.*
(ii) *Every R-module with von Neumann regular endomorphism ring is semisimple.*
(iii) *R is a CSL-ring.*
(iv) *R is isomorphic to a finite product of primary rings.*

2. Preliminaries and notations

(For the terminology and notations used here we refer to [**1**], [**2**].)

Throughout this paper, all rings are associative with identity, and all modules are left unitary modules. If M is a module over a ring R, the endomorphism ring of M is denoted by $\mathrm{End}_R(M)$. The socle of M, i.e. the sum of all simple submodules of M, is denoted by $\mathrm{Soc}(M)$.

A ring R is said to be perfect if it is left and right perfect. Over a perfect ring, every nonzero module has a maximal and a simple submodule.

A ring R is said to be primary, if the factor ring $R/J(R)$, where $J(R)$ denotes the Jacobson radical of R, is simple artinian. Any primary left or right perfect ring is isomorphic to a full matrix ring over a local ring [**2**].

A right or left perfect ring R is said to be primary decomposable, if it is isomorphic to a (finite) product of primary rings. It can be shown that R is primary decomposable, if and only if, every idempotent which is central modulo the Jacobson radical is central.

A ring R is said to be von Neumann regular (abbreviated VNR), if for every $x \in R$ there exists $y \in R$ such that $xyx = x$. An important example of a VNR ring is the endomorphism ring of a semisimple module.

3. The proofs

(i) $\Rightarrow$ (ii) is obvious since every VNR ring is semiprime.

(ii) $\Rightarrow$ (iii). If $\mathrm{End}_R(M)$ is a division ring, then it is VNR. So M is semisimple by hypothesis. Since M is indecomposable, it is therefore simple.

(iv) $\Rightarrow$ (i). It is easy to see that any direct product of a finite number of rings verifying (i) has this property. Hence to show that (iv) implies (i), it suffices to show that every perfect primary ring verifies (i). Let R be such a ring. If M is any nonzero R-module, then M has a maximal submodule N, and a simple submodule S. Since R is primary, R has a unique isomorphism class of simple modules, so there exists an R-module isomorphism $\sigma\colon M/N \to S$. If $\pi\colon M \to M/N$ and $\imath\colon S \to M$ denote respectively the canonical surjection and the canonical injection, then $u = \imath \circ \sigma \circ \pi$ is a nonzero endomorphism of M such that $u(N) = 0$ and $u(M) \subset S$.

Now suppose that M is not semisimple, then M contains a proper essential submodule E which is contained in a maximal submodule N. By what has been proved previously, there exists a nonzero $u \in \mathrm{End}_R(M)$ such that $u(N) = 0$ and $u(M) \subset \mathrm{Soc}(M)$. Since E is essential, we have $\mathrm{Soc}(M) \subset E$ and then $u(\mathrm{Soc}(M)) \subset u(N) = 0$. Now for every $v \in \mathrm{End}_R(M)$, $(u \circ v \circ u)(M) \subset (u \circ v)(\mathrm{Soc}(M)) \subset u(\mathrm{Soc}(M)) = 0$. This proves that $u \circ v \circ u = 0$ for every $v \in \mathrm{End}_R(M)$; so that $\mathrm{End}_R(M)$ is not semiprime.

(iii) $\Rightarrow$ (iv). To prove this implication, we need a preliminary result.

Lemma 3.1. *Let M be a finitely generated module over a perfect ring R. Suppose that $\mathrm{Hom}_R(N, \mathrm{Soc}(M)) = 0$ for every nonsimple submodule N of M. Then $\mathrm{End}_R(M)$ is a division ring.*

Proof: Suppose that $\mathrm{End}_R(M)$ is not a division ring, then there exists $u \in \mathrm{End}_R(M)$ such that u is nonzero and noninvertible. Since M is finitely generated over a perfect ring, u is not injective. Let N be a submodule of M such that $\mathrm{Ker} \subset N$ and $N/\mathrm{Ker}\, u$ is simple. If $v = u|_N$ denotes the restriction of u to N, then $\mathrm{Im}\, v \cong N/\mathrm{Ker}\, v$ so $\mathrm{Im}\, v$ is simple. Thus $\mathrm{Im}\, v \subset \mathrm{Soc}(M)$. This proves that $\mathrm{Hom}(N, \mathrm{Soc}(M)) \neq 0$.

We are now going to prove the implication (iii) $\Rightarrow$ (iv). Suppose on the contrary that R is a CSL-ring which is not primary decomposable. Then there exists an idempotent $e \in R$ central modulo $J = J(R)$ but not central. Either $R(1-e)Re \neq 0$ or $ReR(1-e) \neq 0$. Without loss of generality, we can suppose that $R(1-e)Re \neq 0$. Since $R(1-e)Re \neq J(1-e)Re$, we can pick an element $x \in R(1-e)Re \backslash J(1-e)Re$, and consider the left ideal I maximal with respect to:

$$J(1-e)Re \subset I \subset Re \quad \text{and} \quad x \notin I.$$

Then, the module $M = Re/I$ is finitely generated with simple socle equal to $S = Rx + I/I$. Since $J(1-e)Re \subset I$, we have $J(1-e)M = 0$. Hence $(1-e)M \subset S$. On the other hand, $eR \subset Re + J$, thus $eR(1-e)Re \subset J(1-e)Re$, implying $eS = 0$.

Now let N be a submodule of M such that $\mathrm{Hom}_R(N, S) \neq 0$ and $u\colon N \to S$ a nonzero homomorphism. We have $u(N) = S$ and $u((1-e)N) = (1-e)S \neq 0$. Since $(1-e)N \subset S$, then $u(S) \neq 0$. Consequently $\mathrm{Ker}\, u = 0$ and u is therefore an isomorphism. So N is necessary simple.

By Lemma 3.1, $\mathrm{End}_R(M)$ is a division ring. Since R is a CSL-ring, M is simple. So $M = S$ and $eM = eS = 0$, a contradiction. □

References

[1] F. W. ANDERSON AND K. R. FULLER, *"Rings and categories of modules"*, Graduate Texts in Mathematics **13**, Springer-Verlag, New York, 1974.

[2] C. FAITH, *"Algebra. II. Ring theory"*, Grundlehren der Mathematischen Wissenschaften **191**, Springer-Verlag, Berlin, 1976.

[3] Y. HIRANO AND J. K. PARK, Rings for which the converse of Schur's lemma holds, *Math. J. Okayama Univ.* **33** (1991), 121–131.

[4] R. WARE AND J. ZELMANOWITZ, Simple endomorphism rings, *Amer. Math. Monthly* **77** (1970), 987–989.

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