BOUNDING THE DEGREE OF SOLUTIONS TO PFAFF EQUATIONS

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Abstract

We study hypersurfaces of complex projective manifolds which are invariant by a foliation, or more generally which are solutions to a Pfaff equation. We bound their degree using classical results on logarithmic forms.

1. Introduction

Starting with Poincaré [**Po**] and Painlevé [**Pa**], many mathematicians considered the following problem: given a foliation $\mathcal{F}$ on $\mathbf{C}P^n$ of degree d and a hypersurface $V \subset \mathbf{C}P^n$ invariant by $\mathcal{F}$, is it possible to bound the degree of V by a number $h(d)$ which depends only on d (and not on $\mathcal{F}$)? In such a generality, the answer is clearly negative: for example, the curve $\{x^p = y^q\}$ in $\mathbf{C}P^2$ has degree $\max\{p, q\}$ and it is invariant by the foliation given by $pydx - qxdy = 0$, whose degree is one (see also [**LN**] for other interesting examples). However several positive results have been obtained by those authors and, recently, by [**CL**], [**Ca**], [**Ba**], [**Br**] and [**So**]. The philosophy behind these results is that the failure of an uniform bound $h(d)$ is due to the existence of "bad singularities" of V or of $\mathcal{F}$. For instance, one finds the bound $h(d) = d + 2$ in $\mathbf{C}P^2$, provided that either V has only normal crossing singularities [**CL**], or $\mathcal{F}$ has only nondicritical singularities along V [**Ca**]. Our aim is to attract the attention of the reader to the relation between this problem and these results and some basic properties of logarithmic forms, discovered by Deligne and Bogomolov [**De**], [**Bo**].

We shall work in the context of Pfaff equations, more general than foliations. For our purposes, the simplest definition is the following:

Definition 1.1. Given a complex manifold X and a holomorphic line bundle N on X, a *Pfaff equation* of codimension p, $1 \leq p \leq \dim_{\mathbf{C}}(X) - 1$, is a nontrivial global section σ of $\Omega_X^p \otimes N$, where Ω_X^p denotes the sheaf of holomorphic p-forms on X.

Let us consider the corresponding generalization of the notion of leaf of a foliation. Given a hypersurface V, let $i_V\sigma$ denote the restriction of σ to V, i.e. the section of $\Omega_V^p \otimes N|_V$ obtained by projecting $\sigma|_V$ via $(\Omega_X^p \otimes N)|_V \to \Omega_V^p \otimes N|_V$.

Definition 1.2. Given a Pfaff equation $\sigma \in H^0(X, \Omega_X^p \otimes N)$ and a hypersurface $V \subset X$, we shall say that V is a *solution* to σ if $i_V\sigma \equiv 0$.

Some remarks on these definitions may be useful. Usually one requires that the zero set $(\sigma)_0$ of a Pfaff equation $\sigma \in H^0(X, \Omega_X^p \otimes N)$ has codimension ≥ 2, but this is not really important. Anyway, if $\sigma \in H^0(X, \Omega_X^p \otimes N)$ vanishes on a hypersurface $Z \subset X$, then we can replace σ by $\sigma' := \frac{\sigma}{h} \in H^0(X, \Omega_X^p \otimes N')$, where $N' = N \otimes \mathcal{O}_X(-Z)$ and $h \in H^0(X, \mathcal{O}_X(Z))$ vanishes on Z, and then $\sigma'|_Z \not\equiv 0$. This division of σ by h has no significant consequences, because in the study of Pfaff equations one is more interested in the saturated subsheaf of Ω_X^p generated by σ than in σ itself, and this subsheaf is unchanged by the division.

A foliation $\mathcal{F}$ of codimension p gives rise to a Pfaff equation of codimension p, because locally such a foliation can be seen as the kernel of a holomorphic p-form; in this case, N corresponds to the determinant line bundle of the rank p normal sheaf of $\mathcal{F}$. But the converse is not generally true, except when $p = \dim_{\mathbf{C}}(X) - 1$, because no integrability assumption is done. When the Pfaff equation arises from a foliation $\mathcal{F}$, Definition 1.2 reduces to say that the hypersurface V is saturated by the leaves of $\mathcal{F}$.

In order to state our result in a simple form, let us suppose now that X is a projective manifold with Picard group $\mathrm{Pic}(X) = \mathbf{Z}$, although a more general fact, without restrictions on $\mathrm{Pic}(X)$, will be found in the course of the proof. Let H be the positive generator of $\mathrm{Pic}(X)$. We naturally define the *degree* of a holomorphic line bundle L as the integer $d(L)$ such that $L = d(L)H$ in $\mathrm{Pic}(X)$. For a hypersurface V, the degree $d(V)$ is defined as $d(\mathcal{O}_X(V))$. We recall that projective manifolds with $\mathrm{Pic}(X) = \mathbf{Z}$ are quite abundant: for instance, the classical Noether-Lefschetz theorem states that a generic hypersurface X in $\mathbf{C}P^n$ of high degree has $\mathrm{Pic}(X) = \mathbf{Z}$. We refer to Section 2 for the basic properties of logarithmic forms.

Theorem. *Let X be a complex projective manifold with $\mathrm{Pic}(X) = \mathbf{Z}$ and let $\sigma \in H^0(X, \Omega_X^p \otimes N)$ be a Pfaff equation. Let $V \subset X$ be a normal crossing hypersurface, which is solution to σ. Then*

$$d(V) \leq d(N)$$

and the inequality is strict if V is smooth. Moreover, if $d(V) = d(N)$ then σ is given by a global closed logarithmic p-form on X with poles along V.

When $\dim_{\mathbf{C}}(X) = 2$, we recover results of [**CL**] and [**Ba**]. Let us consider the case when a Pfaff equation on $\mathbf{C}P^n$ arises from a foliation with $\mathrm{codim}_{\mathbf{C}} \mathrm{Sing}(\mathcal{F}) \geq 2$ (and $\mathrm{codim}_{\mathbf{C}}(\sigma)_0 \geq 2$). One usually defines the *degree* $d(\mathcal{F})$ as the degree of the tangency set between the $(n-p)$-dimensional leaves of $\mathcal{F}$ and a generic linear subspace $\Pi \simeq \mathbf{C}P^p$ (remark that this tangency set is a hypersurface of Π). In Lemma 3.2 we show that $d(N) = d(\mathcal{F}) + p + 1$ and we recover the result of [**CL**] in $\mathbf{C}P^2$, since our bound becomes

$$d(V) \leq d(\mathcal{F}) + p + 1.$$

At the end of the paper we shall discuss to which extent the normal crossing hypothesis on V may be weakened.

2. Logarithmic forms

In this section we recall basic facts on logarithmic forms (see, for instance, [**Sa**]).

Let X be a complex manifold and V a hypersurface with at most normal crossing singularities. A *logarithmic p-form* on X with poles along V is a meromorphic p-form ω with polar set $(\omega)_\infty \subset V$ such that ω and $d\omega$ have at most simple poles along V. Equivalently, if $f = 0$ is a local reduced equation of V, then $f\omega$ and $fd\omega$ are holomorphic. Obviously, this condition is equivalent to $f\omega$ and $df \wedge \omega$ being holomorphic.

This definition can be localized on open sets of X; therefore we obtain a (coherent, analytic, locally free) sheaf on X, denoted $\Omega_X^p(\log V)$. Remark that everything makes sense even if $p = 0$ or $p = n = \dim_{\mathbf{C}}(X)$, and $\Omega_X^0(\log V) = \mathcal{O}_X$, $\Omega_X^n(\log V) \simeq \Omega_X^n \otimes \mathcal{O}_X(V)$.

If $(z_1, \ldots, z_n)$ is a local coordinate system around $x = (0, \ldots, 0) \in V$ such that V is locally expressed by $V = \{z_1 \cdot \ldots \cdot z_k = 0\}$, then $\bigoplus_{p=0}^n \Omega_X^p(\log V)$ is locally generated by holomorphic forms and $\{\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}\}$. More precisely, every section ω of $\Omega_X^p(\log V)$ can be locally written as

(1)
$$\omega = \omega_0 + \sum_{j=1}^k \omega_j \wedge \frac{dz_j}{z_j},$$

where ω_0 is a holomorphic p-form and each ω_j is a local section of $\Omega_X^{p-1}(\log V)$. Remark that the exterior product of two logarithmic forms

is still a logarithmic form. We also note that the existence of a decomposition like (1) is strongly dependent on the hypothesis that V has only normal crossing singularities.

In particular, for every $j = 1, \ldots, k$, we can locally decompose ω as

$$\text{(2)} \qquad \omega = \gamma_j + \eta_j \wedge \frac{dz_j}{z_j},$$

where γ_j is a local section of $\Omega_X^p(\log V)$, η_j is a local section of $\Omega_X^{p-1}(\log V)$ and moreover both γ_j and η_j do *not* contain $V_j := \{z_j = 0\}$ in their polar set. The decomposition (1), as well as (2), is not unique; however, a simple computation shows that the restriction of η_j to V_j is *intrinsically* defined by ω, i.e. does not depend on the involved choices. Setting

$$\Gamma_j := V_j \cap (\cup_{i=1, i \neq j}^k V_i),$$

which is a normal crossing hypersurface of V_j, we therefore have a well defined map

$$\text{Res}_{V_j} : \omega \mapsto \eta_j|_{V_j},$$

and $\eta_j|_{V_j} \in \Omega_{V_j}^{p-1}(\log \Gamma_j)$ is called *local residue* of ω along V_j. Summing on j and patching together these local constructions, we finally obtain the *residue map*

$$\text{Res}: \Omega_X^p(\log V) \to \Omega_{\hat{V}}^{p-1}(\log \Gamma),$$

where $\hat{V}$ is the normalization of V and $\Gamma \subset \hat{V}$ is the normal crossing hypersurface induced by V on $\hat{V}$. Note that the kernel of the residue map is exactly Ω_X^p and that there is an exact sequence

$$0 \to \Omega_X^p \to \Omega_X^p(\log V) \to \Omega_{\hat{V}}^{p-1}(\log \Gamma) \to 0.$$

The next lemma was proved in [**De**], as a by-product of a logarithmic Hodge decomposition, but an elementary proof was later found in [**No**]. For sake of completeness, we give a proof, which is even simpler than that of [**No**].

Lemma 2.1. [**De**] *Let X be a complex projective manifold, $V \subset X$ a normal crossing hypersurface and ω a global logarithmic p-form with polar set contained in V. Then ω is closed.*

Proof: In order to prove that a p-form is closed, it is sufficient to prove that the restriction of the p-form to a generic $(p+1)$-dimensional submanifold is closed. Hence we may assume that $n = \dim_{\mathbf{C}}(X) = p + 1$, and the proof will be by induction on p.

The case $p = 0$ is trivial: $\Omega_X^0(\log V) = \mathcal{O}_X$ and a global holomorphic function is constant. Assume now that the lemma has been proved for $(p-1)$-forms.

If $\omega \in H^0(X, \Omega_X^p(\log V))$, then, as in [**No**], we may consider the current T_ω of bidegree $(p, 0)$ defined by

$$T_\omega(\phi) = \int_X \omega \wedge \phi,$$

for every smooth $(1, p+1)$-form ϕ. This is well defined, i.e. the integral is convergent, precisely because ω has logarithmic poles along V: the 2-form $\frac{1}{z} dz \wedge d\bar{z}$ is integrable on the disc. Similarly, we may associate to $d\omega$, which is still logarithmic, a current $T_{d\omega}$ of bidegree $(p+1, 0)$. We then have, in the sense of currents,

$$\partial T_\omega = T_{d\omega},$$

which follows from the fact that if ψ is any smooth $(0, p+1)$-form then $\int_X d(\omega \wedge \psi) = 0$ by Stokes theorem (here $\omega \wedge \psi$ is again a current, and the integral is its value on $d1 \equiv 0$). In particular, we have

$$\partial T_\omega \equiv 0 \Leftrightarrow d\omega = 0.$$

On the other hand, $\overline{\partial} T_\omega$ is *not* zero: a simple computation, based on $\overline{\partial}(\frac{dz}{z}) = 2\pi i \delta_0$ (where δ_0 is the Dirac distribution), shows that [**No**]

$$\overline{\partial} T_\omega = 2\pi i\, T_{\mathrm{Res}(\omega)},$$

where, with a negligible abuse of notation, we identify $T_{\mathrm{Res}(\omega)}$ (a current on $\hat{V}$ of bidegree $(p-1, 0)$) with its direct image in X (a current of bidegree $(p, 1)$).

By induction hypothesis, $\mathrm{Res}(\omega)$ is closed, i.e. $\partial T_{\mathrm{Res}(\omega)} \equiv 0$, and so $\overline{\partial}\partial T_\omega \equiv 0$. By regularity theory, the current ∂T_ω is in fact a *holomorphic* $(p+1)$-form, because it is of bidegree $(p+1, 0)$ and $\overline{\partial}$-closed.

By Stokes Theorem,

$$\int_X \partial T_\omega \wedge \overline{\partial T_\omega} = \int_X d(T_\omega \wedge \overline{\partial T_\omega}) = 0$$

($T_\omega \wedge \overline{\partial T_\omega}$ is a current, being $\overline{\partial T_\omega}$ smooth, and its differential is $\partial T_\omega \wedge \overline{\partial T_\omega}$ because $\overline{\partial}\partial T_\omega \equiv 0$). This forces ∂T_ω to be identically zero, because $\partial T_\omega \wedge \overline{\partial T_\omega} \geq 0$. $\square$

Remark. We have used the projectivity of X only to reduce the problem to $\dim_{\mathbf{C}}(X) = p + 1$. In other words: any logarithmic p-form on any compact complex manifold of dimension $p + 1$ is closed.

In the context of manifolds with $\mathrm{Pic}(X) = \mathbf{Z}$, we shall use the following well-known fact:

Lemma 2.2. *Let X be a complex projective manifold with $\mathrm{Pic}(X) = \mathbf{Z}$ and $V \subset X$ a normal crossing hypersurface. Let ω be a global logarithmic p-form with poles along V and $\mathrm{Res}(\omega) \not\equiv 0$, $1 \leq p \leq n-1$. Then V is not smooth.*

Proof: By contradiction, assume that V is smooth, so that $\eta = \mathrm{Res}(\omega) \not\equiv 0$ is a holomorphic $(p-1)$-form on V. The line bundle $\mathcal{O}_X(V)$ is ample, so that by Kodaira Vanishing Theorem $H^p(X, \mathcal{O}_X(-V)) = 0$, because $p < n$. Hence the restriction map $H^{p-1}(X, \mathcal{O}_X) \to H^{p-1}(V, \mathcal{O}_V)$ is surjective.

The conjugate form $\overline{\eta}$ is a $(0, p-1)$-form on V which is $\overline{\partial}$-closed, because η is ∂-closed. Hence $\overline{\eta}$ defines a class in $H^{p-1}(V, \mathcal{O}_V)$ (à la Dolbeault), which arises from $H^{p-1}(X, \mathcal{O}_X)$, whence it follows that there exists a $\overline{\partial}$-closed $(0, p-1)$-form β on X whose restriction to V is cohomologous to $\overline{\eta}$, that is equal to $\overline{\eta} + \overline{\partial}\gamma$ for some $(0, p-2)$-form γ on V. After extending γ to X and replacing β by $\beta - \overline{\partial}\gamma$, we may and will suppose that β coincides with $\overline{\eta}$ on V.

Let now θ be a Kähler form on X. By considering η as a current T_η of bidegree $(p, 1)$ on X, we may evaluate it on the $(n-p, n-1)$-form $\theta^{n-p} \wedge \beta$:

$$T_\eta(\theta^{n-p} \wedge \beta) = \int_V \eta \wedge \theta^{n-p} \wedge \beta = \int_V \eta \wedge \overline{\eta} \wedge \theta^{n-p}.$$

But $\theta^{n-p} \wedge \beta$ is $\overline{\partial}$-closed and $T_\eta = \frac{1}{2\pi i}\overline{\partial} T_\omega$ is $\overline{\partial}$-exact, so that the integral is zero. Contradiction, because $\eta \wedge \overline{\eta} \wedge \theta^{n-p} \geq 0$ and $\eta \wedge \overline{\eta} \wedge \theta^{n-p} \not\equiv 0$. $\square$

It will be useful to reformulate Lemma 2.1 in a more abstract form, due to Bogomolov [**Bo**], [**Re**]. To this end, we recall the definition of *Kodaira dimension* of a holomorphic line bundle L of a projective variety X, denoted $\kappa(X, L)$ [**Ii**]. Consider the ring

$$R(X, L) := \bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$$

and the homogeneous field of fractions $Q(X, L) := \{\frac{l_i}{l_j} \mid l_i, l_j \in H^0(X, L^{\otimes m}), m \geq 0\}$. Then we define $\kappa(X, L)$ as the transcendence degree of $Q(X, L)$, if $R(X, L) \neq \mathbf{C}$ or $\kappa(X, L) := -\infty$, if $R(X, L) = \mathbf{C}$. One has $\kappa(X, L) \leq n = \dim_{\mathbf{C}}(X)$, and $\kappa(X, L) = n$ if L is ample.

Lemma 2.3. [**Bo**] *Let X be a complex projective n-manifold, $V \subset X$ a normal crossing hypersurface and $L \in \mathrm{Pic}(X)$. If there exists a nontrivial global section σ of $\Omega_X^p(\log V) \otimes L$, then*

$$\kappa(X, L^{-1}) \leq p.$$

Proof: Suppose by contradiction that $\kappa(X, L^{-1}) \geq p + 1$, i.e. for some $m \geq 1$ there exist $p + 2$ global sections $l_0, \ldots, l_{p+1} \in H^0(X, L^{\otimes -m})$ such that the meromorphic functions on X given by $f_i := \frac{l_i}{l_0}$ are algebraically independent.

Let us suppose for a moment that $m = 1$. We can multiply σ by each l_i, obtaining global sections $\omega_i := l_i \sigma \in H^0(\Omega_X^p(\log V))$. Since $\omega_i = f_i \omega_0$, the closedness of each ω_i (Lemma 2.1) gives, for $i = 1, \ldots, p + 1$,

$$df_i \wedge \omega_0 \equiv 0.$$

The nontriviality of ω_0 implies $df_1 \wedge df_2 \wedge \ldots \wedge df_{p+1} \equiv 0$, contradicting the algebraic independence.

The case $m > 1$ is reducible to the case $m = 1$ by passing to a suitable m-fold ramified covering. We refer to [**Re**] for details. □

3. Bounding the degree of solutions

Let $\sigma \in H^0(X, \Omega_X^p \otimes N)$ be a Pfaff equation and $V \subset X$ a normal crossing hypersurface, which is a solution to σ.

We can look at σ as a global holomorphic section of $\Omega_X^p \otimes \mathcal{O}_X(-V) \otimes N \otimes \mathcal{O}_X(V)$, that is, a meromorphic section $\hat{\sigma}$ of $\Omega_X^p \otimes \mathcal{O}_X(-V) \otimes N$ with simple poles $(\hat{\sigma})_\infty \subset V$. Locally, i.e. after local trivialization of $\mathcal{O}_X(-V) \otimes N$, we can see $\hat{\sigma}$ as a meromorphic p-form ω with simple poles $(\omega)_\infty \subset V$ and we assert:

Lemma 3.1. *The meromorphic p-form ω is logarithmic.*

Proof: We have to check that that, if $\{f = 0\}$ is a local reduced equation of V, then $df \wedge \omega$ is holomorphic. From $i_V \sigma \equiv 0$ (Definition 1.2) it follows that $df \wedge \sigma$ is identically zero along V, i.e.

$$df \wedge \sigma = f \cdot \theta,$$

for some regular local section θ of $\Omega_X^{p+1} \otimes N$. Hence $df \wedge \frac{\sigma}{f}$ is a regular section of $\Omega_X^{p+1} \otimes N$ and $df \wedge \omega$ is holomorphic, because (up to local trivialization) $\omega = \frac{\sigma}{f}$. □

Remark that the converse of Lemma 3.1 is also true: if ω is a logarithmic p-form with polar set given by $\{f = 0\}$, then $V = \{f = 0\}$ is a solution to the Pfaff equation defined by $f\omega$. That's the reason for

which the use of logarithmic forms is particularly well adapted to the study of solutions to Pfaff equations.

The meaning of Lemma 3.1 is that σ is in fact a global holomorphic section of $\Omega_X^p(\log V) \otimes \mathcal{O}_X(-V) \otimes N$. Now, Bogomolov's Lemma (Lemma 2.3) gives:

$$\text{(3)} \qquad \kappa(X, \mathcal{O}_X(V) \otimes N^{-1}) \leq p < \dim_{\mathbf{C}}(X),$$

which is the "bound on $d(V)$" we were looking for.

Let us now specialize to the case $\text{Pic}(X) = \mathbf{Z}$. Then

$$\mathcal{O}_X(V) \otimes N^{-1} = lH,$$

where $l = d(V) - d(N) \in \mathbf{Z}$ and H is the positive generator of $\text{Pic}(X) = \mathbf{Z}$. Since $l > 0$ implies $\kappa(X, lH) = \dim_{\mathbf{C}}(X)$, we conclude from (3) that

$$d(V) - d(N) \leq 0$$

as desired. Moreover, $d(V) - d(N) = 0$ means $\mathcal{O}_X(V) \otimes N^{-1} = \mathcal{O}_X$ and therefore the Pfaff equation is globally defined by a (closed!) logarithmic p-form ω with $(\omega)_\infty = V$ and hence $\text{Res}(\omega) \not\equiv 0$. In this case, Lemma 2.2 says that V is not smooth.

In order to apply this result to foliations of $\mathbf{C}P^n$, we remark:

Lemma 3.2. *Let* $\sigma \in H^0(\mathbf{C}P^n, \Omega_{\mathbf{C}P^n}^p \otimes N)$ *be a Pfaff equation associated to a foliation* $\mathcal{F}$ *of* $\mathbf{C}P^n$ *with* $\text{codim}_{\mathbf{C}} \text{Sing}(\mathcal{F}) \geq 2$ *(and* $\text{codim}_{\mathbf{C}}(\sigma)_0 \geq 2$*). Then* $d(N) = d(\mathcal{F}) + p + 1$.

Proof: Take a generic linear subspace $\Pi \simeq \mathbf{C}P^p$ and consider the restriction of σ to Π, denoted $i_\Pi \sigma$, with hypersurface of zeros on Π denoted by $(i_\Pi \sigma)_0$. Observe that $i_\Pi \sigma$ is a global regular section of $K_{\mathbf{C}P^p} \otimes N|_\Pi$ vanishing on $(i_\Pi \sigma)_0$; hence

$$\mathcal{O}_\Pi((i_\Pi \sigma)_0) = K_{\mathbf{C}P^p} \otimes N|_\Pi,$$

where $K_{\mathbf{C}P^p}$ is the canonical line bundle, whose degree is $-(p+1)$. Since $(i_\Pi \sigma)_0$ is the tangency set between $\mathcal{F}$ and Π and $d(\mathcal{F})$ is defined as the degree of $(i_\Pi \sigma)_0$ in Π, we obtain $d(\mathcal{F}) = -(p+1) + d(N)$. $\qquad\square$

Returning to the general inequality (3), we stress that it gives informations whatever $\text{Pic}(X)$ is. Roughly speaking, it says that $\mathcal{O}_X(V)$ is "partially less positive" than N. First of all, let us observe that if $\kappa(X, \mathcal{O}_X(V) \otimes N^{-1}) = r \geq 0$ then (as the proof of Lemma 2.3 shows) on a suitable ramified covering of X the Pfaff equation will be defined by a global (and closed) logarithmic p-form and will have r algebraically independent first integrals. Next, when X is a surface we have the following

fact (probably, a similar statement holds in any dimension). Recall [**Dm**] that a divisor D is *nef* if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$.

Lemma 3.3. *Let M be a line bundle on a projective surface X with $\kappa(X, M) \leq 1$. Then there exists a non-trivial nef divisor D, with real coefficients, such that $M \cdot D \leq 0$.*

Proof: Let $NS_{\mathbf{R}}(X) \subset H^2(X, \mathbf{R})$ be the real Neron-Severi group of X, let $N_{\text{nef}} \subset NS_{\mathbf{R}}(X)$ be the nef cone (i.e., the closure of the ample cone), and let $N_{\text{psef}} \subset NS_{\mathbf{R}}(X)$ be the pseudoeffective cone (i.e., the closure of the effective cone). See for instance [**Dm**] for these notions. Then, by Kleiman criterion (in its dual form), a line bundle M belongs to the interior of N_{psef} if and only if $M \cdot D > 0$ for every $D \in N_{\text{nef}} \setminus \{0\}$. On the other hand, to say that M belongs to the interior of N_{psef} is the same as to say that $\kappa(X, M) = 2$, by [**Dm**, Prop. 6.6]. Whence the result. $\quad\square$

In our case, applied to $M = \mathcal{O}_X(V) \otimes N^{-1}$ this fact gives

$$V \cdot D \leq N \cdot D.$$

On the other hand, one finds in [**Br**] the inequality

$$V \cdot V \leq N \cdot V$$

(and V is not necessarily nef). The relation between these inequalities is not clear to us: one is a "global" statement about the line bundle $\mathcal{O}_X(V) \otimes N^{-1}$ over X, while the other is a "local" statement, i.e. about the restriction of $\mathcal{O}_X(V) \otimes N^{-1}$ to V.

We now discuss some possible extensions.

Given an analytic hypersurface $V \subset X$ whose singularities are worst than normal crossings, we can generalize the definition of logarithmic forms (Section 2) in two ways, which are *not* equivalent in general:

1) Given a meromorphic p-form ω with simple poles along $V = \{f = 0\}$, we say that ω is *weakly logarithmic* if $f d\omega$ (or equivalently $df \wedge \omega$) is holomorphic.
2) Given ω as in 1), we say that it is *strongly logarithmic* if, on a neighbourhood of any $x \in V$, ω belongs to the $\mathcal{O}_X$-module generated by holomorphic forms and the forms $\frac{df_1}{f_1}, \ldots, \frac{df_k}{f_k}$, where $V = \{f_1 \cdot f_2 \ldots f_k = 0\}$, with f_i reduced equations of local branches of V at x.

The previous Lemmata 2.1, 2.3 are still valid for *strongly* logarithmic forms, as was observed in [**No**]. There are at least two ways to see this:

a) one can still define a good residue map for strongly logarithmic forms, and this is sufficient for the proof of Lemma 2.1;

b) one can resolve the singularities of V, $\pi\colon (\tilde{X}, \tilde{V}) \to (X, V)$, and observe that the pull-back by π of a strongly logarithmic form is (strongly) logarithmic.

On the other hand, Lemma 3.1 is still valid but the conclusion is only that ω is *weakly* logarithmic, which is clear from the proof.

It is exactly the discrepancy between weak and strong logarithmicity which is responsible for the failure of the inequality $d(V) \leq d(N)$ or $d(V) \leq d(\mathcal{F}) + p + 1$. For example, the foliation in $\mathbf{C}P^2$ given by $2y dx - 5x dy = 0$ has a solution curve $V = \{x^2 = y^5\}$ and $d(V) > d(\mathcal{F}) + 2$. The meromorphic 1-form $\omega = \frac{1}{x^2 - y^5}(2y dx - 5x dy)$ has simple poles along V and defines $\mathcal{F}$. Observe that ω is weakly logarithmic, but not strongly logarithmic (and not closed). The reader may also verify that a sequence of blow-ups over 0 transforms ω to a meromorphic 1-form which is not logarithmic, in neither sense. Moreover, ω has a well-defined residue along $V \setminus \{0\}$, which is a holomorphic function on $V \setminus \{0\}$, but its extension to V is only meromorphic (it has a pole of order two at 0). In other words, we cannot reasonably define a residue of ω along V, because such a residue should be a holomorphic function on V (hence constant, if V is compact).

These problems are related to the GSV index studied in [**Br**]. Let X be a surface and $V \subset X$ be a curve, solution to a Pfaff equation σ, and let $\{f = 0\}$ be a reduced equation of V on a neighbourhood of $x \in V$. We work locally, hence we identify σ with a holomorphic 1-form. Also, we suppose for a moment that V is irreducible near x. We can find, on a neighbourhood of x, holomorphic functions g and h, prime to f, and a holomorphic 1-form β such that $g\sigma = h df + f\beta$, i.e.

$$\frac{\sigma}{f} = \frac{h}{g}\frac{df}{f} + \frac{\beta}{g}.$$

Again, $\frac{\sigma}{f}$ is weakly logarithmic but (possibly) not strongly logarithmic, for the (possible) presence of g. Outside $x \in V$ the 1-form $\frac{\sigma}{f}$ has residue $\frac{h}{g}|_{V \setminus \{x\}} \in \mathcal{O}_{V \setminus \{x\}}$. This residue has a meromorphic extension to V. Using the formula of the GSV index given in [**Br**], we see that $GSV(\mathcal{F}, V, x)$ is equal to the vanishing order of $\frac{h}{g}$ at x (which is negative if $\frac{h}{g}|_V$ has a pole at x). Therefore, if $GSV(\mathcal{F}, V, x) \geq 0$ (which happens, for instance, if x is a so-called *non-dicritical* singularity [**Br**]), then the residue of $\frac{\sigma}{f}$ along $V \setminus \{x\}$ has a holomorphic extension to V. If V has several irreducible local components $V_1, \ldots, V_k$ at x, one still finds that

if V is non-dicritical, then $\operatorname{Res}(\frac{\sigma}{f}) \in \mathcal{O}_{V\setminus\{x\}}$ has a holomorphic extension to $\bigoplus_{j=1}^{k} \mathcal{O}_{V_j}$ (each $\operatorname{Res}(\frac{\sigma}{f})|_{V_j\setminus\{x\}}$ extends holomorphically to V_j, but the vanishing order at x of $\operatorname{Res}(\frac{\sigma}{f})|_{V_j}$ may depend on j). This is also related to the following fact, contained in the proof of Proposition 6 of [**Br**]: if V is non-dicritical at x and $\pi\colon (\tilde{X},\tilde{V}) \to (X,V)$ is a resolution of V, then $\pi^*(\frac{\sigma}{f})$ is a (strongly) logarithmic 1-form with poles along $\tilde{V}$. Consequently, inequality (3) still holds, i.e. $\kappa(X, \mathcal{O}_X(V) \otimes N^{-1}) \leq 1$, for $\dim_{\mathbf{C}}(X) = 2$ and V a *non-dicritical separatrix* ([**Br**, p. 533]).

Returning to the higher dimensional situation, let us observe that our bound $\kappa(X, \mathcal{O}_X(V) \otimes N^{-1}) \leq p$ obviously holds if $V \subset X$ has the property that every weakly logarithmic p-form with poles along V is strongly logarithmic, by the previous considerations. This happens, for instance, if V has (besides normal crossings) isolated singularities and $p \leq n-2$, by Saito's generalization of the De Rham division lemma [**Sa**].

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