Publicacions Matemàtiques, Vol. **44** (2000), 491–501

SOLVABLE GROUPS WITH MANY *BFC*-SUBGROUPS

O. D. Artemovych

Abstract

We characterize the solvable groups without infinite properly ascending chains of non-BFC subgroups and prove that a non-BFC group with a descending chain whose factors are finite or abelian is a Černikov group or has an infinite properly descending chain of non-BFC subgroups.

0. Introduction

In a series of papers Belyaev-Sesekin [**2**], Belyaev [**3**], Bruno-Phillips [**4**], [**5**], Kuzucuoğlu-Phillips [**12**], Leinen-Puglisi [**13**], Asar [**1**], Leinen [**14**] have obtained the results on mimimal non-FC groups. In particular, in [**2**] are characterized the minimal non-BFC groups, i.e. the non-BFC groups in which every proper subgroup is BFC. Recall that a group G is called a BFC-group if there is a positive integer d such that no element of G has more than d conjugates. Due to the well known result of B. H. Neumann (see e.g. [**16**, Theorem 4.35]) the BFC-groups are precisely the groups with the finite commutator subgroups.

We say that a group G satisfies the minimal condition on non-BFC subgroups (for short Min-$\overline{BFC}$) if for every properly descending series $\{G_n \mid n \in \mathbb{N}\}$ of subgroups of G there exists a number $n_0 \in \mathbb{N}$ such that G_n is a BFC-group for every integer $n \geq n_0$ and a group G satisfies maximal condition on non-BFC subgroups (for short Max-$\overline{BFC}$) if there exists no infinite properly ascending series of non-BFC subgroups in G. Every minimal non-BFC group satisfies Min-$\overline{BFC}$ and Max-$\overline{BFC}$. S. Franciosi, F. de Giovanni and Ya. P. Sysak [**11**] have investigated the locally graded groups with the minimal condition on non-FC subgroups.

In this paper we characterize the solvable groups satisfying Max-$\overline{BFC}$ and Min-$\overline{BFC}$, respectively. Namely, we prove the two following theorems.

2000 *Mathematics Subject Classification.* 20E15, 20F16, 20F24.
Key words. BFC-group, minimal non-BFC group, maximal condition, minimal condition, solvable group.

Theorem 1. *A solvable group G satisfies* Max-$\overline{BFC}$ *if and only if it is of one of the following types:*

(i) *G is a BFC-group;*

(ii) *$G = BU$ is a finitely generated group, where B is a proper torsion normal subgroup of G, U its polycyclic subgroup and $B\langle x\rangle$ is either a BFC-subgroup or a finitely generated subgroup for every element x of U;*

(iii) *$G = DU$ is a locally nilpotent-by-finite group with the torsion commutator subgroup G', where D is a normal divisible abelian p-subgroup, U is a polycyclic subgroup, and if $\langle u\rangle$ acts non-trivially on D for an element u of U, then D is an indecomposable injective $\mathbb{Q}\langle u\rangle$-module and $A\langle u\rangle$ is a BFC-subgroup for every proper submodule A of a $\mathbb{Z}\langle u\rangle$-module D with the action induced by the conjugation of u on D.*

Theorem 2. *Let the group G have a descending series whose factors are finite or abelian. If G satisfies the minimal condition on non-BFC subgroups, then it is a BFC-group or a Černikov group.*

Throughout this paper p is a prime. For a group G, $Z(G)$ will always denote the centre of G, $G', G'', \dots, G^{(n)}$ the terms of derived series of G, $\tau(G)$ the set of all torsion elements of G, $G^p = \langle g^p \mid g \in G\rangle$. In the sequel we will use the following notation:

$\mathbb{Q}$ the rational number field;

$\mathbb{F}_p$ the finite field with p elements;

$\mathbb{Q}_p$ the additive group of all rational numbers whose denominators are p-numbers;

$\mathbb{Z}$ the additive group of all rational integers;

$\mathbb{C}_{p^\infty}$ the quasicyclic p-group;

$R\langle x\rangle$ the group ring of a cyclic group $\langle x\rangle$ over a commutative ring R.

We will also use other standard terminology from [**10**] and [**16**].

1. Solvable groups with Max-$\overline{BFC}$

In this section we study the solvable groups with the maximal condition on non-BFC subgroups.

Lemma 1.1. *Let G be a group satisfying* Max-$\overline{BFC}$ *and H its subgroup. Then:*

(i) H satisfies Max-$\overline{BFC}$*;*
(ii) if H is normal in G, then the quotient group G/H satisfies Max-$\overline{BFC}$*;*
(iii) if H is a normal non-BFC subgroup of G, then G/H satisfies the maximal condition on subgroups.

Proof: Is immediate. □

Lemma 1.2. *Let G be a group which satisfies* Max-$\overline{BFC}$*. If G contains a normal abelian subgroup N with the quasicyclic quotient group G/N, then G is a nilpotent group.*

Proof: We prove this lemma by the same arguments as in the proof of Lemma 2.3 from [**2**]. Since G/N is a quasicyclic p-group for some prime p,

$$G/N = \bigcup\nolimits_{n=1}^{\infty} \langle \overline{a}_n \rangle,$$

where ${\overline{a}_n}^p = \overline{a}_{n-1}$, $\overline{a}_0 = N$. Put $A_n = \langle N, a_n \rangle$. Then $A_n \lhd G$, ${A_n}' \lhd G$ and by Lemma 1.1(iii) A_n is a BFC-subgroup. Hence ${A_n}' \leq Z(G)$ and consequently

$$G' = \bigcup\nolimits_{n=1}^{\infty} {A_n}' \leq Z(G),$$

as desired. □

Lemma 1.3. *If G is a Černikov group with* Max-$\overline{BFC}$*, then it is a BFC-group or the quotient group G/G' is finite.*

Proof: Assume that the quotient group $\overline{G} = G/G'$ is infinite and G is not a BFC-group. Then by Theorem 21.3 of [**10**] $\overline{G} = \overline{D} \times \overline{F}$ is a direct product of the non-trivial divisible part $\overline{D}$ and a reducible subgroup $\overline{F}$. Let D and F be the inverse images of $\overline{D}$ and $\overline{F}$ in G, respectively. By Corollary 2.2 of [**2**] $G' = D'F'$. Since G is not a BFC-group, $\overline{F}$ is a finite group. It is clear that $\overline{D} \cong \mathbb{C}_{p^\infty}$ for some prime p and G has a normal BFC-subgroup N with $G/N \cong \mathbb{C}_{p^\infty}$. By Theorem 1.16 of [**7**] $G = NZ(G)$ and so $G' = N'$, a contradiction with our assumption. The lemma is proved. □

Proposition 1.4. *If a group G satisfies* Max-$\overline{BFC}$*, then it is a BFC-group or the quotient group G/G' is finitely generated.*

Proof: As it is well known $\overline{G} = G/G' = \overline{D} \times \overline{S}$ is a direct product of the divisible part $\overline{D} = D/G'$ and a reducible subgroup $\overline{S} = S/G'$.

(1) First, let $\overline{D}$ be a non-trivial subgroup. Then S and G' are the BFC-subgroups. It is clear that G is a BFC-group or $\overline{D}$ is a quasicyclic group. We suppose that $\overline{D} \cong \mathbb{C}_{p^\infty}$. Let $\overline{F} = F/G'$ be a p-basic subgroup of $\overline{S}$. If $\overline{F} \neq \overline{S}$, then $\overline{G}/\overline{F}$ is a direct product of a quasicyclic p-subgroup and an infinite p-divisible abelian subgroup. By Lemma 2.2 of [**2**] and Lemma 1.1 G is a BFC-group.

Assume that $\overline{F} = \overline{S}$. Then by Lemma 26.1 and Proposition 27.1 from [**10**] $\overline{G}/\overline{F}^p = D^* \times F^*$ is a direct product of a quasicyclic p-subgroup D^* and a p-subgroup F^* of exponent p. Lemma 2.2 of [**2**] implies $G' = D'S'$. If $\overline{F}$ is not a finitely generated subgroup, then in view of Lemma 1.1 D and G are the BFC-groups. Therefore we assume that $\overline{F}$ is a finitely generated subgroup. Since F is a BFC-subgroup, $|G' : D'| < \infty$. By Lemma 1.2 D/G'' is a nilpotent group and so D/D' is a Černikov group. This yields that D is a Černikov group. By Lemmas 1.1 and 1.3 D is a BFC-group and as a consequence G is the ones.

(2) Now let the divisible part $\overline{D}$ is trivial. If $\overline{F} = \overline{S}$, then the quotient group G/G' is finitely generated or $\overline{G}/\overline{F}^p$ is a direct product of infinitely many cyclic subgroups of order p in which case G is a BFC-group. Therefore we assume that $\overline{F} \neq \overline{S}$. If $\overline{F}$ is not finitely generated, in the same manner as above we can prove that G is a BFC-group.

Let $\overline{F}$ be a finitely generated subgroup.

(a) Assume that the quotient group $G_1 = \overline{G}/\overline{F}$ is non-torsion. Then there exists a subgroup $\overline{F}_0$ such that $\overline{F} \leq \overline{F}_0 \leq \overline{G}$ and $\overline{G}/\overline{F}_0$ is torsion-free. As noted in [**6**] (see also [**7**, Chapter 2, §6]) $\overline{G}/\overline{F}_0$ contains a subgroup $\overline{T}/\overline{F}_0$ isomorphic to $\mathbb{Q}_p$. If $\overline{Z}/\overline{F}_0$ is a subgroup of $\overline{T}/\overline{F}_0$ isomorphic to $\mathbb{Z}$, then $\overline{T}/\overline{Z}$ is a quasicyclic p-group, and it follows that G has a normal BFC-subgroup X with $G/X \cong \mathbb{C}_{p^\infty}$. By Lemma 1.2 $G_0 = G/X'F^p$ is a nilpotent group and so by Lemma 26.1 and Proposition 27.1 from [**10**] $G_0/G_0' = F_1 \times K_1$ is a direct product of a finite p-subgroup F_1 and an infinite p-divisible abelian subgroup K_1. Let K_0 be an inverse image of K_1 in G_0. From what is proved above it follows that K_0 has a normal subgroup K^* with $K_0/K^* \cong \mathbb{C}_{p^\infty}$. If $K^* \neq (K^*)^p$, then Theorem 1.16 of [**7**] yields that $K_0/(K^*)^p = \overline{X} \times \overline{Y}$ is a direct product of a quasicyclic p-subgroup $\overline{X}$ and some divisible p-subgroup $\overline{Y}$. Since $\overline{Y}$ is a non-trivial subgroup, it is infinite. Consequently G is a BFC-group. Therefore we suppose that $K^* = (K^*)^p$. As above we can prove that K^* contains a G-invariant subgroup L with $K^*/L \cong \mathbb{C}_{p^\infty}$. Hence $K_0/L \cong \mathbb{C}_{p^\infty} \times \mathbb{C}_{p^\infty}$ and so G is a BFC-group.

(b) Let $G_1 = \overline{G}/\overline{F}$ be an infinite torsion p'-group. Then without loss of generality we can assume that G_1 is an infinite q-group for some prime q different from p. By B we denote a basic subgroup of G_1. If $B = G_1$, then the quotient group G/G' is finitely generated or B is an infinitely generated subgroup in which case G is a BFC-group.

Let $B \neq G_1$. If B is not a finitely generated subgroup, then Lemma 26.1 and Proposition 27.1 of [**10**] give that $G_1/B^q \cong \overline{B} \times \mathbb{C}_{q^\infty}$, where $\overline{B}$ is an infinite abelian q-subgroup of exponent q, and this yields that G is a BFC-group. Therefore we assume that B is a finitely generated subgroup. Then without loss of generality let $B = 1$ and $G_1 \cong \mathbb{C}_{q^\infty}$. We would like to prove that the commutator subgroup G' is torsion. Since the subgroup G'' is finite, without restricting of generality let $G'' = 1$. But then $\hat{F} = F/\tau(G')$ is an abelian subgroup of $\hat{G} = G/\tau(G')$ and from $G_1 \cong \hat{G}/\hat{F}$ it follows that $\hat{G}$ is an abelian group. This means that G' is a torsion subgroup. By Lemma 1.2 G/F' is a nilpotent group and it has the torsion commutator subgroup. So Corollary 3.3 of [**2**] yields that G/F' is a torsion group. Hence G is a torsion group and $\overline{G} \cong \mathbb{C}_{q^\infty} \times M$, where M is a finite subgroup, a contradiction with our assumption.

(c) Finally, if $G_1 = \overline{G}/\overline{F}$ is a torsion group and it has a non-trivial p-subgroup, then without loss of generality we can assume that G_1 is a quasicyclic p-group. As in the line (b) this gives that G is a BFC-group. The proposition is proved. □

Lemma 1.5. *Let $G = B\langle x\rangle$ be a product of a normal abelian torsion-free subgroup B and a cyclic subgroup $\langle x\rangle$. If G satisfies* Max-$\overline{BFC}$*, then it is either an abelian group or a polycyclic group.*

Proof: If F is any finitely generated subgroup of B, then $\langle F, x\rangle$ is a polycyclic subgroup in G and $\langle F, x\rangle = A\langle x\rangle$ for some G-invariant subgroup A of B. Assume that the quotient group G/A is not finitely genereted. Then $A\langle x\rangle$ is a BFC-subgroup in view of Lemma 1.1 and consequently it is abelian. Therefore a non-polycyclic group G is abelian, as desired. □

Lemma 1.6. *If G is a solvable group satisfying* Max-$\overline{BFC}$*, then one of the following conditions holds:*

(i) $G = BU$ is a finitely generated group, where B is a proper torsion normal subgroup of G, U its polycyclic subgroup and $B\langle x\rangle$ is either a BFC-subgroup or a finitely generated subgroup for every element x of U;

(ii) G is a BFC-group;

(iii) $G = DV$ is a product of a normal divisible abelian p-subgroup D and a polycyclic subgroup V.

Proof: Suppose that G is not a BFC-group. Let n be the derived length of G. Then there exists an integer k such that $G^{(k-1)}$ is not a BFC-group, but $G^{(k)}$ is a BFC-group, where $1 \leq k \leq n-1$ and $G^{(0)} = G$. Proposition 1.4 implies that $G^{(k-1)} = G^{(k)}U$ for some polycyclic subgroup U. By Lemma 1.5 $\overline{U} \lhd \overline{G}^{(k-1)}$, where $\overline{G}^{(k-1)} = G^{(k-1)}/\tau(G^{(k)}) = \overline{G}^{(k)}\overline{U}$, and so $\overline{U} = \overline{G}^{(k-1)}$. This means that $G^{(k-1)} = \tau(G^{(k)})U$. We denote $\tau(G^{(k)})$ by B.

(a) First we assume that G is not a finitely generated group. Clearly that there is an element u of U such that $H_1 = G^{(k)}\langle u\rangle$ is a non-BFC group. We would like to prove that $H = B\langle u\rangle$ is the ones. Indeed, if H is a BFC-group, then the quotient group $H_1/H'G^{(k+1)}$ is a nilpotent group and by Theorem 2.26 of [**10**] and Proposition 1.4 it is finitely generated. But then H_1 (and consequently G) is also a finitely generated group, a contradiction. Hence H is a non-BFC group.

(1) Assume that B is an abelian π-subgroup for some set π of primes. If $B = B_1 \times B_2$ is a direct product of an infinite π_1-subgroup B_1 and an infinite π_2-subgroup B_2, where π_1 and π_2 are the disjoint subsets of π such that $\pi = \pi_1 \cup \pi_2$, then it is not difficulty to prove that H is a BFC-group, a contradiction. Thus π is a finite set and $B = P \times S$, where P is an infinite p-subgroup for some prime $p \in \pi$ and S is a finite p'-subgroup. Moreover $P\langle u\rangle$ is a non-BFC group.

(2) If B is not necessary an abelian subgroup, then from the line (1) it follows that B/T is a divisible abelian p-group for some finite H-invariant subgroup T. By Theorem 1.16 of [**7**] there exists a divisible abelian p-subgroup D of B such that $D \leq Z(B)$ and $B = DT$. Thus $G = DV$, where V is a polycyclic subgroup.

(b) Now let G be a finitely generated group. Then $G = BU$ for some polycyclic subgroup U. Suppose that $B\langle x\rangle$ is not a BFC-group for some $x \in U$. If $B\langle x\rangle$ is not finitely generated, then, as in the line (1) and (2), we can prove that $B\langle x\rangle = D_1V_1$, where D_1 is a normal divisible p-subgroup, V_1 is a polycyclic subgroup and $D_1 \leq B$. By Theorem of [**2**] $B\langle x\rangle$ contains a proper non-BFC subgroup K. Since $\overline{D_1K} = D_1K/(D_1 \cap K) = \overline{D}_1 \rtimes \overline{K}$ and $\overline{D}_1$ is a non-trivial divisible p-subgroup, we conclude that D_1K (and consequently G) contains an infinite properly ascending series of type

$$K < K_1 < \cdots < K_n < \cdots ,$$

a contradiction. This means that $B\langle x\rangle$ is a finitely generated subgroup. The lemma is proved. $\square$

Example 1.7. If $G = A \rtimes \langle t\rangle$, where $\langle t\rangle$ is an infinite cyclic subgroup, $A \cong \mathbb{C}_{p^\infty}$ and $a^t = a^{1+p}$ $(a \in A)$, then G satisfies Max-$\overline{BFC}$.

If D is a commutative Dedekind domain, A right D-module, $\mathrm{Spec}(D)$ the set of non-trivial prime ideals of D and $P \in \mathrm{Spec}(D)$, then

$$A_P = \{a \in A \mid aP^n = \{0\} \text{ for some positive integer } n = n(a) \in \mathbb{N}\}$$

is said to be the P-component of A, and A is said to be a D-torsion module if

$$A = \{a \in A \mid Ann(a) \neq \{0\}\}.$$

Lemma 1.8. *Let $G = A \rtimes \langle x\rangle$ be a semidirect product of a normal abelian subgroup A of exponent p and an infinite cyclic subgroup $\langle x\rangle$. If G satisfies* Max-$\overline{BFC}$, *then it is either a finitely generated group or a BFC-group.*

Proof: It is clear that A is a right $\mathbb{F}_p\langle x\rangle$-module with the action determined by the conjugation of x on A. Assume that G is not neither a finitely generated group nor a BFC-group. Then A is a $\mathbb{F}_p\langle x\rangle$-torsion module and by Proposition 2.4 of [**8**, §8.2]

$$A = \sum\nolimits^{\oplus}_{P \in \mathrm{Spec}(\mathbb{F}_p\langle x\rangle)} A_P$$

is a module direct sum of its P-component A_P. Without loss of generality we can suppose that $|A : A_Q| < \infty$ for some $Q \in \mathrm{Spec}(\mathbb{F}_p\langle x\rangle)$. Let B be a basic submodule of A_Q. By our hypothesis $B = A_Q$. Since B can be written as a direct product of two infinite G-invariant subgroup of infinite index, we obtain that $B \rtimes \langle x\rangle$ (and consequently G) is a BFC-group, a contradiction. The lemma is proved. $\square$

Proposition 1.9. *If G is a non-"finitely generated" non-BFC solvable group satisfying* Max-$\overline{BFC}$, *then:*

(1) G is a locally nilpotent-by-finite group;

(2) $G = BU$ is a product of a normal divisible abelian p-subgroup B and a polycyclic subgroup U;

(3) $B\langle u\rangle$ is a BFC-subgroup for an element $u \in U$ if and only if $u \in C_U(B)$;

(4) if $B\langle u\rangle$ is a non-BFC subgroup for some element $u \in U$, then $[B, \langle u\rangle] = B$;

(5) if $B\langle u\rangle$ is a non-BFC subgroup for some element $u \in U$, then B is an indecomposable injective $\mathbb{Q}\langle u\rangle$-module;

(6) if $B\langle u\rangle$ is a non-BFC subgroup for some $u \in U$, then $A\langle u\rangle$ is a BFC-subgroup for every proper $\mathbb{Z}\langle u\rangle$-submodule A of B, where the action is induced by the conjugation of u on B;

(7) G contains a normal subgroup H of finite index in which every non-BFC subgroup is subnormal;

(8) G' is a torsion subgroup of G.

Proof: (1) Is obvious.

(2) Follows from Lemma 1.6.

(3) Assume that $H = B\langle u\rangle$ is a BFC-subgroup for some element $u \in U$. If u has a finite order, then $H/H'\langle u\rangle$ is a divisible group and by Theorem 1.16 of [**7**] $H = Z(H)H'\langle u\rangle$. Consequently $Z(H)$ is a subgroup of finite index in H and H is an abelian group.

Let u be an element of infinite order. Since the subgroup $H'\langle u^s\rangle$ and the quotient group $B\langle u^s\rangle/(H'\langle u^s\rangle)'$ are nilpotent for some integer s, $B\langle u^s\rangle$ is a nilpotent group by Hall theorem [**16**, Theorem 2.27]. But then $B\langle u^s\rangle$ is an abelian group and therefore as proved above $H/(Z(H) \cap \langle u\rangle)$ is abelian. This yields that H is an abelian group.

(4) If $B\langle u\rangle$ is a non-BFC subgroup for some element u of U and $[B, \langle u\rangle] \neq B$, then $T = [B, \langle u\rangle]\langle u\rangle$ is a BFC-subgroup. Since $B\langle u\rangle/T'$ is a nilpotent group, it is abelian, a contradiction.

(5) It is clear that B is a right $\mathbb{Q}\langle u\rangle$-module with the action induced by the conjugation of u on B. Furthermore, B is a divisible $\mathbb{Q}\langle u\rangle$-module and therefore it is injective (see e.g. [**11**, Theorem 5.28]). By Theorem 2.5 of [**15**] B has a decomposition as a module direct sum of indecomposable injective $\mathbb{Q}\langle u\rangle$-submodules. Since $B\langle u\rangle$ satisfies Max-$\overline{BFC}$, B is an indecomposable module.

(6) Let $B\langle u\rangle$ be a non-BFC group and A a proper submodule of a right $\mathbb{Z}\langle u\rangle$-module B, where the action is induced by the conjugation of u on B. By F we denote a basic subgroup of A. If $A = F$, then $A\langle u\rangle$ is either a polycyclic group or a BFC-group in view of Lemmas 1.8 and 1.5. Therefore we assume that $F \neq A$. Since B is an indecomposable $\mathbb{Q}\langle u\rangle$-module, we conclude that F is an infinite group. But then A/A^p is also infinite and so $A\langle u\rangle/A^p$ is a BFC-group by Lemma 1.8. This yields that $A\langle u\rangle$ is the ones.

(7) If V is a nilpotent subgroup of finite index in U and K is any non-BFC subgroup of DV, then $D \leq K$. Hence K is a subnormal subgroup of DV.

(8) Is obvious. The proposition is proved. □

Proof of Theorem 1: ($\Rightarrow$) Follows from Proposition 1.9.

($\Leftarrow$) Suppose that K is a non-BFC subgroup of a non-BFC group G.

Let G be a group of type (ii) and $\overline{BK} = BK/B'(B \cap K) = \overline{B} \rtimes \overline{K}$. Since BK is a finitely generated subgroup, $\overline{S} = (\overline{B} \cap \overline{S}) \rtimes \overline{K}$, where $\overline{S}$ is a subgroup of $\overline{BK}$ which contains $\overline{K}$, and $\overline{BK}$ satisfies the maximal condition on normal subgroups by Theorem 5.34 of [**10**], we conclude that every properly ascending series of type $\overline{K} < \overline{K}_1 < \cdots < \overline{K}_n < \cdots$ is finite. This means that BK (and consequently G) satisfies Max-$\overline{BFC}$.

If G is a group of type (iii), then it is clear that $K = (K \cap D)F$, where $F = \langle u_1, \ldots, u_t \rangle$ is some finitely generated subgroup. Assume that $K_i = (K \cap D)\langle u_i \rangle$ has the finite commutator subgroup ${K_i}'$ for all i $(1 \leq i \leq t)$. Since the subgroup $\langle {K_1}', \ldots, {K_t}', F \rangle$ is a finitely generated and $\langle {K_1}', \ldots, {K_t}', F \rangle = K_0 F$ for some finite F-invariant subgroup $K_0 \leq K \cap D$, $(K/K_0)' = (FK_0/K_0)'$ is a finite subgroup and therefore K is a BFC-subgroup, a contradiction. Hence $(K \cap D)\langle u \rangle$ is non-BFC subgroup for some $u \in F$ and by our hypothesis $D = K \cap D \leq K$. The theorem is proved. □

Corollary 1.10. *A solvable group G satisfies* Max-$\overline{BFC}$ *if and only if it is of one of the following types:*

(i) G is a BFC-group;

(ii) $G = BU$ is a finitely generated group, where B is a proper torsion normal subgroup of G, U its polycyclic subgroup and $B\langle x \rangle$ is either a BFC-subgroup or a finitely generated subgroup for every element x of U;

(iii) $G = DU$ is a product of a normal divisible abelian p-subgroup D and a polycyclic subgroup U with $D \leq \bigcap\{H \mid H$ is a non-BFC subgroup of $G\}$.

2. Groups with Min-$\overline{BFC}$

In this section we prove that a group which have a descending series with abelian or finite factors and satisfying Min-$\overline{BFC}$ is either a BFC-group or a Černikov group.

Lemma 2.1. *If G is a non-perfect group in which every proper normal subgroup is a BFC-subgroup, then G is a BFC-group or $G = G'\langle x \rangle$, where $x^{p^n} \in G'$ for some prime p and some positive integer n.*

Proof: By Theorem 21.3 of [**10**] $\overline{G} = G/G' = \overline{D} \times \overline{S}$ is a direct product of the divisible part $\overline{D}$ and a reducible subgroup $\overline{S}$. Let $\overline{B}$ be a p-basic subgroup of $\overline{S}$. If $\overline{B}$ is not a finitely generated subgroup, then by Lemma 26.1 and Proposition 27.1 of [**10**] $\overline{S}/\overline{B}^p = B_1 \times S_1$ is a direct product of an infinite abelian subgroup B_1 of exponent p and a p-divisible subgroup S_1. By Corollary 2.1 of [**2**] G is a BFC-group.

Let $\overline{B}$ be a finitely generated subgroup. If $\overline{D}$ is a non-trivial subgroup or $\overline{B} = \overline{S}$, then Lemma 26.1, Proposition 27.1 of [**10**] and Lemma 2.2 of [**2**] yield that G is a BFC-group. Finally, from $\overline{B} = \overline{S}$ and $\overline{D} = \overline{1}$ in view of Corollary 2.1 of [**2**] it follows that G is a BFC-group or G/G' is a cyclic p-group for some prime p, as desired. □

Proof of Theorem 2: Assume that G is neither a BFC-group nor a Černikov group. Since G satisfies Min-$\overline{BFC}$, we may invoke [**9**, Theorem 2.2] and obtain in this way that G is an FC-group. Choose

$$G = G_0 \geq G_1 \geq \cdots \geq G_n$$

such that every G_i is not a BFC-subgroup ($i = 0, \ldots, k-1$), while every proper normal subgroup of G_n is a BFC-group. Since G has a descending series whose factors are finite or abelian, there exists a normal subgroup N in G_n such that G_n/N is finite or abelian. From Lemma 2.1, the quotient G_n/N is finite in both cases. Hence there exists a finite subset F of G_n such that $G_n = NF$, and every element in G_n is of the form hf for suitable $h \in N$, $f \in F$. However, since G is an FC-group, every $f \in F$ has just finitely many conjugates in G_n. And for $h \in N$ the number of conjugates of h in G_n is bounded by $|G_n : N||N : C_N(h)|$ (note that N is a BFC-group). Hence G_n itself becomes a BFC-group. This contradiction shows that G must be a BFC-group or a Černikov group. □

Acknowledgements. I am grateful to the referee whose remarks helped me to improve the exposition of this paper.

References

[1] A. O. Asar, Barely transitive locally nilpotent p-groups, *J. London Math. Soc. (2)* **55(2)** (1997), 357–362.

[2] V. V. Belyaev and N. F. Sesekin, Infinite groups of Miller-Moreno type, *Acta Math. Acad. Sci. Hungar.* **26(3-4)** (1975), 369–376.

[3] V. V. Belyaev, Groups of Miller-Moreno type, *Siberian Math. J.* **19** (1978), 350–360.

[4] B. Bruno and R. E. Phillips, Minimal non-FC groups, *Abstracts Amer. Math. Soc.* **2** (1980), 565.

[5] B. BRUNO AND R. E. PHILLIPS, On minimal conditions related to Miller-Moreno type groups, *Rend. Sem. Mat. Univ. Padova* **69** (1983), 153–168.

[6] S. N. ČERNIKOV, Addendum to the paper "On the theory of complete groups.", *Mat. Sbornik N.S.* **22(64)** (1948), 455–456 (Russian).

[7] S. N. ČERNIKOV, *"Groups with the prescribed properties of systems of subgroups"*, Nauka, Moskow, 1984 (Russian).

[8] P. M. COHN, *"Free rings and their relations"*, London Mathematical Society Monographs **2**, Academic Press, London, 1971.

[9] S. FRANCIOSI, F. DE GIOVANNI AND Y. P. SYSAK, Groups with many FC-subgroups, *J. Algebra* **218(1)** (1999), 165–182.

[10] L. FUCHS, *"Infinite abelian groups. Vol. II"*, Pure and Applied Mathematics **36-II**, Academic Press, New York, 1973.

[11] L. A. KURDACHENKO AND I. Y. SUBBOTIN, Modules over Dedekind domains, National University of Los Angeles, 1996.

[12] M. KUZUCUOĞLU AND R. E. PHILLIPS, Locally finite minimal non-FC-groups, *Math. Proc. Cambridge Philos. Soc.* **105(3)** (1989), 417–420.

[13] F. LEINEN AND O. PUGLISI, Unipotent finitary linear groups, *J. London Math. Soc. (2)* **48(1)** (1993), 59–76.

[14] F. LEINEN, A reduction theorem for perfect locally finite minimal non-FC groups, *Glasg. Math. J.* **41(1)** (1999), 81–83.

[15] E. MATLIS, Injective modules over Noetherian rings, *Pacific J. Math.* **8** (1958), 511–528.

[16] D. J. S. ROBINSON, *"Finiteness conditions and generalized soluble groups. Part 1"*, Ergebnisse der Mathematik und ihrer Grenzgebiete **62**, Springer-Verlag, New York, 1972.

Department of Algebra and Topology
Faculty of Mechanics and Mathematics
Ivan Franko National University of Lviv
University St 1
79000 Lviv
Ukraine
E-mail address: `topos@prima.franko.lviv.ua`

Primera versió rebuda el 18 de novembre de 1999,
darrera versió rebuda el 17 de maig de 2000.