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ON A RESULT OF PEETRE ABOUT INTERPOLATION OF OPERATOR SPACES

FERNANDO COBOS* AND TERESA SIGNES**

*Dedicated to Professor Jaak Peetre with affection
and deep admiration for his useful and creative work
during the last 40 years*

Abstract

We establish interpolation formulæ for operator spaces that are components of a given quasi-normed operator ideal. Sometimes we assume that one of the couples involved is quasi-linearizable, some other times we assume injectivity or surjectivity in the ideal. We also show the necessity of these suppositions.

1. Introduction

Let $\overline{B} = (B_0, B_1)$ be a Banach couple. The Peetre K -functional is defined by

$$K(t, b) = \inf\{\|b_0\|_{B_0} + t\|b_1\|_{B_1} : b = b_0 + b_1, b_i \in B_i\}$$

where $t > 0$ and $b \in B_0 + B_1$. This function plays a major role in Interpolation Theory (see e.g. [15], [1], [21]); it can be computed explicitly in many concrete cases, and sometimes it coincides with well-known objects in Analysis. For example, in the case of the couple of Lebesgue spaces (L_1, L_∞) , one can prove that

$$K(t, f) = \int_0^t f^*(s) ds$$

where f^* is the non-increasing rearrangement of f on $(0, \infty)$. For the couple (C^0, C^1) , formed by the space C^0 of bounded uniformly continuous functions on the real line and the space C^1 of functions with

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derivatives in C^0 , it turns out that

$$K(t, f) \approx w(t, f),$$

that is to say, $K(t, f)$ is equivalent to the modulus of continuity $w(t, f)$ of f .

A Banach couple $\overline{B} = (B_0, B_1)$ is said to be *quasi-linearizable* if for any $b \in B_0 + B_1$ one can get in a linear way an optimum decomposition of b for the K -functional. We give the precise definition and examples in Section 2. This notion was introduced by Peetre in [16] and [17]. It is rather restrictive, but many interesting couples satisfy it (see [21]). Moreover, the idea of quasi-linearization is important for the computation of the K -functional in other special instances (see, for example, [12]).

Peetre proved in [16] that if \overline{B} is quasi-linearizable then for any Banach space A and any positive measurable function $f = f(t)$, the inclusion

$$(1.1) \quad \mathcal{L}(A, \overline{B}_{f, \infty; K}) \hookrightarrow (\mathcal{L}(A, B_0), \mathcal{L}(A, B_1))_{f, \infty; K}$$

is valid, with embedding constant depending only on \overline{B} . Here the space $\overline{B}_{f, \infty; K}$ is defined similarly to the real interpolation space $\overline{B}_{\theta, \infty}$ realized as a K -space, but replacing the function t^θ by a more general function $f(t)$ (see Section 2 for more details).

The interest of formula (1.1) is due to its being a converse inclusion to the one given by the interpolation property of the real method, namely

$$(\mathcal{L}(A, B_0), \mathcal{L}(A, B_1))_{\theta, 1} \hookrightarrow \mathcal{L}(A, \overline{B}_{\theta, 1}).$$

It turns out that quasi-linearizable couples \overline{B} can be characterized as those for which (1.1) holds (see [16]).

The aim of this paper is to continue the research on embedding (1.1) in several directions. After reviewing some basic notions in Section 2, we show in Section 3 that (1.1) remains valid if we replace the class of bounded linear operators by any quasi-normed operator ideal \mathcal{J} . It is also possible to substitute $\overline{B}_{f, \infty; K}$ by any intermediate space B with respect to \overline{B} . In other words, it holds

$$(1.2) \quad \mathcal{J}(A, B) \hookrightarrow (\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{\psi, \infty; K}.$$

Here $\psi = \psi(t)$ is a function associated in a natural way to B and \overline{B} . We study then the “dual” situation in which the couple \overline{B} is in the front, and we prove that

$$(1.3) \quad \mathcal{J}(B, A) \hookrightarrow (\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{\rho^*, \infty; K}$$

where $\rho = \rho(t)$ is another function naturally associated to B and \overline{B} and $\rho^*(t) = 1/\rho(t^{-1})$. Moreover, the validity of (1.3) implies the quasi-linearizability of \overline{B} . Other embeddings that arise interpolating by the J -method are established in Section 3 as well.

In Section 4 we combine the previous results with some arguments of bilinear type to identify some interpolation spaces between spaces of bounded operators, spaces of nuclear operators and spaces of tensor products. Previous results in this direction are due to Kouba [11] and to Ovchinnikov [13] and [14]. We also establish some interpolation formulæ for approximation numbers and some embeddings for operator ideals defined by approximation numbers.

Finally, in Section 5, we return to the J -formulæ, and we show that it is possible to dispense with the quasi-linearizability of \overline{B} provided the operator ideal \mathcal{J} is injective or surjective, and that these conditions on \mathcal{J} are essential for the results.

2. Preliminaries

Let $\overline{B} = (B_0, B_1)$ be a Banach couple, that is, B_0 and B_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. We equip $B_0 + B_1$ [respectively $B_0 \cap B_1$] with the norm $K(1, \cdot)$ [respectively $J(1, \cdot)$] where for $t > 0$

$$K(t, \cdot) = K(t, \cdot; B_0, B_1) \text{ and } J(t, \cdot) = J(t, \cdot; B_0, B_1)$$

are the Peetre functionals, defined by

$$K(t, b) = \inf\{\|b_0\|_{B_0} + t\|b_1\|_{B_1} : b = b_0 + b_1, b_i \in B_i\}$$

and

$$J(t, b) = \max\{\|b\|_{B_0}, t\|b\|_{B_1}\}.$$

A Banach couple $\overline{B} = (B_0, B_1)$ is said to be quasi-linearizable if there exist two families of operators $V_j(t) \in \mathcal{L}(B_0 + B_1, B_j)$, $j = 0, 1$, $0 < t < \infty$ and a constant k such that

$$(2.1) \quad V_0(t) + V_1(t) = I \text{ (identity mapping in } B_0 + B_1),$$

$$(2.2) \quad \|V_0(t)b\|_{B_0} \leq kt^j \|b\|_{B_j}, \quad b \in B_j \quad (j = 0, 1),$$

$$(2.3) \quad \|V_1(t)b\|_{B_1} \leq kt^{j-1} \|b\|_{B_j}, \quad b \in B_j \quad (j = 0, 1),$$

(see [16] or [17]).

For instance, the couples $(L_p(w_0), L_p(w_1))$ of weighted L_p -spaces, or $(L_p(\mathbb{R}^n), W_p(\mathbb{R}^n))$ satisfy this condition. More generally, if A is a Banach space and $D(\Lambda)$ is the domain of the infinitesimal generator

of a strongly continuous semi-group of operators in A , then the couple $(A, D(A))$ is quasi-linearizable. Another example is the couple of Besov spaces $(B_{p,q}^{s_0}(\mathbb{R}^n), B_{p,q}^{s_1}(\mathbb{R}^n))$. On the other hand, if $p_0 \neq p_1$, the couple (L_{p_0}, L_{p_1}) is not quasi-linearizable. See [16] and [21] for more details.

We say that a Banach space B is intermediate with respect to the couple $\overline{B} = (B_0, B_1)$ if $B_0 \cap B_1 \hookrightarrow B \hookrightarrow B_0 + B_1$, with continuous inclusions. The position of B with respect to \overline{B} can be described by the functions

$$(2.4) \quad \psi(t) = \psi(t, B; \overline{B}) = \sup\{K(t, b) : \|b\|_B = 1\}$$

and

$$(2.5) \quad \rho(t) = \rho(t, B; \overline{B}) = \inf\{J(t, b) : b \in B_0 \cap B_1, \|b\|_B = 1\}.$$

These functions have been introduced by Cobos, Cwikel and Matos in [2] and they are variants of functions studied by Dmitriev [5] and Pustylnik [20].

We have $0 < \rho(t), \psi(t) < \infty$ for all $t > 0$; the functions $\psi(t)$ and $\rho(t)$ being non-decreasing, while $\psi(t)/t, \rho(t)/t$ are non-increasing. As an example, let B be a rearrangement invariant space on a σ -finite measure space (Ω, μ) and let φ_B be its fundamental function, i.e. $\varphi_B(t) = \|\chi_E\|_B$ where $E \subseteq \Omega$ with $\mu(E) = t$. Then B is an intermediate space with respect to the couple (L_1, L_∞) and

$$\psi(t, B; L_1, L_\infty) = \rho(t, B; L_1, L_\infty) = t/\varphi_B(t) \quad (\text{see [20]}).$$

A class \mathcal{J} of bounded linear operators between Banach spaces is said to be an operator ideal if each component $\mathcal{J} \cap \mathcal{L}(A, B) = \mathcal{J}(A, B)$ is a linear subspace of $\mathcal{L}(A, B)$ that contains the finite rank operators and satisfies that $STR \in \mathcal{J}(E, F)$ whenever $R \in \mathcal{L}(E, A), T \in \mathcal{J}(A, B)$ and $S \in \mathcal{L}(B, F)$.

A non-negative function $\tau: \mathcal{J} \rightarrow [0, \infty)$ is called a quasi-norm on \mathcal{J} if τ has the following properties:

$$(2.6) \quad \tau(h \otimes b) = \|h\|_{A^*} \|b\|_B \quad \text{for } h \in A^* \text{ and } b \in B.$$

$$(2.7) \quad \tau(S + T) \leq c(\tau(S) + \tau(T)) \quad \text{for } S, T \in \mathcal{J}(A, B),$$

where $c = c_\tau \geq 1$ is a constant.

$$(2.8) \quad \tau(STR) \leq \|S\|_{B, F} \tau(T) \|R\|_{E, A} \quad \text{for } R \in \mathcal{L}(E, A),$$

$T \in \mathcal{J}(A, B)$ and $S \in \mathcal{L}(B, F)$.

It follows from (2.8) that $\tau(\lambda T) = |\lambda|\tau(T)$, and (2.6) and (2.8) yield that $\|T\| \leq \tau(T)$ (see [18]). Sometimes we write $\tau(T) = \tau(T_{A,B})$ to emphasize that T is considered as acting from A into B .

A quasi-normed operator ideal is an operator ideal \mathcal{J} equipped with a quasi-norm τ so that all components $\mathcal{J}(A, B)$ are complete with respect to the induced metric. If $c_\tau = 1$, then $[\mathcal{J}, \tau]$ is said to be a normed operator ideal.

Clearly, the class of all bounded linear operators with the operator norm $[\mathcal{L}, \|\cdot\|]$ is an example of a normed operator ideal. Another example is the ideal of nuclear operators $[\mathcal{N}, \|\cdot\|_1]$. Recall that $T \in \mathcal{L}(A, B)$ is said to be nuclear if there are sequences $(h_n) \subset A^*$ and $(y_n) \subset B$ such that $Tx = \sum_{n=1}^\infty h_n(x)y_n$ and $\sum_{n=1}^\infty \|h_n\|_{A^*} \|y_n\|_B < \infty$. The norm on \mathcal{N} is given by

$$\|T\|_1 = \inf \left\{ \sum_{n=1}^\infty \|h_n\|_{A^*} \|y_n\|_B \right\}$$

where the infimum is taken over all representations $T = \sum_{n=1}^\infty h_n \otimes y_n$ as above. More examples will be given in Sections 4 and 5. We refer to [18] and [4] for other details on operator ideals.

In successive sections we shall interpolate operator spaces that are components of a given ideal, so they are quasi-Banach spaces. The concept of quasi-Banach couple $\overline{X} = (X_0, X_1)$ is analogous to the Banach case, but replacing Banach spaces by quasi-Banach spaces. The K - and J -functionals are defined in the same way. Now $X_0 \cap X_1$ and $X_0 + X_1$ are only quasi-Banach spaces.

If $\overline{X} = (X_0, X_1)$ is a quasi-Banach couple, $0 < \theta < 1$ and $0 < q \leq \infty$, the real interpolation space $\overline{X}_{\theta,q} = (X_0, X_1)_{\theta,q}$ consists of all $x \in X_0 + X_1$ which have a finite quasi-norm

$$\begin{aligned} \|x\|_{\theta,q} &= \left(\int_0^\infty (t^{-\theta} K(t,x))^q \frac{dt}{t} \right)^{1/q} && \text{if } 0 < q < \infty, \\ \|x\|_{\theta,q} &= \sup_{t>0} \{ t^{-\theta} K(t,x) \} && \text{if } q = \infty \end{aligned}$$

(see [1] or [21]).

For some results we shall need to replace in the former definition the function t^θ by a general positive function $g: (0, \infty) \rightarrow (0, \infty)$. The space $\overline{X}_{g,\infty;K} = (X_0, X_1)_{g,\infty;K}$ is formed of all $x \in X_0 + X_1$ which have a finite quasi-norm

$$\|x\|_{g,\infty;K} = \sup_{t>0} \left\{ \frac{K(t,x)}{g(t)} \right\}.$$

It is clear that $\overline{X}_{g,\infty;K} \hookrightarrow X_0 + X_1$, but it might happen that $\overline{X}_{g,\infty;K} = \{0\}$. Indeed, for any $x \in X_0 + X_1$, it follows from $\min\{1, t\}\|x\|_{X_0+X_1} \leq K(t, x)$ that

$$\sup_{t>0} \left\{ \frac{\min\{1, t\}}{g(t)} \right\} \|x\|_{X_0+X_1} \leq \|x\|_{g,\infty;K}.$$

Consequently, the space $\overline{X}_{g,\infty;K}$ reduces to $\{0\}$ if $\sup_{t>0} \left\{ \frac{\min\{1, t\}}{g(t)} \right\} = \infty$. Note that Peetre’s embedding (1.1) is trivial in this case. For this reason we shall assume in the following that the function g satisfies

$$(2.9) \quad \sup_{t>0} \left\{ \frac{\min\{1, t\}}{g(t)} \right\} < \infty.$$

This condition guarantees that $X_0 \cap X_1 \hookrightarrow \overline{X}_{g,\infty;K}$ because

$$\|x\|_{g,\infty;K} = \sup_{t>0} \left\{ \frac{K(t, x)}{g(t)} \right\} \leq \sup_{t>0} \left\{ \frac{\min\{1, t\}}{g(t)} \right\} \|x\|_{X_0 \cap X_1}.$$

So $(X_0, X_1)_{g,\infty;K}$ is an intermediate space with respect to \overline{X} . Note that the function ψ (see (2.4)) associated to $(X_0, X_1)_{g,\infty;K}$ and \overline{X} satisfies

$$(2.10) \quad \psi(t) = \psi(t, \overline{X}_{g,\infty;K}; \overline{X}) \leq g(t), \quad t > 0.$$

For $g(t) = t^\theta$ it holds $\overline{X}_{g,\infty;K} = \overline{X}_{\theta,\infty}$.

We shall also work with J -spaces. Let $f: (0, \infty) \rightarrow (0, \infty)$ be any positive function. Assume that the constant in the triangle inequality for X_j is c_j . Put $c = \max\{c_0, c_1\}$ and define p by the equation $(2c)^p = 2$. If $0 < q \leq p$, the space $\overline{X}_{f,q;J} = (X_0, X_1)_{f,q;J}$ consists of all sums $x = \sum_{m=-\infty}^\infty x_m$ (convergence in $X_0 + X_1$) where $(x_m) \subset X_0 \cap X_1$ and

$$\left(\sum_{m=-\infty}^\infty \left(\frac{J(2^m, x_m)}{f(2^m)} \right)^q \right)^{1/q} < \infty.$$

We put

$$\|x\|_{f,q;J} = \inf \left\{ \left(\sum_{m=-\infty}^\infty \left(\frac{J(2^m, x_m)}{f(2^m)} \right)^q \right)^{\frac{1}{q}} : x = \sum_{m=-\infty}^\infty x_m \right\}.$$

The difficulty now is that the functional $\|\cdot\|_{f,q;J}$ is, in general, only a semi-quasi-norm. Indeed, for any $x \in X_0 \cap X_1$ and any $m \in \mathbb{Z}$, we have

$$\|x\|_{f,q;J} \leq \frac{J(2^m, x)}{f(2^m)} \leq \frac{\max\{1, 2^m\}}{f(2^m)} \|x\|_{X_0 \cap X_1}.$$

Hence

$$\|x\|_{f,q;J} \leq \frac{1}{\sup_{m \in \mathbb{Z}} \left\{ f(2^m) \min\left\{1, \frac{1}{2^m}\right\} \right\}} \|x\|_{X_0 \cap X_1}.$$

In other words, if $\sup_{t>0} \{f(t) \min\{1, \frac{1}{t}\}\} = \infty$ then $\|x\|_{f,q;J} = 0$ for every $x \in X_0 \cap X_1$. To avoid this obstruction we shall assume in the following that f satisfies

$$(2.11) \quad \sup_{t>0} \left\{ f(t) \min\left\{1, \frac{1}{t}\right\} \right\} < \infty.$$

Then it is clear that $X_0 \cap X_1$ is densely and continuously embedded in $\overline{X}_{f,q;J}$. On the other hand, taking into account that the triangle inequality is valid in $X_0 + X_1$ with the constant c , if $x = \sum_{m=-\infty}^{\infty} x_m$ it follows from [1, Lemma 3.10.2], that

$$\begin{aligned} \|x\|_{X_0+X_1} &\leq 2^{1/p} \left(\sum_{m=-\infty}^{\infty} \|x_m\|_{X_0+X_1}^p \right)^{\frac{1}{p}} \\ &\leq 2^{1/p} \left(\sum_{m=-\infty}^{\infty} K(1, x_m)^q \right)^{\frac{1}{q}} \\ &\leq 2^{1/p} \left(\sum_{m=-\infty}^{\infty} \left(\min\left\{1, \frac{1}{2^m}\right\} J(2^m, x_m) \right)^q \right)^{\frac{1}{q}} \\ &\leq 2^{1/p} \sup_{m \in \mathbb{Z}} \left\{ f(2^m) \min\left\{1, \frac{1}{2^m}\right\} \right\} \left(\sum_{m=-\infty}^{\infty} \left(\frac{J(2^m, x_m)}{f(2^m)} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Since the supremum is finite by (2.11) we get that $\overline{X}_{f,q;J} \hookrightarrow X_0 + X_1$.

Let now $\rho(t) = \rho(t, \overline{X}_{f,q;J}; \overline{X})$ be the function ρ associated to the J -space $\overline{X}_{f,q;J}$ and the couple \overline{X} (see (2.5)). One can check easily that

$$(2.12) \quad f(2^m) \leq \rho(2^m), \quad m \in \mathbb{Z}.$$

Moreover, if f is non-decreasing, it follows that

$$(2.13) \quad f(t) \leq 2\rho(t), \quad t > 0.$$

Let us also point out that if $f(t) = t^\theta$ then the equivalence theorem (see [1] or [21]) yields that $(X_0, X_1)_{f,q;J} = (X_0, X_1)_{\theta,q}$ with equivalence of quasi-norms. The reason for the restriction on q in the definition of $\overline{X}_{f,q;J}$ is that we are working with very general functions f . If we impose

stronger conditions on f , then $\overline{X}_{f,q;J}$ makes sense for $0 < q \leq \infty$ and still is an intermediate space with respect to \overline{X} . The same happens with the K -spaces $\overline{X}_{g,q;K}$ (see [15]). We observe that $\overline{X}_{g,\infty;K}$ and $\overline{X}_{f,q;J}$ are complete.

Given any positive function f , we put $f^*(t) = 1/f(t^{-1})$. It turns out that f satisfies (2.9) (resp. (2.11)) if and only if f^* satisfies (2.11) (resp. (2.9)). If $f(t) = t^\theta$, then $f^*(t) = f(t) = t^\theta$.

Observe also that if B is an intermediate space with respect to the couple \overline{B} , then both functions $\psi(t, B; \overline{B})$ and $\rho(t, B; \overline{B})$ satisfy (2.9) and (2.11).

3. Operator spaces and quasi-linearizable couples

We start with an estimate for the norms of the operators $V_j(t)$ associated to a quasi-linearizable couple when they are considered as operators with B as domain or B as target space, where B is any intermediate space.

Lemma 3.1. *Let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple, let B be an intermediate space with respect to \overline{B} and let $\psi(t) = \psi(t, B; \overline{B})$ and $\rho(t) = \rho(t, B; \overline{B})$ be the functions associated to B and \overline{B} . Then*

- a) $\|V_j(t)\|_{B, B_j} \leq kt^{-j}\psi(t)$, $j = 0, 1$, $0 < t < \infty$;
- b) $\|V_0(t)\|_{B_1, B} \leq (1+k)\frac{t}{\rho(t)}$, $0 < t < \infty$,
- $\|V_1(t)\|_{B_0, B} \leq (1+k)\frac{1}{\rho(t)}$, $0 < t < \infty$.

Proof: Since $V_j(t) \in \mathcal{L}(B_0 + B_1, B_j)$ and $B \hookrightarrow B_0 + B_1$, it is clear that $V_j(t) \in \mathcal{L}(B, B_j)$. In order to estimate the norm, first note that by (2.2) and (2.3)

$$\|V_j(t)b\|_{B_j} \leq kt^{-j}K(t, b).$$

So, if $b \in B$ with $\|b\|_B = 1$, $\|V_j(t)b\|_{B_j} \leq kt^{-j}\psi(t)$.

We go on to establish b). Assume $j = 0$. By the definition of ρ and properties (2.1) to (2.3), for $b \in B_1$ we obtain

$$\begin{aligned} \|V_0(t)b\|_B &\leq \frac{J(t, V_0(t)b)}{\rho(t)} = \frac{1}{\rho(t)} \max\{\|V_0(t)b\|_{B_0}, t\|b - V_1(t)b\|_{B_1}\} \\ &\leq (1+k)t\frac{\|b\|_{B_1}}{\rho(t)}. \end{aligned}$$

The case $j = 1$ can be treated analogously. □

We can now extend the result by Peetre [16, Satz 3.1].

Theorem 3.2. *Let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple, let B be an intermediate space with respect to \overline{B} and let $\psi(t)$ be the ψ -function associated to B and \overline{B} . Assume $[\mathcal{J}, \tau]$ is a quasi-normed operator ideal and let A be any Banach space. Then*

$$\mathcal{J}(A, B) \hookrightarrow (\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{\psi, \infty; K}.$$

Moreover, there is a constant d depending only on \overline{B} such that

$$\|T\|_{\psi, \infty; K} \leq d\tau(T_{A, B}).$$

Proof: Given any $T \in \mathcal{J}(A, B)$ and any $t > 0$, we have by (2.1), $T = V_0(t)T + V_1(t)T$ with $V_j(t)T \in \mathcal{J}(A, B_j)$ due to the ideal property of \mathcal{J} . Whence, using Lemma 3.1/a, we derive that

$$\begin{aligned} \|T\|_{\psi, \infty; K} &= \sup_{t>0} \left\{ \frac{K(t, T)}{\psi(t)} \right\} \\ &\leq \sup_{t>0} \left\{ \frac{\tau([V_0(t)T]_{A, B_0}) + t\tau([V_1(t)T]_{A, B_1})}{\psi(t)} \right\} \\ &\leq \sup_{t>0} \left\{ \frac{\tau(T_{A, B})\|V_0(t)\|_{B, B_0} + t\tau(T_{A, B})\|V_1(t)\|_{B, B_1}}{\psi(t)} \right\} \\ &\leq 2k\tau(T_{A, B}). \quad \square \end{aligned}$$

Observe that $(\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))$ is a quasi-Banach couple because $\mathcal{J}(A, B_j) \hookrightarrow \mathcal{L}(A, B_0 + B_1)$, $j = 0, 1$. When the couple \overline{B} is in the front, we need to assume that \overline{B} is *regular*, i.e. $B_0 \cap B_1$ is dense in B_0 and in B_1 , in order to guarantee that $(\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))$ is a quasi-Banach couple. Indeed, under this extra supposition we have

$$\mathcal{J}(B_j, A) \hookrightarrow \mathcal{L}(B_j, A) \hookrightarrow \mathcal{L}(B_0 \cap B_1, A), \quad j = 0, 1.$$

The next result refers to this “dual” case when \overline{B} is in the front.

Theorem 3.3. *Let $\overline{B} = (B_0, B_1)$ be a regular quasi-linearizable couple, let B be an intermediate space with respect to \overline{B} with $B_0 \cap B_1$ dense in B , and let $\rho(t)$ be the ρ -function associated to B and \overline{B} . Assume $[\mathcal{J}, \tau]$ is a quasi-normed operator ideal and let A be any Banach space. Then*

$$\mathcal{J}(B, A) \hookrightarrow (\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{\rho^*, \infty; K},$$

where $\rho^*(t) = 1/\rho(t^{-1})$. Moreover, there is a constant d depending only on \overline{B} such that

$$\|T\|_{\rho^*, \infty; K} \leq d\tau(T_{B, A}).$$

Proof: Density of $B_0 \cap B_1$ in B implies that $\mathcal{J}(B, A)$ is also continuously embedded in $\mathcal{L}(B_0 \cap B_1, A)$, so we can compare the space $\mathcal{J}(B, A)$ with $(\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{\rho^*, \infty; K}$.

Given any $T \in \mathcal{J}(B, A)$, we can split T as $T = TV_1(t) + TV_0(t)$ with $TV_1(t) \in \mathcal{J}(B_0, A)$ and $TV_0(t) \in \mathcal{J}(B_1, A)$. So, according to Lemma 3.1/b, we get

$$\begin{aligned} \|T\|_{\rho^*, \infty; K} &= \sup_{t > 0} \{ \rho(t) K(t^{-1}, T) \} \\ &\leq \sup_{t > 0} \{ \rho(t) [\tau(T_{B,A}) \|V_1(t)\|_{B_0, B} + t^{-1} \tau(T_{B,A}) \|V_0(t)\|_{B_1, B}] \} \\ &\leq 2(1 + k) \tau(T_{B,A}). \quad \square \end{aligned}$$

Assume now that $g: (0, \infty) \rightarrow (0, \infty)$ is a function satisfying (2.9) and choose $B = (B_0, B_1)_{g, \infty; K}$. As we pointed out in (2.10), $\psi(t, B; \overline{B}) \leq g(t)$ for $t > 0$. Hence

$$(\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{\psi, \infty; K} \hookrightarrow (\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{g, \infty; K}.$$

This remark allows us to derive from Theorem 3.2 the following consequence:

Corollary 3.4. *Let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple and let g be a positive function satisfying (2.9). If $[\mathcal{J}, \tau]$ is any quasi-normed operator ideal and A is any Banach space, then*

$$\mathcal{J}(A, (B_0, B_1)_{g, \infty; K}) \hookrightarrow (\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{g, \infty; K}.$$

Moreover, there is a constant d depending only on \overline{B} such that

$$\|T\|_{g, \infty; K} \leq d \tau(T_{A, \overline{B}, g, \infty; K}).$$

On the other hand, Theorem 3.3 and (2.13) yield:

Corollary 3.5. *Let $\overline{B} = (B_0, B_1)$ be a regular quasi-linearizable couple and let f be a non-decreasing positive function satisfying (2.11). If $[\mathcal{J}, \tau]$ is any quasi-normed operator ideal and A is any Banach space, then*

$$\mathcal{J}((B_0, B_1)_{f, 1; J}, A) \hookrightarrow (\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{f^*, \infty; K}.$$

Moreover, there exists a constant d depending only on \overline{B} such that

$$\|T\|_{f^*, \infty; K} \leq d \tau(T_{\overline{B}, f, 1; J, A}).$$

Writing down Corollary 3.4 for the ideal $\mathcal{J} = \mathcal{L}$ of all bounded linear operators we recover Peetre’s result [16, Satz 3.1] (recall that, as we pointed out in Section 2, the embedding (1.1) is trivial if condition (2.9) is

not satisfied). Peetre proved in [16, Satz 3.2], that the validity of Corollary 3.4 for $\mathcal{J} = \mathcal{L}$ forces \overline{B} to be quasi-linearizable. Hence without this assumption Theorem 3.2 and Corollary 3.4 are not true in general. Next we show that the same happens with Theorem 3.3 and Corollary 3.5.

Theorem 3.6. *Let $\overline{B} = (B_0, B_1)$ be a regular Banach couple. If there is a constant d such that for any Banach space A and any non-decreasing positive function f satisfying (2.11) it holds*

$$\mathcal{L}((B_0, B_1)_{f,1;J}, A) \hookrightarrow (\mathcal{L}(B_0, A), \mathcal{L}(B_1, A))_{f^*, \infty; K}$$

with

$$\|T\|_{f^*, \infty; K} \leq d \|T\|_{\overline{B}_{f,1;J}, A},$$

then \overline{B} must be quasi-linearizable.

Proof: Fix $s > 0$ and put $f(t) = \min\{1, \frac{t}{s}\}$. Then f is non-decreasing, satisfies (2.11) and $\overline{B}_{f,1;J} = (B_0 \cap B_1, J(s, \cdot))$ with

$$(3.1) \quad J(s, b) \leq \|b\|_{f,1;J} \leq 2J(s, b).$$

Indeed, if $b = \sum_{m=-\infty}^{\infty} b_m$ is any J -representation of b , we have

$$\begin{aligned} J(s, b) &\leq \sum_{m=-\infty}^{\infty} J(s, b_m) \\ &\leq \sum_{m=-\infty}^{\infty} \max\{1, \frac{s}{2^m}\} J(2^m, b_m) = \sum_{m=-\infty}^{\infty} \frac{J(2^m, b_m)}{f(2^m)}. \end{aligned}$$

This gives the first inequality in (3.1). The second one follows from the choice

$$b_m = \begin{cases} 0, & \text{if } m \neq n \\ b, & \text{if } m = n, \end{cases}$$

where $2^n \leq s < 2^{n+1}$, because

$$\|b\|_{f,1;J} \leq \frac{J(2^n, b)}{f(2^n)} \leq \frac{s}{2^n} J(s, b) \leq 2J(s, b).$$

Take next $A = \overline{B}_{f,1;J}$ and $T = I_{\overline{B}_{f,1;J}}$ the identity mapping of $\overline{B}_{f,1;J}$. According to our assumption, there are operators $V_1(s) \in \mathcal{L}(B_0, \overline{B}_{f,1;J})$ and $V_0(s) \in \mathcal{L}(B_1, \overline{B}_{f,1;J})$ such that

$$V_0(s) + V_1(s) = T = I_{B_0 \cap B_1}$$

and

$$\begin{aligned} \|V_1(s)\|_{B_0, \overline{B}_{f,1;J}} + s^{-1}\|V_0(s)\|_{B_1, \overline{B}_{f,1;J}} &< 2K(s^{-1}, T) \leq 2 \sup_{t>0} \frac{K(t, T)}{f^*(t)} \\ &\leq 2d\|T\|_{\overline{B}_{f,1;J}, \overline{B}_{f,1;J}} = 2d. \end{aligned}$$

Therefore

$$\begin{aligned} \|V_0(s)b\|_{B_0} &\leq J(s, V_0(s)b) \leq 2ds\|b\|_{B_1}, \quad b \in B_1, \\ \|V_0(s)b\|_{B_1} &\leq s^{-1}J(s, V_0(s)b) \leq 2d\|b\|_{B_1}, \quad b \in B_1, \end{aligned}$$

and similarly

$$\begin{aligned} \|V_1(s)b\|_{B_1} &\leq 2ds^{-1}\|b\|_{B_0}, \quad b \in B_0, \\ \|V_1(s)b\|_{B_0} &\leq 2d\|b\|_{B_0}, \quad b \in B_0. \end{aligned}$$

Thus, if $b \in B_0 \cap B_1$, we obtain

$$(3.2) \quad \|V_0(s)b\|_{B_0} \leq \|b\|_{B_0} + \|V_1(s)b\|_{B_0} \leq (1 + 2d)\|b\|_{B_0}$$

and in the same way

$$(3.3) \quad \|V_1(s)b\|_{B_1} \leq (1 + 2d)\|b\|_{B_1}.$$

Taking into account that \overline{B} is regular, and so $B_0 \cap B_1$ is dense in $B_0 + B_1$, as well, it is not hard to check that the operators $V_j(s)$ can be extended boundedly to operators (denoted by the same letters) $V_j(s) \in \mathcal{L}(B_0 + B_1, B_j)$ such that (3.2) holds for any $b \in B_0$, (3.3) for any $b \in B_1$ and $V_0(s) + V_1(s) = I$ in $B_0 + B_1$. The proof is complete. \square

The remainder of this section is devoted to interpolation by the J -method. We denote by c_τ the constant in the quasi-triangle inequality (2.7).

Theorem 3.7. *Let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple, let B be an intermediate space with respect to \overline{B} and let $\rho(t)$ be the ρ -function associated to B and \overline{B} . Assume that A is any Banach space, that $[\mathcal{J}, \tau]$ is a quasi-normed operator ideal and let q be defined by the equation $(2c_\tau)^q = 2$. Then*

$$(\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{\rho, q; J} \hookrightarrow \mathcal{J}(A, B).$$

Proof: Let $T \in \mathcal{J}(A, B_0) \cap \mathcal{J}(A, B_1)$. The decomposition

$$T = V_1(t)T + V_0(t)T$$

yields that $T \in \mathcal{J}(A, B)$ with

$$\begin{aligned} \tau(T_{A,B}) &\leq c_\tau(\|V_1(t)\|_{B_0,B}\tau(T_{A,B_0}) + \|V_0(t)\|_{B_1,B}\tau(T_{A,B_1})) \\ &\leq c_\tau(1+k)\frac{1}{\rho(t)}(\tau(T_{A,B_0}) + t\tau(T_{A,B_1})) \\ &\leq 2c_\tau(1+k)\frac{1}{\rho(t)}J(t, T), \end{aligned}$$

where we have used again Lemma 3.1 to estimate $\|V_1(t)\|_{B_0,B}$ and $\|V_0(t)\|_{B_1,B}$. Suppose now that $T \in (\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{\rho,q;J}$ and let $T = \sum_{m=-\infty}^\infty T_m$ be any J -representation of T . Using [1, Lemma 3.10.2] and the previous estimates, we obtain that

$$\begin{aligned} \tau(T_{A,B}) &\leq 2^{\frac{1}{q}} \left(\sum_{m=-\infty}^\infty (\tau(T_m))^q \right)^{\frac{1}{q}} \\ &\leq 2^{\frac{1}{q}} 2c_\tau(1+k) \left(\sum_{m=-\infty}^\infty \left(\frac{J(2^m, T_m)}{\rho(2^m)} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Consequently, $T \in \mathcal{J}(A, B)$ with $\tau(T_{A,B}) \leq 2^{\frac{1}{q}} 2c_\tau(1+k)\|T\|_{\rho,q;J}$. □

Theorem 3.8. *Let $\overline{B} = (B_0, B_1)$ be a regular quasi-linearizable couple, let B be an intermediate space with respect to \overline{B} with $B_0 \cap B_1$ dense in B and let $\psi(t)$ be the ψ -function associated to B and \overline{B} . If A is any Banach space, $[\mathcal{J}, \tau]$ is a quasi-normed operator ideal and q is defined by $(2c_\tau)^q = 2$, then*

$$(\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{\psi^*,q;J} \hookrightarrow \mathcal{J}(B, A).$$

Proof: Let $T \in \mathcal{J}(B_0, A) \cap \mathcal{J}(B_1, A)$ and decompose it in the way $T = TV_0(t) + TV_1(t)$. It is clear that $T \in \mathcal{J}(B, A)$ with

$$\begin{aligned} \tau(T_{B,A}) &\leq c_\tau[\tau(T_{B_0,A})\|V_0(t)\|_{B,B_0} + \tau(T_{B_1,A})\|V_1(t)\|_{B,B_1}] \\ &\leq c_\tau k\psi(t) \left[\tau(T_{B_0,A}) + \frac{1}{t}\tau(T_{B_1,A}) \right] \leq 2c_\tau k \frac{J(t^{-1}, T)}{\psi^*(t^{-1})}. \end{aligned}$$

Now the result follows by combining this estimate with [1, Lemma 3.10.2] as we did in Theorem 3.7. □

The next result follows from Theorems 3.7 and 3.8 by using (2.10) and (2.12). We put $(B_0, B_1)_{g,\infty;K}^o$ for the closure of $B_0 \cap B_1$ in $(B_0, B_1)_{g,\infty;K}$.

Corollary 3.9. *Let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple and let A be a Banach space. Suppose that $[\mathcal{J}, \tau]$ is a quasi-normed operator ideal and defined q by $(2c_\tau)^q = 2$.*

a) *If f satisfies (2.11), then*

$$(\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{f,q;J} \hookrightarrow \mathcal{J}(A, (B_0, B_1)_{f,1;J}).$$

b) *If \overline{B} is regular and g satisfies (2.9), then*

$$(\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{g^*,q;J} \hookrightarrow \mathcal{J}((B_0, B_1)_{g,\infty;K}^o, A).$$

4. Some applications

We begin identifying some interpolation spaces.

Theorem 4.1. *Let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple and let A be a Banach space.*

a) *If g is positive and satisfies (2.9), then*

$$\mathcal{L}(A, (B_0, B_1)_{g,\infty;K}) = (\mathcal{L}(A, B_0), \mathcal{L}(A, B_1))_{g,\infty;K}$$

(equivalent norms).

b) *If \overline{B} is regular and f is a non-decreasing positive function satisfying (2.11) then*

$$\mathcal{L}((B_0, B_1)_{f,1;J}, A) = (\mathcal{L}(B_0, A), \mathcal{L}(B_1, A))_{f^*,\infty;K}$$

(equivalent norms).

Proof: First we prove b). By Corollary 3.5, we have that $\mathcal{L}(\overline{B}_{f,1;J}, A) \hookrightarrow (\mathcal{L}(B_0, A), \mathcal{L}(B_1, A))_{f^*,\infty;K}$. In order to establish the converse inclusion let $T \in (\mathcal{L}(B_0, A), \mathcal{L}(B_1, A))_{f^*,\infty;K}$ and $m \in \mathbb{Z}$. Since $K(2^m, T) \leq f^*(2^m)\|T\|_{f^*,\infty;K}$ we can find a decomposition $T = T_0^m + T_1^m$ of T with $T_j^m \in \mathcal{L}(B_j, A)$ and

$$\|T_0^m\|_{B_0,A} + 2^m\|T_1^m\|_{B_1,A} \leq 2f^*(2^m)\|T\|_{f^*,\infty;K}.$$

Then, for any $b \in B_0 \cap B_1$, it follows that

$$\begin{aligned} \|Tb\|_A &\leq \|T_0^m b\|_A + \|T_1^m b\|_A \leq \|T_0^m\|_{B_0,A}\|b\|_{B_0} + 2^m\|T_1^m\|_{B_1,A} \frac{\|b\|_{B_1}}{2^m} \\ &\leq (\|T_0^m\|_{B_0,A} + 2^m\|T_1^m\|_{B_1,A})J(2^{-m}, b) \\ &\leq 2f^*(2^m)\|T\|_{f^*,\infty;K}J(2^{-m}, b) = 2\|T\|_{f^*,\infty;K} \frac{J(2^{-m}, b)}{f(2^{-m})}. \end{aligned}$$

If now $b \in \overline{B}_{f,1;J}$ and $b = \sum_{m=-\infty}^{\infty} b_m$ is a J -representation of b , we get

$$\|Tb\|_A \leq \sum_{m=-\infty}^{\infty} \|Tb_m\|_A \leq 2\|T\|_{f^*,\infty;K} \sum_{m=-\infty}^{\infty} \frac{J(2^m, b_m)}{f(2^m)}.$$

This implies $\|T\|_{\overline{B}_{f,1;J},A} \leq 2\|T\|_{f^*,\infty;K}$ and ends the proof of b). The proof of a) is more easy. The inclusion $(\mathcal{L}(A, B_0), \mathcal{L}(A, B_1))_{g,\infty;K} \hookrightarrow \mathcal{L}(A, (B_0, B_1)_{g,\infty;K})$ follows from a similar argument to the one given before, but that now can be simplified because we work with K -spaces only. The converse inclusion follows from Corollary 3.4. \square

Our next result deals with nuclear operators.

Theorem 4.2. *Let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple, let A be a Banach space and assume that f is a positive function satisfying (2.11). Then*

$$\mathcal{N}(A, (B_0, B_1)_{f,1;J}) = (\mathcal{N}(A, B_0), \mathcal{N}(A, B_1))_{f,1;J}$$

(equivalent norms).

Proof: As we pointed out in Section 2, $[\mathcal{N}, \|\cdot\|_1]$ is a normed operator ideal so we can apply Corollary 3.9/a with $q = 1$. We obtain that

$$(\mathcal{N}(A, B_0), \mathcal{N}(A, B_1))_{f,1;J} \hookrightarrow \mathcal{N}(A, (B_0, B_1)_{f,1;J}).$$

Conversely, let $T \in \mathcal{N}(A, \overline{B}_{f,1;J})$. We can find sequences $(h_n) \subset A^*$ and $(b_n) \subset \overline{B}_{f,1;J}$ such that

$$T = \sum_{n=1}^{\infty} h_n \otimes b_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|h_n\|_{A^*} \|b_n\|_{f,1;J} \leq 2\|T\|_1.$$

On the other hand, each b_n can be written as $b_n = \sum_{m=-\infty}^{\infty} b_m^n$ with $(b_m^n) \subset B_0 \cap B_1$ and

$$\sum_{m=-\infty}^{\infty} \frac{J(2^m, b_m^n)}{f(2^m)} \leq 2\|b_n\|_{f,1;J}.$$

Put $S_n = \sum_{m=-\infty}^{\infty} h_n \otimes b_m^n$. For each $n \in \mathbb{N}$, the operator S_n belongs to the space $(\mathcal{N}(A, B_0), \mathcal{N}(A, B_1))_{f,1;J}$, because

$$\sum_{m=-\infty}^{\infty} \frac{J(2^m, h_n \otimes b_m^n)}{f(2^m)} \leq 2\|h_n\|_{A^*} \|b_n\|_{f,1;J}.$$

Since $T = \sum_{n=1}^{\infty} S_n$, we conclude that $T \in (\mathcal{N}(A, B_0), \mathcal{N}(A, B_1))_{f,1;J}$ with

$$\|T\|_{f,1;J} \leq \sum_{n=1}^{\infty} \|S_n\|_{f,1;J} \leq 2 \sum_{n=1}^{\infty} \|h_n\|_{A^*} \|b_n\|_{f,1;J} \leq 4\|T\|_1.$$

The proof is complete. □

Assume next that the function g belongs to the class \mathcal{P}^{+-} (see [10]). This means that $g(t)$ is non-decreasing, $g(t)/t$ is non-increasing with $\bar{g}(t) = \sup_{s>0} \{g(ts)/g(s)\}$ finite for every $t > 0$ and $\bar{g}(t) = o(\max\{1, t\})$ as $t \rightarrow 0$ and $t \rightarrow \infty$. Then it is well-known that the spaces $(B_0, B_1)_{g,q;K}$ and $(B_0, B_1)_{g,q;J}$ coincide, and that for regular couples the following duality formula holds

$$((B_0, B_1)_{g,\infty;K}^o)^* = (B_0^*, B_1^*)_{g^*,1;J}.$$

Arguing as in Theorem 4.2 but using Corollary 3.9/b, we obtain

Theorem 4.3. *Let $\bar{B} = (B_0, B_1)$ be a regular quasi-linearizable couple, let A be a Banach space and assume that $g \in \mathcal{P}^{+-}$. Then*

$$\mathcal{N}((B_0, B_1)_{g,\infty;K}^o, A) = (\mathcal{N}(B_0, A), \mathcal{N}(B_1, A))_{g^*,1;J}$$

(equivalent norms).

Previous results in this direction are due to Ovchinnikov [13] and [14]. His approach works for regular couples of Hilbert spaces (H_0, H_1) , (G_0, G_1) and gives that

$$\begin{aligned} (\mathcal{L}(H_0, G_0), \mathcal{L}(H_1, G_1))_{\theta,\infty} &= \mathcal{L}((H_0, H_1)_{\theta,1}, (G_0, G_1)_{\theta,\infty}), \\ (\mathcal{N}(H_0, G_0), \mathcal{N}(H_1, G_1))_{\theta,1} &= \mathcal{N}((H_0, H_1)_{\theta,\infty}^o, (G_0, G_1)_{\theta,1}). \end{aligned}$$

Note that each one of the couples (H_0, H_1) , (G_0, G_1) is quasi-linearizable. The reason is that any regular couple of Hilbert spaces is isomorphic to a couple $(\ell_2(G_m), \ell_2(2^{-m}G_m))$ formed by vector valued ℓ_2 -spaces (see [6]), so the quasi-linearizability follows from the next result.

Lemma 4.4. *Let $1 \leq p \leq \infty$ and let $(G_m)_{m \in \mathbb{Z}}$ be a sequence of Banach spaces. Then the couple $(\ell_p(G_m), \ell_p(2^{-m}G_m))$ is quasi-linearizable.*

Proof: Let $x = (x_m)_{m \in \mathbb{Z}} \in \ell_p(G_m) + \ell_p(2^{-m}G_m)$. Then

$$\begin{aligned} K(t, x)^p &\approx \inf \left\{ \sum_{m=-\infty}^{\infty} \|y_m\|_{G_m}^p + t^p \sum_{m=-\infty}^{\infty} 2^{-mp} \|z_m\|_{G_m}^p : \right. \\ &\quad \left. x = y + z, y \in \ell_p(G_m), z \in \ell_p(2^{-m}G_m) \right\} \\ &= \sum_{m=-\infty}^{\infty} \min\{1, t^p 2^{-mp}\} \|x_m\|_{G_m}^p \\ &= \sum_{m \in Z_1} \|x_m\|_{G_m}^p + t^p \sum_{m \in Z_2} 2^{-mp} \|x_m\|_{G_m}^p \end{aligned}$$

where $Z_1 = \{m \in \mathbb{Z} : 2^m \leq t\}$ and $Z_2 = \mathbb{Z} \setminus Z_1$. This suggest to define for each $t > 0$

$$V_0(t)x = (v_m), \quad V_1(t)x = (w_m)$$

where

$$v_m = \begin{cases} x_m & \text{if } m \in Z_1 \\ 0 & \text{otherwise} \end{cases}, \quad w_m = \begin{cases} 0 & \text{if } m \in Z_1 \\ x_m & \text{otherwise} \end{cases}.$$

A direct computation shows that conditions (2.1), (2.2) and (2.3) are fulfilled with $k = 1$. □

It is well-known the relationship between $\mathcal{N}(A, B)$ and the projective tensor product $A^* \hat{\otimes} B$, so one may guess that the results of Section 3 are also useful to interpolate projective tensor products, or even injective tensor products $A \check{\otimes} B$. In fact, by means of similar arguments to those described in Theorems 3.2 and 4.1, one can establish

Theorem 4.5. *Let $\overline{B} = (B_0, B_1)$ be a regular quasi-linearizable couple, let A be a Banach space and assume that $g \in \mathcal{P}^{+-}$. Then*

$$A \check{\otimes} (B_0, B_1)_{g, \infty; K}^o = (A \check{\otimes} B_0, A \check{\otimes} B_1)_{g, \infty; K}^o$$

(equivalent norms).

The result for the projective tensor product reads

Theorem 4.6. *Let $\overline{B} = (B_0, B_1)$ be a regular quasi-linearizable couple and let A be a Banach space such that $(A \hat{\otimes} B_0, A \hat{\otimes} B_1)$ is also a Banach couple. If f is any positive function satisfying (2.11), then*

$$A \hat{\otimes} (B_0, B_1)_{f, 1; J} = (A \hat{\otimes} B_0, A \hat{\otimes} B_1)_{f, 1; J}$$

(equivalent norms).

The proof follows similar lines to those in Theorem 4.2.

Previous results on interpolation of tensor products are due to Kouba [11]. He found conditions on the regular Banach couples (A_0, A_1) and (B_0, B_1) so that the following complex interpolation formulæ

$$\begin{aligned}
 [A_0, A_1]_\theta \check{\otimes} [B_0, B_1]_\theta &= [A_0 \check{\otimes} B_0, A_1 \check{\otimes} B_1]_\theta, \\
 [A_0, A_1]_\theta \hat{\otimes} [B_0, B_1]_\theta &= [A_0 \hat{\otimes} B_0, A_1 \hat{\otimes} B_1]_\theta
 \end{aligned}$$

are valid. His conditions has nothing to be with the quasi-linearizability, but they refers to the notions of type 2 and 2-convexity.

Next we focus our attention on operator ideals defined by approximation numbers. Recall that for $T \in \mathcal{L}(A, B)$ and $n = 1, 2, \dots$, the n -th approximation number is given by

$$a_n(T) = a_n(T_{A,B}) = \inf\{\|T - L\| : L \in \mathcal{L}(A, B), \text{rank } L < n\}.$$

For $0 < p < \infty$, the quasi-normed operator ideals $[\mathcal{L}_p^{(a)}, \sigma_p]$ generated by the approximation numbers and the sequence space ℓ_p are defined by

$$\mathcal{L}_p^{(a)}(A, B) = \left\{ T \in \mathcal{L}(A, B) : \sigma_p(T) = \left(\sum_{n=1}^{\infty} a_n(T)^p \right)^{\frac{1}{p}} < \infty \right\}$$

(see [18] and [19]).

It follows from [8, Theorem 3.3.4], that approximation numbers have no unrestricted interpolation properties. However, working with quasi-linearizable couples, we can use ideas of Section 3 to establish the following formulæ:

Lemma 4.7. *Let A be a Banach space, let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple, let B be an intermediate space with respect to \overline{B} and let $\psi(t)$ and $\rho(t)$ be the functions associated to B and \overline{B} .*

a) *If $T \in \mathcal{L}(A, B_0 \cap B_1)$ and $n_0, n_1 \in \mathbb{N}$, then*

$$a_{n_0+n_1-1}(T_{A,B}) \leq 2(1+k)a_{n_0}(T_{A,B_0})\rho^* \left(\frac{a_{n_1}(T_{A,B_1})}{a_{n_0}(T_{A,B_0})} \right).$$

b) *If \overline{B} is regular, $T \in \mathcal{L}(B_0 + B_1, A)$ and $n_0, n_1 \in \mathbb{N}$, then*

$$a_{n_0+n_1-1}(T_{B,A}) \leq 2ka_{n_0}(T_{B_0,A})\psi \left(\frac{a_{n_1}(T_{B_1,A})}{a_{n_0}(T_{B_0,A})} \right).$$

Proof: Let $T \in \mathcal{L}(A, B_0 \cap B_1)$. Splitting T as $T = V_1(t)T + V_0(t)T$, using Lemma 3.1 and additivity of approximation numbers, we get that

$$\begin{aligned} a_{n_0+n_1-1}(T_{A,B}) &\leq a_{n_0}([V_1(t)T]_{A,B}) + a_{n_1}([V_0(t)T]_{A,B}) \\ &\leq \|V_1(t)\|_{B_0,B} a_{n_0}(T_{A,B_0}) + \|V_0(t)\|_{B_1,B} a_{n_1}(T_{A,B_1}) \\ &\leq (1+k) \left[\frac{a_{n_0}(T_{A,B_0})}{\rho(t)} + \frac{t a_{n_1}(T_{A,B_1})}{\rho(t)} \right]. \end{aligned}$$

If $a_{n_j}(T_{A,B_j}) \neq 0$, for $j = 0, 1$, then the choice $t = a_{n_0}(T_{A,B_0})/a_{n_1}(T_{A,B_1})$ yields a). If $a_{n_j}(T_{A,B_j}) = 0$, for $j = 0$ or $j = 1$, then $a_{n_0+n_1-1}(T_{A,B}) = 0$ as well, because $\text{rank } T < n_j$. The proof of b) is similar, using now the splitting $T = TV_0(t) + TV_1(t)$. \square

See [7] for some remarks on the connection between this result and the study of approximation numbers of embeddings from Besov spaces into spaces of Lipschitz type.

We are now ready to establish embedding formulæ for $\mathcal{L}_p^{(a)}$ -ideals.

Theorem 4.8. *Let A be a Banach space and let $\overline{B} = (B_0, B_1)$ be a quasi-linearizable couple. Assume that $0 < p_0, p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \max\{1, \frac{1}{p}\} + \frac{1}{p}$. Then the following holds:*

- a) $(\mathcal{L}_{p_0}^{(a)}(A, B_0), \mathcal{L}_{p_1}^{(a)}(A, B_1))_{\theta, q} \hookrightarrow \mathcal{L}_p^{(a)}(A, (B_0, B_1)_{\theta, 1})$.
- b) Moreover, if \overline{B} is regular,

$$(\mathcal{L}_{p_0}^{(a)}(B_0, A), \mathcal{L}_{p_1}^{(a)}(B_1, A))_{\theta, q} \hookrightarrow \mathcal{L}_p^{(a)}((B_0, B_1)_{\theta, \infty}^o, A).$$

Proof: Let $T \in \mathcal{L}_{p_0}^{(a)}(A, B_0) \cap \mathcal{L}_{p_1}^{(a)}(A, B_1)$. Using Lemma 4.7/a with $B = (B_0, B_1)_{\theta, 1} = (B_0, B_1)_{f, 1; J}$ where $f(t) = t^\theta$, we have that

$$a_{2n-1}(T_{A, (B_0, B_1)_{\theta, 1}}) \leq 2^{1+\theta} (1+k) a_n(T_{A, B_0})^{1-\theta} a_n(T_{A, B_1})^\theta.$$

Whence

$$\begin{aligned} \sigma_p(T_{A, (B_0, B_1)_{\theta, 1}}) &\leq 2^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} a_{2n-1}(T_{A, (B_0, B_1)_{\theta, 1}})^p \right)^{\frac{1}{p}} \\ &\leq 2^{\frac{\theta p + p + 1}{p}} (1+k) \left(\sum_{n=1}^{\infty} a_n(T_{A, B_0})^{(1-\theta)p} a_n(T_{A, B_1})^{\theta p} \right)^{\frac{1}{p}} \\ &\leq 2^{\frac{\theta p + p + 1}{p}} (1+k) \sigma_{p_0}(T_{A, B_0})^{1-\theta} \sigma_{p_1}(T_{A, B_1})^\theta \\ &\leq 2^{\frac{\theta p + p + 1}{p}} (1+k) 2^{-\theta m} J(2^m, T) \end{aligned}$$

where $m \in \mathbb{Z}$ is arbitrary.

Take now $T \in (\mathcal{L}_{p_0}^{(a)}(A, B_0), \mathcal{L}_{p_1}^{(a)}(A, B_1))_{\theta, q}$ and let $T = \sum_{m=-\infty}^{\infty} T_m$ be any J -representation of T with $(T_m) \subset \mathcal{L}_{p_0}^{(a)}(A, B_0) \cap \mathcal{L}_{p_1}^{(a)}(A, B_1)$. Since the constant in the triangle inequality for σ_p is $2^{\frac{1}{p}} \max\{2^{\frac{1}{p}-1}, 1\}$ and $(2 \cdot 2^{1/p} \max\{2^{1/p-1}, 1\})^q = 2$, we derive from our previous estimates that

$$\begin{aligned} \sigma_p(T_{A, (B_0, B_1)_{\theta, 1}}) &\leq 2^{\frac{1}{q}} \left(\sum_{m=-\infty}^{\infty} \sigma_p(T_m)^q \right)^{\frac{1}{q}} \\ &\leq C \left(\sum_{m=-\infty}^{\infty} \left(\frac{J(2^m, T_m)}{2^{\theta m}} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

This implies a).

A similar argument, but using Lemma 4.7/b and (2.10), proves b). \square

Remark 4.9. In fact, Lemma 4.7 is valid for any additive s function \mathbf{s} in the sense of Pietsch [18] and [19], and Theorem 4.8 works for operator ideals generated by any additive s -function.

5. Injectivity and surjectivity

Let $[\mathcal{J}, \tau]$ be a quasi-normed operator ideal. An operator $S \in \mathcal{L}(A, B)$ belongs to the surjective hull \mathcal{J}^{sur} if there are a Banach space E and an operator $T \in \mathcal{J}(E, B)$ so that $S(\mathcal{U}_A) \subseteq T(\mathcal{U}_E)$ where \mathcal{U}_A and \mathcal{U}_E stand for the closed unit balls of A and E , respectively. The function

$$\tau^{\text{sur}}(S) = \inf\{\tau(T) : S(\mathcal{U}_A) \subseteq T(\mathcal{U}_E)\}$$

defines a quasi-norm in \mathcal{J}^{sur} and $[\mathcal{J}^{\text{sur}}, \tau^{\text{sur}}]$ turns out to be a quasi-normed operator ideal (see [18]). It is clear that $c_{\tau^{\text{sur}}} \leq c_{\tau}$.

An operator $S \in \mathcal{L}(A, B)$ is said to belong to the injective hull \mathcal{J}^{inj} if there are a Banach space F and an operator $T \in \mathcal{J}(A, F)$ such that

$$\|Sx\|_B \leq \|Tx\|_F \text{ for all } x \in A.$$

The quasi-norm on \mathcal{J}^{inj} is given by

$$\tau^{\text{inj}}(S) = \inf\{\tau(T) : \|Sx\|_B \leq \|Tx\|_F, x \in A\}$$

and $[\mathcal{J}^{\text{inj}}, \tau^{\text{inj}}]$ is a quasi-normed operator ideal (see [18]). Again $c_{\tau^{\text{inj}}} \leq c_{\tau}$.

If $[\mathcal{J}, \tau] = [\mathcal{J}^{\text{sur}}, \tau^{\text{sur}}]$ (resp. $[\mathcal{J}, \tau] = [\mathcal{J}^{\text{inj}}, \tau^{\text{inj}}]$), then the quasi-normed operator ideal \mathcal{J} is called surjective (resp. injective). Of course $[\mathcal{L}, \|\cdot\|]$ satisfies these two conditions. Other examples are compact operators $[\mathcal{K}, \|\cdot\|]$ and weakly compact operators $[\mathcal{W}, \|\cdot\|]$. A normed operator ideal which is injective but fails to be surjective is the ideal $[\Pi, \pi]$

of all absolutely summing operators. Recall that $T \in \mathcal{L}(A, B)$ is absolutely summing if there is a constant $c > 0$ such that for every finite set $\{a_j\}_{j=1}^n \subseteq A$

$$\sum_{j=1}^n \|Ta_j\|_B \leq c \sup \left\{ \sum_{j=1}^n |f(a_j)| : f \in \mathcal{U}_{A^*} \right\}.$$

We write $\pi(T)$ for the least constant c for which the above inequality holds. We refer to [18] and [4] for more details on these notions.

Given two Banach spaces E, F , we denote by $E \oplus F$ the direct sum of E and F , normed by $\|(x, y)\| = \max\{\|x\|_E, \|y\|_F\}$.

We shall use some ideas developed in [3] to establish the following theorems.

Theorem 5.1. *Let $\overline{B} = (B_0, B_1)$ be a regular Banach couple, let B be an intermediate space with respect to \overline{B} with $B_0 \cap B_1$ dense in B and let $\psi(t)$ be the ψ -function associated to B and \overline{B} . If A is any Banach space, $[\mathcal{J}, \tau]$ is a quasi-normed operator ideal and q is defined by $(2c_\tau)^q = 2$, then*

$$(\mathcal{J}^{\text{sur}}(B_0, A), \mathcal{J}^{\text{sur}}(B_1, A))_{\psi^*, q; \mathcal{J}} \hookrightarrow \mathcal{J}^{\text{sur}}(B, A).$$

Proof: Let $T \in \mathcal{J}^{\text{sur}}(B_0, A) \cap \mathcal{J}^{\text{sur}}(B_1, A)$. Take any $\epsilon > 0$, and let E_j ($j = 0, 1$) be Banach spaces and $R_j \in \mathcal{J}(E_j, A)$ so that $T(\mathcal{U}_{B_j}) \subseteq R_j(\mathcal{U}_{E_j})$ and $\tau(R_j) \leq (1 + \epsilon)\tau^{\text{sur}}(T_{B_j, A})$.

Given any $b \in \mathcal{U}_B$, we can find $b_j \in B_j$ such that $b = b_0 + b_1$ and $\|b_0\|_{B_0} + t\|b_1\|_{B_1} \leq (1 + \epsilon)\psi(t)$. Hence

$$\mathcal{U}_B \subseteq (1 + \epsilon)\psi(t)\mathcal{U}_{B_0} + (1 + \epsilon)\frac{\psi(t)}{t}\mathcal{U}_{B_1}.$$

Consider the Banach space $E_0 \oplus E_1$ and the operator

$$R(x, y) = (1 + \epsilon)\psi(t)R_0x + (1 + \epsilon)\frac{\psi(t)}{t}R_1y.$$

Then $R \in \mathcal{J}(E_0 \oplus E_1, A)$ and

$$T(\mathcal{U}_B) \subseteq (1 + \epsilon)\psi(t)R_0(\mathcal{U}_{E_0}) + (1 + \epsilon)\frac{\psi(t)}{t}R_1(\mathcal{U}_{E_1}) \subseteq R(\mathcal{U}_{E_0 \oplus E_1}).$$

Consequently,

$$\begin{aligned} \tau^{\text{sur}}(T_{B, A}) &\leq \tau(R) \leq c_\tau(1 + \epsilon)^2\psi(t)[\tau^{\text{sur}}(T_{B_0, A}) + \frac{1}{t}\tau^{\text{sur}}(T_{B_1, A})] \\ &\leq 2c_\tau(1 + \epsilon)^2\frac{J(t^{-1}, T)}{\psi^*(t^{-1})}. \end{aligned}$$

Combining this estimate with [1, Lemma 3.10.2], the result follows. \square

Since $[\mathcal{J}, \tau] \hookrightarrow [\mathcal{J}^{\text{sur}}, \tau^{\text{sur}}]$, we also have that

$$(\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{\psi^*, q; J} \hookrightarrow \mathcal{J}^{\text{sur}}(B, A).$$

This means that the assumption that \overline{B} is quasi-linearizable in Theorem 3.8 and Corollary 3.9/b can be eliminated, provided we take $[\mathcal{J}^{\text{sur}}, \tau^{\text{sur}}]$ as the last operator space.

Writing down Theorem 5.1 for $B = (B_0, B_1)_{\theta, \infty}^{\circ}$ and \mathcal{J} surjective we conclude:

Corollary 5.2. *Let $\overline{B} = (B_0, B_1)$ be a regular Banach couple and let A be a Banach space. If $[\mathcal{J}, \tau]$ is a surjective quasi-normed operator ideal, $(2c_{\tau})^q = 2$ and $0 < \theta < 1$, then*

$$(\mathcal{J}(B_0, A), \mathcal{J}(B_1, A))_{\theta, q} \hookrightarrow \mathcal{J}((B_0, B_1)_{\theta, \infty}^{\circ}, A).$$

Next we show by means of an example that surjectivity is essential in the former result. We shall use the norm ideal Π which is not surjective.

Example 5.3. Let $\Omega = c_0 \oplus \ell_1 \oplus \ell_1$ and let $\overline{A} = (A_0, A_1)$ be the Banach couple constructed by Garling and Montgomery-Smith in [9, Theorem 2]. Then A_j is isometric to ℓ_1 ; the projection $P: \Omega \rightarrow c_0$ given by $P(x, y, z) = x$ belongs to $\mathcal{L}(\overline{A}_{\theta, \infty}^{\circ}, c_0)$ and the embedding $J: c_0 \rightarrow \Omega$ given by $Jx = (x, 0, 0)$ belongs to $\mathcal{L}(c_0, \overline{A}_{\theta, \infty}^{\circ})$. Moreover, it is easy to check that \overline{A} is regular.

Let now $R \in \mathcal{L}(c_0, \ell_2)$ be the operator defined by $R(\zeta_n) = (\zeta_n/n)$. A direct computation shows that $R \notin \Pi(c_0, \ell_2)$. Put $T = RP$. Since A_j is isometric to ℓ_1 , Grothendieck’s theorem implies that

$$T \in \Pi(A_0, \ell_2) \cap \Pi(A_1, \ell_2).$$

However, T does not belongs to $\Pi(\overline{A}_{\theta, \infty}^{\circ}, \ell_2)$ because $TJ = RPJ = R$ which is not absolutely summing.

We consider now the injective hull.

Theorem 5.4. *Let $\overline{B} = (B_0, B_1)$ be a Banach couple, let B be an intermediate space with respect to \overline{B} and let $\rho(t)$ be the ρ -function associated to B and \overline{B} . Assume that A is any Banach space, that $[\mathcal{J}, \tau]$ is a quasi-normed operator ideal and let q be defined by $(2c_\tau)^q = 2$. Then*

$$(\mathcal{J}^{\text{inj}}(A, B_0), \mathcal{J}^{\text{inj}}(A, B_1))_{\rho, q; \mathcal{J}} \hookrightarrow \mathcal{J}^{\text{inj}}(A, B).$$

Proof: Let $T \in \mathcal{J}^{\text{inj}}(A, B_0) \cap \mathcal{J}^{\text{inj}}(A, B_1)$. Given any $\epsilon > 0$ we can find Banach spaces F_j ($j = 0, 1$) and operators $R_j \in \mathcal{J}(A, F_j)$ with $\|Tx\|_{B_j} \leq \|R_j x\|_{F_j}$ for all $x \in A$, and $\tau(R_j) \leq (1 + \epsilon)\tau^{\text{inj}}(T_{A, B_j})$. Given $t > 0$, put $Rx = (R_0x, tR_1x)$. Then $R \in \mathcal{J}(A, F_0 \oplus F_1)$. It follows from

$$\|T(x)\|_B \leq \frac{J(t, Tx)}{\rho(t)} \leq \frac{1}{\rho(t)} \max\{\|R_0x\|_{F_0}, t\|R_1x\|_{F_1}\} = \frac{1}{\rho(t)} \|Rx\|_{F_0 \oplus F_1}$$

that $T \in \mathcal{J}^{\text{inj}}(A, B)$ with

$$\tau^{\text{inj}}(T_{A, B}) \leq \frac{1}{\rho(t)} \tau(R) \leq \frac{c_\tau}{\rho(t)} (\tau(R_0) + t\tau(R_1)) \leq \frac{2c_\tau(1 + \epsilon)}{\rho(t)} J(t, T).$$

Now we conclude the result by appealing to [1, Lemma 3.10.2]. □

Since $[\mathcal{J}, \tau] \hookrightarrow [\mathcal{J}^{\text{inj}}, \tau^{\text{inj}}]$, Theorem 5.4 shows that we can eliminate the assumption of quasi-linearizability in Theorem 3.7 and Corollary 3.9/a provided we take $\mathcal{J}^{\text{inj}}(A, B)$ as the last operator space. In particular, we have

Corollary 5.5. *Let $\overline{B} = (B_0, B_1)$ be a Banach couple and let A be a Banach space. If $[\mathcal{J}, \tau]$ is an injective quasi-normed operator ideal, $(2c_\tau)^q = 2$ and $0 < \theta < 1$, then*

$$(\mathcal{J}(A, B_0), \mathcal{J}(A, B_1))_{\theta, q} \hookrightarrow \mathcal{J}(A, (B_0, B_1)_{\theta, 1}).$$

Working with the dual ideal $[\Pi^d, \pi^d]$ of absolutely summing operators, it is not hard to derive from Example 5.3 that injectivity is essential in Corollary 5.5.

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Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid
Spain
E-mail address: cobos@eucmax.sim.ucm.es
E-mail address: Tesera_Signes@mat.ucm.es

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