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UNIQUENESS OF KÄHLER-EINSTEIN CONE METRICS

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Abstract

The purpose of this paper is to describe a method to construct a Kähler metric with cone singularity along a divisor and to illustrate a type of maximum principle for these incomplete metrics by showing that Kähler-Einstein metrics are unique in geometric Hölder spaces.

1. Introduction

Outline of results. We show that if M is a compact complex manifold of complex dimension two or greater, and D a divisor with one irreducible component and if the cohomology class $C_1(K_M) + \alpha C_1(O(D))$, for $\alpha \in (0, 1)$, contains a positive representative, then we can construct an initial Kähler cone metric ω with cone angle α. Here K_M denotes the canonical bundle of the manifold and $O(D)$ the line bundle associated to the divisor. This metric is incomplete along the divisor.

Functions describing geometric quantities will often be continuous on all of M, but may achieve nonsmooth extrema over the divisor. We develope a generalized maximum principle for such functions. The technique is illustrated by proving uniqueness of Kähler-Einstein cone metrics. For the existence of such metrics, [**JM**], more special function spaces are needed, which simultaneously yield refined regularity properties at the divisor, but for uniqueness it suffices to work within the larger geometric Hölder spaces. It is useful to provide an example of the technique in this more general and more geometrically intuitive setting because the method has other applications. In [**J**], for example, it is used to prove a Schwarz Lemma for Kähler metrics with cone singularities.

Background. Singular spaces are of interest in differential geometry, algebraic geometry, and in analysis. They occur naturally in many settings. For example, many algebraic varieties are not smooth. Differential geometers interested in special metrics on smooth manifolds will naturally study the moduli space of such, and singularities often develope at the boundary of the moduli space. Interesting analytic features arise in the resolution of geometric problems, and this paper is an example of such.

Uniqueness of partial differential equations, and also the estimates needed to prove existence of Kähler-Einstein metrics, often rely upon the maximum principle. The importance of the maximum principle is that it enables one to deduce, from elliptic inequalities, *a priori* estimates on solutions of differential equations. In the case of a singular or noncompact space, direct application of the maximum principle may be impossible. There is a close relationship between singular and noncompact spaces, because removal of the singular set leaves a noncompact space. One obvious difficulty in applying the maximum principle is that a maximum may simply fail to exist. An instance of this may be found in the paper of Cheng and Yau [**CY**] wherein they prove existence of Kähler-Einstein metrics on pseudoconvex domains. Indeed, a nontrivial step along the way is a generalized maximum principle which asserts, roughly speaking, the existence of a sequence of points approaching the boundary for which the first derivatives of the solution go to zero and the Hessian becomes negative semidefinite. In some cases, the problem can be circumvented. Incomplete metrics on the complement of a divisor were studied by Tian and Yau, [**TY**], but restrictions on the cone angle make it possible to pass to a finite branched cover and apply techniques similar to the smooth, compact case.

The construction described below yields a metric with singularity along a divisor. Because the metric is incomplete, it is not possible to regard the singular set as being out at infinity; it is reached in finite time. If a function achieves an extremum over the divisor, the singular background metric allows the extremum to be achieved nonsmoothly or with a cusp shape. In some sense, this phenomenon is the opposite of that encountered by Cheng and Yau; their function did not achieve a maximum but had the correct shape, while ours has a maximum but with the wrong shape. The maximum principle is really an analysis of the shape of a function, and the technique described below consists in using a barrier function to push the maximum off the divisor and into the interior where it will be achieved smoothly and with the correct shape. Naturally, this barrier function must be chosen so that the resulting estimates are uniform.

2. Cone metrics

Cone singularities are very natural singularities and have been studied from different points of view by many people. In the Riemannian context one might consult, for example the papers of Cheeger, among them [**C**]. The point of view there is to construct a singular space by forming a cone over a given compact Riemannian manifold, and the author describes the analysis of such a space. The special case of a sphere has also been considered by Troyanov in [**Tr1**], who has also studied cone singularities from the point of view of Riemann surfaces and conformal maps; see also [**Tr2**].

In this section we describe a construction of a Kähler cone metric with negative curvature on a compact complex manifold. An important application of this construction is to use it as an initial approximation to a Kähler-Einstein cone metric and then prove existence by perturbing away from it. In joint work with Rafe Mazzeo, [**JM**], we used this approach to prove the existence of Kähler-Einstein metrics with cone singularities at least for certain cone angles. To construct this cone metric, we need to assume some global conditions, namely, we suppose that M contains a smooth divisor D with one irreducible component, and fix a constant α with $0 < \alpha < 1$. This constant will be referred to as the *cone angle*. Now let K_M denote the canonical bundle of M, and $O(D)$ the line bundle associated to the divisor D. In order to make sure that what we construct is really a metric we need to assume that

$$C_1(K_M) + \alpha C_1(O(D)) \in H^2_{\mathrm{DR}}(M)$$

contains a positive definite real, closed $(1,1)$ form. Some choices of M and D that satisfy this are smooth algebraic varieties V in CP^n described as the zero locus of a homogeneous polynomial of degree $k > n+1$, and $D = V \cap H$, where H is a hyperplane section of CP^n that intersects V in one smooth irreducible component.

We can now proceed to construct the Kähler cone metric. Let s be a defining section of $O(D)$. Make provisional choices of a smooth volume function V on M and a Hermitian metric $\|\cdot\|$ in $O(D)$ and write down

$$\hat{V} = \frac{V}{\|s\|^{2\alpha}(1 - \|s\|^{2(1-\alpha)})^2}.$$

In the denominator, we only want the first term to vanish, so multiply s by a constant if necessary so that $\|s\| \le \delta < 1$; this is no problem because s is a smooth section defined on all of the compact manifold M. We would like to make sense of this as a singular Kähler potential so that our cone metric will be given by $\omega = i\partial\overline{\partial}\log\hat{V}$, so we compute directly

$$\begin{aligned}\omega &\stackrel{\text{def}}{=} i\partial\overline{\partial}\log\hat{V}\\ &= i\partial\overline{\partial}\log V - i\alpha\partial\overline{\partial}\log\|s\|^2 - 2i\partial\overline{\partial}\log(1-\|s\|^{2(1-\alpha)}).\end{aligned}$$

Noting that a volume function on M is the same thing as a metric h in the anticanonical line bundle K_M^{-1}, and writing $\Theta(K_M^{-1})$ for the curvature of this metric, the first term here is

$$i\partial\overline{\partial}\log V = i\partial\overline{\partial}\log h(K_M^{-1}) = -i\partial\overline{\partial}\log h(K_M) = i\Theta(K_M).$$

How can we interpret the second term? Locally, suppose that e_0 is a nonvanishing holomorphic section of $O(D)$, so that $s = s_0e_0$ for a holomorphic function s_0. Then

$$-i\alpha\partial\overline{\partial}\log\|s\|^2 = -i\alpha\partial\overline{\partial}\log|s_0|^2 - i\alpha\partial\overline{\partial}\log\|e_0\|^2.$$

This is actually independent of the local choices, because any other choice e_1 of nonvanishing section would give $s = s_1e_1 = (s_1g_{10})e_0 = s_0e_0$ and

$$\partial\overline{\partial}\log|s_0|^2 = \partial\overline{\partial}\log|s_1|^2 + \partial\overline{\partial}\log|g_{10}|^2 = \partial\overline{\partial}\log|s_1|^2 + 0.$$

This term gives a singular current supported over the divisor, since

$$\partial\overline{\partial}\log|s_0|^2 = \pi\delta ds_0 \wedge d\overline{s}_0.$$

We denote this current by T_D. So we have

$$\begin{aligned}\omega = i\partial\overline{\partial}\log\hat{V} = i\Theta(K_M) &+ i\alpha\Theta(O(D))\\ &- 2\pi\alpha T_D - 2i\partial\overline{\partial}\log(1-\|s\|^{2(1-\alpha)}).\end{aligned}$$

Direct computation shows that the last term also produces a singularity; it looks like $\|s\|^{-2\alpha}$ times a bounded form.

Now the assumption that $2\pi(C_1(K_M)) + \alpha C_1(O(D))) > 0$ comes into play, because the sum of the first two terms, $i\Theta(K_M) + i\alpha\Theta(O(D))$, is a representative of this cohomology class. Therefore, the initial choices of volume function on M and metric in $O(D)$ can be made in such a way that the sum of the first three terms is positive definite. In fact, in local holomorphic coordinates $(z, w_2, \dots, w_n)$ in which $D = \{z = 0\}$, the sum of the first three terms is equivalent to

$$\sqrt{-1}\left(\frac{1}{|z|^{2\alpha}}dz \wedge d\overline{z} + \sum_2^n dw_i \wedge d\overline{w}_i\right).$$

So ω consists of a positive definite metric on $\Omega \stackrel{\text{def}}{=} M \setminus D$ together with a singular term supported by D. ω is a current on all of M, and a genuine metric on Ω but we just refer to it as a singular metric on M. More descriptively, we say it has a cone singularity and that α is the cone angle, because in the z direction, the surface with metric $(1/|z|^{2\alpha})dz \wedge d\overline{z}$ is a cone.

Let us now explain what it means to say that such a metric is Kähler-Einstein. Since ω defines a metric on Ω, one may compute the Ricci curvature ρ of this metric, and then extend as a current to all of M. A local expression for ρ is

$$\rho = -\sqrt{-1}\partial\overline{\partial} \log \det g_{i\overline{j}},$$

and again choosing local coordinates $(z, w_2, \ldots, w_n)$ for which D appears as the zero set of z, this is

$$\rho = -i\partial\overline{\partial} \log |z|^{-2\alpha} b,$$

where b is a smooth nonzero bounded function which makes sense on all of M, or

$$\rho = i\alpha\partial\overline{\partial} \log |z|^2 - \partial\overline{\partial} \log b = 2\pi\alpha T_D - i\partial\overline{\partial} \log b.$$

Since $-\omega$ and ρ both contain the singular term $2\pi\alpha T_D$ supported by the divisor, the correct Kähler -Einstein condition is:

$$\rho = -\omega$$

as currents on all of M, and pointwise on Ω.

As in the smooth case, the existence of a Kähler-Einstein cone metric may be reformulated analytically as the existence of a solution u to the Monge-Ampère equation. This is the same equation as in the smooth case, except in this context a Kähler-Einstein metric is determined by a solution u over the noncompact set Ω.

The derivation too is the same as in appearance as in the smooth case; one has only to remember that everything must be interpreted in the sense of currents and distributions. Excellent references for the smooth case are the exposé of Bourguignon, [**B**], and also the lecture notes of Siu, [**S**]. With ω the original Kähler cone metric, we consider new metrics of the form

$$\omega' \stackrel{\text{def}}{=} \omega + i\partial\overline{\partial}u$$

with $u \in C^{2,\delta}_g(\Omega)$, the so-called geometric Hölder space, the definition of which will be given in the next section. For the moment, suffice it to say that u is a function that has $2+\delta$ covariant or geometric derivatives bounded on Ω. Then u must satisfy

$$\frac{\det(g_{i\bar{\jmath}} + \partial_i \partial_{\bar{\jmath}} u)}{\det(g_{i\bar{\jmath}})} = e^{f+u};$$

we also require $\omega + i\partial\overline{\partial}u > 0$ so that the solution is a metric.

3. Geometric Hölder spaces

A natural setting in which we solve nonlinear problems on Ω is the geometric Hölder spaces $C_g^{k,\delta}(\Omega)$. These spaces consist of the functions which are continuous on all of M and whose covariant derivatives up to order $k+\delta$ are bounded on Ω with respect to the singular cone metric ω constructed above. Namely, $C_g^k(\Omega)$ consists in functions u which are continuous on all of M and for which

$$\sup_\Omega |u| + \cdots + \sup_\Omega \|\nabla^k u\|_g$$

is finite. Note that the singular metric appears in this expression twice —both in the covariant derivative ∇ and in the norm $\|\cdot\|$. For the Hölder part we first define $C_g^{0,\delta}(\Omega)$ to consist in those functions u for which

$$\sup_\Omega |u| + \sup_{p \neq q \in \Omega} \frac{|u(z, w_2, \dots, w_n) - u(z_0, w_{2,0}, \dots, w_{n,0})|}{|z|^{\alpha\delta}|z - z_0|^\delta + |w_2 - w_{2,0}|^\delta + \cdots + |w_n - w_{n,0}|^\delta}$$

is bounded. Here $p = (z, w_2, \dots, w_n)$, and $q = (z_0, w_{2,0}, \dots, w_{n,0})$. Successive $C_g^{k,\delta}(\Omega)$ are then obtained by replacing the numerator by $|X_1 \dots X_k u(p) - X_1 \dots X_k u(q)|$ where the X_i are vector fields for which $\|X_i\|$ is bounded on Ω. The definitions of $C_g^k(\Omega)$ and $C_g^{k,\delta}(\Omega)$ are consistent with each other because $\|\nabla u\|_g$ is bounded if and only if for every vector field X there is a constant C so that

$$|Xu| \leq C\|X\|_g.$$

4. Maximum principle technique

To obtain the uniqueness result, in the next section we will use the maximum principle to show that the difference between two solutions, $u_1 - u_2$, must be zero. We demonstrate the idea here by explaining how to obtain a C^0 estimate. This is also an important step in the proof of existence, [**JM**]. Then in the following section a refinement gives uniqueness.

We begin by recalling how C^0 estimates on solutions of the Monge-Ampère equation for negative first Chern class were obtained in the smooth case by Aubin, [**A1**] and [**A2**] and by Yau, [**Y**]. If u is a solution of the Monge-Ampère equation, then locally

$$\frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u)}{\det(g_{i\bar{j}})} = e^{f+u}.$$

Remember that f is determined by the original geometry. At a point P where u achieves a maximum, $(\partial_i \partial_{\bar{j}} u)$ is a negative semidefinite Hermitian matrix, and so at this point,

$$e^{f+u}(P) = \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u)}{\det(g_{i\bar{j}})}(P) \leq 1,$$

and so $f(P)+u(P) \leq 0$. Therefore, for all x, we have $u(x) \leq \max\{-f(x)\}$. In our singular case, if a maximum of u occurs over D, it could have a cusp shape, but because it is an element of the function space $C_g^{2,\delta}(\Omega)$, we know exactly how fast the derivatives can blow up. So our modification is to add a function F which just fails to be in this space, and show that uniform control is maintained.

Put $v = u + F$ for an unknown function F to be determined. Then $u = v - F$ so the Monge Ampère equation becomes

$$\frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} v - \partial_i \partial_{\bar{j}} F)}{\det(g_{i\bar{j}})} = e^{f+u}.$$

Suppose v achieves a maximum on Ω. Then at that point, $(\partial_i \partial_{\bar{j}} v)$ is negative semidefinite, so that

$$\frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} v - \partial_i \partial_{\bar{j}} F)}{\det(g_{i\bar{j}})} \leq \frac{\det(g_{i\bar{j}} - \partial_i \partial_{\bar{j}} F)}{\det(g_{i\bar{j}})}$$

or

$$e^{f+u} \leq \frac{\det(g_{i\bar{j}} - \partial_i \partial_{\bar{j}} F)}{\det(g_{i\bar{j}})},$$

that is,

$$e^{v} \leq e^{-f+F} \cdot \frac{\det(g_{i\bar{j}} - \partial_i \partial_{\bar{j}} F)}{\det(g_{i\bar{j}})}.$$

If the right hand side can be bounded, then we will have obtained a bound for v and hence of u. So we can write down the conditions that the choice of F must satisfy. They are:

1. $\operatorname{Max} v$ occurs on Ω.
2. $\max u \leq \max v$.
3. F is uniformly bounded.
4. For some C, $Cg_{i\bar{j}} \leq \partial_i \partial_{\bar{j}} F$, uniformly.

Let s be a section with $\|s\| \leq 1$ and put $F = \|s\|^{2\beta}$ for a positive power β to be determined in a moment. Then $v = u + F$ will agree with u along D, and $\|s\|^{2\beta}$ will be a function which is initially increasing in directions perpendicular to the divisor. If F increases more rapidly than any element of $C_g^{2,\delta}(\Omega)$, then v will achieve a maximum on Ω.

We compare the gradient of F to that of functions in $C_g^{2,\delta}(\Omega)$; if the gradient is unbounded with respect to the flat cone metric ω_0 then it will also be unbounded with respect to ω_g.

Write $\|s\|^{2\beta} = |z|^{2\beta}\|e\|^{2\beta}$ locally, with e a basis section for $O(D)$, or $\|s\|^{2\beta} = |z|^{2\beta} b$. Because $\|e\|$ is bounded away from zero, b is smooth.

In diagonal coordinates at one point

$$\langle \operatorname{grad} f, \operatorname{grad} f \rangle_g = \sum \left| \frac{\partial f}{\partial z_i} \right|^2 g^{i\bar{i}}.$$

We compute the first term $i = 1$, since it corresponds to the singular $(z, \bar{z})$ direction

$$\frac{\partial}{\partial z}(|z|^{2\beta} b) \frac{\partial}{\partial \bar{z}}(|z|^{2\beta} b) g_0^{11} = \beta^2 b^2 |z|^{2+4(\beta-1)+2\alpha} + \cdots .$$

This is unbounded if $2 + 4(\beta - 1) + 2\alpha < 0$ or $2\beta < 1 - \alpha$. Because F rises more steeply than u can fall, $\max u + F$ occurs over Ω. We now show that $i\partial\bar{\partial} F \geq C\omega_g$ for some C, noting first the formula $\partial\bar{\partial} e^f = e^f(\partial\bar{\partial} f + \partial f \wedge \bar{\partial} f)$. Then

$$\begin{aligned} i\partial\bar{\partial}\|s\|^{2\beta} &= i\partial\bar{\partial} e^{\beta \log \|s\|} \\ &= i\|s\|^{2\beta}(\beta\partial\bar{\partial} \log \|s\|^2 + \beta^2 \partial \log \|s\|^2 \wedge \bar{\partial} \log \|s\|^2) \\ &\geq i\|s\|^{2\beta} \beta \partial\bar{\partial} \log \|s\|^2, \end{aligned}$$

since for a real-valued form h, $\partial h \wedge \bar{\partial} h \geq 0$.

But $i\partial\bar{\partial} \log \|s\|^2 = -R(\| \cdot \|)$, so we need to show that there exists C such that

$$-\beta \|s\|^{2\beta} R(\| \cdot \|) \geq C\omega_g$$

or

$$\beta \|s\|^{2\beta} R(\| \cdot \|) \leq -C\omega_g.$$

Since $\|s\| \leq 1$ and $R(\| \cdot \|)$ is bounded, there is such a constant C.

It is interesting to note that C depends on the original metric and on the curvature of the line bundle, while in the smooth case of course it only depends on the metric of M. That makes sense, because we had some freedom in how to choose the metric in the line bundle $O(D)$.

In the above instance, all that was needed was *some* bound for the maximum and the minimum of u. That is not good enough to obtain a uniqueness result, for that would only show that the difference between two solutions was at most one. So we need a sharpening of this technique.

5. Uniqueness of Kähler-Einstein cone metrics

Now turning to the uniqueness question, the main result is:

Theorem. *Suppose that $u \in C_g^{2,\delta}(\Omega)$ is a solution to the Monge-Ampère equation*

$$(\omega + i\partial\overline{\partial}u)^n = e^{f+u}\omega^n,$$

with $\omega + i\partial\overline{\partial}u$ positive definite, where ω is equivalent to

$$\sqrt{-1}\left(\frac{1}{|z|^{2\alpha}}dz \wedge d\overline{z} + \sum_2^n dw_i \wedge d\overline{w}_i\right)$$

at the divisor. Then u is unique.

To prove this, we suppose that u_1 and u_2 are two solutions to the Monge-Ampère equation, lying in $C_g^{2,\delta}(\Omega)$. That is, on the interior Ω,

$$\begin{aligned}(\omega + i\partial\overline{\partial}u_1)^n &= e^{f+u_1}\omega^n \quad \text{and}\\ (\omega + i\partial\overline{\partial}u_2)^n &= e^{f+u_2}\omega^n,\end{aligned}$$

with both solution metrics positive definite. If a maximum or minimum of the difference $u_2 - u_1$ occurs over the interior, then these can be handled as in the smooth case. So we imagine that these extrema occur over the divisor. Since this equation is nonlinear, u_2-u_1 is not a solution, so the first step is to find a nonlinear equation for which $u_2 - u_1$ *is* a solution. Similarly to the exposé of Bourguignon, [**B**], equating the two expressions for $e^f\omega^n$ gives

$$(\omega + i\partial\overline{\partial}u_1)^n = e^{u_1-u_2}(\omega + i\partial\overline{\partial}u_1 + i\partial\overline{\partial}(u_2 - u_1))^n.$$

Setting $u = u_2 - u_1$ and $\omega_1 = \omega + i\partial\overline{\partial}u_1$, this reads

$$e^u\omega_1^n = (\omega_1 + i\partial\overline{\partial}u)^n.$$

This is the nonlinear equation solved by the difference of the two solutions.

Now we define for every positive integer k, $v_k = u + F_k$, where $F_k = 1/k \cdot F = 1/k\|s\|^{2\beta}$, with β again chosen such that $2\beta < 1 - \alpha$. Then for each k, $u = v_k - F_k$, so that

$$e^u \omega_1^n = (\omega_1 + i\partial\overline{\partial}(v_k - F_k))^n$$

everywhere in the interior Ω. Recall that the exponent β was chosen in such a way that $F = \|s\|^{2\beta}$ just failed to be in the same function space as u so that F rose more steeply than any possible solution u, and therefore the sum $u + F$ had to achieve a maximum in the interior. Multiplying F by the nonzero constant cannot change that, and so each v_k will also achieve a maximum at a point over the interior, say at P_k. We may rewrite the equation above locally as

$$e^u \det(g'_{i\overline{j}}) = \det(g'_{i\overline{j}} + \partial_i \partial_{\overline{j}}(v_k - F_k)),$$

where $g'_{i\overline{j}}$ denotes the metric given by ω_1. At P_k, since $(g'_{i\overline{j}})$ is positive definite and $(\partial_i\partial_{\overline{j}}(v_k - F_k))$ is symmetric, there are coordinates which simultaneously diagonalize these two matrices at the point. So at P_k, this equation is

$$e^{u(P_k)} g'_{1\overline{1}} \cdots g'_{n\overline{n}} = (g'_{1\overline{1}} + (v_1)_{1\overline{1}} - (F_k)_{1\overline{1}}) \cdots (g'_{n,\overline{n}} + (v_k)_{n\overline{n}} - (F_k)_{n\overline{n}}).$$

For each i, at the maximum point P_k,

$$0 < g'_{i\overline{i}} + (v_k)_{i\overline{i}} - (F_k)_{i\overline{i}} \le g'_{i\overline{i}} - (F_k)_{i\overline{i}}.$$

As shown in the previous section, there exists a constant C such that $i\partial\overline{\partial}F \ge C\omega_1$; the argument works equally well for the solution metric ω_1 as for the original ω. Therefore, $i\partial\overline{\partial}F_k \ge \frac{1}{k}C\omega_1$, so that for each i, $(F_k)_{i\overline{i}} \ge 1/kCg'_{i\overline{i}}$, and $g'_{i\overline{i}} - (F_k)_{i\overline{i}} < (1 - \frac{1}{k}C)g'_{i\overline{i}}$. Then at P_k,

$$e^{u(P_k)} g'_{1\overline{1}} \cdots g'_{n\overline{n}} \le \left(1 - \frac{1}{k}C\right)^n g'_{1\overline{1}} \cdots g'_{n\overline{n}},$$

so that

$$e^{u(P_k)} \le \left(1 - \frac{1}{k}C\right)^n.$$

Now we must relate $u(x)$ to $u(P_k)$, since P_k is a maximum point of v_k, not of u itself. For any $x \in M$,

$$\begin{aligned} u(x) &= v_k(x) - F_k(x) \\ &\le v_k(P_k) - F_k(x) \\ &\le v_k(P_k) = u(P_k) + F_k(P_k) \\ &\le u(P_k) + \frac{1}{k}. \end{aligned}$$

But as $k \to \infty$, we have $u(P_k)$ arbitrarily small by the above, and $1/k \to 0$. So $u(x) \leq 0$.

A similar argument applied at the minimum of u, should it occur over the divisor, shows that $u(x) \geq 0$. If either occurs over the interior, then the usual maximum principle applies. So the difference between the solutions, $u(x)$, is identically zero. This concludes the proof.

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