MAXIMAL NON-JAFFARD SUBRINGS OF A FIELD

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Abstract

A domain R is called a maximal non-Jaffard subring of a field L if $R \subset L$, R is not a Jaffard domain and each domain T such that $R \subset T \subseteq L$ is Jaffard. We show that maximal non-Jaffard subrings R of a field L are the integrally closed pseudo-valuation domains satisfying $\dim_v R = \dim R + 1$. Further characterizations are given. Maximal non-universally catenarian subrings of their quotient fields are also studied. It is proved that this class of domains coincides with the previous class when R is integrally closed. Moreover, these domains are characterized in terms of the altitude formula in case R is not integrally closed. An example of a maximal non-universally catenarian subring of its quotient field which is not integrally closed is given (Example 4.2). Other results and applications are also given.

0. Introduction

Throughout this paper, $R \hookrightarrow S$ denotes an extension of commutative integral domains, $\mathrm{qf}(R)$ the quotient field of an integral domain R, R' the integral closure of R in its quotient field, and $\mathrm{tr.\,deg}[S : R]$ the transcendence degree of $\mathrm{qf}(S)$ over $\mathrm{qf}(R)$. If $\mathrm{tr.\,deg}[S : R] = 0$, we say as in [**4**] that S is algebraic over R. We recall that a ring R of finite (Krull) dimension n is a *Jaffard ring* if its *valuative dimension* (the limit of the sequence $(\dim R[X_1, \dots, X_n] - n,\ n \in \mathbf{N})$) $\dim_v R$, is also n. Prüfer domains and Noetherian domains are Jaffard domains. The notion of Jaffard ring is not a local property and thus we say that R is a *locally Jaffard ring* if R_P is a Jaffard ring for each prime ideal P of R. We assume familiarity with these concepts as in [**1**], [**6**], [**14**].

A. R. Wadsworth in [22] considered pairs of domains $R \subset S$ such that each domain T between R and S (that is $R \subseteq T \subseteq S$) is Noetherian. In [21], S. Visweswaran noticed that if $S = k[y_1, \ldots, y_t]$ is an affine domain (where k is a field) having Krull dimension $n > 0$ and if I is a nonzero proper ideal of S and D a subring of k, then the ring $R = D + I$ may be not Noetherian, but each ring T such that $R \subset T \subseteq S$ is Noetherian. In view of this, he introduced the following definition: "Let A be a subring of a Noetherian ring B. Then A is said to be a maximal non-Noetherian subring of B if A is non-Noetherian and any subring of B that properly contains A is Noetherian". S. Visweswaran characterized when $D + I$ is a maximal non-Noetherian subring of $k[y_1, \ldots, y_t]$ ([21, Proposition 2.1]). On the other hand, A. Ayache and P.-J. Cahen in [4] studied the domains R such that each domain contained between R and its quotient field $\mathrm{qf}(R)$ is Jaffard; that is, the domains whose integral closure R' is a Prüfer domain ([4, Théorème 2.6]). They are said to be domains satisfying absolutely the altitude inequality formula. Our purpose here is to complete this circle of ideas by dealing with *maximal non-Jaffard subrings of a field L*; that is, the domains R where R is a non-Jaffard domain and each ring T, $R \subset T \subseteq L$ is Jaffard. First, we show that if R is a maximal non-Jaffard subring of a field L, then $L = \mathrm{qf}(R)$. Hence, we may restrict ourselves to the case where $L = \mathrm{qf}(R)$.

Our concern in Section 1 is primarily with maximal non-Jaffard subrings. Our initial line of inquiry was suggested by pointing out a relationship between this kind of domains and pseudo-valuation domains (for short PVDs), which are closely related to valuation domains and have been deeply studied by several authors, notably J. R. Hedstrom, E. G. Houston [17], D. E. Dobbs and M. Fontana [13]. The main result of Section 1 is Theorem 1.4, which states that R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$ if and only if R is an integrally closed PVD and $\dim_v R = \dim R + 1$. Among the several interesting consequences ot this theorem, we only point out that if R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$, then R is a residually integrally closed domain (in the sense that R/P is integrally closed for each prime ideal P of R). The last part of this section is devoted to a deeper study of these rings, for instance, Theorem 1.7 states that R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$ if and only if R is a local non-Jaffard domain with nonzero maximal ideal M and for each ring T such that $R \subset T \subseteq \mathrm{qf}(R)$ and for each prime ideal Q of T, if $Q \neq M$, then $R/(Q \cap R) \subset T/Q$ is an algebraic extension, and if $Q = M$, then $\mathrm{tr.\,deg}[T/M : R/M] = 1$. If $\mathcal{P}$ is a property which may be possessed by rings (extensions) such as locally (totally) Jaffard, (stably) strong S, universally catenarian, we say that

R is a maximal non-$\mathcal{P}$ subring of a field L, if R is a non-$\mathcal{P}$ domain and each ring T such as $R \subset T \subseteq L$ is $\mathcal{P}$. We prove a result analogous to [**6**, Théorème 5.1]; we show in Theorem 1.8 that the following statements are equivalent:

  (i) R is a maximal non-Jaffard subring of qf(R);
 (ii) R is a maximal non-locally Jaffard subring of qf(R);
(iii) R is a maximal non-totally Jaffard subring of qf(R);
 (iv) R is a maximal non-strong S subring of qf(R);
  (v) R is a maximal non-stably strong S subring of qf(R).

Recall from [**4**] that a domain R satisfies absolutely the altitude formula; if each overring of R satisfies the altitude formula, equivalently, each overring of R is universally catenarian. Of course these domains satisfy absolutely the altitude inequality formula. However, the converse is not true [**4**, Exemple 5.1]. Section 2 is concluded with the study of *maximal non-universally catenarian subrings of their quotient field*. We begin by showing that this class of rings is larger than the class of domains which are maximal non-Jaffard subrings of their quotient fields; and if R is assumed to be integrally closed, then these two classes of domains coincide. As Example 4.2 reveals, Theorem 2.3 is the best possible. This example depends ultimately on the pullback techniques in [**15**]. It points out the importance of the "integrally closed" hypothesis in Proposition 2.2 and suggests the need of characterizing the maximal non-universally catenarian subrings R, when R is not integrally closed. We prove the following (Theorem 2.3):

Let R be a non integrally closed domain. Then R is a maximal non-universally catenarian subring of qf(R) *if and only if R' is a Prüfer domain, R does not satisfy the altitude formula, and the extension $T \subseteq T'$ satisfies the altitude formula for each proper overring T of R.*

Section 3 finds necessary and sufficient conditions for certain pullbacks to be maximal non-Jaffard (resp., non-universally catenarian) subrings of their quotient fields. Section 4 is devoted to the investigation of several examples illuminating the earlier sections' results, which in some cases are shown to be the best-possible.

It cannot be expected that the typical reader is conversant with all this article's references. Therefore in order to shorten this introduction, we have chosen to recall relevant definitions and facts as needed throughout the article. Any unexplained material is standard as in [**16**], [**19**].

1. Maximal non-Jaffard subrings of a field

Let R be a domain contained in a field L. We say that R is a *maximal non-Jaffard subring of L* if R is not Jaffard and each ring T such that $R \subset T \subseteq L$, is Jaffard.

First of all, we establish the following.

Proposition 1.1. *Let R be a domain and L a field containing R. If R is a maximal non-Jaffard subring of L, then $L = \mathrm{qf}(R)$.*

Proof: First notice that L is algebraic over R. Indeed, if not then there exists an element X of L trancendental over R. Hence each overring of $R[X]$ became Jaffard, that is $R'[X]$ is a Prüfer domain, which is impossible [**4**, Théorème 2.6]. Now our task is to prove that $\mathrm{qf}(R) = L$. Assume that $\mathrm{qf}(R) \subset L$ and let $\alpha \in L \setminus \mathrm{qf}(R)$. Then α is algebraic over R. Thus there exists an element $r \in R$ such that $r\alpha$ is integral over R. Thus $R \subset R[r\alpha]$ is an integral extension. But $R[r\alpha]$ is a Jaffard domain. Hence, according to [**1**, Proposition 1.1], R is a Jaffard domain, the desired contradiction to complete the proof. $\square$

As a direct consequence of Proposition 1.1, the study of maximal non-Jaffard subrings of a field L can be reduced to the case where $L = \mathrm{qf}(R)$.

Now, notice that if R is a maximal non-Jaffard subring of its quotient field, then R is integrally closed. Indeed, if $R \neq R'$, then R' is a Jaffard domain, and hence so is R (since $R \subset R'$ is an integral extension [**1**, Proposition 1.1]) which is impossible.

In this section, we collect more information on this kind of domains and we characterize them in terms of pseudo-valuation domains. We begin by presenting some terminology. Let B be an integral domain, I an ideal of B, and D a subring of B/I. Consider the pullback construction of commutative rings:

$$
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
B & \longrightarrow & B/I
\end{array}
$$

Following [**9**], we say that R is the domain of the (B, I, D) construction and we set $R := (B, I, D)$. Next we consider the case where I is assumed to be maximal. Denoting by M the ideal I, T the domain B, K the field T/M, and $\varphi : T \longrightarrow K$ the natural epimorphism.

We next recall a few wellknown properties about pullbacks to be used in our next theorems and examples. First, M is a common ideal of both R and T, $M = (R : T) = \{x \in T \mid xT \subset R\}$, and $R/M \simeq D$. For each $P \in \mathrm{Spec}(R)$ with $M \subseteq P$, there is a (unique) $Q \in \mathrm{Spec}(D)$ such that $\varphi^{-1}(Q) = P$; and $\varphi^{-1}(D_Q) = R_P$. If T is local, then each prime ideal of R compares with M, and thus $\dim R = \dim D + \dim T$. Also R is local if and only if both D and T are local (cf. [**15**, Theorem 1.4, Proposition 2.1]).

The last construction to be noted here concerns the notion of a pseudo-valuation domain (for short, a PVD), which was introduced by Hedstrom and Houston [**17**] and has been studied subsequently in [**2**], [**11**], [**12**] and [**18**]. A domain R is said to be a PVD in case each prime ideal P of R is strongly prime, in the sense that whenever $x, y \in \mathrm{qf}(R)$ satisfy $xy \in P$, then either $x \in P$ or $y \in P$, equivalently, in case R has a (uniquely determined) valuation overring V such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ as sets, equivalently (by [**2**, Proposition 2.6]) in case R is a pullback of the form $V \times_K k$ where V is a valuation domain with residue field K and k is a subfield of K. As the terminology suggests, any valuation domain is a PVD [**17**, Proposition 1.1]. Although the converse is false [**17**, Example 2.1], any PVD must, at least, be local [**17**, Corollary 1.3].

Before stating Theorem 1.4, we establish a lemma which serves both to motivate this theorem and to dispatch the difficult implication in its proof. First it is convenient to recall that given a ring extension $R \subset S$, (R, S) is said to be a Jaffard pair [**7**], if each ring T between R and S is Jaffard.

Lemma 1.2. *Let $k \subset K$ be an extension of fields. Then the following statements are equivalent.*

(i) (k, K) *is a Jaffard pair;*
(ii) $\mathrm{tr.\,deg}[K : k] \leq 1$.

Proof: (ii)$\Rightarrow$(i) Assume that $\mathrm{tr.\,deg}[K : k] \leq 1$ and let T be a ring between k and K. We have $\dim_v T \leq \dim_v k + \mathrm{tr.\,deg}[T : k]$ (cf. [**4**, Lemme 1.1]). Thus $\dim_v T \leq 1$. If T is not a field, then $\dim T = \dim_v T = 1$, so T is a Jaffard domain.

(i)$\Rightarrow$(ii) If $\mathrm{tr.\,deg}[K : k] \geq 2$, let X, Y be two transcendental algebraically independent elements of K over k. Then, the domain $T = k + Yk(X)[Y]_{(Y)}$ is contained between k and K, and we have $\dim T = 1$ and $\dim_v T = 2$ [**1**, Proposition 2.5]. Thus T is not a Jaffard domain, which contradicts assertion (i). Hence $\mathrm{tr.\,deg}[K : k] \leq 1$. $\square$

Now we establish the following usefull result.

Lemma 1.3. *Let V be a valuation domain with maximal ideal M and residue field K. Let D be a subring of K and $R := (V, M, D)$. Then the overrings of R are exactly the overrings of V and the intermediate domains between R and V.*

Proof: Let S be an overring of R such that $V \nsubseteq S$. Thus there exists $v \in V$ and $v \notin S$. Our task is to show that $S \subseteq V$. Let $s \in S$, assume that $s \notin V$. Since V is a valuation domain, then $\frac{1}{s} \in M$. Hence $\frac{v}{s} \in M$. Therefore $v = \frac{v}{s}s \in S$, the desired contradiction. $\qquad\square$

As Lemma 1.3 is to be used repeatedly in the proof of some of the next theorems, we make use of it without any reference or comment.

Theorem 1.4. *Let R be a domain. Then the following statements are equivalent:*

(i) *R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$;*
(ii) *R is an integrally closed PVD with maximal ideal M and $\mathrm{tr.deg}[V/M : R/M] = 1$, where V is the associated valuation domain of R;*
(iii) *R is an integrally closed PVD and $\dim_v R = \dim R + 1$.*

Proof: (i)$\Rightarrow$(ii) We have already observed that R is integrally closed. Now we claim that there exists a valuation overring V of R and a nonzero prime ideal Q of V such that $R = R_P + QV_Q$, where $P = Q \cap R$. Indeed, assume that $R \subset R_P + QV_Q$ for each valuation overring V of R and each nonzero prime ideal Q of V. Then $R_P + QV_Q$ is a Jaffard domain. Hence $R \subset V$ is a residually algebraic extension [**1**, Proposition 2.5 (b)]. Thus R' is a Prüfer domain ([**16**, Theorem 19.15]), and therefore R is a Jaffard domain. Now $R = R_P + QV_Q$ for some valuation overring V of R and nonzero prime ideal Q of V. Thus $R := (V_Q, QV_Q, R_P/PR_P)$. Hence R is a PVD (cf. [**2**]).

Denote QV_Q by M. Then M is the unique maximal ideal of R. Notice that $M = M \cap R = QV_Q \cap R = Q \cap R = P$. Our task is to show that $\mathrm{tr.deg}[V/M : R/M] = 1$. Let D be a domain such that $R/M \subset D \subset V/M$ and consider the ring $T := (V, M, D)$. Then $R \subset T \subset V$, so T is a Jaffard domain. Hence D is a Jaffard domain [**1**, Theorem 2.6]. Thus each intermediate domain between R/M and V/M is Jaffard. Hence by Lemma 1.2, we get $\mathrm{tr.deg}[V/M.R/M] \leq 1$. The extension $R/M \subset V/M$ can not be algebraic since R is not Jaffard [**1**, Proposition 2.5 (b)]. Thus $\mathrm{tr.deg}[V/M : R/M] = 1$.

(ii)$\Rightarrow$(i) It is clear that R is not a Jaffard domain [**1**, Proposition 2.5 (b)]. Now, let T be a domain such that $R \subset T \subseteq \mathrm{qf}(R)$. Then either T is an overring of V, so it is Jaffard, or T is an intermediate domain between R and V, so $T := (V, M, D)$, where $R/M \subset D \subseteq V/M$. According to Lemma 1.2, D is a Jaffard domain, and since R is integrally closed, then $\mathrm{tr.deg}[V/M : D] = 0$. Thus T is a Jaffard domain [**1**, Theorem 2.6]. Hence R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$.

(ii)$\Leftrightarrow$(iii) We have $\dim R = \dim V$ and $\dim_v R = \dim V + \mathrm{tr.deg}[V/M : R/M]$ (cf. [**1**, Proposition 2.5 (a)]). Thus $\dim_v R = \dim R + 1$ if and only if $\mathrm{tr.deg}[V/M : R/M] = 1$. $\square$

We turn now to point out the connection with integrally closed domains. A special result is that if R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$, then R/P is integrally closed for each prime ideal P of R. We begin by noticing that if R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$, then for each non-maximal prime ideal P of R, R_P is a valuation domain since $V \subset R_P$. Nevertheless we have the following:

Proposition 1.5. *Let R be a maximal non-Jaffard subring of $\mathrm{qf}(R)$. Then for each non maximal prime ideal P of R, R/P is a maximal non-Jaffard surbring of its quotient field.*

Proof: Notice that R/P is not a Jaffard domain. Indeed, consider the domain $R + PR_P$. If R/P is a Jaffard domain, then so is $R + PR_P$ [**1**, Theorem 2.6 (b)]. Thus $R \subset R + PR_P$. But R is a PVD, so $R := (V, M, k)$ ([**2**]), where V is the associated valuation domain and $k = R/M$. Since $P \subset M$, then there exists a prime ideal Q of V such that $R_P = V_Q$. Thus $PR_P = QV_Q = Q \subseteq M$. Therefore $R + PR_P = R + Q \subseteq R + M = R$. This contradicts the fact that $R \subset R + PR_P$. Consequently R/P is not a Jaffard domain. Now, let D be a domain such that $R/P \subset D \subseteq \mathrm{qf}(R/P)$ and consider the ring $T := (R_P, PR_P, D)$. Since $R/P \subset D \subseteq \mathrm{qf}(R/P)$, then $R \subset T \subseteq \mathrm{qf}(R)$. Hence T is a Jaffard domain, and thus so is D [**1**, Theorem 2.6]. Therefore R/P is a maximal non-Jaffard subring of its quotient field. $\square$

Remark 1.6. By the previous proposition, we conclude that if R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$, then R/P is integrally closed for each prime ideal P of R. The converse does not hold. More precisely, in Example 4.1 we construct for each $n, m \in \mathbf{N}$ such that $n + 1 \leq m \leq 2n + 1$, $n \geq 2$ a residually integrally closed local domain R such that $\dim R = n$, $\dim R[X] = m$, and R is not a maximal non-Jaffard subring of its quotient field.

The next result provides another characterization of a domain R which is a maximal non-Jaffard subring of its quotient field.

Theorem 1.7. *Let R be domain. Then the following statements are equivalent:*

(i) *R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$;*
(ii) *R is local non-Jaffard domain with nonzero maximal ideal M, and for each ring T such that $R \subset T \subseteq \mathrm{qf}(R)$ and for each $Q \in \mathrm{Spec}(T)$: If $Q \neq M$, then $R/(Q \cap R) \subset T/Q$ is algebraic, and if $Q = M$, then $\mathrm{tr.\,deg}[T/M : R/M] = 1$.*

Proof: (i)$\Rightarrow$(ii) By Theorem 1.4, R is local since it is a PVD. Now let T be a domain such that $R \subset T \subseteq \mathrm{qf}(R)$. Then either T is an overring of V, where V is the associated valuation domain of R or T is an intermediate domain between R and V. In the first case, let $Q \in \mathrm{Spec}(T)$ such that $Q \neq M$, then $R/(Q \cap R) \subset V/(Q \cap V) \subset T/Q$. But $\mathrm{tr.\,deg}[V/(Q \cap V) : R/(Q \cap R)] = 0$ since $Q \cap R \neq M$. Hence $\mathrm{tr.\,deg}[T/Q : R/(Q \cap R)] = \mathrm{tr.\,deg}[T/Q : V/(Q \cap V)] + \mathrm{tr.\,deg}[V/(Q \cap V) : R/(Q \cap R)] = 0$. Now, if $Q = M$, then $T = V$. Hence $\mathrm{tr.\,deg}[T/M : R/M] = \mathrm{tr.\,deg}[V/M : R/M] = 1$ (by Theorem 1.4). In the second case, T shares the ideal M with R and V. If $Q \in \mathrm{Spec}(T)$ and $Q \neq M$, then we discuss two cases:

Case 1: If $M \not\subseteq Q$, then $T_Q = R_{Q \cap R}$, so $R/(Q \cap R) \subset T/Q$ is algebraic.

Case 2: If $M \subset Q$, then $R + QT_Q \neq R$, so $R + QT_Q$ is a Jaffard domain. Thus $R/(Q \cap R) \subset T/Q$ is an algebraic extension ([**1**, Theorem 2.6]).

Now, if $Q = M$, then $\mathrm{tr.\,deg}[T/M : R/M] = \mathrm{tr.\,deg}[V/M : R/M] - \mathrm{tr.\,deg}[V/M : T/M] = 1$.

(ii)$\Rightarrow$(i) Let T be a domain such that $R \subset T \subseteq \mathrm{qf}(R)$, and let S be an overring of T. If $Q \in \mathrm{Spec}(S)$, then either $Q \neq M$, so by hypothesis, we have $\mathrm{tr.\,deg}[S/Q : R/(Q \cap R)] = 0$. Thus $\mathrm{tr.\,deg}[S/Q : T/(Q \cap T)] = 0$. Or $Q = M$, so $\mathrm{tr.\,deg}[S/M : R/M] = 1$. Hence $\mathrm{tr.\,deg}[S/M : T/M] = \mathrm{tr.\,deg}[S/M : R/M] - \mathrm{tr.\,deg}[T/M : R/M] = 0$. Therefore $T \subseteq S$ is a residually algebraic extension for each overring S of T. Thus T satisfies absolutely the altitude inequality formula [**4**, Théorème 2.6]. In particular, T is a Jaffard domain. Thus R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$. $\qquad\square$

Recall that a ring R is said to be a strong S-domain if for each pair of consecutive prime ideals $P \subset Q$ of R, the extended primes $P[X] \subset Q[X]$ are consecutive. The most natural examples of strong S-domains are arbitrary Noetherian domains [**19**, Theorem 68]. Dispite the above

material, the class of strong S-domains is not very stable, for instance with respect to polynomial extensions. Following [20], we say that R is a stably strong S-domain if $R[X_1, \ldots, X_n]$ is a strong S-domain for each nonnegative integer n.

A domain R is said to be totally Jaffard if R/P is a locally Jaffard domain for each prime ideal P of R (cf. [10]).

We establish the following result.

Theorem 1.8. *Let R be a domain. Then the following statements are equivalent:*

  (i) *R is a maximal non-Jaffard subring of* qf(R);
 (ii) *R is a maximal non-locally Jaffard subring of* qf(R);
(iii) *R is a maximal non-totally Jaffard subring of* qf(R);
 (iv) *R is a maximal non-strong S subring of* qf(R);
  (v) *R is a maximal non-stably strong S subring of* qf(R).

Proof: It follows readily from [6, Théorème 5.1]. $\square$

2. Maximal non-universally catenarian subrings of their quotient fields

First recall that an extension $R \subseteq S$ of integral domains satisfies the altitude inequality formula (resp., the altitude formula), if for each prime ideal Q of S, if we set $P = Q \cap R$, we have $htQ + \operatorname{tr.deg}[S/Q : R/P] \leq htP + \operatorname{tr.deg}[S : R]$ (resp., $htQ + \operatorname{tr.deg}[S/Q : R/P] = htP + \operatorname{tr.deg}[S : R]$). A domain R is said to satisfy the altitude inequality formula (resp., the altitude formula) if $R \subseteq S$ satisfies the altitude inequality formula (resp., the altitude formula) for each finite type R-algebra S containing R.

A ring R is said to be catenarian in case, for each pair $P \subset Q$ of prime ideals of R, all saturated chains of primes from P to Q have a common finite length. We shall say that R is universally catenarian if the polynomial rings $R[X_1, \ldots, X_n]$ are catenarian for each positive integer n [8]. Notice that if R is universally catenarian, then it satisfies the altitude formula [8, Theorem 5.1].

In [4], A. Ayache and P.-J. Cahen studied the domains R such that each domain T between R and qf(R) satisfies the altitude formula. These domains are said to satisfy absolutely the altitude formula. They established that R satisfies absolutely the altitude formula if and only if R' is a Prüfer domain and $R \subseteq R'$ satisfies the altitude formula ([4, Théorème 3.3]). Naturally these domains satisfy absolutely the altitude inequality formula, but the converse does not hold (cf. [4, Exemple 5.1]).

We say that a domain R is a maximal subring of $\mathrm{qf}(R)$ not satisfying the altitude formula, if R does not satisfy the altitude formula and each domain T such that $R \subset T \subseteq \mathrm{qf}(R)$ satisfies the altitude formula. We show in the following that if R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$, then it is maximal not satisfying the altitude formula and the converse is not true (see Example 4.2). But first a key lemma:

Lemma 2.1. *Let R be a maximal subring of $\mathrm{qf}(R)$ not satisfying the altitude formula. Then the following statements are equivalent:*

  (i) *R is integrally closed;*
 (ii) *R is not a Jaffard domain;*
(iii) *The extension $R \subseteq R'$ satisfies the altitude formula.*

Proof: (i)$\Rightarrow$(ii) Assume that R is a Jaffard domain. Since each proper overring of R satisfies the altitude formula, then R' is a Prüfer domain ([**4**, Théorème 2.6]). Hence $R = R'$ satisfies the altitude formula, the desired contradiction.

(ii)$\Rightarrow$(i) If R is not integrally closed, then R' satisfies the altitude formula. Hence R' is a Jaffard domain and so is R [**1**, Proposition 1.1], a contradiction.

(i)$\Rightarrow$(iii) This is trivial.

(iii)$\Rightarrow$(i) Assume that $R \neq R'$. Since R' satisfies the altitude formula and $R \subset R'$ satisfies the altitude formula, then by [**14**, Proposition 2.6], R also satisfies the altitude formula. This is the desired contradiction to complete the proof. □

Proposition 2.2. *Let R be an integrally closed domain. Then the following statements are equivalent:*

  (i) *R is a maximal non-universally catenarian subring of $\mathrm{qf}(R)$;*
 (ii) *R is a maximal subring of $\mathrm{qf}(R)$ not satisfying the altitude formula;*
(iii) *R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$.*

Proof: (i)$\Leftrightarrow$(ii) It follows readily from [**4**, Théorème 3.8] without the assumption R is integrally closed.

(ii)$\Rightarrow$(iii) Each proper overring of R satisfies the altitude formula; hence R is Jaffard. By Lemma 2.1, R is not a Jaffard domain. Thus R is a maximal non-Jaffard subring of its quotient field.

(iii)$\Rightarrow$(i) Since R is not a Jaffard domain, then it is not universally catenarian [**8**, Corollary 3.3]. By Theorem 1.4, R is a PVD. Thus there exists a valuation overring V of R such that $R := (V, M, k)$, where M is the maximal ideal of V and k is a subfield of $K = V/M$. Let T be a domain such that $R \subset T \subseteq \mathrm{qf}(R)$. If T is an overring of V,

then it is universally catenarian since it is a valuation domain. If not, $T := (V, M, D)$, where D is a domain properly contained between k and K. We have $\operatorname{tr.deg}[K : D] = 0$. Thus $\dim_v D \leq \operatorname{tr.deg}[D : k] = 1$ (cf. [**4**, Lemme 1.1]). Hence, either D is a field or a one-dimensional Jaffard domain. Therefore by [**8**, Corollary 6.3], D is universally catenarian. According to [**3**, Corollary 2.3], T is a universally catenarian domain, which completes the proof. $\square$

The case when R is not integrally closed will now be given.

Theorem 2.3. *Let R be a non integrally closed domain. Then the following statements are equivalent:*

(i) *R is a maximal non-universally catenarian subring of $\operatorname{qf}(R)$;*

(ii) *R' is a Prüfer domain, R does not satisfy the altitude formula, and $T \subseteq T'$ satisfies the altitude fromula for each T such that $R \subset T \subseteq \operatorname{qf}(R)$.*

Proof: (i)$\Rightarrow$(ii) We have $R \neq R'$. Then each overring of R' is universally catenarian. Thus by [**4**, Théorème 3.3], R' is a Prüfer domain. Now, let T be an overring of R such that $R \neq T$. Then T satisfies absolutely the altitude formula. Hence $T \subseteq T'$ satisfies the altitude formula (cf. [**4**, Théorème 3.3]).

(ii)$\Rightarrow$(i) Let T be a ring such that $R \subset T \subseteq \operatorname{qf}(R)$. Then $R' \subseteq T' \subseteq \operatorname{qf}(R)$. Thus T' is a Prüfer domain since it is an overring of the Prüfer domain R'. On the other hand, $T \subseteq T'$ satisfies the altitude formula. Thus by [**8**, Theorem 6.1], T is universally catenarian. Since R does not satisfy the altitude formula, then it is not universally catenarian. Hence R is a maximal non-universally catenarian subring of $\operatorname{qf}(R)$. $\square$

3. Transfer results

In this section, we begin by determining when a pullback R is a maximal non-Jaffard subring of its quotient field. We recall some notation for conductors. If R is a domain and I and J are R-submodules of $\operatorname{qf}(R)$, then $(I : J) = \{x \in \operatorname{qf}(R) \mid xJ \subset I\}$. If R is a PVD with associated valuation domain V and maximal ideal M, assume that $R \neq V$, then M is not a principal ideal of R and $V = (M : M)$ [**2**, Proposition 2.3], and by [**2**, Lemma 2.4], we get $V = (R : M) = (M : M)$.

We establish the following theorem.

Theorem 3.1. *Let T be a local domain with maximal ideal M, and D a subring of the field $K = T/M$. Let $R := (T, M, D)$. Then the following hold.*

(a) *If D is a field, then R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$ if and only if T is a PVD satisfying absolutely the altitude inequality formula, D is algebraically closed in $(M : M)/M$, and $\mathrm{tr.\,deg}[K : D] = 1$.*

(b) *If D is not a field, then R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$ if and only if T is a valuation domain, D is a maximal non-Jaffard subring of its quotient field, and $\mathrm{qf}(D) = K$.*

Proof: (a) Assume that R is a maximal non-Jaffard subring of its quotient field. Since T properly contains R, then T is a Jaffard domain. Thus T satisfies absolutely the altitude inequality formula (since each proper overring of T is a proper overring of R, so it is Jaffard). The ring R is a PVD, so there exists a valuation domain V with M as a maximal ideal such that $R := (V, M, D)$. Now, T is an overring of R, thus T is a PVD (since $\mathrm{Spec}(R) = \mathrm{Spec}(T)$). On the other hand, we have:

$$\begin{aligned}
\mathrm{tr.\,deg}[K : D] &= \dim_v R - \dim_v T &&(\text{cf. } [\mathbf{1}, \text{Proposition 2.5}]) \\
&= (\dim R + 1) - \dim T &&(\text{cf. Theorem 1.4}) \\
&= 1 &&(\text{since } \dim R = \dim T).
\end{aligned}$$

Now, $V = (M : M)$. Since R is integrally closed (Theorem 1.4), then D is algebraically closed in $V/M = (M : M)/M$ (cf. [$\mathbf{9}$, Proposition 2]).

Conversely, R is not a Jaffard domain since $D \subset K$ is not an algebraic extension [$\mathbf{1}$, Proposition 2.5]. The ring T is a PVD, so there is a valuation domain W with maximal ideal M such that $T := (W, M, K)$. But $R := (T, M, D)$. Hence R is a PVD with associated valuation domain $W = (M : M)$. Furthermore, $\dim_v R - \dim R = \dim_v T + \mathrm{tr.\,deg}[K : D] - \dim T = \mathrm{tr.\,deg}[K : D] = 1$. Since D is algebraically closed in W/M, then R is integrally closed. Thus by Theorem 1.4, R is a maximal non-Jaffard subring of $\mathrm{qf}(R)$.

(b) Assume that R is a maximal non-Jaffard subring of its quotient field. Then R is a PVD. Hence there exists a valuation domain V with m as a maximal ideal such that $R := (V, m, k')$. We show under the assumption on D that T must be an overring of V. Indeed, assume that $R \subset T \subseteq V$, then T shares the ideal m with R and V, so $T := (V, m, T/m)$. Since T is local with M as a maximal ideal, then $m \subseteq M$. On the other hand, we have $M \subseteq m$ (since R is local with maximal ideal M). Thus $m = M$ and $R/M = R/m = D$ is a field, a contradiction. Therefore T is an overring of V, so T is a valuation domain. According to Proposition 1.5, the domain D is a maximal non-Jaffard subring of its quotient field.

Assume now that $\mathrm{qf}(D) \subset K$ and let $a \in K \setminus \mathrm{qf}(D)$. Since $\mathrm{tr.\,deg}[K : D] = (\dim_v R - \dim R) - (\dim_v T - \dim T) - (\dim_v D - \dim D) = 0$, then a is algebraic over $\mathrm{qf}(D)$. Hence there exists $d \in D$ such that ad is integral over D. The element $ad \notin D$ because $a \notin \mathrm{qf}(D)$. On the other hand, $D \subset D[ad]$ is an integral extension. Hence $D[ad]$ is not a Jaffard domain because D is not a Jaffard domain ([**1**, Proposition 1.1]). Let $S := (V, m, D[ad])$. The ring S is such that $R \subset S \subset T$ and S is not a Jaffard domain, which contradicts the fact that R is a maximal non-Jaffard subring of its quotient field. Therefore $\mathrm{qf}(D) = K$. Conversely, since $R := (T, M, D)$ and T is a valuation domain, then each proper overring S of R is either an overring of T, hence it is Jaffard or a ring contained between R and T, and in the latter case $S := (T, M, D_1)$, where D_1 is a ring such that $D \subset D_1 \subset K = \mathrm{qf}(D)$. Hence D_1 is a Jaffard domain, and thus so is S ([**1**, Theorem 2.6]). $\square$

We close this section by studying when a pullback domain which is not integrally closed is a maximal non-universally catenarian subring of its quotient field. First we show that such domains are local.

Lemma 3.2. *Let R be maximal non-universally catenarian subring of $\mathrm{qf}(R)$. Then R is local.*

Proof: The case when R is integrally closed follows from Theorem 1.4 and Proposition 2.2. Suppose that R is not integrally closed. By Lemma 2.1, the extension $R \subset R'$ does not satisfy the altitude formula. Thus there exists a prime ideal Q of R' such that $ht_{R'}Q < ht_R P$, where $P = Q \cap R$. Let $N = R \setminus P$. The extension $R_P \subset N^{-1}R' = (R_P)'$ does not satisfy the altitude formula, which implies by Theorem 2.3 (ii) that $R = R_P$, as required. $\square$

Before presenting our next theorem, we establish the following.

Proposition 3.3. *Let T be a local domain with maximal ideal M, D a subring of the field $K = T/M$, and $R := (T, M, D)$. Assume that R is not integrally closed. If R is a maximal non-universally catenarian subring of $\mathrm{qf}(R)$, then D is a Jaffard domain which is maximal non-universally catenarian subring of its quotient field and T' is a Prüfer domain.*

Proof: The domain T is a proper overring of R. Hence each proper overring of T is universally catenarian. Thus by [**4**, Théorème 3.3], T' is a Prüfer domain. By Lemma 2.1, R is a Jaffard domain. Hence D is a Jaffard domain and $D \subset K$ is an algebraic extension (cf. [**1**, Theorem 2.6]). According to [**3**, Corollary 2.3], D is not universally

catenarian. Now, let D_1 be a domain such that $D \subset D_1 \subseteq \mathrm{qf}(D)$ and set $R_1 := (T, M, D_1)$. R_1 is a proper overring of R, so it is universally catenarian. Hence $D_1 \simeq R_1/M$ is also universally catenarian. This yields that D is a maximal non-universally catenarian subring of its quotient field. □

Theorem 3.4. *Let T be a Prüfer domain, M a maximal ideal of T, D a subring of the field $K = T/M$, and $R := (T, M, D)$. Assume that R is not integrally closed. Then the following statements are equivalent:*

(i) *R is a maximal non-universally catenarian subring of its quotient field;*

(ii) *T is a valuation domain with maximal ideal M, D is a maximal non-universally catenarian subring of K (that is D is not universally catenarian and each domain D_1 such that $D \subset D_1 \subseteq K$ is universally catenarian), and D is a Jaffard domain.*

Proof: (i)⇒(ii) By Lemma 3.2, R is local. Then so is T. Thus T is a valuation domain with maximal ideal M. On the other hand, the domain R' is universally catenarian, in particular it is Jaffard [**8**, Corollary 3.3]. Hence R is a Jaffard domain [**1**, Proposition 1.1]. Thus $D \subset K$ is an algebraic extension and D is a Jaffard domain [**1**, Theorem 2.6]. Since R does not satisfy the altitude formula, then so does D [**5**, Lemme 3.6].

Now, since R' is a Prüfer domain, then so is D' [**4**, Proposition 2.2 and Théorème 2.6]. It remains to show that if D_1 is a ring such that $D \subset D_1 \subseteq K$, then D_1 is a universally catenarian domain. For this, D_1' is universally catenarian since $D' \subset D_1'$ is an algebraic extension ([**4**, Corollaire 3.9]). By [**8**, Theorem 6.1], it suffices to show that $D_1 \subset D_1'$ satisfies the altitude formula. Let $R_1 := (T, M, D_1)$, $R_2 := (T, M, D_1')$. It is clear that $R_1 \subseteq R_2 \subseteq R_1'$. The extension $R_1 \subseteq R_1'$ satisfies the altitude formula (Theorem 2.3); $R_1 \subseteq R_2$ and $R_2 \subseteq R_1'$ satisfy the altitude inequality formula since R_1 and R_2 are locally Jaffard domains. $R_2 \subseteq R_1'$ is a lying-over extension. Thus by [**4**, Lemme 3.1], $R_1 \subseteq R_2$ satisfies the altitude formula. Now, let $Q_1'/M \in \mathrm{Spec}(D_1')$ and set $Q_1 = Q_1' \cap R_1$. We have $ht_{R_2} Q_1' = ht_{R_1} Q_1$. Thus $ht_{D_1'}(Q_1'/M) = ht_{R_2} Q_1' - ht_{R_2} M = ht_{R_1} Q_1 - ht_{R_1} M = ht_{D_1}(Q_1/M)$ (because M is a divided prime in R_1 and R_2 and hence is comparable with all primes of R_1 and R_2). Therefore $D_1 \subset D_1'$ satisfies the altitude formula.

(ii)⇒(i) Notice that $D \subseteq K$ is an algebraic extension since D is a maximal non-universally catenarian subring of K. Now let R_1 be a domain such that $R \subset R_1 \subseteq \mathrm{qf}(R)$. If R_1 is an overring of T, then it is universally catenarian. If not, we get $R \subset R_1 \subseteq T$. Hence $R_1 := (T, M, D_1)$, where $D \subset D_1 \subseteq K$. Hence R_1 is universally catenarian [**3**, Corollary 2.3].

Now, since D is Jaffard and it is maximal non-universally catenarian subring of K, then $D \neq D'$ and $D \subset D'$ does not satisfy the altitude formula (Lemma 2.1). Hence $R \subset R'$ does not satisfy the altitude formula. Thus R does not satisfy the altitude formula (because if not, R satisfies absolutely the altitude formula and hence the extension $R \subset R'$ satisfies the altitude formula). Therefore R is a maximal subring of qf(R) not satisfying the altitude formula. Thus R is a maximal non-universally catenarian subring of its quotient field, the desired conclusion. $\square$

If we leave out the assumption "T is a valuation domain" in the previous proposition, the implication (ii)$\Rightarrow$(i) fails to be true. (See Example 4.3).

4. Examples and counterexamples

This section is concerned with examples showing the limits of the results established in the previous sections.

Example 4.1. For all integers n, $m \in \mathbf{N}$ satisfying $n + 1 \leq m \leq 2n + 1$, there exists a non-Jaffard local domain R such that $\dim R = n$, $\dim R[X] = m$ and R is a residually integrally closed domain.

If we assume that $m \neq n + 1$, then R need not be a maximal non-Jaffard subring of its quotient field.

Write $m = n+1+t$. Observe that $0 \leq t \leq n$. Let $K = k(X_1, X_2, \dots)$, where k is a field and $X_1, X_2, \dots$ is an infinite number of indeterminates over k. Let $Y_1, \dots, Y_t, Z_1, \dots, Z_t, Z_{t+1}, \dots, Z_n$ be ideterminates over K. Let $D_0 = K$. For $1 \leq j \leq t$, let $D_j = D_{j-1} + M_j$, where $M_j = Z_j K(Y_1, Z_1, \dots, Y_{j-1}, Z_{j-1}, Y_j)[Z_j]_{(Z_j)}$ is the maximal ideal of the valuation domain $V_j = K(Y_1, Z_1, \dots, Y_{j-1}, Z_{j-1}, Y_j)[Z_j]_{(Z_j)}$. For $t+1 \leq j \leq n$, let $D_j = D_{j-1} + M_j$, where $M_j = Z_j K(Y_1, Z_1, \dots, Y_t, Z_t, Z_{t+1}, \dots, Z_{j-1})[Z_j]_{(Z_j)}$ is the maximal ideal of the valaution domain

$$V_j = K(Y_1, Z_1, \dots, Y_t, Z_t, Z_{t+1}, \dots, Z_{j-1})[Z_j]_{(Z_j)}.$$

Notice that for any j such that $1 \leq j \leq n$, the set of the nonzero prime ideals of D_j is $\{M_s + M_{s+1} + \cdots + M_j \mid s = 1, \dots, j\}$ which is linearly ordered by inclusion. Observe also that $D_j = K + M_1 + \cdots + M_j$.

Let $R = D_n$, R is a local integral domain with maximal ideal $M = M_1 + \cdots + M_n$. We have $\dim R = n$, $\dim R[X] = m$ and $\dim_v R = n + t$ (cf. [**1**]). The domain R is such that R/P is integrally closed for each $P \in \mathrm{Spec}(R)$. Indeed, R is integrally closed [**9**, Proposition 2]. Let P be a nonzero prime ideal of R, there exists $1 \leq s \leq n$ such that $P = M_s + M_{s+1} + \cdots + M_n$. We have $R/P = (D_{s-1} + M_s + \cdots +$

$M_n)/(M_s + \cdots + M_n) \simeq D_{s-1}$, which is integrally closed. If we assume that $m > n + 1$, R can not be a maximal non-Jaffard subring of $\mathrm{qf}(R)$. Since if not, then we get $\dim_v R = \dim R + 1 = n + 1$ (Theorem 1.4). But $\dim R[X] = m > n + 1 = \dim_v R$, which is impossible.

The next example illustrates the fact that Proposition 2.2 does not hold if R is not integrally closed.

Example 4.2. This example provides a domain R such that:

(a) R is not integrally closed.
(b) R satisfies absolutely the altitude inequality formula.
(c) R is a maximal non-universally catenarian subring of its quotient field.

Let V, W be two incomparable valuation domains with the same quotient field L, with maximal ideals M_1 and N_1, respectively. Assume that $\dim V = 2$, $\dim W = 1$, and that $V/M_1 \simeq W/N_1 \simeq K$. Thus $S = V \cap W$ is a semilocal Prüfer domain with maximal ideals $M = M_1 \cap S$ and $N = N_1 \cap S$. Set $R := (S, I, K)$, where $I = M \cap N$. We have $\dim S = 2$ and $\dim R = 1$ (cf. [**9**, Corollaire 2]). The integral closure R' of R is equal to S [**9**, Proposition 2]. Hence R' is a Prüfer domain. Thus R satisfies absolutely the altitude inequality formula [**4**, Théorème 2.6].

We have $ht_S N + \mathrm{tr.\,deg}[S/N : R/(N \cap R)] = 1 < ht_R(N \cap R) + \mathrm{tr.\,deg}[S : R] = ht_R I = 2$. Thus the finite type extension $R \subset S$ does not satisfy the altitude formula. Therefore R does not satisfy the altitude formula. Now let T be a domain such that $R \subset T \subseteq \mathrm{qf}(R)$. Our task is to show that T satisfies the altitude formula. We claim that T' is a Prüfer domain (since it is an overring of R'). By [**14**, Proposition 2.6], it will suffices to show that the extension $T \subseteq T'$ satisfies the altitude formula. Notice that T' is an overring of S; then $T' \in \{S, V, W, V_{Q'_1}, V_{Q'_1} \cap W, \mathrm{qf}(S)\}$ ([**16**]), where Q'_1 is the unique prime ideal of S such that $(0) \subset Q'_1 \subset M$.

Let Q' be a nonzero prime ideal of T'. Set $Q_1 = Q' \cap S$, $Q = Q' \cap T$, and $P = Q' \cap R$. Our task is to show that $htQ' = htQ$. Two cases are then possible:

Case 1: If $Q_1 \neq N$, then $htQ_1 = htP$. On the other hand, $htQ' = htQ_1$ since the extension $S \subseteq T'$ is residually algebraic and satisfies the altitude formula. Thus $htQ' = htP$. The domain R is locally Jaffard; hence the extension $R \subset T$ satisfies the altitude inequality formula ([**4**, Théorème 1.5]), in particular, $htQ \leq htP$; which yields $htQ \leq htQ'$. But $T \subset T'$ is an integral extension, so it satisfies INC. Thus $htQ' \leq htQ$. Therefore $htQ' = htQ$.

Case 2: If $Q_1 = N$, then Q_1 is a maximal ideal of S. Hence Q is a maximal ideal of T'. We discuss the following cases:

(i) If $T' = W$, then T is local with $\dim T' = \dim T = htQ' = 1$. Thus Q is a maximal ideal of T. Hence $htQ' = htQ$.

(ii) The case when either $T' = V$ or $T' = V_{Q_1'}$ is impossible since N does not lift in V.

(iii) If $T' = V_{Q_1'} \cap W$, then T' is semilocal with maximal ideals $Q_1'T_{Q_1'}'$ and NT_N'. Thus $\dim T' = 1 = htQ$, and since $\dim T = \dim T' = 1$, then $htQ = htQ' = 1$.

(iv) If $T' = S$, in this case we have the following inclusions $R \subset T \subseteq T' = S$. Thus $R/I \subset T/I \subseteq S/I$; that is $K \subset T/I \subseteq K \times K$. Then necessarily we get $T/I = K \times K$. Indeed, $K \times K$ is a K-vector space of dimension 2. Thus T/I is K-vector space of dimension at most 2, and since $T/I \neq K$, then $T/I = K \times K$. Therefore $T = S = T'$ and $Q' = Q$.

As stated earlier, if we leave out the assumption "T is a valuation domain" in Theorem 3.4, the following example shows, among other facts that the implication (ii)$\Rightarrow$(i) fails.

Example 4.3. Denote by D the domain R constructed in Example 4.2. Let $L = \mathrm{qf}(D)$ and X, Y be two indeterminates over L. Set $V_1 = L(Y) + XL(Y)[X]_{(X)}$ and $V_2 = L + YL[Y]_{(Y)} + (X+1)L(Y)[X]_{(X+1)}$. Then V_1 and V_2 are two incomparable valuation domains with maximal ideals $M_1 = XL(Y)[X]_{(X)}$ and $M_2 = YL[Y]_{(Y)} + (X+1)L(Y)[X]_{(X+1)}$, respectively. We have $\dim V_1 = 1$ and $\dim V_2 = 2$. The domain $T_1 = V_1 \cap V_2$ is a semilocal Prüfer domain with maximal ideals $M' = M_1 \cap T_1$ and $N' = M_2 \cap T_1$. Let $M = M' \cap N'$ and $T := (T_1, M, L)$. Let $R := (T, M, D)$. Then we have the following properties:

(α) D is a Jaffard domain which is a maximal non-universally catenarian subring of its quotient field. (See Example 4.2).

(β) T' is a Prüfer domain since $T' = T$.

(γ) $D \subset T/M$ is an algebraic extension.

(δ) T does not satisfy the altitude formula (because the finite type extension $T \subset T_1$ does not satisfy the altitude formula). Hence R is not a maximal non-universally catenarian subring of its quotient field.

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