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Analysis of $L1$-difference methods for time-fractional nonlinear parabolic problems with delay

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Abstract

This work is concerned with numerical solutions of time-fractional nonlinear parabolic problems by a class of $L1$-difference methods. The analysis of $L1$ methods for time-fractional nonlinear problems with delay is limited mainly due to the lack of a fundamental Gronwall type inequality. We establish such a fundamental inequality for the $L1$ approximation to the Caputo fractional derivative. In terms of the Gronwall type inequality, we will provide error estimates of a fully discrete linearized difference scheme for this kind of problems.

Key words: time-fractional nonlinear parabolic problems with delay, $L1$-difference scheme, error estimates, Gronwall inequality, linearized schemes

1 Introduction

We study numerical solutions of the time-fractional nonlinear parabolic equation with delay

$$ {}^{C}D_t^{\alpha} u - \frac{\partial^2 u}{\partial x^2} = f(x,t,u,u(x,t-s)), \;\; (x,t) \in \Omega \times (0,T], \tag{1a} $$

with the following initial and boundary conditions

$$ u(x,t) = \phi(x,t), \qquad (x,t) \in \Omega \times [-s,0], \tag{1b} $$
$$ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{1c} $$

where $\Omega = [a,b]$ and $s > 0$ is a fixed delay parameter. The fractional derivative ${}^{C}D_t^{\alpha}$ of order $0 < \alpha \le 1$ is defined in Caputo sense.

In the past decades, developing effective numerical methods and rigorous numerical analysis for the time-fractional PDEs have been a hot research spot, see e.g. [2]. Numerical methods can be roughly divided into two categories: indirect and direct methods. The former is based on the solution of an integro-differential equation by some proper numerical schemes since time-fractional differential equations can be reformulated into integro-differential equations in general, while the latter is based on a direct (such as piecewise polynomial) approximation to the time-fractional derivative [1]. Direct methods are more popular in practical computations due to its ease of implementation. One of the most commonly used direct methods is the so-called $L1$-scheme, which can be viewed as a piecewise linear approximation to the fractional derivative and which has been widely applied for solving various time-fractional PDEs [3]. However, numerical analysis for direct methods is limited, even for a simple linear model (1) where $s = 0$ and $f(u) = L_0 u$.

The analysis of $L1$-type methods for the linear model was studied by several authors, while the convergence and error estimates were obtained under the assumption that $L_0 \leq 0$ in general, see [4]. Recently, this condition was improved in [6], in which a time-fractional nonlinear predator-prey model was studied by an $L1$ finite difference scheme and $f(u)$ was assumed to satisfy a global Lipschitz condition. The stability and convergence were proved under the assumption $T^\alpha < \frac{1}{L\Gamma(1-\alpha)}$ where L denotes the Lipschitz constant. This restriction condition implies that the scheme is convergent and stable only locally in time.

It is well known that the classical Gronwall inequality plays an important role in analysis of parabolic PDEs ($\alpha = 1$) and the analysis of corresponding numerical methods also relies heavily on the discrete counterpart of this inequality. Clearly, the analysis of $L1$-type numerical methods for time-fractional nonlinear differential equations ($0 < \alpha < 1$) has not been well done mainly due to the lack of such a fundamental inequality. In [5], the authors aimed to present the numerical analysis for several fully discrete $L1$ Galerkin FEMs for the general nonlinear equation 1 at $s = 0$ with any given $T > 0$. The key to their analysis is to establish a Gronwall type inequality for a positive sequence satisfying

$$D_\tau^\alpha \omega^k \leq \lambda_1 \omega^k + \lambda_2 \omega^{k-1} + g^k.$$

As an extension to the work of [5], the main purpose of our contribution is to discuss the numerical analysis for fully discrete L_1 difference schemes for the general nonlinear equation (1) with fixed delay with any given $T > 0$. Our analysis is done by constructing a new Gronwall type inequality for a positive sequence satisfying

$$D_\tau^\alpha \omega^k \leq \lambda_1 \omega^{k-1} + \lambda_2 \omega^{k-2} + \lambda_3 \omega^{k-n} + g^k,$$

where D_τ^α denotes an $L1$ approximation to ${}^C D_t^\alpha$, and λ_1, λ_2 and λ_3 are all positive constants.

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