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Comprehensive Numerical Schemes for Computational Electromagnetism

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Jaroslav Chovan

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Summary

The wide application of technologies in recent mechanical, electric and biomedical systems calls for materials and structures with nonconventional properties. Naturally, also theoretical understanding of the material behaviour and mathematical modelling is required.

This thesis is devoted to the mathematical modelling of electromagnetism in solid structures. Development of electromagnetism in matter is done with emphasis on material effects which are ascribed to the nonlinear constitutive relations and memory in time. Many electromagnetic phenomena are nonlinear (e.g. ferromagnetic materials). Memory effects occur in most of them and show up through dispersion and dissipation.

Throughout this thesis we do not assume any deformation of the body. The investigated system is characterized by a space-time region. This region is occupied by fields which arise from Maxwell's equations in the appropriate form.

In order to achieve the self-containedness of the thesis, we present some basic mathematical background in Chapter 1. Essential definitions and theorems from the field of functional analysis are followed by the definition of important function spaces for scalar and vector functions. Then, we briefly introduce the theory of monotone operators and partial differential equations. Lastly, we present several key inequalities and identities followed by a short introduction to Maxwell's equations. We split our thesis in two separate parts. In the first part we develop several mathematical models of the induction heating phenomena and provide a rigorous mathematical analysis of them. In the second part we recover a time dependent source term appearing in nonlinear Maxwell's equations from a single boundary measurement on a part of the considered boundary.

Part I

In Chapter 2 we develop a mathematical model for the electromagnetic induction heating of a continuous medium at rest. The model consists of two parts. The first part (electromagnetic) is described by Maxwell's equations. We assume a nonlinear relation between the magnetic field and the magnetic induction field. Moreover, the electric conductivity of the material is supposed to be temperature dependent. The second part (heat transfer) is determined by the nonlinear heat transfer equation. Induction heating processes create Joule heat in the material. This term acts as a heat source in the heat transfer equation, therefore, in order to control it, we apply a truncation function. The coupling between the two parts of our model is provided through the electric conductivity function on the one hand and through the Joule heating term on the other hand. The continuous system of equations is then discretized in time and basic energy estimates are obtained. We then use the Rothe method to prove the existence of a global solution to the whole system. The nonlinearities are overcome by the theory of monotone operators and the technique of Minty-Browder. This chapter is based on the article [68] which has been published in the journal of *Applicable Analysis*.

Another mathematical model of the induction heating process is presented in Chapter 3. The nonlinear relation between the magnetic field and the magnetic induction field describes the monotone behaviour of these fields. Generally, we can say that either the magnetic field depends on the magnetic induction field or vice versa. The latter relation was adopted in previous chapter. In this chapter we take into account the former nonlinear relation between these two fields, i.e. the magnetic field depends on the magnetic induction field. This approach leads to a different equation for the electromagnetic part of our model. The thermal part is modeled in the same fashion as in Chapter 2. Methods and techniques from the previous chapter are used to guarantee the existence of a global solution. The article [22] which has been published in the *Journal of Computational and Applied Mathematics* has been an inspiration for this chapter.

A mathematical model of the induction hardening is derived in Chapter 4. The domains considered in Chapters 2 and 3 are very simple. In this chapter the domain consists of a sphere where the electromagnetic field is present, a coil which is connected to a source of an alternating electric current and a workpiece which is subject to be heated by the process of the electromagnetic induction. We take into account that the magnetic permeability might behave differently in various materials e.g. in the air or in the workpiece. This assumption requires a subtle mathematical analysis. We consider a vector-scalar potential formulation of Maxwell's equation to grasp the electromagnetic part of our model. Evolution of temperature in the coil and the workpiece is modelled with the same truncated nonlinear heat transfer equation as in Chapters 2 and 3. This formulation yields a system of three coupled equations. We semi-discretize (time discretization) these equations and use the Rothe method to show the convergence of Rothe's functions towards a weak solution of the whole system. To supplement the theoretical results, we provide a simple numerical simulation. This chapter has been encouraged by the article [21] that has been published in the journal of *Computer Methods in Applied Mechanics and Engineering*.

Part II

In Chapter 5 we investigate a hyperbolic Maxwell's equation with an unknown time dependent source term. We start with a brief introduction to inverse source problems. In some applications, such as chiral media, meta-material, nonlinear optics or geophysics the solution values at present time strongly depend on the past. This phenomenon is expressed with a memory term. We consider a generalized nonlinear Ohm's law with memory. The time dependent part of the source term is reconstructed from a single boundary measurement over a part of the boundary. We discretize the equation in time and propose a numerical scheme (implicit Euler scheme) that provides us with a solution at each time step. This scheme is obtained by the application of Rothe's method. We then prove the existence of a global solution. In the case of a regular solution, we also prove its uniqueness. The *Numerical experiment* section contains an academic example which demonstrates the convergent behaviour of the proposed scheme. This chapter has been heavily influenced by an article that has been submitted to the *Journal of Computational and Applied Mathematics*.

The results and findings of this thesis are concluded in Chapter 6 where we discuss the possible improvements and potential objects of interest for the future research.

Samenvatting

De brede toepassing van technologie in nieuwe mechanische, elektrische en biomedische systemen vereist materialen en structuren met niet-conventionele eigenschappen. Uiteraard zijn ook theoretisch begrip van het gedrag van het materiaal en wiskundige modellering nodig.

Deze thesis is toegewijd aan de wiskundige modellering van elektromagnetisme in vaste structuren. Elektromagnetisme wordt ontwikkeld met de nadruk op materiaaleffecten die toegeschreven zijn aan de niet-lineaire constitutieve relaties en aan gebeurtenissen uit het verleden. Vele elektromagnetische verschijnselen zijn niet-lineair (bijvoorbeeld ferromagnetische materialen). Bij de meeste van deze verschijnselen treedt de invloed van het verleden op via disseminatie en dissipatie.

Doorheen deze thesis veronderstellen we geen vervorming van het lichaam en bijgevolg wordt het onderzochte systeem gekarakteriseerd door een gebied dat vast is in tijd en ruimte. Deze regio wordt voorzien van velden die ontstaan uit een geschikte vorm van de Maxwell vergelijkingen.

De wiskundige achtergrond die nodig is om deze thesis vlot te b stellen we voor in Hoofdstuk 1. Essentiële definities en stellingen uit de functionaalanalyse worden gevolgd door de definities van belangrijke functieruimten voor scalaire functies en vectorfuncties. Vervolgens introduceren we kort de theorie van monotone operatoren en partiële differentiaalvergelijkingen. Tot slot vermelden we verschillende noodzakelijke gelijkheden en ongelijkheden en geven we een korte introductie op de Maxwell vergelijkingen. Deze thesis bestaat uit twee afzonderlijke delen. In het eerste deel ontwikkelen we verschillende wiskundige modellen voor de (electromagnetische) inductieverhitting en voorzien we een strikte wiskundige analyse ervan. In het tweede deel vinden we een tijdsafhankelijke bronterm in niet-lineaire Maxwell vergelijkingen terug uit één enkele meting op een deel van de rand van het beschouwde domein.

Deel I

In Hoofdstuk 2 ontwikkelen we een wiskundig model voor de elektromagnetische inductieverhitting van een continu medium in rust. Het model bestaat uit twee delen. Het eerste deel (elektromagnetisch) wordt beschreven door Maxwell vergelijkingen. We veronderstellen een niet-lineaire relatie tussen het magnetisch veld en het magnetisch inductieveld. Bovendien wordt verondersteld dat de elektrische geleidbaarheid van het materiaal tijdsafhankelijk is. Het tweede deel (warmteoverdracht) wordt bepaald door de niet-lineaire warmtevergelijking. Inductie verhittingsprocessen creëren Joule warmte in het materiaal. Deze term gedraagt zich als een warmtebron in de warmtevergelijking. Bijgevolg passen we een truncatiefunctie in het rechterlid toe om deze term onder controle te houden. De koppeling tussen de twee delen van het model wordt gegeven door de elektrische conductiviteitsfunctie enerzijds en door de Joule warmte term anderzijds. Het continue systeem van vergelijkingen wordt vervolgens gediscretiseerd in de tijd en basisafschattingen voor de energie worden bekomen. Nadien gebruiken we de Rothe methode om het bestaan van een globale oplossing van het volledige systeem te bewijzen. De niet-lineariteiten worden behandeld via de theorie van monotone operatoren en de techniek van Minty-Browder. Dit hoofdstuk is gebaseerd op het artikel [68] dat gepubliceerd is in het tijdschrift *Applicable Analysis*.

Een ander wiskundig model voor het inductie verhittingsproces wordt gepresenteerd in Hoofdstuk 3. De niet-lineaire relatie tussen het magnetisch veld en het magnetisch inductieveld beschrijft het monotone gedrag van deze velden. Algemeen kunnen we stellen dat ofwel het magnetisch veld afhangt van het magnetisch inductieveld of andersom. De laatste relatie werd aangenomen in het vorige hoofdstuk. In dit hoofdstuk beschouwen we de eerste niet-lineaire relatie tussen deze twee velden, d.i. het magnetisch veld hangt af van het magnetisch inductieveld. Deze benadering leidt tot een andere vergelijking voor het elektromagnetisch deel van ons model. Het thermisch deel wordt gemodelleerd op dezelfde manier als in Hoofdstuk 2. Methoden en technieken van het vorige hoofdstuk worden gebruikt om het bestaan van een globale oplossing te garanderen. Het artikel [22], dat gepubliceerd is in het tijdschrift *Journal of Computational and Applied Mathematics*, heeft als inspiratie gediend voor dit hoofdstuk.

Een wiskundig model van inductieverharding wordt afgeleid in Hoofdstuk 4. De domeinen die beschouwd werden in Hoofdstukken 2 en 3 waren zeer eenvoudig. In dit hoofdstuk bestaat het domein uit een bol, waarbinnen het elektromagnetisch veld aanwezig is, een spoel die verbonden is met een bron van alternerende elektrische stroom en een object dat verwarmd moet worden via het proces van de elektromagnetische inductie. We houden er rekening mee dat magnetische permeabiliteit zich mogelijks anders gedraagt in verschillende materialen, d.i. in de lucht of in het object. Deze veronderstelling vereist een subtiele wiskundige analyse. We beschouwen een vector-scalaire potentiaalvorm van de Maxwell vergelijkingen om het elektromagnetisch deel van ons model te beschrijven. De temperatuursevolutie in de veer en in het object wordt gemodelleerd met dezelfde niet-lineaire warmtevergelijking met getrunceerd rechterlid als in Hoofdstukken 2 en 3. Deze formulering leidt tot een systeem van drie gekoppelde vergelijking. We discretiseren deze vergelijkingen in de tijd en gebruiken de Rothe methode om de convergentie van de Rothe functies naar een zwakke oplossing van het volledige systeem aan te tonen. Om de theoretische resultaten te ondersteunen voeren we een eenvoudige numerieke simulatie uit. Dit hoofdstuk is gebaseerd op het artikel [21] dat gepubliceerd is in het tijdschrift *Computer Methods in Applied Mechanics and Engineering*.

Deel II

In hoofdstuk 5 onderzoeken we een hyperbolische Maxwell vergelijking met een onbekende tijdsafhankelijke bronterm. We starten met een korte introductie op inverse bronproblemen. In sommige toepassingen, zoals in chirale media, metamateriaal, niet-lineaire optiek of geofysica, de waarden van de oplossing in het heden sterk af van het verleden. Dit verschijnsel wordt uitgedrukt met een term die afhangt van de waarden van de oplossing op vorige tijdstippen. We beschouwen een veralgemeende niet-lineaire wet van Ohm met dergelijke term. Het tijdsafhankelijk deel van de bronterm wordt gereconstrueerd op basis van een enkele meting over een deel van de rand. We discretiseren de vergelijking in tijd en stellen een numeriek schema (impliciet Euler schema) voor dat ons een oplossing verschaft op elk tijdstip. Dit schema wordt onmiddellijk geïmpliceerd door de toepassing van de Rothe methode. Nadien bewijzen we het bestaan van een globale oplossing. In het geval van een reguliere oplossing bewijzen we ook de uniciteit ervan. De sectie *Numerical experiment* bevat een academisch voorbeeld dat het convergentiegedrag van het voorgestelde schema demonstreert. Dit hoofdstuk werd sterk beïnvloed door een artikel dat ingediend is bij het tijdschrift *Journal of Computational and Applied Mathematics*.

De resultaten en bevindingen van deze thesis worden samengevat in Hoofdstuk 6. We bespreken mogelijke verbeteringen en potentiële onderwerpen voor verder onderzoek.

Chapter 1

Mathematical background

To enhance the coherence of this thesis, we provide the reader with some essential definitions and important theorems from the fields of functional analysis, Sobolev spaces, monotone operators, partial differential equations and Maxwell's equations. We also state several key inequalities and integral identities which are used in the sequel of the thesis. The reader is expected to be familiar with the following topics: linear algebra, real mathematical analysis and the theory of Lebesgue measure and integration.

Let us further make a small remark about notations used in the following chapters. Positive constants are expressed with symbols $\varepsilon, C_\varepsilon$ and C. These constants depend only on a priori known data and are either very small (ε) or quite large (C_ε, C). We do not distinguish between subsequences and original sequences in the convergence sections of coming chapters. The right hand side and the left hand side of an equation are denoted as r.h.s. and l.h.s., respectively. This is done solely to sustain the clarity and readability of the thesis.

1.1 Functional analysis

The main references, among many others in this section, are [4, 48, 51, 52, 67, 87, 89] and [90].

Definition 1.1 (Normed vector Space)**.** *A vector space X is said to be a normed vector space if to every $x \in X$ there is associated a nonnegative real number $\|x\|$, called the norm of x, in such way that the following holds:*

(a) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X,$

(b) $\|\alpha x\| = |\alpha| \, \|x\|$ *if* $x \in X$ *and* α *is a scalar,*

(c) $\|x\| > 0$ *if* $x \neq 0$.

Every normed vector space may be regarded as metric space, in which the distance $d(x, y)$ *between* x *and* y *is* $\|x - y\|$.

Definition 1.2 (Cauchy sequence)**.** *A sequence* $\{x_n\}_{n=0}^{\infty}$ *in a metric space* X *is called Cauchy if for every* $\varepsilon > 0$ *there exists a nonnegative integer* N *such that* $d(x_m, x_n) < \varepsilon$ *for all* $m, n \geq N$.

Definition 1.3 (Complete space)**.** *A metric space* X *is called complete if every Cauchy sequence* $\{x_n\}_{n=0}^{\infty}$, $x_n \in X$ *converges to* $x \in X$.

Definition 1.4 (Banach space)**.** *Every complete normed vector space is a Banach space.*

Definition 1.5 (Inner product space)**.** *A vector space* X *is said to be an inner product space if to every* $x, y \in X$ *there is a real or complex number* $(x, y)_X$ *associated, called the inner product of* x *and* y*, in such way that the following is true:*

(a) $(x, y) = \overline{(y, x)} \quad \forall x, y \in X,$

(b) $(cx, y) = c\,(x, y)$ *if* $x, y \in X$ *and* c *is a scalar,*

(c) $(x, x) \geq 0$ *if* $x \in X$ *and* $(x, x) = 0$ *only if* $x = 0$.

Given an inner product we define the norm as $\|x\|_X = (x, x)^{1/2}$ *for all* $x \in X$.

Definition 1.6 (Hilbert space)**.** *Every inner product space which is complete with respect to the norm induced by the inner product is called a Hilbert space.*

Definition 1.7 (Density)**.** *A set* Y *is called dense in a normed vector space* X *if its closure is the whole space* X*, i.e.* $\overline{Y} = X$.

Definition 1.8 (Separability)**.** *A normed vector space* X *is called separable if it contains a countable set of points that is dense.*

Definition 1.9 (Compactness)**.** *A set* Y *in a normed vector space* X *is called relatively compact if and only if (iff) every sequence in* Y *contains a convergent subsequence. If* Y *is also closed then it is called compact.*

Definition 1.10 (Linear bounded functional)**.** *A linear functional on a normed vector space* X *is a mapping* $F : X \to \mathbb{R}$ *(or* $\mathbb{C}$*), that satisfies the following properties:*

(a) $F(x+y) = F(x) + F(y) \quad \forall x, y \in X,$

(b) $F(cx) = cF(x)$ *if* $x \in X$ *and* c *is a scalar.*

F is said to be bounded iff there exists a constant $C > 0$ such that $|F(x)| \leq C \|x\|_X$ for all $x \in X$.

If X is an inner product space and y is some fixed vector in X then the mapping $X \to F$ defined as $x \to (x, y)_X$ represents a linear functional on X.

Definition 1.11 (Dual space)**.** *Given any normed vector space X, the dual space X^* is defined as the set of all linear functionals on X. The norm in X^* is defined as*

$$\|F\|_{X^*} = \sup_{\|x\|_X \leq 1,\ x \neq 0} \frac{|F(x)|}{\|x\|_X}.$$

Theorem 1.1 (Hahn-Banach)**.** *Let X be normed vector space over $\mathbb{R}$, and let Y be a linear subspace of X. Suppose that $f \in Y^*$ then f can be extended to a linear functional $F \in X^*$ with $\|F\|_{X^*} = \|f\|_{Y^*}$.*

Definition 1.12 (Strong convergence)**.** *A sequence $\{x_n\}_{n=0}^{\infty}$ in a normed vector space X is said to be strongly convergent (or convergent in the norm) if there is an $x \in X$ such that*

$$\lim_{n \to \infty} \|x_n - x\|_X = 0.$$

This is written as $x_n \to x$. The element x is called the strong limit of $\{x_n\}_{n=0}^{\infty}$, and we say that $\{x_n\}_{n=0}^{\infty}$ converges strongly to x.

Definition 1.13 (Weak convergence)**.** *A sequence $\{x_n\}_{n=0}^{\infty}$ in a normed vector space X is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X^*$ the following holds*

$$\lim_{n \to \infty} f(x_n) = f(x).$$

This is written as $x_n \rightharpoonup x$. The element x is called the weak limit of $\{x_n\}_{n=0}^{\infty}$, and we say that $\{x_n\}_{n=0}^{\infty}$ converges weakly to x.

Definition 1.14 (Reflexivity)**.** *Let X be a normed vector space and X^{**} denote the second dual vector space of X. The canonical map $X \to X^{**} : x \to \hat{x}$ defined by $\hat{x}(f) = f(x)$, $x \in X$, $f \in X^*$ gives an isometric linear isomorphism from X into X^{**}. The space X is called reflexive if this map is surjective.*

Theorem 1.2 (Weak compactness of reflexive spaces)**.** *Assume that X is a reflexive Banach space and $\{x_n\}_{n=0}^{\infty}$, $x_n \in X$ a bounded sequence. Then there exists a subsequence $\{x_{n_l}\}_{n_l=0}^{\infty}$ and $x \in X$ such that $x_{n_l} \rightharpoonup x$.*

Lemma 1.1 (Lax-Milgram)**.** *(see [55, Lemma 2.21]) Let X be a Hilbert space and assume that A is a bilinear functional and F is a continuous linear functional that satisfies:*

(a) $A(v,v) \geq \alpha \|v\|_X^2$ *for some* $\alpha > 0$ *and* $\forall v \in X$,

(b) $|A(v,w)| \leq C \|v\|_X \|w\|_X$ *for some positive* $C \in \mathbb{R}$ *and* $\forall v, w \in X$.

Then there is a unique $u \in X$ such that $A(u,v) = F(v)$, $\forall v \in X$ and the stability estimate $\|u\|_X^2 \leq \frac{|F(u)|}{\alpha}$ holds.

Definition 1.15 (Equicontinuity)**.** *Let (X,d) be a compact metric space. Then the space $C(X)$ is a vector space consisting of all continuous functions $f : X \to \mathbb{R}$. The space $C(X)$ is equipped with the norm $\|f\| := \max\{|f(x)| \mid x \in X\}$. The family $\mathcal{F} \subset C(X)$ is called equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends only on ε) such that for $x, y \in X$:*

$$d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F}.$$

Definition 1.16 (Equiboundedness)**.** *The family $\mathcal{F} \subset C(X)$ is called equibounded if there exists a positive constant $M < \infty$ such that $|f(x)| \leq M$ for each $x \in X$ and each $f \in \mathcal{F}$.*

Definition 1.17 (Gâteaux differential)**.** *Suppose that X and Y are Banach spaces, $U \subset X$ is open and let $F : X \to Y$ be a given map. The Gâteaux differential $DF(u;\psi)$ of F at $u \in U$ in the direction $\psi \in X$ is defined as*

$$DF(u;\psi) = \lim_{\tau \to 0} \frac{F(u + \tau\psi) - F(u)}{\tau} = \frac{d}{d\tau} F(u + \tau\psi)\Big|_{\tau=0}.$$

If the limit exists for all $\psi \in X$, then F is Gâteaux differentiable at u.

1.2 Function spaces

Throughout this chapter the symbol Ω always represents a bounded domain in $\mathbb{R}^n$ with $n \geq 1$. The boundary of Ω is denoted as $\partial\Omega$ and is assumed to be Lipschitz continuous.

Definition 1.18. *$L^p(\Omega)$, $1 \leq p < \infty$, is the set of all measureable functions u in Ω such that the norm*

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \ dx \right)^{1/p}$$

is finite. $L^p(\Omega)$ is a Banach space.

Definition 1.19. $L^p_{loc}(\Omega)$, $1 \le p < \infty$, *is the set of all measurable functions* u *in* Ω *such that* $\int_{\Omega'} |u(x)|^p \ dx < \infty$ *for any subdomain* $\Omega' \subset \Omega$ *such that* $\overline{\Omega'} \subset \Omega$.

Definition 1.20. $L^\infty(\Omega)$ *is the set of all bounded measurable functions* u *in* Ω *with the norm defined as*

$$\|u\|_{L^\infty(\Omega)} = ess \sup_{x \in \Omega} |u(x)| .$$

At this point we introduce a multi-index notation for partial derivatives

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\alpha_i \in \mathbb{N}_0$ for $i = 1, \dots, n$.

Definition 1.21. $C_0^\infty(\Omega)$ *is the class of functions* u *in* Ω *such that:*

(a) u *is infinitely smooth, which means that* $\partial^\alpha u$ *is uniformly continuous in* $\overline{\Omega}$, $\forall \alpha$,

(b) u *is compactly supported, i.e.* $supp(u)$ *is a compact subset of* Ω.

Theorem 1.3 (Lebesgue's dominated convergence theorem)**.** *Let* $\{f_n\}$ *be a sequence of Lebesgue measurable functions* $f_n : \Omega \to \mathbb{R}$. *Assume that* f_n *converges almost everywhere (a.e.) in* Ω *to a measurable function* f. *Moreover, assume that for* $n \in \mathbb{N}$ *the function* f_n *is dominated by a function* $g \in L^p(\Omega)$, *i.e.* $|f_n| \le g$ *a.e. Then, all* f_n *and* f *are in* $L^p(\Omega)$ *and the sequence* $\{f_n\}$ *converges to* f *in the sense of* L^p, *i.e.*

$$\lim_{n \to \infty} \|f_n - f\|_{L^p(\Omega)} = 0.$$

Definition 1.22 (Weak derivative)**.** *Suppose that* $u, v \in L^1_{loc}(\Omega)$, *and*

$$\int_\Omega u(x) \partial^\alpha \eta(x) \ dx = (-1)^{|\alpha|} \int_\Omega v(x) \eta(x) \ dx, \quad \forall \eta \in C_0^\infty(\Omega).$$

Then v *is called the weak (or distributional) partial derivative of* u *in* Ω *and is denoted by* $\partial^\alpha u$.

Definition 1.23 ($W^{l,p}(\Omega)$ space)**.** *Suppose that* $u \in L^p(\Omega)$ *and that there exist weak derivatives* $\partial^\alpha u$ *for any* α *with* $|\alpha| \le l$, *such that*

$$\partial^\alpha u \in L^p(\Omega), \quad |\alpha| \le l.$$

Then we say that $u \in W^{l,p}(\Omega)$. *This space is called the Sobolev space and is equipped with the standard norm:*

$$\|u\|_{W^{l,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \le l} |\partial^\alpha u|^p \ dx \right)^{1/p}.$$

Remark 1.1. *If* $p = 2$, *the space* $W^{l,2}(\Omega)$ *is a Hilbert space and is usually denoted as* $H^l(\Omega)$. *The inner product for* $u, v \in H^l(\Omega)$ *is defined as*

$$(u, v) = \int_\Omega \sum_{|\alpha| \le l} \partial^\alpha u(x) \overline{\partial^\alpha v(x)} \ dx.$$

Definition 1.24 ($W_0^{l,p}(\Omega)$ space)**.** *The closure of* $C_0^\infty(\Omega)$ *in the norm of* $W^{l,p}(\Omega)$ *is denoted by* $W_0^{l,p}(\Omega)$.

Definition 1.25 ($H^1(\Omega)/\mathbb{R}$ space)**.** *The quotient space* $H^1(\Omega)/\mathbb{R}$ *is defined as*

$$H^1(\Omega)/\mathbb{R} = \{u \in H^1(\Omega) : \int_{\partial\Omega} u \ dx = 0\}.$$

Definition 1.26 (Continuous embedding)**.** *Let* X *and* Y *be two Banach spaces. We say that* X *is embedded into* Y *and write* $X \hookrightarrow Y$, *if for any* $u \in X$, *we have* $u \in Y$ *and* $\|u\|_Y \le C \|u\|_X$ *where the constant* C *is nonnegative and does not depend on* $u \in X$.
We define the embedding operator $J : X \to Y$ *which takes* $u \in X$ *into the same element* u *considered as an element of* Y.

Remark 1.2. *The fact that* $X \hookrightarrow Y$ *is equivalent to the fact that the embedding operator* $J : X \to Y$ *is continuous linear operator.*

$$\textit{If} \quad \|u\|_Y \le C \|u\|_X, \quad \forall u \in X, \quad \textit{then} \quad \|J\|_{X \to Y} \le C.$$

Definition 1.27 (Compact embedding)**.** *If* $X \hookrightarrow Y$ *and the embedding operator* $J : X \to Y$ *is a compact operator, then we say that* X *is compactly embedded into* Y.

Remark 1.3. *The compactness of operator* J *is equivalent to the fact that any bounded set in* X *is a compact set in* Y.

Theorem 1.4 (Embedding theorem for $W^{l,p}(\Omega)$)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain.*

(a) *If* $p \ge 1$, $1 \le q < \infty$, $0 \le r \le l$, $l - r - \frac{n}{p} + \frac{n}{q} \ge 0$, *then* $W^{l,p}(\Omega) \hookrightarrow W^{r,q}(\Omega)$. *If* $l - r - \frac{n}{p} + \frac{n}{q} > 0$, *then this embedding is compact.*

(b) If $p(l-r) > n$, then $W^{l,p}(\Omega) \hookrightarrow C^r(\overline{\Omega})$ and this embedding is compact.

Definition 1.28 (Bochner space)**.** *Let X be a Banach space and $0 < T < \infty$. The space $L^p((0,T);X)$ consists of all measurable functions $u : [0,T] \to X$ with*

(a) $\|u\|_{L^p((0,T);X)} := \left(\int_0^T \|u(t)\|_X^p \; dt\right)^{1/p} < \infty$ for $1 \le p < \infty$, and

(b) $\|u\|_{L^\infty((0,T);X)} := ess \sup_{0\le t\le T} \|u(t)\|_X < \infty$ for $p = \infty$.

The space $L^p((0,T);X)$ is also called a Bochner space.

Definition 1.29. *The space $C([0,T];X)$ comprises all continuous functions $u : [0,T] \to X$ with*

$$\|u\|_{C([0,T];X)} := \max_{0\le t\le T} \|u(t)\|_X < \infty.$$

Definition 1.30. *Assume that X is a reflexive Banach space. By $C_w((0,T);X)$ we denote the set of all $u : (0,T) \to X$ satisfying $\langle f, u(t)\rangle \in C(0,T)$ for all $f \in X^*$.*

Theorem 1.5 (Generalized Arzelà-Ascoli theorem)**.** *Let X and Y be reflexive Banach spaces and let the embedding $X \hookrightarrow Y$ be compact.*

(i) If the sequence $\{u_n\}_{n=0}^\infty$, $u_n : (0,T) \to X$ is equibounded and equicontinuous. Then there exists $u \in C_w((0,T);X) \cap L^\infty((0,T);X)$ and a subsequence $\{u_{n_k}\}_{n_k=0}^\infty$ such that $u_{n_k} \rightharpoonup u(t)$ in X for all $t \in (0,T)$.

(ii) If the sequence $\{u_n\}_{n=0}^\infty$, $u_n : (0,T) \to X$ is equibounded and $u_n : (0,T) \to Y$ is equicontinuous, then there exists $u \in C((0,T);Y) \cap L^\infty((0,T);X)$ and a subsequence $\{u_{n_k}\}_{n_k=0}^\infty$ such that $u_{n_k} \to u$ in $C((0,T);Y)$ and $u_{n_k} \rightharpoonup u$ in X for a.e. $t \in (0,T)$.

1.3 Sobolev spaces for vector fields

The inner product in $L^2(\Omega)$ can be simply extended to vector functions. Suppose that $\boldsymbol{u} = (u_1, u_2, u_3) \in \mathbf{L}^2(\Omega) := (\mathrm{L}^2(\Omega))^3$ and $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbf{L}^2(\Omega)$. Then, we define the inner product in $\mathbf{L}^2(\Omega)$ as

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sum_{i=1}^3 u_i \overline{v_i} \, \mathrm{d}x.$$

Norm in this space is induced by its inner product, i.e.

$$\|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} = \left(\int_\Omega \sum_{i=1}^3 |u_i|^2 \; \mathrm{d}x\right)^{1/2}.$$

Definition 1.31 (The curl and the divergence operator)**.** *Let* $\boldsymbol{v} \in (\mathbf{C}_0^\infty(\Omega))^*$*. Then the curl operator is defined as*

$$\nabla \times \boldsymbol{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

And the divergence operator is defined as

$$\nabla \cdot \boldsymbol{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}.$$

Definition 1.32. *The standard Sobolev spaces for vector fields* $\mathbf{H}^1(\Omega)$*,* $\mathbf{H}(\mathbf{curl}\,;\Omega)$ *and* $\mathbf{H}(\mathrm{div};\Omega)$ *are defined as*

$$\begin{aligned} \mathbf{H}^1(\Omega) &:= \{\boldsymbol{u} \in \mathbf{L}^2(\Omega) : \nabla \boldsymbol{u} \in (\mathrm{L}^2(\Omega))^{3\times 3}\}, \\ \mathbf{H}(\mathbf{curl}\,;\Omega) &:= \{\boldsymbol{u} \in \mathbf{L}^2(\Omega) : \nabla \times \boldsymbol{u} \in \mathbf{L}^2(\Omega)\}, \\ \mathbf{H}(\mathrm{div};\Omega) &:= \{\boldsymbol{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \boldsymbol{u} \in \mathrm{L}^2(\Omega)\} \end{aligned}$$

with the associated norms

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} &= \left(\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \boldsymbol{u}\|^2_{(\mathrm{L}^2(\Omega))^{3\times 3}} \right)^{1/2}, \\ \|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} &= \left(\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \times \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \right)^{1/2}, \\ \|\boldsymbol{u}\|_{\mathbf{H}(\mathrm{div};\Omega)} &= \left(\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|^2_{\mathrm{L}^2(\Omega)} \right)^{1/2} \end{aligned}$$

where $\|\nabla \boldsymbol{u}\|^2_{(\mathrm{L}^2(\Omega))^{3\times 3}} = \sum_{i=1}^{3} \|\nabla u_i\|^2_{\mathbf{L}^2(\Omega)} = \sum_{j=1}^{3} \sum_{i=1}^{3} \left\|\partial_{x_j} u_i\right\|^2_{\mathrm{L}^2(\Omega)}$.

Theorem 1.6. *Suppose that* $\boldsymbol{n}$ *is a unit outward normal vector on the boundary* $\partial\Omega$*. Then*

$$\begin{aligned} \mathbf{H}_0(\mathbf{curl}\,;\Omega) &= \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega) : \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\}, \\ \mathbf{H}_0(\mathrm{div},\Omega) &= \{\boldsymbol{u} \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

The following spaces are commonly associated with solutions of problems derived from Maxwell's equations.

Definition 1.33.

$$\begin{aligned} \boldsymbol{X}_N &= \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{n} \times \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega\}, \\ \boldsymbol{X}_{N,0} &= \{\boldsymbol{u} \in \boldsymbol{X}_N : \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega\}. \end{aligned}$$

Theorem 1.7 (Friedrich's inequality for vector fields, [30]). *Assume that* Ω *is also simply-connected. Then there exists a positive constant* C *such that:*

$$\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \le C \|\nabla \times \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{X}_{N,0}.$$

Now, we define norms for the spaces mentioned above.

Definition 1.34. *Norm in the space* $\boldsymbol{X}_N$ *is a usual graph norm, i.e.*

$$\|\boldsymbol{u}\|_{\boldsymbol{X}_N} = \|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \times \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} .$$

Using the Friedrich's inequality above we furnish the space $\boldsymbol{X}_{N,0}$ *with the norm*

$$\|\boldsymbol{u}\|_{\boldsymbol{X}_{N,0}} = \|\nabla \times \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} .$$

The following theorem is crucial for the mathematical approach used in this thesis. For more details, we refer the reader to [35, Theorem 3.7] and [3, Theorem 2.12].

Theorem 1.8. *The space* $\boldsymbol{X}_N$ *and the space* $\boldsymbol{X}_{N,0}$ *are continuously embedded into* $\mathbf{H}^1(\Omega)$.

Remark 1.4. *The embedding above holds true also for a convex domain* Ω*. For more details, we refer the reader to [3, Theorem 2.17].*

Theorem 1.9. *The embedding of* $\boldsymbol{X}_N$ *into* $\mathbf{L}^2(\Omega)$ *is compact.*

For more details about Sobolev spaces for scalar and vector valued functions we refer the reader to [31, 49, 55] and [73].

1.4 Monotone operators

Let X be a real Banach space. In this section we present the main theorem on monotone operators from [91] (for more details on monotone operators we refer the reader to [56, 79]). First, let us introduce the subject of the study that is the following operator equation

$$Au = b \quad \text{for } u \in X, \tag{1.1}$$

where $A : X \to X^*$. Theory of monotone operators is sometimes used to overcome the nonlinearities in partial differential equations so the existence of a unique solution can be provided. More in depth theory can be found in the work of Minty and Browder (see [54] and [18]) and in Zeidler [91]. Before we introduce the main theorem it is necessary to present some key definitions.

Definition 1.35 (Monotone operator)**.** *A is called monotone iff*

$$\langle Au - Av, u - v \rangle \geq 0 \quad \forall u, v \in X.$$

Definition 1.36 (Strictly monotone operator)**.** *A is called strictly monotone iff*

$$\langle Au - Av, u - v \rangle > 0 \quad \forall u, v \in X \quad \textit{with } u \neq v.$$

Definition 1.37 (Strongly monotone operator)**.** *A is called strongly monotone iff there is a* $C > 0$ *such that*

$$\langle Au - Av, u - v \rangle \geq C \|u - v\|_X^2 \quad \forall u, v \in X.$$

Definition 1.38 (Coercive operator)**.** *A is called coercive iff*

$$\lim_{\|u\|_X \to \infty} \frac{\langle Au, u \rangle}{\|u\|_X} = +\infty.$$

Definition 1.39 (Demicontinuous operator)**.** *A is said to be demicontinuous iff*

$$u_n \to u \quad \textit{as} \quad n \to \infty$$

implies $Au_n \rightharpoonup Au$ *as* $n \to \infty$.

Definition 1.40 (Hemicontinuous operator)**.** *A is said to be hemicontinuous iff the real function*

$$t \mapsto \langle A(u + tv), w \rangle$$

is continuous on $[0, 1]$ *for all* $u, v, w \in X$.

Theorem 1.10 (Main theorem on monotone operators)**.** *Let* $A : X \to X^*$ *be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space* X. *Then the following assertions hold:*

(a) For each $b \in X^*$, *equation* (1.1) *has a solution.*

(b) If the operator A is strictly monotone then equation (1.1) *has a unique solution.*

(c) If A is strictly monotone then the inverse operator $A^{-1} : X^* \to X$ *exists. This operator is strictly monotone, demicontinuous, and bounded.*
If A is strongly monotone then A^{-1} *is Lipschitz continuous.*

1.4.1 Potential of a vector field

Since potentials of vector fields will be used throughout Chapters 2, 3 and 4 we would like to introduce the formal definition of it. Moreover, we present a theorem that provides a crucial inequality which is used to obtain necessary a priori estimates in the following chapters.

Definition 1.41. *We say that $\Phi_{\boldsymbol{H}}(\boldsymbol{x})$ is a potential of the vector field $\boldsymbol{H}(\boldsymbol{x})$ if the Gâteaux differential of $\Phi_{\boldsymbol{H}}(\boldsymbol{x})$ in the direction $\boldsymbol{y} \in \Omega$ is $\boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{y}$, i.e.*

$$D\Phi_{\boldsymbol{H}}(\boldsymbol{x};\boldsymbol{y}) = \lim_{\tau \to 0} \frac{\Phi_{\boldsymbol{H}}(\boldsymbol{x}+\tau\boldsymbol{y}) - \Phi_{\boldsymbol{H}}(\boldsymbol{x})}{\tau} = \boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{y} \quad \text{for any } \boldsymbol{x},\boldsymbol{y} \in \Omega.$$

Theorem 1.11 (from [79])**.** *Let $\Phi_{\boldsymbol{H}}(\boldsymbol{y})$ be the potential introduced above and let $\boldsymbol{H}$ be a strictly monotone vector field. Assume that $\Phi_{\boldsymbol{H}}(\boldsymbol{y})$ is twice Gâteaux differentiable and also that the following holds*

$$D^2\Phi_{\boldsymbol{H}}(\boldsymbol{y};\boldsymbol{h},\boldsymbol{h}) = \lim_{\tau \to 0} \frac{D\Phi_{\boldsymbol{H}}(\boldsymbol{x}+\tau\boldsymbol{h};\boldsymbol{h}) - D\Phi_{\boldsymbol{H}}(\boldsymbol{x};\boldsymbol{h})}{\tau} \geq 0 \quad \text{for any } \boldsymbol{y},\boldsymbol{h} \in \Omega.$$

Then

$$\boldsymbol{H}(\boldsymbol{x}) \cdot (\boldsymbol{x}-\boldsymbol{y}) \geq \Phi_{\boldsymbol{H}}(\boldsymbol{x}) - \Phi_{\boldsymbol{H}}(\boldsymbol{y}) \quad \text{for any } \boldsymbol{x},\boldsymbol{y} \in \Omega.$$

1.5 Partial differential equations

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and some of its partial derivatives. There are many books which provide an excellent introduction to the theory of PDE. For that reason, we name only few. Great insight can be obtained in [17, 28, 60] or in [80].

Definition 1.42. *Let $k \geq 1$, $k \in \mathbb{N}$ and let Ω denote an open subset of $\mathbb{R}^n$ for some $n \geq 1, n \in \mathbb{N}$. Then an expression of the form*

$$F(\partial^k u(x), \partial^{k-1}u(x), \ldots, \partial u(x), u(x), x) = 0 \quad (x \in \Omega) \tag{1.2}$$

is called a k^{th}-order partial differential equation where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and

$$u : \Omega \to \mathbb{R}$$

is the unknown.

We introduce four basic classifications of partial differential equations.

Definition 1.43. *(a) The PDE* (1.2) *is called linear if it has the form*

$$\sum_{|\alpha|\leq k} \omega_\alpha(x)\partial^\alpha u = f(x)$$

for given functions ω_α ($|\alpha| \leq k$), f. This linear PDE is homogeneous if $f \equiv 0$.

(b) The PDE (1.2) *is semilinear if it has the form*

$$\sum_{|\alpha|=k} \omega_\alpha(x)\partial^\alpha u + \omega_0\left(\partial^{k-1}u, \dots, \partial u, u, x\right) = 0.$$

(c) The PDE (1.2) *is quasilinear if it has the form*

$$\sum_{|\alpha|=k} \omega_\alpha\left(\partial^{k-1}u, \dots, \partial u, u, x\right)\partial^\alpha u + \omega_0\left(\partial^{k-1}u, \dots, \partial u, u, x\right) = 0.$$

(d) The PDE (1.2) *is fully nonlinear if it depends nonlinearly upon the highest order derivatives.*

If some u satisfies (1.2) we say that u solves the PDE. In general, partial differential equations have infinitely many solutions. In order to obtain a unique solution we need to apply additional constraints in the form of *initial* and *boundary conditions.*

Initial conditions are used in evolution (partial) differential equations. These equations can be used to model various physical phenomena. If the initial state of our solution is given then they describe how the solution evolves in time. To guarantee a unique solution we need to prescribe how the solution looks like at the initial time ($t = 0$).

If the PDE (1.2) is paired together with a set of additional constraints on the boundary of Ω then we speak about a boundary value problem. There are several types of *boundary conditions.* We list them in the example below.

Example 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ for $n \geq 1, n \in \mathbb{N}$. The boundary of Ω is denoted as $\partial\Omega$. Suppose that function $g : \partial\Omega \to \mathbb{R}$ is given. Let $\partial\Omega_1$ and $\partial\Omega_2$ be subsets of $\partial\Omega$ such that $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ and $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ and let n denote a unit outward normal vector associated with the boundary $\partial\Omega$. Then we can interpret different boundary conditions as:*

$$u = g \quad \text{on } \partial\Omega, \tag{Dirichlet}$$

$$\frac{\partial u}{\partial n} = g \quad on \ \partial\Omega, \tag{Neumann}$$

$$au + b\frac{\partial u}{\partial n} = g \quad on \ \partial\Omega, \ a, b \neq 0 \ and \, a, b \in \mathbb{R}, \tag{Robin}$$

$$u = g \quad on \quad \partial\Omega_1 \quad \frac{\partial u}{\partial n} = g \quad on \ \partial\Omega_2. \tag{Mixed}$$

If the PDE (1.2) associated with one of the *boundary conditions* above has a unique solution u, then we say that it is a *classical solution*. This solution has continuous derivatives up to k-th order. However, in some cases a *classical solution* cannot be provided. Then we need to reformulate the problem and look for the solution in a broader function space. This is usually done by multiplying (1.2) with a test function from an appropriate function space. Then integrating over the domain Ω, taking into account all boundary conditions and using the Green theorem, we obtain a new formulation of the original problem, called the *variational formulation*. An advantage of the *variational formulation* is that it weakens the assumptions on our solution u, i.e. it does not need to be as regular as a *classical solution*. If u satisfies the variational formulation for all test functions we say that u is a *weak solution* of the original problem.

1.6 Some important (in)equalities and identities

The following inequalities and identities are used extensively in theoretical parts of the thesis and form the cornerstone of our modus operandi.

Discrete Hölder's inequality. Suppose that $n \in \mathbb{N}$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers then

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q}.$$

Remark 1.5. If $p = q = 2$ then the inequality above is called Cauchy's inequality.

Continuous Hölder's inequality. Suppose that $n \in \mathbb{N}$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are integrable functions on $\Omega \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) then

$$\int_\Omega |f(x)g(x)| \, \mathrm{d}x \leq \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p} \left(\int_\Omega |g(x)|^q \, \mathrm{d}x \right)^{1/q}.$$

Remark 1.6. If $p = q = 2$ then the inequality above is called Cauchy-Schwarz's inequality.

Abel's summation rule. Assume that $a_0, a_1, \ldots, a_n$ is a set of real numbers then

$$\sum_{i=1}^{n}(a_i - a_{i-1})a_i = \frac{1}{2}\left(a_n^2 - a_0^2 + \sum_{i=1}^{n}(a_i - a_{i-1})^2\right).$$

Young's inequality. Assume that a and b are nonnegative real numbers and p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ with $p \geq 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Young's inequality with ε. Taking $p = 2 = q$, $a = \frac{a}{\sqrt{\varepsilon}}$ and $b = \sqrt{\varepsilon}b$ in the inequality above for some $\varepsilon > 0$, we obtain

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$$

or

$$ab \leq C(\varepsilon')a^2 + \varepsilon' b^2$$

where $\varepsilon' = 2\varepsilon$ and $C(\varepsilon') = \frac{1}{4\varepsilon'}$.

Curl identity. If ϕ is a scalar valued function and $\boldsymbol{F}$ is a vector field then

$$\nabla \times (\phi \boldsymbol{F}) = \nabla\phi \times \boldsymbol{F} + \phi\nabla \times \boldsymbol{F}.$$

Lemma 1.2 (Discrete version of Grönwall's lemma, from [6])**.** *Let $\{y_n\}$ and $\{g_n\}$ be nonnegative sequences and C a nonnegative constant. If*

$$y_n \leq C + \sum_{i=0}^{n-1} g_i y_i \quad \text{for } n \geq 0,$$

then

$$y_n \leq C \exp\left(\sum_{i=0}^{n-1} g_i\right) \quad \text{for } n \geq 0.$$

Lemma 1.3 (Continuous version of Grönwall's lemma, from [6])**.** *Let y and g be nonnegative integrable functions and C a nonnegative constant. If*

$$y(t) \leq C + \int_0^t g(s)y(s)\ ds \quad \text{for } t \geq 0,$$

then

$$y(t) \leq C \exp\left(\int_0^t g(s)\ ds\right) \quad \text{for } t \geq 0.$$

Theorem 1.12 (Green's integral identity for vector fields)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$ with $\boldsymbol{n}$ being a unit outward normal vector associated with the Lipschitz continuous boundary $\partial\Omega$. Suppose that $\boldsymbol{f} \in \mathbf{H}(\mathbf{curl}\,;\Omega)$ and $\boldsymbol{g} \in \mathbf{H}^1(\Omega)$, then*

$$\int_{\partial\Omega} (\boldsymbol{n} \times \boldsymbol{f}) \cdot \boldsymbol{g}\ dx = \int_{\Omega} \nabla \times \boldsymbol{f} \cdot \boldsymbol{g}\ dx - \int_{\Omega} \boldsymbol{f} \cdot \nabla \times \boldsymbol{g}\ dx.$$

Theorem 1.13 (Poincaré-Wirtinger inequality, from [1])**.** *Assume that $1 \leq p \leq \infty$ and that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then there exists a positive constant C, depending only on Ω and p such that for every $u \in W^{1,p}(\Omega)$*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{\mathbf{L}^p(\Omega)}$$

where

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x)\ dx.$$

Theorem 1.14 (Nečas inequality, from [26] or [57])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Moreover, let $\Gamma \subset \partial\Omega$ be a part of the boundary with $|\Gamma| > 0$. Then*

$$\|w\|^2_{L^2(\Gamma)} \leq \varepsilon \|\nabla w\|^2 + C_\varepsilon \|w\|^2, \quad \forall w \in H^1(\Omega), \quad 0 < \varepsilon < \varepsilon_0.$$

Theorem 1.15 (Mean value theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function which is also differentiable on the open interval (a, b). Then there is a real number $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

1.7 Maxwell's equations

Maxwell's equations were introduced by Maxwell in 1873 in his publication *Treatise on Electricity and Magnetism* [53]. These equations consist of two pairs of coupled integral (partial differential) equations relating six fields, two of which model sources of electromagnetism.

Let Ω be a bounded domain in $\mathbb{R}^3$ and let Σ be any surface. Then $\partial\Omega$ and $\partial\Sigma$ denote boundaries of Ω and Σ, respectively. The integral version of Maxwell's equations are as follows:

$$\int_{\partial\Omega} \boldsymbol{D} \cdot \mathrm{d}s = \int_\Omega \rho\ \mathrm{d}x, \qquad \text{(Gauss' law)}$$

$$\int_{\partial\Omega} \boldsymbol{B} \cdot \mathrm{d}s = 0, \qquad \text{(Gauss' law for the magnetic charge)}$$

$$\int_{\partial\Sigma} \boldsymbol{E} \cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \boldsymbol{B} \cdot \mathrm{d}s, \qquad \text{(Faraday's law)}$$

$$\int_{\partial\Sigma} \boldsymbol{H} \cdot \mathrm{d}l = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \boldsymbol{D} \cdot \mathrm{d}s + \int_{\Sigma} \boldsymbol{J} \cdot \mathrm{d}s \qquad \text{(Ampère's law)}$$

where

$\boldsymbol{E} =$ Electric field intensity vector,

$\boldsymbol{B} =$ Magnetic field flux density vector (Magnetic induction field),

$\boldsymbol{H} =$ Magnetic field intensity vector,

$\boldsymbol{D} =$ Electric field flux density vector (Displacement current),

and the current and charge sources are described by

$\boldsymbol{J} =$ Electric current flux density vector,

$\rho =$ Electric charge density.

The first equation above says that the electric flux leaving a volume is proportional to the charge inside. The second one states that there are no magnetic monopoles, i.e. the total magnetic flux through a closed surface is zero. The meaning of the third equation is that the voltage induced in a closed circuit is proportional to the rate of change of the magnetic flux it encloses. Lastly, the fourth equation says that the magnetic field induced around a closed loop is proportional to the electric current plus displacement current (rate of change of electric field) it encloses.

We can rewrite Maxwell's equations in their differential form as follows:

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{1.3}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.4}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \tag{1.5}$$

$$\nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D} + \boldsymbol{J}. \tag{1.6}$$

The above equations are known as *point form* because each equality is true at every point in space. From now on this will be our reference point regarding Maxwell's equations since all mathematical models contained in this thesis are obtained from the equations above.

1.7.1 Constitutive relations

The electric and magnetic flux densities $\boldsymbol{D}, \boldsymbol{B}$ are related to the field intensities $\boldsymbol{E}, \boldsymbol{H}$ via the constitutive relations whose precise form depends on the material in which the fields exist. In vacuum, they take their simplest form:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E},$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H},$$

where ε_0, μ_0 are the permittivity and permeability of vacuum.

More generally, constitutive relations may be inhomogeneous, anisotropic, non-linear, frequency dependent (dispersive), or all of the above. In inhomogeneous materials the permittivity ε and permeability μ depend on the location within the material:

$$\boldsymbol{D}(\boldsymbol{x}, t) = \varepsilon(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}, t),$$
$$\boldsymbol{B}(\boldsymbol{x}, t) = \mu(\boldsymbol{x})\boldsymbol{H}(\boldsymbol{x}, t).$$

In anisotropic materials ε and μ depend on the x, y, z direction and the constitutive relations may be written component-wise in matrix (or tensor) form:

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \quad \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}.$$

In nonlinear materials, ε and μ may depend on the fields $\boldsymbol{E}$ and $\boldsymbol{H}$, i.e.

$$\boldsymbol{D} = \varepsilon(\boldsymbol{E})\boldsymbol{E},$$
$$\boldsymbol{B} = \mu(\boldsymbol{H})\boldsymbol{H}.$$

In a conducting material the electromagnetic field itself gives rise to currents. If the field strengths are not large, we can assume that Ohm's law holds so that:

$$\boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{F} \tag{1.7}$$

where σ is called the electric conductivity and it is a non-negative function of position. The vector function $\boldsymbol{F}$ describes the applied current density.

1.7.2 Boundary and interface conditions

Equations (1.3)-(1.6) are not a complete classical description of the electromagnetic field since the equations do not hold at boundaries between different materials where either μ or ε are discontinuous (for instance a steel-air interface). For that reason we need to define interface conditions.

Let us consider the case of two materials with different electric and magnetic properties separated by a surface S with a unit normal vector $\boldsymbol{n}$ pointing from the region of the first material to the region of the second material (see Fig. 1.1).

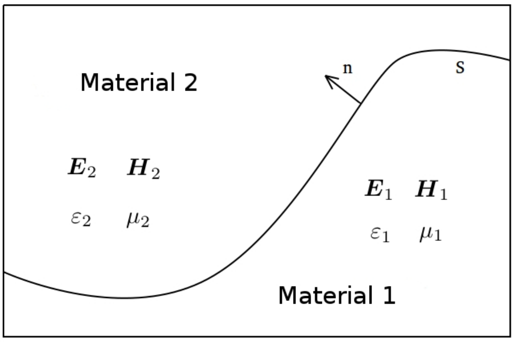

Figure 1.1: Geometry of two materials with different electromagnetic properties.

For $\nabla \times \boldsymbol{E}$ to be well defined, we must have the tangential component of the electric field to be continuous across S and so $\boldsymbol{n} \times \boldsymbol{E}$ is continuous across S. Thus, if $\boldsymbol{E}_1$ denotes the limiting value of the electric field as S is approached from the region of the first material and $\boldsymbol{E}_2$ denotes the limit of the field from the other region we must have

$$\boldsymbol{n} \times \boldsymbol{E}_1 = \boldsymbol{n} \times \boldsymbol{E}_2 \quad \text{on } S.$$

On the other hand, for $\boldsymbol{B}$ to have well defined divergence, the normal components of $\boldsymbol{B}$ must be continuous across S so that

$$\boldsymbol{n} \cdot \boldsymbol{B}_1 = \boldsymbol{n} \cdot \boldsymbol{B}_2 \quad \text{on } S.$$

The continuity conditions above hold for any electromagnetic field. However, we cannot assume that the analogue holds for the magnetic field. In general,

$$\boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_S$$

where this relation defines the tangential vector field $\boldsymbol{J}_S$, the surface current density. In most cases the magnetic field has continuous tangential components (i.e. $\boldsymbol{J}_S = \boldsymbol{0}$). This is true unless the surface S models a thin conductive layer giving rise to the conductive boundary condition. Hence, it is usually assumed that

$$\boldsymbol{n} \times \boldsymbol{H}_1 = \boldsymbol{n} \times \boldsymbol{H}_2 \quad \text{on } S.$$

The presence of singularities in the electric charge density ρ may cause jumps in the normal component of $\boldsymbol{D}$. It holds that

$$\boldsymbol{n} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_S \quad \text{on } S$$

where ρ_S denotes the surface charge density.

A particularly important case occurs when one of the materials discussed above is a perfect conductor. Looking at Ohm's law, we see that if $\sigma \to \infty$ and we want $\boldsymbol{J}$ stay bounded then $\boldsymbol{E} \to \boldsymbol{0}$. In other words, this suggests that in a perfect conductor the electric field vanishes. Therefore, if the second material in Fig. 1.1 is a perfect conductor then $\boldsymbol{E}_2 = \boldsymbol{0}$ and we obtain the perfect conducting boundary condition for $\boldsymbol{E}_1$,

$$\boldsymbol{n} \times \boldsymbol{E}_1 = \boldsymbol{0} \quad \text{on } S.$$

1.7.3 Potential formulation

Potential formulation is quite common for electromagnetic problems. It efficiently reduces the number of variables and simplifies the whole system (1.3)-(1.6). Let us make an additional assumption on Ω and assume that it is also simply-connected. Since equation (1.4) is valid in the whole domain Ω, we have a vector potential $\boldsymbol{A} \in \mathbf{H}(\mathbf{curl}\,; \Omega)$ such that:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{in } \Omega. \tag{1.8}$$

Combining the equation above with equation (1.5), we arrive at

$$\nabla \times (\boldsymbol{E} + \partial_t \boldsymbol{A}) = \boldsymbol{0} \quad \text{in } \Omega.$$

Now, using [35, Theorem 2.9] we obtain a scalar potential $\phi \in H^1(\Omega)$ such that:

$$\boldsymbol{E} + \partial_t \boldsymbol{A} = -\nabla \phi \quad \text{in } \Omega$$

or

$$\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla \phi \quad \text{in } \Omega. \tag{1.9}$$

Maxwell's equations and constitutive laws specify $\boldsymbol{E}$ and $\boldsymbol{B}$ uniquely, they do not specify $\boldsymbol{A}$ and ϕ uniquely. The nonuniqueness of the potentials is summarized by gauge transformations. Suppose that we have two vector potentials $\boldsymbol{A}_1, \boldsymbol{A}_2$ and two scalar potentials ϕ_1, ϕ_2 satisfying equations (1.8) and (1.9), respectively. The relation between these potentials can be then expressed through a scalar function $q \in H^1(\Omega)$ in the following way:

$$\boldsymbol{A}_1 = \boldsymbol{A}_2 + \nabla q,$$

$$\phi_1 = \phi_2 - \partial_t q.$$

The nonuniqueness of $\boldsymbol{A}$ and ϕ enables us to put additional constraints which have no physical significance, but results in mathematical convenience through a process called gauge-fixing. Gauge-fixing usually consists of imposing an additional constraint in the form of a linear differential operator acting on $\boldsymbol{A}$ and ϕ.

One of these gauge-fixings is called the *Coulomb gauge.* Suppose that

$$\nabla \cdot \boldsymbol{A}_1 = 0 \quad \text{in } \Omega.$$

Then, we have

$$\nabla \cdot \boldsymbol{A}_1 - \nabla \cdot \boldsymbol{A}_2 = \nabla \cdot (\boldsymbol{A}_1 - \boldsymbol{A}_2) = \nabla \cdot \nabla q = 0.$$

The function q is now constrained to be a harmonic function. This, together with suitable boundary conditions on $\boldsymbol{A}$, force q to be a constant. Therefore, using the Coulomb gauge, we specify $\boldsymbol{A}$ uniquely.

Below, we propose two important theorems from [35] which will be used later in Chapter 4.

Theorem 1.16. *Assume that Ω is a simply-connected Lipschitz domain. Let $\boldsymbol{n}$ be a unit outward normal vector associated with boundary $\partial\Omega$. Then each function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfying*

$$\nabla \cdot \boldsymbol{u} = 0 \quad \textit{in } \Omega$$

has at most one divergence-free vector potential $\boldsymbol{A} \in \mathbf{H}(\mathbf{curl}\,;\Omega)$ that satisfies:

$$\boldsymbol{n} \times \boldsymbol{A} = \mathbf{0} \quad \textit{on } \partial\Omega.$$

Theorem 1.17. *Assume that Ω is a simply-connected Lipschitz domain. Then a function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfies:*

$$\nabla \times \boldsymbol{u} = \mathbf{0} \quad \textit{in } \Omega$$

iff there exists a unique function $\phi \in H^1(\Omega)/\mathbb{R}$ such that:

$$\boldsymbol{u} = -\nabla \phi.$$

Since we offer only a compressed introduction to the electromagnetism in continuous media, we refer the reader to M. Fabrizio and A. Morro [29] and to L. D. Landau and E. M. Lifshitz [50]. These books present a detailed derivation of mathematical models of electromagnetic solids.

Part I

Mathematical modelling of the induction heating phenomena

Chapter 2

Solvability of the induction heating model including nonlinear magnetic field flux density and restrained Joule heat

This chapter is based on the article [68] that has been published in the journal of *Applicable Analysis.*

2.1 Introduction

Induction heating is a non-contact heating process. A large alternating current is passing through the coil, which is known as the work coil. This generates a very intense and rapidly changing magnetic field in the space within the work coil. The workpiece to be heated is placed within this intense alternating magnetic field. The alternating magnetic field induces a current flow in the conductive workpiece. This is known as eddy current. These currents flow against the electrical resistivity of the material, generating precise and localized heat. This heating occurs with both magnetic and non-magnetic materials and is often referred to as the *Joule heat.*

Induction heating is frequently used in industrial applications such as metal hardening and preheating for forging operations. The investigation of an induction heating system usually relies upon a series of expensive, long, and complicated experiments. The mathematical analysis and numerical simulation for induction heating play an important role in the designing process.

Induction heating involves two different types of physics: electromagnetism and heat transfer. Some material properties are temperature dependent. Hence, their attributes change when heat is applied. In such event, we consider the two physical phenomena coupled. The analysis of this full system is quite complicated. In some situations (when the electric field is given by a certain special time-harmonic form) one can decouple Maxwell's equations from the heat problem and then study the heat equation alone. The microwave heating process is partly based on this approximation.

Numerous articles are devoted to the induction heating phenomena. Most of them present various numerical schemes for computation, e.g. [2, 11, 12, 27, 32, 44, 75]. Nevertheless, only a few articles perform theoretical study about the well-posedness of the problem, e.g. [13, 83, 84, 85, 86]. The common feature of all these works are linear constitutive laws in Maxwell's equations. The most extensive study has been carried out in [38, 39], and [16] where authors have proven the existence of a weak solution to the mathematical model with linear dependent magnetic induction field $\boldsymbol{B}$, i.e. $\boldsymbol{B} = \mu \boldsymbol{H}$ with μ being a positive constant.

In this chapter we derive and investigate a mathematical model of induction heating. We assume that the electric conductivity present in the Ohm law is temperature dependent. Moreover, the constitutive relation between the fields $\boldsymbol{B}$ and $\boldsymbol{H}$ is assumed to be nonlinear.

2.2 Derivation of the mathematical model

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain which is occupied by an electromagnetic material (ferromagnetic). The boundary $\partial\Omega$ is assumed to be Lipschitz continuous. The symbol $\boldsymbol{n}$ stands for the outer normal vector associated with $\partial\Omega$. The electromagnetic phenomena are modeled with Maxwell's equations (1.3)-(1.6).

As in eddy current problems the change of electric displacement in time is not significant the term $\partial_t \boldsymbol{D}$ in (1.6) is not included. We adopt the Ohm law in the same form as in (1.7), i.e.

$$\boldsymbol{J} = \sigma(\mathcal{T})\boldsymbol{E} + \boldsymbol{F} \tag{2.1}$$

where $\sigma(\mathcal{T})$ denotes the temperature dependent electric conductivity. This function is strictly positive and bounded from above and below, i.e. there exist positive

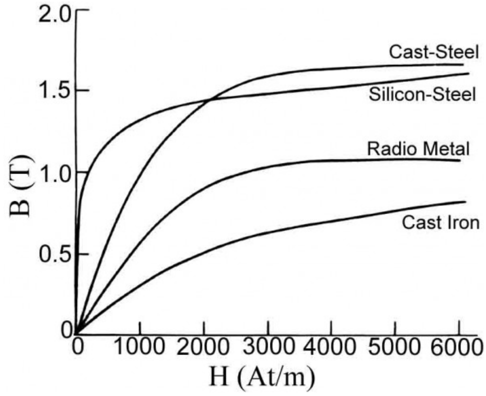

Figure 2.1: Magnetization curve.

constants σ_* and σ^* such that

$$0 < \sigma_* \leq \sigma(s) \leq \sigma^* < \infty, \quad \forall s \geq 0. \tag{2.2}$$

We also introduce the function $\gamma := 1/\sigma$ which is also bounded (i.e. $0 < \gamma_* \leq \gamma \leq \gamma^* < \infty$). According to (2.1), we express $\boldsymbol{E}$ in the following way

$$\boldsymbol{E} = \boldsymbol{J}/\sigma - \boldsymbol{F}/\sigma := \gamma \boldsymbol{J} - \gamma \boldsymbol{F}. \tag{2.3}$$

Magnetization curve for ferromagnets shows a monotone character (see Fig. 2.1). Therefore, we assume a nonlinear relation between the vector fields $\boldsymbol{B}$ and $\boldsymbol{H}$. It can be expressed either as

$$\boldsymbol{B} := \boldsymbol{B}(\boldsymbol{H}) \tag{2.4}$$

or

$$\boldsymbol{H} := \boldsymbol{H}(\boldsymbol{B}). \tag{2.5}$$

We adopt a nonlinear relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ in the form of (2.4). Then, after eliminating other variables in (1.5) and (1.6), we conclude that the magnetic field $\boldsymbol{H}$ is determined by the solution of the following nonlinear PDE:

$$\partial_t \boldsymbol{B}(\boldsymbol{H}) + \nabla \times (\gamma(\mathcal{T}) \nabla \times \boldsymbol{H}) = \nabla \times (\gamma(\mathcal{T}) \boldsymbol{F}) \quad \text{a.e. in} \quad Q_T := \Omega \times (0, T) \tag{2.6}$$

with the initial condition

$$\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}) \quad \text{in} \quad \Omega. \tag{2.7}$$

We assume that the boundary $\partial\Omega$ is a perfect conductor. This means that the tangential component of $\boldsymbol{H}$ vanishes on the boundary, i.e.

$$\boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega. \tag{2.8}$$

Looking at the Gauss' law for the magnetic charge (1.4), we see that

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{H}(0)) = 0 \quad \text{in} \quad \Omega. \tag{2.9}$$

The choice of which form of the nonlinear constitutive relation between $\boldsymbol{H}$ and $\boldsymbol{B}$ is adopted is very important. As we can see, the adoption of (2.4) has led to the nonlinear PDE (2.6) with nonlinear term being under the time derivative. However, as we will see later in Chapter 3, the adoption of (2.5) moves the nonlinearity under the *curl* operator. Let us demonstrate the importance of this choice in the following example.

Example 2.1. *The relation between the two vector fields* $\boldsymbol{H}$ *and* $\boldsymbol{B}$ *can be expressed by a nonlinear vector field* $\boldsymbol{M}$*, i.e. if* $\boldsymbol{B} = \boldsymbol{M}(\boldsymbol{H})$*, then* $\boldsymbol{H} = \boldsymbol{M}^{-1}(\boldsymbol{B})$ *(assuming that* $\boldsymbol{M}^{-1}$ *exists). Suppose that*

$$\boldsymbol{M}(\boldsymbol{x}) = m(|\boldsymbol{x}|)\boldsymbol{x} \quad \forall \boldsymbol{x} \in \mathbb{R}^3,$$

where $0 \leq m(s)$ *for* $s \geq 0$*. Then, we have*

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{B}(\boldsymbol{x}) = m(|\boldsymbol{x}|)\boldsymbol{x} \\ |\boldsymbol{y}| &= m(|\boldsymbol{x}|)\,|\boldsymbol{x}| \\ g^{-1}(|\boldsymbol{y}|) &= |\boldsymbol{x}|\,, \quad (g(s) = m(s)s). \end{aligned}$$

Now, we define $\boldsymbol{M}^{-1}$ *as follows*

$$\boldsymbol{M}^{-1}(\boldsymbol{y}) = \frac{g^{-1}(|\boldsymbol{y}|)}{|\boldsymbol{y}|}\boldsymbol{y}.$$

The nonlinear nature of $\boldsymbol{M}$ *and* $\boldsymbol{M}^{-1}$ *is characterized by functions* $m(|\boldsymbol{x}|)$ *and* $g^{-1}(|\boldsymbol{x}|)$*, respectively. Let us model the nonlinear relation between* $\boldsymbol{B}$ *and* $\boldsymbol{H}$ *using the function* $m(|\boldsymbol{x}|) = |\boldsymbol{x}|^4$*. Then the inverse function of* $g(|\boldsymbol{x}|)$ *equals to* $g^{-1}(|\boldsymbol{x}|) = |\boldsymbol{x}|^{1/5}$*. We can see these two functions pictured in Fig. 2.2. Looking at the function* g^{-1}*, we observe that it is rather steep in the neighbourhood of* 0*. Therefore, in order to avoid unstable numerical schemes, we would suggest using the relation* $\boldsymbol{B} = \boldsymbol{M}(\boldsymbol{H}) = |\boldsymbol{H}|^4\,\boldsymbol{H}$ *instead of* $\boldsymbol{H} = \boldsymbol{M}^{-1}(\boldsymbol{B}) = |\boldsymbol{B}|^{1/5}\,\boldsymbol{B}$*. However, it all depends on the choice of the vector field* $\boldsymbol{M}$ *(or* $\boldsymbol{M}^{-1}$*). In some cases it is better to use the former relation (2.4) while for the other cases the second relation (2.5) works better.*

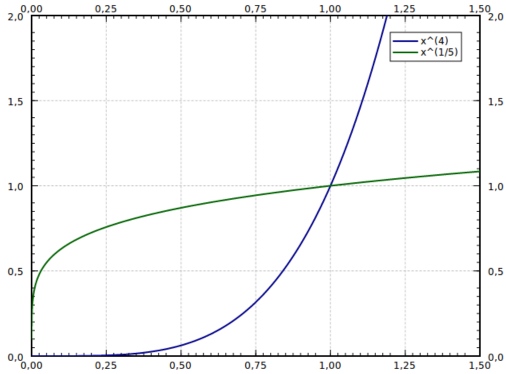

Figure 2.2: Functions $m(x)$ and $g^{-1}(x)$.

The problem defined by (2.6), (2.7) and (2.8) models the electromagnetic part of our induction heating model.

The equation to be solved for the heat transfer in Ω is the following diffusion PDE:

$$\rho c_p \partial_t \mathcal{T} = \nabla \cdot (\lambda \nabla \mathcal{T}) + Q \tag{2.10}$$

where ρ is the material mass density, c_p the specific heat and λ the thermal conductivity of the conductor. The term Q acts as a source (in our case it is a source of heat). *Joule heat* generated by eddy currents is responsible for the heating up. Hence, it plays the role of Q in (2.10) and it is expressed in the following way:

$$Q = \gamma \left| \boldsymbol{J} \right|^2 = \gamma \left| \nabla \times \boldsymbol{H} \right|^2 .$$

For the sake of simplicity we assume a homogeneous Dirichlet boundary condition

$$\mathcal{T}(\boldsymbol{x}, t) = 0 \quad \text{on} \quad \partial\Omega.$$

To guarantee well-posedness we need to introduce an initial condition:

$$\mathcal{T}(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega.$$

Equation (2.10) is nonlinear because all three coefficients ρ, c_p and λ are, in general, temperature dependent. However, the temperature dependences of ρ and c_p are usually weak. Therefore, we neglect it. To overcome the nonlinear term λ in equation (2.10), we introduce the *Kirchhoff transformation* [34]

$$u(\boldsymbol{x}, t) = u_0(\boldsymbol{x}) + \frac{1}{\lambda(u_0(\boldsymbol{x}))} \int_{u_0(\boldsymbol{x})}^{\mathcal{T}(\boldsymbol{x},t)} \lambda(s) \, \mathrm{d}s.$$

After the application of the Kirchhoff transformation to equation (2.10), we obtain the following

$$\left(\frac{\lambda_0 \rho c_p}{\lambda(u)}\right) \partial_t u + \nabla \cdot (\lambda_0 \nabla u) = \gamma(u) \left|\nabla \times \boldsymbol{H}\right|^2 .$$

We define the real continuous function $\theta(u)$ as

$$\theta'(u) = \frac{\lambda_0 \rho c_p}{\lambda(u)} \quad \text{for all } u \in \mathbb{R}.$$

Now, we rewrite the PDE above together with boundary and initial conditions:

$$\begin{aligned} \partial_t \theta(u) - \nabla \cdot (\lambda_0 \nabla u) = \gamma(u) \left|\nabla \times \boldsymbol{H}\right|^2 \quad \text{in} \quad Q_T, \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \text{in} \quad \Omega, \\ u(\boldsymbol{x}, t) = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.11}$$

The function $\lambda_0(\boldsymbol{x})$ is supposed to be strictly positive and bounded, i.e.

$$0 < \lambda_* \leq \lambda_0 \leq \lambda^* < \infty \quad \text{for some positive constants } \lambda_*, \lambda^*. \tag{2.12}$$

If C and θ_* are positive constants then the function $\theta(u)$ is assumed to obey the following

$$\theta(0) = 0, \quad 0 < \theta_* \leq \theta'(s), \quad |\theta(s)| \leq C(1 + |s|) \quad \text{for all } s \in \mathbb{R}. \tag{2.13}$$

Coupling between the electromagnetic and heat transfer part of our model is provided through the term $\gamma(u)$ in (2.6) and the *Joule heat* term in (2.11). Especially the source term in the heat equation has to be treated carefully. Induction heating may cause temperature blow-ups in a workpiece under some circumstances. In many applications the whole process is controlled by the current flowing through the induction coil. This current is shut down after a given time which prohibits the temperature blowing-up. This process is expressed by the application of a truncation to the *Joule heat* term in (2.11). The truncation function (see illustration in Fig. 2.3) is defined as

$$\mathcal{R}_r(x) := \begin{cases} r & \text{if } x > r, \\ x & \text{if } |x| \leq r, \\ -r & \text{if } x < -r \end{cases} \tag{2.14}$$

where r is a positive constant. We truncate the *Joule heat* term in (2.11) such that the heat transfer equation becomes:

$$\partial_t \theta(u) - \nabla \cdot (\lambda_0 \nabla u) = \mathcal{R}_r(\gamma(u) \left|\nabla \times \boldsymbol{H}\right|^2) \quad \text{in} \quad Q_T,$$

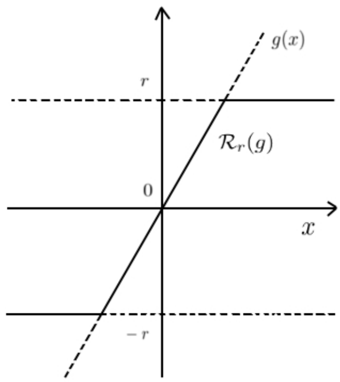

Figure 2.3: Illustration of the truncation function $\mathcal{R}_r$ applied on a general function g.

$$\begin{aligned} u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}) \quad \text{in} \quad \boldsymbol{x}\in\Omega, \\ u(\boldsymbol{x},t) &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.15}$$

The equation above coupled with the model of the electromagnetic part (2.6), (2.7) and (2.8) form the induction heating model.

2.2.1 Weak formulation

In order to write the variational formulation of the coupled system introduced in the previous section we need to multiply (2.6) by a vector function $\boldsymbol{\varphi}$ from the space $\mathbf{H}_0(\mathbf{curl}\,;\Omega)$. Then, integrate it over the whole domain Ω, apply the boundary condition (2.8) and use the Green integral identity for vector fields, cf. Theorem 1.12. Identical steps can also be applied to (2.15), then, the weak formulation of (2.6) and (2.15) reads as:
Find $\boldsymbol{H}\in L^2((0,T);\mathbf{H}_0(\mathbf{curl}\,;\Omega))$ and $u\in C([0,T];\mathrm{L}^2(\Omega))\cap L^\infty((0,T);H_0^1(\Omega))$ with $\partial_t u\in L^2((0,T);L^2(\Omega))$ such that

$$(\partial_t \boldsymbol{B}(\boldsymbol{H}),\boldsymbol{\varphi}) + (\gamma(u)\nabla\times\boldsymbol{H},\nabla\times\boldsymbol{\varphi}) = (\gamma(u)\boldsymbol{F},\nabla\times\boldsymbol{\varphi}), \tag{2.16}$$

$$\boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}) \quad \text{in } \Omega,$$

$$(\partial_t\theta(u),\psi) + (\lambda_0\nabla u,\nabla\psi) = \left(\mathcal{R}_r\left(\gamma(u)\left|\nabla\times\boldsymbol{H}\right|^2\right),\psi\right), \tag{2.17}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \text{in } \Omega$$

holds true for any $\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and $\psi\in H_0^1(\Omega)$.

2.2.2 Assumptions

The vector field $\boldsymbol{B}$ is supposed to be potential, hemicontinuous and strongly monotone. We recall the Definition 1.41 and denote the potential of $\boldsymbol{B}$ as $\Phi_{\boldsymbol{B}}$, i.e. $\mathrm{D}\,\Phi_{\boldsymbol{B}} = \boldsymbol{B}$. Throughout this chapter we assume that

$$\begin{aligned}
(\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) &\geq b_* \left|\boldsymbol{x} - \boldsymbol{y}\right|^2, && b_* > 0 \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, \\
\boldsymbol{B}(\boldsymbol{0}) &= \boldsymbol{0}, && \\
|\boldsymbol{B}(\boldsymbol{x})| &\leq C(1 + |\boldsymbol{x}|) && \forall \boldsymbol{x} \in \mathbb{R}^3, \\
\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{x})) &\geq C_0 \left|\boldsymbol{x}\right|^2 && \forall \boldsymbol{x} \in \mathbb{R}^3.
\end{aligned} \tag{2.18}$$

The strong monotonicity of $\boldsymbol{B}$ implies its invertibility and Lipschitz continuity of the inverse field $\boldsymbol{B}^{-1}$, namely

$$b_* \left|\boldsymbol{x} - \boldsymbol{y}\right|^2 \leq (\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \leq |\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})| \, |\boldsymbol{x} - \boldsymbol{y}|$$

and

$$b_* \left|\boldsymbol{x} - \boldsymbol{y}\right| \leq \left|\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})\right|. \tag{2.19}$$

This can be rewritten as

$$\left|\boldsymbol{B}^{-1}(\boldsymbol{x}) - \boldsymbol{B}^{-1}(\boldsymbol{y})\right| \leq \frac{1}{b_*} \left|\boldsymbol{x} - \boldsymbol{y}\right|. \tag{2.20}$$

The potential $\Phi_{\boldsymbol{B}}$ is strictly convex, cf. [79, Theorem 5.1] . Moreover, following Theorem 1.11, we have

$$\Phi_{\boldsymbol{B}}(\boldsymbol{x}) - \Phi_{\boldsymbol{B}}(\boldsymbol{y}) \geq \boldsymbol{B}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3. \tag{2.21}$$

A similar inequality is valid also for $\Phi_{\boldsymbol{B}^{-1}}$

$$\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{x}) - \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{y}) \geq \boldsymbol{B}^{-1}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3. \tag{2.22}$$

By the chain rule we deduce that

$$\frac{d}{dt} \Phi_{\boldsymbol{B}^{-1}} \left(\boldsymbol{B}(\boldsymbol{x})\right) = \boldsymbol{B}^{-1} \left(\boldsymbol{B}(\boldsymbol{x})\right) \cdot \frac{d\boldsymbol{B}(\boldsymbol{x})}{dt} = \boldsymbol{x} \cdot \frac{d\boldsymbol{B}(\boldsymbol{x})}{dt}. \tag{2.23}$$

An easy calculation gives

$$\begin{aligned}
\Phi_{\boldsymbol{B}}(\boldsymbol{x}) &= \int_0^1 \boldsymbol{B}(t\boldsymbol{x}) \cdot \boldsymbol{x} \,\mathrm{d}t = \int_0^1 t^{-1} \boldsymbol{B}(t\boldsymbol{x}) \cdot t\boldsymbol{x} \,\mathrm{d}t \\
&\overset{(2.18a)}{\geq} \int_0^1 t^{-1} b_* \left|t\boldsymbol{x}\right|^2 \,\mathrm{d}t = \frac{b_*}{2} \left|\boldsymbol{x}\right|^2,
\end{aligned} \tag{2.24}$$

$$\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{x}) = \int_0^1 \boldsymbol{B}^{-1}(t\boldsymbol{x}) \cdot \boldsymbol{x} \, \mathrm{d}t = \int_0^1 t^{-1}\boldsymbol{B}^{-1}(t\boldsymbol{x}) \cdot \boldsymbol{B}\left(\boldsymbol{B}^{-1}(t\boldsymbol{x})\right) \, \mathrm{d}t$$
$$\overset{(2.18a)}{\geq} \int_0^1 t^{-1} b_* \left|\boldsymbol{B}^{-1}(t\boldsymbol{x})\right|^2 \, \mathrm{d}t \geq 0 \tag{2.25}$$

and

$$\begin{aligned} \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{x}) &= \int_0^1 \boldsymbol{B}^{-1}(t\boldsymbol{x}) \cdot \boldsymbol{x} \, \mathrm{d}t &&\leq \int_0^1 \left|\boldsymbol{B}^{-1}(t\boldsymbol{x})\right| |\boldsymbol{x}| \, \mathrm{d}t \\ &\overset{(2.20)}{\leq} \frac{1}{b_*}\int_0^1 |t\boldsymbol{x}| \, |\boldsymbol{x}| \, \mathrm{d}t &&= \frac{1}{2b_*} |\boldsymbol{x}|^2 . \end{aligned} \tag{2.26}$$

Example 2.2. *A typical example of a magnetic field that satisfies* (2.18) *could be* $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{x} + b(|\boldsymbol{x}|)\boldsymbol{x}$ *where the real function* b *obeys:*

$$0 \leq b_1 \leq b(s) \leq b_2, \quad 0 \leq (b(s)s)' \leq C.$$

One can easily check that

$$\begin{aligned} &(\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|^2 + (b(|\boldsymbol{x}|)\boldsymbol{x} - b(|\boldsymbol{y}|)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \\ &= |\boldsymbol{x} - \boldsymbol{y}|^2 + b(|\boldsymbol{x}|) |\boldsymbol{x}|^2 + b(|\boldsymbol{y}|) |\boldsymbol{y}|^2 - b(|\boldsymbol{x}|)\boldsymbol{x} \cdot \boldsymbol{y} - b(|\boldsymbol{y}|)\boldsymbol{y} \cdot \boldsymbol{x} \\ &\geq |\boldsymbol{x} - \boldsymbol{y}|^2 + b(|\boldsymbol{x}|) |\boldsymbol{x}|^2 + b(|\boldsymbol{y}|) |\boldsymbol{y}|^2 - b(|\boldsymbol{x}|) |\boldsymbol{x}| |\boldsymbol{y}| - b(|\boldsymbol{y}|) |\boldsymbol{x}| |\boldsymbol{y}| \\ &= |\boldsymbol{x} - \boldsymbol{y}|^2 + (b(|\boldsymbol{x}|) |\boldsymbol{x}| - b(|\boldsymbol{y}|) |\boldsymbol{y}|) (|\boldsymbol{x}| - |\boldsymbol{y}|) \\ &\geq |\boldsymbol{x} - \boldsymbol{y}|^2 \end{aligned}$$

and

$$\Phi_{\boldsymbol{B}}(\boldsymbol{x}) = \frac{1}{2} |\boldsymbol{x}|^2 + \int_0^{|\boldsymbol{x}|} b(s)s \; ds.$$

Indeed, for the Gâteaux derivative in the direction $\boldsymbol{y}$*, we have*

$$d\Phi_{\boldsymbol{B}}(\boldsymbol{x};\boldsymbol{y}) = \lim_{t\to 0} \frac{\Phi_{\boldsymbol{B}}(\boldsymbol{x} + t\boldsymbol{y}) - \Phi_{\boldsymbol{B}}(\boldsymbol{x})}{t} = \boldsymbol{x} \cdot \boldsymbol{y} + b(|\boldsymbol{x}|)\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{B}(\boldsymbol{x}) \cdot \boldsymbol{y}.$$

The inverse field $\boldsymbol{B}^{-1}$ *for* $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{x} + b(|\boldsymbol{x}|)\boldsymbol{x}$ *is defined as*

$$\boldsymbol{B}^{-1}(\boldsymbol{y}) = \frac{g^{-1}(|\boldsymbol{y}|)}{|\boldsymbol{y}|}\boldsymbol{y} \qquad \text{with } g(s) = [1 + b(s)]s.$$

Then, using the mean value theorem, we may write for some $\eta \in (0, |t\boldsymbol{x}|)$

$$\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{x}) = \int_0^1 \boldsymbol{B}^{-1}(t\boldsymbol{x}) \cdot \boldsymbol{x} \; dt = \int_0^1 g^{-1}(|t\boldsymbol{x}|) \, |\boldsymbol{x}| \; dt$$

$$= \int_0^1 \left(g^{-1}\right)'(\eta)\,|\boldsymbol{x}|^2\, t\ dt.$$

Because $1 \le g'(s) \le L_g$, *we get*

$$\frac{1}{2L_g}|\boldsymbol{x}|^2 \le \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{x}).$$

Further, we have

$$\frac{b_*^2}{2L_g}|\boldsymbol{x}|^2 \overset{(2.19)}{\le} \frac{1}{2L_g}|\boldsymbol{B}(\boldsymbol{x})|^2 \le \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{x})).$$

Since $\theta(0) = 0$ and it is also monotonically increasing, we define its potential by

$$\Phi_\theta(z) = \int_0^z \theta(s)\,\mathrm{d}s \quad \text{for any } z \in \mathbb{R}.$$

The potential defined above is clearly convex. If θ is also Lipschitz continuous with the coefficient L_θ, then the following inequalities hold true, cf. [70]

$$\frac{\theta^2(z)}{2L_\theta} \le \Phi_\theta(z) \le \frac{z^2 L_\theta}{2}, \tag{2.27}$$

$$\theta(z_1)(z_2 - z_1) \le \Phi_\theta(z_2) - \Phi_\theta(z_1) \le \theta(z_2)(z_2 - z_1) \tag{2.28}$$

for any $z, z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$.

2.3 Time discretization

This part is devoted to the time discretization of (2.16) and (2.17). We design a nonlinear time-discrete approximation scheme based on the method of lines (implicit Euler scheme or Rothe's method), cf. [45, 61, 65]. Consider a number of time steps $n \in \mathbb{N}$. We introduce a time discretization of $[0, T]$ in the following sense:

$$[0, T] = \bigcup_{0 \le i \le n-1} [t_i, t_{i+1}], \quad \text{where} \ \ t_i = i\tau,\ 0 \le i \le n,\ n\tau = T.$$

The value of any function f at t_i is denoted as f_i. To approximate the time derivative of f at t_i, we use the backward Euler method, i.e.

$$\partial_t f(t_i) \approx \delta f_i = \frac{f_i - f_{i-1}}{\tau}.$$

Using this notations, we approximate the variational formulations (2.16) and (2.17) as

$$(\delta \boldsymbol{B}(\boldsymbol{h}_i), \boldsymbol{\varphi}) + (\gamma(u_{i-1})\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) = (\gamma(u_{i-1})\boldsymbol{F}_i, \nabla \times \boldsymbol{\varphi}) \tag{2.29}$$

$$(\delta\theta(u_i), \psi) + (\lambda_0 \nabla u_i, \nabla \psi) = \left(\mathcal{R}_r\left(\gamma(u_{i-1})\left|\nabla \times \boldsymbol{h}_i\right|^2\right), \psi\right), \tag{2.30}$$

at every time step t_i for $i = 1, \ldots, n$, $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and $\psi \in H_0^1(\Omega)$.

Let us make a short comment on the suggested scheme. The time continuous problem is approximated by a recurrent system of steady-state settings. Taking $\gamma(u_{i-1})$ in (2.29), we have decoupled this relation from (2.30). Therefore, the pseudoalgorithm reads as:

Algorithm 1 Implicit Euler

Require: $\boldsymbol{h}_0, u_0, n$
1: **for** $i = 1, i \leq n$ **do**
2: $\quad \boldsymbol{h}_i \leftarrow$ Solve: (2.29)
3: $\quad u_i \leftarrow$ Solve: (2.30)
4: $\quad i \leftarrow i + 1$
5: **return** $\{\boldsymbol{h}_1, u_1\}, \ldots, \{\boldsymbol{h}_n, u_n\}$

The main difficulty for this algorithm is the convergence proof of $\{\boldsymbol{h}_i, u_i\}$ towards the solution $\{\boldsymbol{H}, u\}$. The most difficult term to handle is $\gamma(u_{i-1})\left|\nabla \times \boldsymbol{h}_i\right|^2$ which appears in argument of the truncation function $\mathcal{R}_r$. Therefore, in order to pass to the limit in the truncated term, we have to show the a.e. pointwise convergence of u_i and $\nabla \times \boldsymbol{h}_i$.

The existence of a unique solution $\{\boldsymbol{h}_i, u_i\}$ of (2.29) and (2.30) at each time step t_i for $i = 1, \ldots, n$ is guaranteed by Theorem 1.10. We compile our statements in the following Lemma.

Lemma 2.1. *Let (2.13) and (2.18) hold true. Moreover, assume that* $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$, $u_0 \in \mathrm{L}^2(\Omega)$, $\boldsymbol{F} : [0, T] \to \mathbf{L}^2(\Omega)$, $\boldsymbol{F} \in L^2((0, T); \mathbf{L}^2(\Omega))$. *Then there exists a uniquely determined pair* $\boldsymbol{h}_i \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ *and* $u_i \in H_0^1(\Omega)$ *solving* (2.29) *and* (2.30) *for any* $i = 1, \ldots, n$.

Proof. Let us define the operators $F_\gamma : \mathbf{H}_0(\mathbf{curl}\,;\Omega) \to (\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*$ and $G : H_0^1(\Omega) \to \left(H_0^1(\Omega)\right)^*$

$$\langle F_\gamma(\boldsymbol{h}), \boldsymbol{\varphi}\rangle := \left(\frac{\boldsymbol{B}(\boldsymbol{h})}{\tau}, \boldsymbol{\varphi}\right) + (\gamma \nabla \times \boldsymbol{h}, \nabla \times \boldsymbol{\varphi}),$$

$$\langle G(u), \psi \rangle := \left(\frac{\theta(u)}{\tau}, \psi \right) + (\lambda_0 \nabla u, \nabla \psi) .$$

We need to show that these operators are strictly monotone, coercive and hemicontinuous.

Hemicontinuity follows from continuity of $\boldsymbol{B}$ and θ. The next step is to show monotonicity of both operators. Since $\boldsymbol{B}$ is supposed to be strongly monotone (which also implies strict monotonicity), we have for some positive constant C and $\tau \in (0,1)$

$$\begin{aligned}
\langle F_\gamma(\boldsymbol{h}_1) - F_\gamma(\boldsymbol{h}_2), \boldsymbol{h}_1 - \boldsymbol{h}_2 \rangle &= \left(\frac{\boldsymbol{B}(\boldsymbol{h}_1) - \boldsymbol{B}(\boldsymbol{h}_2)}{\tau}, \boldsymbol{h}_1 - \boldsymbol{h}_2 \right) \\
&\quad + (\gamma(\nabla \times \boldsymbol{h}_1 - \nabla \times \boldsymbol{h}_2), \nabla \times \boldsymbol{h}_1 - \nabla \times \boldsymbol{h}_2) \\
&\geq \frac{b_*}{\tau} \|\boldsymbol{h}_1 - \boldsymbol{h}_2\|^2 + \gamma_* \|\nabla \times \boldsymbol{h}_1 - \nabla \times \boldsymbol{h}_2\|^2 \\
&\geq C \|\boldsymbol{h}_1 - \boldsymbol{h}_2\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)} > 0
\end{aligned}$$

for any $\boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$, $\boldsymbol{h}_1 \neq \boldsymbol{h}_2$. Thus, the operator F_γ is strictly monotone. To show that also G is strictly monotone, we use (2.13) and the Mean value Theorem 1.15. Then, we have for $\tau \in (0,1)$, some positive constant C and $\xi \in (0,1)$

$$\begin{aligned}
\langle G(u_1) - G(u_2), u_1 - u_2 \rangle &= \left(\frac{\theta'[u_1 + \xi(u_2 - u_1)]}{\tau}, |u_1 - u_2|^2 \right) \\
&\quad + (\lambda_0(\nabla u_1 - \nabla u_2), \nabla u_1 - \nabla u_2) \\
&\geq \theta_* \|u_1 - u_2\|^2 + \lambda_* \|\nabla u_1 - \nabla u_2\|^2 \\
&\geq C \|u_1 - u_2\|^2_{H^1(\Omega)} > 0
\end{aligned}$$

for any $u_1, u_2 \in H^1_0(\Omega)$, $u_1 \neq u_2$. Coercivity of these operators is guaranteed since $\boldsymbol{B}(\mathbf{0}) = \mathbf{0}$ and $\theta(0) = 0$. We have

$$\begin{aligned}
\langle F_\gamma(\boldsymbol{h}), \boldsymbol{h} \rangle &= \left(\frac{\boldsymbol{B}(\boldsymbol{h}) - \boldsymbol{B}(\mathbf{0})}{\tau}, \boldsymbol{h} - \mathbf{0} \right) + (\gamma \nabla \times \boldsymbol{h}, \nabla \times \boldsymbol{h}) \geq C \|\boldsymbol{h}\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)} , \\
\langle G(u), u \rangle &= \left(\frac{\theta(u) - \theta(0)}{\tau}, u - 0 \right) + (\lambda_0 \nabla u, \nabla u) \geq C \|u\|^2_{H^1(\Omega)} .
\end{aligned}$$

Thus,

$$\lim_{\|\boldsymbol{h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \to \infty} \frac{\langle F_\gamma(\boldsymbol{h}), \boldsymbol{h} \rangle}{\|\boldsymbol{h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}} \geq +\infty \quad \text{and} \quad \lim_{\|u\|_{H^1(\Omega)} \to \infty} \frac{\langle G(u), u \rangle}{\|u\|_{H^1(\Omega)}} \geq +\infty.$$

We have shown that the operators F_γ and G are strictly monotone, hemicontinuous and coercive. Now, we rewrite our pseudoalgorithm scheme with the operator notation as follows

Algorithm 2 Implicit Euler

Require: $\boldsymbol{h}_0 \in \mathbf{H}_0(\mathbf{curl}\,;\Omega), u_0 \in H_0^1(\Omega), n \in \mathbb{N}$

1: **for** $i = 1, i \leq n$ **do**

2: $\quad \boldsymbol{h}_i \leftarrow$ Solve: $\left\langle F_{\gamma(u_{i-1})}(\boldsymbol{h}_i), \boldsymbol{\varphi} \right\rangle = \left(\frac{\boldsymbol{B}(\boldsymbol{h}_{i-1})}{\tau}, \boldsymbol{\varphi} \right) + (\gamma(u_{i-1})\boldsymbol{F}_i, \nabla \times \boldsymbol{\varphi})$ ▷ Since the r.h.s. is from $\mathbf{H}_0(\mathbf{curl}\,;\Omega)^*$, we can use Theorem 1.10

3: $\quad u_i \leftarrow$ Solve: $\langle G(u_i), \psi \rangle = \left(\frac{\theta(u_{i-1})}{\tau}, \psi \right) + \left(\mathcal{R}_r \left(\gamma(u_{i-1}) \left| \nabla \times \boldsymbol{h}_i \right|^2 \right), \psi \right)$ ▷ Since the r.h.s. is from $H_0^1(\Omega)^*$, we can use Theorem 1.10

4: $\quad i \leftarrow i + 1$

5: **return** $\{\boldsymbol{h}_1, u_1\}, \ldots, \{\boldsymbol{h}_n, u_n\}$

□

2.3.1 A priori energy estimates

This section is devoted to the introduction of several lemmas which provide the basic energy estimates for functions $\boldsymbol{h}_i$ and u_i.

Lemma 2.2. *Let the assumptions of Lemma 2.1 be satisfied. Then there exists a positive constant C such that*

$$(i) \quad \max_{1 \leq j \leq n} \|\boldsymbol{h}_j\|^2 + \sum_{i=1}^{n} \|\nabla \times \boldsymbol{h}_i\|^2 \tau \leq C,$$

$$(ii) \quad \sum_{i=1}^{n} \|\delta \boldsymbol{B}(\boldsymbol{h}_i)\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \tau \leq C.$$

Proof. (i) Set $\boldsymbol{\varphi} = \boldsymbol{h}_i \tau$ in (2.29) and sum it up for $i = 1, \ldots, j$ to obtain

$$\sum_{i=1}^{j} (\boldsymbol{B}(\boldsymbol{h}_i) - \boldsymbol{B}(\boldsymbol{h}_{i-1}), \boldsymbol{h}_i) + \sum_{i=1}^{j} (\gamma(u_{i-1}) \nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{h}_i)\, \tau = \\ = \sum_{i=1}^{j} (\gamma(u_{i-1}) \boldsymbol{F}_i, \nabla \times \boldsymbol{h}_i)\, \tau.$$

For the first term we deduce that

$$\begin{aligned}\sum_{i=1}^{j}\left(\boldsymbol{B}(\boldsymbol{h}_i)-\boldsymbol{B}(\boldsymbol{h}_{i-1}),\boldsymbol{h}_i\right) &= \sum_{i=1}^{j}\left(\boldsymbol{B}(\boldsymbol{h}_i)-\boldsymbol{B}(\boldsymbol{h}_{i-1}),\boldsymbol{B}^{-1}\left(\boldsymbol{B}(\boldsymbol{h}_i)\right)\right)\\ &\overset{(2.22)}{\geq} \sum_{i=1}^{j}\int_\Omega\left[\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{h}_i))-\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{h}_{i-1}))\right]\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega\left[\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{h}_j))-\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{H}_0))\right]\,\mathrm{d}\boldsymbol{x}\\ &\overset{(2.18d)}{\geq} C_0\left\|\boldsymbol{h}_j\right\|^2-\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{H}_0))\\ &\overset{(2.26),(2.18c)}{\geq} C_0\left\|\boldsymbol{h}_j\right\|^2-C.\end{aligned}$$

With the previously made assumption (2.2), we write

$$\sum_{i=1}^{j}\left(\gamma(u_{i-1})\nabla\times\boldsymbol{h}_i,\nabla\times\boldsymbol{h}_i\right)\tau\geq\gamma_*\sum_{i=1}^{j}\left\|\nabla\times\boldsymbol{h}_i\right\|^2\tau.$$

Using Cauchy's and Young's inequalities, we get that

$$\sum_{i=1}^{j}\left(\gamma(u_{i-1})\boldsymbol{F}_i,\nabla\times\boldsymbol{h}_i\right)\tau\leq\sum_{i=1}^{j}\left\|\boldsymbol{F}_i\right\|\left\|\nabla\times\boldsymbol{h}_i\right\|\tau\leq C_\varepsilon+\varepsilon\sum_{i=1}^{j}\left\|\nabla\times\boldsymbol{h}_i\right\|^2\tau.$$

Putting all partial results above together and fixing a sufficiently small $0<\varepsilon$, we arrive at

$$\left\|\boldsymbol{h}_j\right\|^2+\sum_{i=1}^{j}\left\|\nabla\times\boldsymbol{h}_i\right\|^2\tau\leq C.$$

(ii) From (2.29), we have for any $\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl}\,;\Omega)$ that

$$\begin{aligned}(\delta\boldsymbol{B}(\boldsymbol{h}_i),\boldsymbol{\varphi}) &= (\gamma(u_{i-1})\boldsymbol{F}_i,\nabla\times\boldsymbol{\varphi})-(\gamma(u_{i-1})\nabla\times\boldsymbol{h}_i,\nabla\times\boldsymbol{\varphi})\\ &\leq C\left(\left\|\boldsymbol{F}_i\right\|+\left\|\nabla\times\boldsymbol{h}_i\right\|\right)\left\|\boldsymbol{\varphi}\right\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}\end{aligned}$$

which gives

$$\left\|\delta\boldsymbol{B}(\boldsymbol{h}_i)\right\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\leq C\left(\left\|\boldsymbol{F}_i\right\|+\left\|\nabla\times\boldsymbol{h}_i\right\|\right)$$

and

$$\sum_{i=1}^{n}\left\|\delta\boldsymbol{B}(\boldsymbol{h}_i)\right\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\tau\leq C\sum_{i=1}^{n}\left\|\nabla\times\boldsymbol{h}_i\right\|^2\tau\leq C,$$

where we used the previously obtained estimate result from (i). □

Lemma 2.3. *Let the assumptions of Lemma 2.1 be fulfilled. In addition, suppose that* (2.9) *is satisfied as well. Then*

$$\nabla \cdot (\boldsymbol{B}(\boldsymbol{h}_i)) = 0,$$

for any $i = 1, \dots, n$.

Proof. Set $\boldsymbol{\varphi} = \nabla \varPhi \in \mathbf{H}_0(\mathbf{curl}\,; \Omega)$, for any $\varPhi \in C_0^\infty(\overline{\Omega})$ in (2.29). Since $\nabla \times \nabla \varPhi = 0$ we have

$$(\delta \boldsymbol{B}(\boldsymbol{h}_i), \nabla \varPhi) = 0.$$

We use the Green identity to obtain the following

$$(\nabla \cdot \boldsymbol{B}(\boldsymbol{h}_i), \varPhi) = (\nabla \cdot \boldsymbol{B}(\boldsymbol{H}_0), \varPhi)$$

for any $i = 1, \dots, n$ and $\varPhi \in C_0^\infty(\overline{\Omega})$. The space $C_0^\infty(\overline{\Omega})$ is dense in the space $H_0^1(\Omega)$. As a result, the above equality is true for all $\varPhi \in H_0^1(\Omega)$ or, in other words, $\nabla \cdot \boldsymbol{B}(\boldsymbol{h}_i) = \nabla \cdot \boldsymbol{B}(\boldsymbol{H}_0) = 0$ in $H^{-1}(\Omega) = (H_0^1(\Omega))^*$ for any $i = 1, \dots, n$. □

Lemma 2.4. *Let the assumptions of Lemma 2.1 be satisfied. Moreover, assume that* $u_0 \in H_0^1(\Omega)$ *and that* $\tau \le \tau^* < +\infty$. *Then there exists a positive constant* C_r *which depends on the parameter* r *of the cut-off function* $\mathcal{R}_r$ *such that*

$$
\begin{aligned}
(i) &\quad \max_{1\le j\le n} \|u_j\|^2 + \sum_{i=1}^n \|\nabla u_i\|^2 \tau \le C_r,\\
(ii) &\quad \sum_{i=1}^n \|\delta u_i\|^2 \tau + \max_{1\le j\le n} \|\nabla u_j\|^2 + \sum_{i=1}^n \|\nabla u_i - \nabla u_{i-1}\|^2 \le C_r,\\
(iii) &\quad \max_{1\le j\le n} \|\delta\theta(u_j)\|_{H^{-1}(\Omega)} \le C_r.
\end{aligned}
$$

Proof. (*i*) Set $\psi = u_i \tau$ in (2.30) and sum it up for $i = 1, \dots, j$ to obtain that

$$
\begin{aligned}
\sum_{i=1}^j (\theta(u_i) - \theta(u_{i-1}), u_i) &+ \sum_{i=1}^j (\lambda_0 \nabla u_i, \nabla u_i)\, \tau\\
&= \sum_{i=1}^j \left(\mathcal{R}_r\left(\gamma(u_{i-1})\, |\nabla \times \boldsymbol{h}_i|^2\right), u_i\right) \tau.
\end{aligned}
$$

We bound the first term on the left-hand side using the technique from Lemma 2.2.

$$\sum_{i=1}^j (\theta(u_i) - \theta(u_{i-1}), u_i) = \int_\Omega \sum_{i=1}^j (\theta(u_i) - \theta(u_{i-1}))\theta^{-1}(\theta(u_i))\ \mathrm{d}\boldsymbol{x}$$

$$\overset{(2.28)}{\geq} \int_\Omega \sum_{i=1}^{j} \left[\Phi_{\theta^{-1}}(\theta(u_i)) - \Phi_{\theta^{-1}}(\theta(u_{i-1}))\right] \; \mathrm{d}\boldsymbol{x}$$

$$= \int_\Omega \Phi_{\theta^{-1}}(\theta(u_j)) \; \mathrm{d}\boldsymbol{x} - \int_\Omega \Phi_{\theta^{-1}}(\theta(u_0)) \; \mathrm{d}\boldsymbol{x}$$

$$\overset{(2.27)}{\geq} C_0 \left\| u_j \right\|^2 - C.$$

Using Cauchy's and Young's inequalities, we estimate

$$\sum_{i=1}^{j} \left(\mathcal{R}_r \left(\gamma(u_{i-1}) \left| \nabla \times \boldsymbol{h}_i \right|^2 \right), u_i \right) \tau \leq C_r + C \sum_{i=1}^{j} \left\| u_i \right\|^2 \tau.$$

The rest of the proof is obtained from the application of the Grönwall Lemma 1.2, since $\lambda_* \leq \lambda_0$.

(ii) We set $\psi = \delta u_i \tau$ in (2.30) and sum again for $i = 1, \ldots, j$

$$\sum_{i=1}^{j} \left(\delta\theta(u_i), \delta u_i\right) \tau + \sum_{i=1}^{j} \left(\lambda_0 \nabla u_i, \nabla u_i - \nabla u_{i-1}\right)$$
$$= \sum_{i=1}^{j} \left(\mathcal{R}_r \left(\gamma(u_{i-1}) \left| \nabla \times \boldsymbol{h}_i \right|^2 \right), \delta u_i \right) \tau.$$

Using Abel's summation, we have

$$\sum_{i=1}^{j} \left(\lambda_0 \nabla u_i, \nabla u_i - \nabla u_{i-1}\right) = \int_\Omega \lambda_0 \sum_{i=1}^{j} (\nabla u_i - \nabla u_{i-1}) \cdot \nabla u_i \; \mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega \frac{\lambda_0}{2} \left(\left|\nabla u_j\right|^2 - \left|\nabla u_0\right|^2 + \sum_{i=1}^{j} \left|\nabla u_i - \nabla u_{i-1}\right|^2 \right) \; \mathrm{d}\boldsymbol{x}$$
$$\geq \frac{\lambda_*}{2} \left(\left\|\nabla u_j\right\|^2 - \left\|\nabla u_0\right\|^2 + \sum_{i=1}^{j} \left\|\nabla u_i - \nabla u_{i-1}\right\|^2 \right).$$

From the Mean value Theorem 1.15, we deduce that for some $\xi \in (0, 1)$

$$\sum_{i=1}^{j} \left(\delta\theta(u_i), \delta u_i\right) \tau = \sum_{i=1}^{j} \left(\theta'(u_i + \xi(u_{i-1} - u_i))[u_i - u_{i-1}], \delta u_i\right) \geq \theta_* \sum_{i=1}^{j} \left\|\delta u_i\right\|^2 \tau.$$

Finally, the r.h.s. is estimated via Cauchy's and Young's inequalities similarly as in part (i)

$$\sum_{i=1}^{j} \left(\mathcal{R}_r \left(\gamma(u_{i-1}) \left| \nabla \times \boldsymbol{h}_i \right|^2 \right), \delta u_i \right) \tau \leq C_r + \varepsilon \sum_{i=1}^{j} \left\| \delta u_i \right\|^2 \tau.$$

If we take $\varepsilon \in (0, \theta_*)$ and collect all estimates above then it brings us to

$$(\theta_* - \varepsilon) \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \frac{\lambda_*}{2} \|\nabla u_j\|^2 + \frac{\lambda_*}{2} \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \le C_r + \frac{\lambda^*}{2} \|\nabla u_0\|^2$$

and

$$\sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \le C_r$$

which is the desired estimate (ii).

(iii) Using (2.30) we deduce

$$\begin{aligned} (\delta\theta(u_i), \psi) &= -(\lambda_0 \nabla u_i, \nabla \psi) + \left(\mathcal{R}_r \left(\gamma(u_{i-1}) \left|\nabla \times \boldsymbol{h}_i\right|^2\right), \psi\right) \\ &\le C \|\nabla u_i\| \|\nabla \psi\| + C_r \|\psi\| \\ &\overset{(ii)}{\le} C_r \|\psi\|_{H^1(\Omega)}. \end{aligned}$$

Thus,

$$\|\delta\theta(u_i)\|_{H^{-1}(\Omega)} = \sup_{\substack{\psi \in H_0^1(\Omega) \\ \|\psi\|_{H^1(\Omega)} \le 1}} (\delta\theta(u_i), \psi) \le C_r.$$

□

2.4 The existence of a global solution

The existence proof of a weak solution $\{\boldsymbol{H}, u\}$ of (2.16) and (2.17) is provided in this section. We split the proof into two parts. In the first part we use well known results from the functional analysis valid in evolution problems containing the evolution (Gelfand's) triple (cf. [45, 66]). In this manner, we establish the convergence results for the approximate solution of temperature. In the second part we profit from the monotone character of the nonlinear vector field $\boldsymbol{B}$ and use the method of Minty-Browder (more details in [28, 79]) to overcome the nonlinearity when passing to the limit. Finally, we prove the existence of a weak solution of (2.16) and (2.17).

We construct piece-wise linear and piece-wise constant in time functions in the following manner.

$$\begin{aligned} \overline{f_n}(0) &= f_n(0) = f_0, & \\ \overline{f_n}(t) &= f_i & \text{for } t \in (t_{i-1}, t_i], \\ f_n(t) &= f_{i-1} + (t - t_{i-1})\delta f_i & \text{for } t \in (t_{i-1}, t_i]. \end{aligned}$$

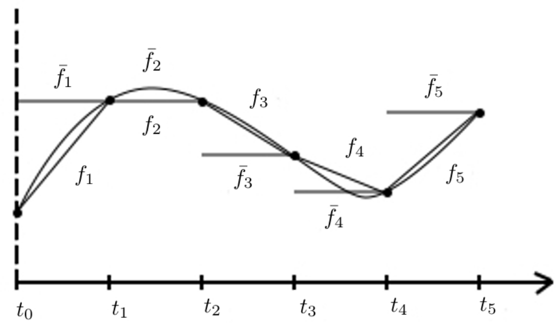

Figure 2.4: Rothe's functions of a general function $f(t)$.

These functions are also called Rothe's functions. We include a simple example for better interpretation in Fig. 2.4. Using this new notations, we rewrite (2.29) and (2.30) in a continuous form for the whole time interval $[0, T]$ as follows

$$(\partial_t \boldsymbol{B}_n, \boldsymbol{\varphi}) + \left(\overline{\gamma_n}(t-\tau)\nabla \times \overline{\boldsymbol{h}_n}, \nabla \times \boldsymbol{\varphi}\right) = \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla \times \boldsymbol{\varphi}\right), \tag{2.31}$$

$$(\partial_t \theta_n, \psi) + (\lambda_0 \nabla \overline{u_n}, \nabla \psi) = \left(\mathcal{R}_r\left(\overline{\gamma_n}(t-\tau)\left|\nabla \times \overline{\boldsymbol{h}_n}\right|^2\right), \psi\right) \tag{2.32}$$

for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,; \Omega)$ and $\psi \in H_0^1(\Omega)$. Please note that

$$\delta\theta(u_i) = \partial_t \left\{\theta(u_{i-1}) + (t - t_{i-1})\delta\theta(u_i)\right\} = \partial_t \theta_n(t) \quad \text{for } t \in (t_{i-1}, t_i]$$

and also

$$\delta \boldsymbol{B}(\boldsymbol{h}_i) = \partial_t \left\{\boldsymbol{B}(\boldsymbol{h}_{i-1}) + (t - t_{i-1})\delta \boldsymbol{B}(\boldsymbol{h}_i)\right\} = \partial_t \boldsymbol{B}_n(t) \quad \text{for } t \in (t_{i-1}, t_i].$$

Before we proceed to the existence theorem itself let us introduce Lemma 1.3.13 from [45] which represents a powerful tool to work within the proof of the existence theorem.

Lemma 2.5. *Let V, Y be reflexive Banach spaces and let the embedding $V \hookrightarrow Y$ be compact. If the estimates*

$$\int_0^T \|\partial_t u_n(t)\|_Y^2 \, dt \le C, \quad \max_{t\in[0,T]} \|\overline{u_n}(t)\|_V \le C$$

hold for all $n \ge n_0 > 0$ then there exist $u \in C([0,T]; Y) \cap L_\infty((0,T); V)$ with $\partial_t u \in L^2((0,T); Y)$ (u is differentiable a.e. in $(0,T)$) and a subsequence $\{u_{n_k}\}$ of

$\{u_n\}$ such that

$$
\begin{array}{lll}
u_{n_k} \to u & in & C([0,T];Y), \\
u_{n_k}(t) \rightharpoonup u(t) & in & V \quad for\ all \quad t \in [0,T], \\
\overline{u_{n_k}}(t) \rightharpoonup u(t) & in & V \quad for\ all \quad t \in [0,T], \\
\partial_t u_{n_k} \rightharpoonup \partial_t u & in & L^2((0,T);Y).
\end{array}
$$

Proof. Detailed proof can be found in [45]. □

Theorem 2.1. *Suppose that all assumptions of Lemma 2.1 are fulfilled. Moreover, assume (2.9), $u_0 \in H_0^1(\Omega)$, and γ is a global Lipschitz continuous function and also $\boldsymbol{F}$ is Lipschitz continuous in time. Then there exist $u \in C([0,T];L^2(\Omega)) \cap L^\infty((0,T);H_0^1(\Omega))$ with $\partial_t u \in L^2((0,T);L^2(\Omega))$ and $\boldsymbol{H} \in L^2((0,T);\mathbf{H}_0(\mathbf{curl}\,;\Omega))$ and subsequences of u_n and $\boldsymbol{h}_n$ (still denoted as n) such that*

$$
\begin{array}{rl}
(i) & u_n \to u,\ \overline{u_n} \to u \ \ in \ \ C([0,T];L^2(\Omega)), \\
(ii) & \overline{u_n}(t) \rightharpoonup u(t) \ \ in \ \ H_0^1(\Omega), \forall t \in [0,T], \\
(iii) & \overline{\gamma_n} \to \gamma(u),\ \overline{\gamma_n}(t-\tau) \to \gamma(u) \ \ in \ \ L^2((0,T);L^2(\Omega)), \\
(iv) & \overline{\theta_n} - \theta_n \to 0 \ \ in \ \ C\left([0,T], H^{-1}(\Omega)\right), \\
(v) & \overline{\theta_n} \to \theta(u) \ \ in \ \ L^2((0,T);L^2(\Omega)), \\
(vi) & \overline{\boldsymbol{B}_n} - \boldsymbol{B}_n \to 0 \ \ in \ \ L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)), \\
(vii) & \overline{\boldsymbol{h}_n} \rightharpoonup \boldsymbol{H},\ \boldsymbol{B}_n \rightharpoonup \boldsymbol{B}(\boldsymbol{H}) \ \ in \ \ L^2((0,T);\mathbf{L}^2(\Omega)), \\
(viii) & \overline{\boldsymbol{h}_n} \to \boldsymbol{H} \ in \ L^2((0,T);\mathbf{L}^2(\Omega)), \\
(ix) & \overline{\boldsymbol{h}_n} \rightharpoonup \boldsymbol{H} \ in \ L^2((0,T);\mathbf{H}_0(\mathbf{curl}\,;\Omega)), \\
(x) & \partial_t \boldsymbol{B}_n \rightharpoonup \partial_t \boldsymbol{B}(\boldsymbol{H}) \ \ in \ \ L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)), \\
(xi) & \overline{\boldsymbol{F}_n} \to \boldsymbol{F} \ \ in \ \ L^2((0,T);\mathbf{L}^2(\Omega)), \\
(xii) & u \ and\ \boldsymbol{H} \ solve\ (2.16), \\
(xiii) & \overline{\boldsymbol{h}_n} \to \boldsymbol{H} \ \ in \ \ L^2((0,T);\mathbf{H}(\mathbf{curl}\,;\Omega)), \\
(xiv) & u \ and\ \boldsymbol{H} \ solve\ (2.17).
\end{array}
$$

Proof. Lemma 2.4 implies that $\max_{t\in[0,T]} \|\overline{u_n}\|_{H^1(\Omega)} + \int_0^T \|\partial_t u_n\|^2 \, \mathrm{d}t \le C_r$. Therefore, we use Lemma 2.5 with $V = H_0^1(\Omega)$ and $Y = L^2(\Omega)$ to prove (i) and (ii).

(iii) From the part (i) we get that $u_n, \overline{u_n} \to u$ *a.e.* in Q_T. The Lipschitz continuity of the function γ yields

$$\begin{aligned}\lim_{n\to\infty}\int_0^T \|\overline{\gamma_n}(t)-\gamma(u(t)\|^2 \ \mathrm{d}t &= \lim_{n\to\infty}\int_0^T \|\gamma(\overline{u_n}(t))-\gamma(u(t)\|^2 \ \mathrm{d}t\\ &\leq C\lim_{n\to\infty}\int_0^T \|\overline{u_n}(t)-u(t)\|^2 \ \mathrm{d}t\\ &= 0\end{aligned}$$

and

$$\int_0^T \|\overline{\gamma_n}(t-\tau)-\overline{\gamma_n}(t)\|^2 \ \mathrm{d}t \leq C\tau^2\int_0^T \|\partial_t u_n\|^2 \ \mathrm{d}t = \mathcal{O}\left(\tau^2\right).$$

Therefore,

$$\lim_{n\to\infty}\int_0^T \|\overline{\gamma_n}(t-\tau)-\gamma(u(t)\|^2 \ \mathrm{d}t = 0.$$

(iv) Again, we use Lemma 2.4 to obtain

$$\left|\left(\overline{\theta_n}-\theta_n,\psi\right)\right| \leq \tau\left|(\partial_t\theta_n,\psi)\right| \leq \tau\left\|\partial_t\theta_n\right\|_{H^{-1}(\Omega)}\|\psi\|_{H^1(\Omega)} \leq C_r\tau\|\psi\|_{H^1(\Omega)},$$

which implies (iv).

(v) The assertion follows from (i), from the continuity of θ, and Lebesgue's dominated convergence Theorem 1.3.

(vi) Lemma 2.2 implies that

$$\left|\left(\overline{\boldsymbol{B}_n}-\boldsymbol{B}_n,\boldsymbol{\varphi}\right)\right| \leq \tau\left|\ (\partial_t\boldsymbol{B}_n,\boldsymbol{\varphi})\ \right| \leq \tau\left\|\partial_t\boldsymbol{B}_n\right\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}.$$

Thus, by Lemma 2.2, we get

$$\left|\int_0^T\left(\overline{\boldsymbol{B}_n}-\boldsymbol{B}_n,\boldsymbol{\varphi}\right)\right| \mathrm{d}t = \mathcal{O}\left(\tau\right)\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}$$

and $\left\|\overline{\boldsymbol{B}_n}-\boldsymbol{B}_n\right\|_{L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega))} = \mathcal{O}\left(\tau\right)\to 0$ as n goes to infinity.

(vii) Thanks to estimates in Lemma 2.2 and Theorem 1.2, we obtain a weak convergence for $\overline{\boldsymbol{h}_n}$ and $\boldsymbol{B}_n$ in $L^2((0,T);\mathbf{L}^2(\Omega))$, i.e.

$$\overline{\boldsymbol{h}_n}\rightharpoonup\boldsymbol{H},\quad \boldsymbol{B}_n\rightharpoonup\boldsymbol{U}\quad \text{in } L^2((0,T);\mathbf{L}^2(\Omega))$$

where $\boldsymbol{U}$ is from $L^2((0,T);\mathbf{L}^2(\Omega))$. Moreover, the results of Lemma 2.2 and Lemma 2.3 yield

$$\int_0^T \left\|\overline{\boldsymbol{h}_n}\right\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \ \mathrm{d}t \leq C,$$

$$\int_0^T \left(\|\partial_t \boldsymbol{B}_n\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} + \|\nabla\cdot \boldsymbol{B}_n\|^2 + \|\boldsymbol{B}_n\|^2 \right)\,\mathrm{d}t \le C.$$

Thus, by applying [69, Lemma 3.1(iii)], we get

$$\lim_{n\to\infty} \int_0^T \left(\boldsymbol{B}_n, \xi\overline{\boldsymbol{h}_n}\right)\,\mathrm{d}t = \int_0^T (\boldsymbol{U}, \xi\boldsymbol{H})\,\mathrm{d}t \tag{2.33}$$

for any $\xi \in C_0^\infty(\overline{\Omega})$. The following inequality is true for any $\boldsymbol{\omega} \in L^2((0,T);\mathbf{L}^2(\Omega))$ and any non-negative $\xi \in C_0^\infty(\overline{\Omega})$ thanks to monotone character of $\boldsymbol{B}$. Please take into account that $\overline{\boldsymbol{B}_n} = \boldsymbol{B}(\overline{\boldsymbol{h}_n}) = \boldsymbol{B}(\boldsymbol{h}_i)$ for $t \in (t_{i-1}, t_i]$. Looking at (2.18), we have for any $\boldsymbol{w} \in L^2((0,T);\mathbf{L}^2(\Omega))$ and $\xi \in C_0^\infty(\overline{\Omega})$

$$\int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}_n}) - \boldsymbol{B}(\boldsymbol{\omega}), \xi(\overline{\boldsymbol{h}_n} - \boldsymbol{\omega})\right)\,\mathrm{d}t \ge 0 \tag{2.34}$$

We split the integral above into four integrals, i.e.

$$\begin{aligned} P_1 &:= \int_0^T \left(\overline{\boldsymbol{B}_n}, \xi\overline{\boldsymbol{h}_n}\right)\,\mathrm{d}t, & P_2 &:= \int_0^T \left(\overline{\boldsymbol{B}_n}, \xi\boldsymbol{\omega}\right)\,\mathrm{d}t,\\ P_3 &:= \int_0^T \left(\boldsymbol{B}(\boldsymbol{\omega}), \xi\overline{\boldsymbol{h}_n}\right)\,\mathrm{d}t, & P_4 &:= \int_0^T (\boldsymbol{B}(\boldsymbol{\omega}), \xi\boldsymbol{\omega})\,\mathrm{d}t. \end{aligned}$$

We rewrite the first integral P_1 in the following manner

$$P_1 = \int_0^T \left(\boldsymbol{B}_n, \xi\overline{\boldsymbol{h}_n}\right)\,\mathrm{d}t + \int_0^T \left(\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n, \xi\overline{\boldsymbol{h}_n}\right)\,\mathrm{d}t.$$

Now, using (vi) and Lemma 2.2, we conclude that

$$\begin{aligned} \int_0^T \left(\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n, \xi\overline{\boldsymbol{h}_n}\right) &\le C \int_0^T \left\|\overline{\boldsymbol{h}_n}\right\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \int_0^T \left\|\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n\right\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\\ &\overset{n\to\infty}{\to} 0. \end{aligned}$$

Thus, using (2.33), we have

$$\lim_{n\to\infty} P_1 = \int_0^T (\boldsymbol{U}, \xi\boldsymbol{H})\,\mathrm{d}t.$$

The space $L^2((0,T);\mathbf{C}^\infty(\Omega))$ is dense in $L^2((0,T);\mathbf{L}^2(\Omega))$. Thus, for any $\varepsilon > 0$ there exists $\boldsymbol{\omega}_\varepsilon \in L^2((0,T);\mathbf{C}^\infty(\Omega))$ such that $\|\boldsymbol{\omega} - \boldsymbol{\omega}_\varepsilon\|_{L^2((0,T);\mathbf{L}^2(\Omega))} \le \varepsilon$. Let us now investigate the following identity

$$P_2 = \int_0^T \left(\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n, \xi\boldsymbol{\omega}_\varepsilon\right)\,\mathrm{d}t + \int_0^T \left(\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n, \xi(\boldsymbol{\omega} - \boldsymbol{\omega}_\varepsilon)\right)\,\mathrm{d}t$$

$$+\int_0^T (\boldsymbol{B}_n, \xi\boldsymbol{\omega}) \; \mathrm{d}t.$$

Using (vi) and the statement above, we bound the first two terms of P_2 as follows

$$\begin{aligned}\left|\int_0^T \left(\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n, \xi\boldsymbol{\omega}_\varepsilon\right)\right| &\leq C\int_0^T \left\|\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n\right\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \int_0^T \|\boldsymbol{\omega}_\varepsilon\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \\ &\leq C\varepsilon \int_0^T \left\|\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n\right\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \\ &\overset{n\to\infty}{\to} 0\end{aligned}$$

and

$$\begin{aligned}\left|\int_0^T \left(\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n, \xi(\boldsymbol{\omega} - \boldsymbol{\omega}_\varepsilon)\right)\right| &\leq C\int_0^T \left\|\overline{\boldsymbol{B}_n} - \boldsymbol{B}_n\right\|^2 \int_0^T \|\boldsymbol{\omega} - \boldsymbol{\omega}_\varepsilon\|^2 \\ &\leq C\varepsilon.\end{aligned}$$

Since $\boldsymbol{\omega} \in L^2((0,T);\mathbf{L}^2(\Omega))$, we have

$$\lim_{n\to\infty}\int_0^T (\boldsymbol{B}_n, \xi\boldsymbol{\omega}) \; \mathrm{d}t = \int_0^T (\boldsymbol{U}, \xi\boldsymbol{\omega}) \; \mathrm{d}t.$$

Therefore, we pass to the limit in

$$\lim_{n\to\infty}\left|P_2 - \int_0^T (\boldsymbol{B}_n, \xi\boldsymbol{\omega}) \; \mathrm{d}t\right| \leq C\varepsilon \overset{\varepsilon\to 0}{\to} 0.$$

Hence,

$$\lim_{n\to\infty} P_2 = \int_0^T (\boldsymbol{U}, \xi\boldsymbol{\omega}) \; \mathrm{d}t.$$

Passing to the limit for $n \to \infty$ in the remaining terms, we obtain

$$\lim_{n\to\infty} P_3 = \int_0^T (\boldsymbol{B}(\boldsymbol{\omega}), \xi\boldsymbol{H}) \; \mathrm{d}t, \quad \lim_{n\to\infty} P_4 = \int_0^T (\boldsymbol{B}(\boldsymbol{\omega}), \xi\boldsymbol{\omega}) \; \mathrm{d}t.$$

Returning to (2.34), we see that

$$\lim_{n\to\infty}\int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}_n}) - \boldsymbol{B}(\boldsymbol{\omega}), \xi(\overline{\boldsymbol{h}_n} - \boldsymbol{\omega})\right) \; \mathrm{d}t = \int_0^T (\boldsymbol{U} - \boldsymbol{B}(\boldsymbol{\omega}), \xi(\boldsymbol{H} - \boldsymbol{\omega})) \; \mathrm{d}t \geq 0.$$

Since $\boldsymbol{\omega}$ has been chosen arbitrarily, we set $\boldsymbol{\omega} = \boldsymbol{H} + \varepsilon \boldsymbol{q}$ where $\varepsilon > 0$ and $\boldsymbol{q} \in L^2((0,T);\mathbf{L}^2(\Omega))$. Then we have

$$\int_0^T (\boldsymbol{U} - \boldsymbol{B}(\boldsymbol{H} + \varepsilon \boldsymbol{q}), \xi \boldsymbol{q}) \ \mathrm{d}t \leq 0.$$

Now, passing with ε to 0 yields

$$\int_0^T (\boldsymbol{U} - \boldsymbol{B}(\boldsymbol{H}), \xi \boldsymbol{q}) \ \mathrm{d}t \leq 0.$$

And, since $\boldsymbol{q}$ has been chosen arbitrarily, we set it to $\boldsymbol{q} = -\boldsymbol{q}$. Hence, also the reverse inequality holds true. That implies the following

$$\int_0^T (\boldsymbol{U} - \boldsymbol{B}(\boldsymbol{H}), \xi \boldsymbol{q}) \ \mathrm{d}t = 0.$$

This is true for any $\boldsymbol{q} \in L^2((0,T);\mathbf{L}^2(\Omega))$ and any non-negative $\xi \in C_0^\infty(\overline{\Omega})$. Therefore, $\boldsymbol{U} = \boldsymbol{B}(\boldsymbol{H})$ a.e. in $\Omega \times (0,T)$, i.e. $\boldsymbol{B}_n \rightharpoonup \boldsymbol{B}(\boldsymbol{H})$ in $L^2((0,T);\mathbf{L}^2(\Omega))$.

$(viii)$ Let $\xi \in C_0^\infty(\overline{\Omega})$ be non-negative. It holds that

$$\int_0^T \big(\boldsymbol{B}(\overline{\boldsymbol{h}_n}) - \boldsymbol{B}(\boldsymbol{H}), \xi(\overline{\boldsymbol{h}_n} - \boldsymbol{H})\big) \ \mathrm{d}t \geq b_* \int_0^T \Big(\xi, \big|\overline{\boldsymbol{h}_n} - \boldsymbol{H}\big|^2\Big) \ \mathrm{d}t.$$

Passing to the limit for $n \to \infty$, using Lemma 3.1 (div-curl lemma) from [69], we get in a similar way as in (vii)

$$0 = \lim_{n\to\infty} \int_0^T \big(\boldsymbol{B}(\overline{\boldsymbol{h}_n}) - \boldsymbol{B}(\boldsymbol{H}), \xi(\overline{\boldsymbol{h}_n} - \boldsymbol{H})\big) \ \mathrm{d}t \geq b_* \lim_{n\to\infty} \int_0^T \Big(\xi, \big|\overline{\boldsymbol{h}_n} - \boldsymbol{H}\big|^2\Big) \ \mathrm{d}t.$$

This relation is valid for any non-negative $\xi \in C_0^\infty(\overline{\Omega})$, which implies that

$$\overline{\boldsymbol{h}_n} \to \boldsymbol{H} \qquad \text{in} \qquad L^2((0,T);\mathbf{L}^2(\Omega)).$$

(ix) Take any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$. It holds that

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \big(\nabla \times \overline{\boldsymbol{h}_n}, \boldsymbol{\varphi}\big) \ \mathrm{d}t &= \lim_{n\to\infty} \int_0^T \big(\overline{\boldsymbol{h}_n}, \nabla \times \boldsymbol{\varphi}\big) \ \mathrm{d}t \\ &= \int_0^T (\boldsymbol{H}, \nabla \times \boldsymbol{\varphi}) \ \mathrm{d}t = \int_0^T (\nabla \times \boldsymbol{H}, \boldsymbol{\varphi}) \ \mathrm{d}t.\end{aligned}$$

(x) Thanks to Lemma 2.2 and reflexivity of $L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega))$, we get the existence of $\boldsymbol{z} \in L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega))$ such that

$$(\boldsymbol{B}_n(t), \boldsymbol{\varphi}) - (\boldsymbol{B}_n(0), \boldsymbol{\varphi}) = \int_0^t (\partial_s \boldsymbol{B}_n, \boldsymbol{\varphi}) \ \mathrm{d}s \xrightarrow{n\to\infty} \int_0^t (\boldsymbol{z}, \boldsymbol{\varphi}) \ \mathrm{d}s. \tag{2.35}$$

Moreover, we have an estimate

$$\begin{aligned}|(\boldsymbol{B}_n(t),\boldsymbol{\varphi})| &= \left|(\boldsymbol{B}_n(0),\boldsymbol{\varphi}) + \int_0^t (\partial_s \boldsymbol{B}_n(s),\boldsymbol{\varphi}) \ \mathrm{d}s\right| \\ &\le C\,\|\boldsymbol{\varphi}\| + C\,\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \int_0^t \|\partial_s \boldsymbol{B}_n(s)\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \ \mathrm{d}s \\ &\overset{\textit{Lemma 2.2}}{\le} C\,\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}\,.\end{aligned}$$

Thus, the sequence $\boldsymbol{B}_n(t)$ is equibounded in $\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)$ for any $n \in \mathbb{N}$. This sequence is also equicontinuous in the same space.

$$\begin{aligned}|(\boldsymbol{B}_n(t_1),\boldsymbol{\varphi}) - (\boldsymbol{B}_n(t_2),\boldsymbol{\varphi})| &= \left|\int_{t_2}^{t_1} (\partial_s \boldsymbol{B}_n(s),\boldsymbol{\varphi}) \ \mathrm{d}s\right| \\ &\le \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \int_{t_2}^{t_1} \|\partial_s \boldsymbol{B}_n(s)\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \ \mathrm{d}s \\ &\le \sqrt{|t_1 - t_2|}\,\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \int_0^{\mathcal{T}} \|\partial_s \boldsymbol{B}_n\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \\ &\overset{\textit{Lemma 2.2}}{\le} C\sqrt{|t_1 - t_2|}\,\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}\,.\end{aligned}$$

Considering $(viii)$ and applying the modification of Arzelà-Ascoli Theorem 1.5-(i), we obtain

$$\lim_{n\to\infty} (\boldsymbol{B}_n(t),\boldsymbol{\varphi}) = (\boldsymbol{B}(\boldsymbol{H}(t)),\boldsymbol{\varphi})\,,$$

for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and for any $t \in [0,T]$. Looking back at (2.35), we see that

$$\begin{aligned}\int_0^t (\partial_s \boldsymbol{B}(\boldsymbol{H}(s)),\boldsymbol{\varphi}) \ \mathrm{d}s &= \boldsymbol{B}(\boldsymbol{H}(t)) - \boldsymbol{B}(\boldsymbol{H}(0)) \\ &= \lim_{n\to\infty}\{\boldsymbol{B}_n(t) - \boldsymbol{B}_n(0)\} = \int_0^t (\boldsymbol{z},\boldsymbol{\varphi}) \ \mathrm{d}s.\end{aligned}$$

Therefore, $\boldsymbol{z} = \partial_t \boldsymbol{B}(\boldsymbol{H})$ a.e. in $\Omega \times (0,T)$.

(xi) Thanks to the Lipschitz continuity of $\boldsymbol{F}$, we have

$$\|\boldsymbol{F}(t_1) - \boldsymbol{F}(t_2)\| \le C\,|t_1 - t_2|$$

for any $t_1, t_2 \in [0,T]$. Therefore,

$$\int_0^T \left\|\overline{\boldsymbol{F}_n}(t) - \boldsymbol{F}(t)\right\|^2 \ \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\boldsymbol{F}(t_i) - \boldsymbol{F}(t)\|^2 \ \mathrm{d}t \le C\tau^2 \overset{n\to\infty}{\longrightarrow} 0.$$

(xii) We start from (2.31) considering $\boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(\overline{\Omega})$. Let us integrate in time to obtain the following

$$\int_0^\eta (\partial_t \boldsymbol{B}_n, \boldsymbol{\varphi}) + \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{h}_n}, \nabla \times \boldsymbol{\varphi} \right) = \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \overline{\boldsymbol{F}_n}, \nabla \times \boldsymbol{\varphi} \right).$$

In virtue of the parts (iii), (ix), (x), and (xi) we easily pass to the limit for $n \to \infty$ to arrive at

$$\int_0^\eta (\partial_t \boldsymbol{B}(\boldsymbol{H}), \boldsymbol{\varphi}) \ \mathrm{d}t + \int_0^\eta (\gamma(u) \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\varphi}) \ \mathrm{d}t = \int_0^\eta (\gamma(u) \boldsymbol{F}, \nabla \times \boldsymbol{\varphi}) \ \mathrm{d}t.$$

Taking into account the density of $\boldsymbol{C}_0^\infty(\overline{\Omega})$ in $\mathbf{H}_0(\mathbf{curl}\,; \Omega)$, and differentiating in time, we conclude this part of the proof.

($xiii$) We use strict positiveness of the function γ and (2.31) to prove that $\nabla \times \overline{\boldsymbol{h}_n} \to \nabla \times \boldsymbol{H}$ in $L^2((0,T); \mathbf{L}^2(\Omega))$. Following ($viii$), we see that $\overline{\boldsymbol{h}_n} \to \boldsymbol{H}$ a.e. in Q_T. Take any $\eta \in (0,T)$ for which $\overline{\boldsymbol{h}_n}(\eta) \to \boldsymbol{H}(\eta)$ a.e. in Ω. Please note that the set of such η is dense in $(0,T)$.

Now, let us examine the following inequality

$$\begin{aligned} 0 \le \gamma_* \int_0^\eta \int_\Omega \left| \nabla \times \overline{\boldsymbol{h}_n} - \nabla \times \boldsymbol{H} \right|^2 &\le \int_0^\eta \int_\Omega \overline{\gamma_n}(t-\tau) \left| \nabla \times \overline{\boldsymbol{h}_n} - \nabla \times \boldsymbol{H} \right|^2 \\ &= \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{h}_n}, \nabla \times \overline{\boldsymbol{h}_n} \right) + \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{H} \right) \\ &\quad - 2 \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{h}_n}, \nabla \times \boldsymbol{H} \right). \end{aligned} \tag{2.36}$$

In the grounds of (iii) and the Lebesgue dominated Theorem 1.3, we get

$$\lim_{n \to \infty} \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{H} \right) \ \mathrm{d}t = \int_0^\eta (\gamma(u) \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{H}) \ \mathrm{d}t. \tag{2.37}$$

According to (iii), (ix), and the Lebesgue dominated Theorem 1.3, we deduce

$$\lim_{n \to \infty} \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{h}_n}, \nabla \times \boldsymbol{H} \right) \ \mathrm{d}t = \int_0^\eta (\gamma(u) \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{H}) \ \mathrm{d}t. \tag{2.38}$$

Using (2.31), we are allowed to say that ($\eta \in (t_{j-1}, t_j]$)

$$\begin{aligned} &\int_0^\eta \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{h}_n}, \nabla \times \overline{\boldsymbol{h}_n} \right) = \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \overline{\boldsymbol{F}_n}, \nabla \times \overline{\boldsymbol{h}_n} \right) - \int_0^\eta \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n} \right) \\ &= \int_0^\eta \left(\overline{\gamma_n}(t-\tau) \overline{\boldsymbol{F}_n}, \nabla \times \overline{\boldsymbol{h}_n} \right) - \int_0^{t_j} \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n} \right) - \int_{t_j}^\eta \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n} \right) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\eta \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla \times \overline{\boldsymbol{h}_n}\right) - \sum_{i=1}^{j} \left(\boldsymbol{B}(\boldsymbol{h}_i) - \boldsymbol{B}(\boldsymbol{h}_{i-1}), \boldsymbol{h}_i\right) - \int_{t_j}^\eta \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n}\right) \\
&= \int_0^\eta \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla \times \overline{\boldsymbol{h}_n}\right) - \sum_{i=1}^{j} \left(\boldsymbol{B}(\boldsymbol{h}_i) - \boldsymbol{B}(\boldsymbol{h}_{i-1}), \boldsymbol{B}^{-1}\left(\boldsymbol{B}(\boldsymbol{h}_i)\right)\right) \\
&- \int_{t_j}^\eta \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n}\right) \\
&\overset{(2.22)}{\leq} \int_0^\eta \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla \times \overline{\boldsymbol{h}_n}\right) + \sum_{i=1}^{j} \int_\Omega \left[\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{h}_{i-1})) - \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{h}_i))\right] \\
&- \int_{t_j}^\eta \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n}\right) \\
&= \int_0^\eta \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla \times \overline{\boldsymbol{h}_n}\right) + \int_\Omega \left[\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{H}_0)) - \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{h}_j))\right] \\
&- \int_{t_j}^\eta \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n}\right) \\
&= \int_0^\eta \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla \times \overline{\boldsymbol{h}_n}\right) + \int_\Omega \left[\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{H}_0)) - \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\overline{\boldsymbol{h}_n}(\eta))\right] \\
&- \int_{t_j}^\eta \left(\partial_t \boldsymbol{B}_n, \overline{\boldsymbol{h}_n}\right). \qquad (2.39)
\end{aligned}
$$

Collecting (2.36)-(2.39) and passing to the limit for $n \to \infty$, we get

$$
\begin{aligned}
\gamma_* \lim_{n\to\infty} \int_0^\eta \left\|\nabla \times \overline{\boldsymbol{h}_n} - \nabla \times \boldsymbol{H}\right\|^2 &\leq \int_0^\eta (\gamma(u)\boldsymbol{F}, \nabla \times \boldsymbol{H}) \\
&+ \int_\Omega \left[\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{h}_0)) - \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{H}(\eta))\right] - \int_0^\eta (\gamma(u)\nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{H}) \\
&= \int_0^\eta (\gamma(u)\boldsymbol{F}, \nabla \times \boldsymbol{H}) - \int_0^\eta \int_\Omega \frac{d\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B}(\boldsymbol{H}))}{dt} \\
&- \int_0^\eta (\gamma(u)\nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{H}) \qquad (2.40) \\
&\overset{(2.23)}{=} \int_0^\eta (\gamma(u)\boldsymbol{F}, \nabla \times \boldsymbol{H}) - \int_0^\eta (\partial_t \boldsymbol{B}(\boldsymbol{H})), \boldsymbol{H}) \\
&- \int_0^\eta (\gamma(u)\nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{H}) \\
&\overset{(xii)}{=} 0.
\end{aligned}
$$

This relation is valid for any $\eta \in (0, T)$ for which $\overline{\boldsymbol{h}_n}(\eta) \to \boldsymbol{H}(\eta)$ a.e. in Ω. Due to the fact that the set of such η is dense in $(0,T)$, we conclude that $\nabla \times \overline{\boldsymbol{h}_n} \to \nabla \times \boldsymbol{H}$ in $L^2((0,T);\mathbf{L}^2(\Omega))$.

(xiv) To show that $\boldsymbol{H}$ and u solve (2.17), we integrate (2.32) in time

$$\begin{aligned} &\left(\overline{\theta_n}(t),\psi\right) - (\theta_n(0),\psi) + \left(\theta_n(t) - \overline{\theta_n}(t),\psi\right) + \int_0^t (\lambda_0 \nabla \overline{u_n}, \nabla\psi)\ \mathrm{d}s \\ &= \int_0^t \left(\mathcal{R}_r\left(\overline{\gamma_n}(s-\tau)\left|\nabla \times \overline{\boldsymbol{h}_n}\right|^2\right),\psi\right)\ \mathrm{d}s. \end{aligned} \tag{2.41}$$

According to (iv), we see that

$$\lim_{n\to\infty}\left(\theta_n(t) - \overline{\theta_n}(t),\psi\right) = 0 \qquad \text{for every } t \in (0,T).$$

Due to $(iii), (xiii)$, and the fact that the function $\mathcal{R}_r$ is continuous and bounded we apply Lebesgue's dominated convergence Theorem 1.3 to pass to the limit on the r.h.s. and obtain

$$\lim_{n\to\infty}\int_0^t \left(\mathcal{R}_r\left(\overline{\gamma_n}(s-\tau)\left|\nabla \times \overline{\boldsymbol{h}_n}\right|^2\right),\psi\right)\ \mathrm{d}s = \int_0^t \left(\mathcal{R}_r\left(\gamma(u)\left|\nabla \times \boldsymbol{H}\right|^2\right),\psi\right)\ \mathrm{d}s.$$

Combination of the convergence results above, (ii), and (v) let us pass to the limit for $n \to \infty$ in the variational equation (2.41). Thus, we have

$$(\theta(u(t)),\psi) - (\theta(u(0)),\psi) + \int_0^t (\lambda_0 \nabla u, \nabla\psi) = \int_0^t \left(\mathcal{R}_r\left(\gamma(u)\left|\nabla \times \boldsymbol{H}\right|^2\right),\psi\right).$$

Differentiation with respect to the time variable yields (2.17) which also concludes the proof.

□

Chapter 3

Solvability of the induction heating model containing nonlinear magnetic field and controlled Joule heat

This chapter is based on the article [22] that has been published in
Journal of Computational and Applied Mathematics.

3.1 Derivation of the mathematical model

As we mentioned in the previous chapter, the nonlinear constitutive relation between the fields $\boldsymbol{H}$ and $\boldsymbol{B}$ can be understood in two ways. The adoption of the relation (2.4) was the subject of Chapter 2. The adoption of the other relation (2.5), i.e. $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{B})$ will be investigated in this chapter.

We consider the same domain Ω with the Lipschitz continuous boundary $\partial\Omega$. The boundary is associated with a unit outward normal vector $\boldsymbol{n}$. Again, the cornerstone of our model is Maxwell's equations (1.3)-(1.6). We consider the Ohm law as in (2.1), i.e.

$$\boldsymbol{E} = \boldsymbol{J}/\sigma + \boldsymbol{F}/\sigma = \gamma\boldsymbol{J} + \gamma\boldsymbol{F}$$

where σ, γ and $\boldsymbol{F}$ are same as in Chapter 2. By the adoption of the relation above and (2.5), we can eliminate other variables in (1.3)-(1.4) and we conclude that the magnetic induction field $\boldsymbol{B}$ is determined by the solution of the following nonlinear PDE:

$$\partial_t \boldsymbol{B} + \nabla \times (\gamma(u) \nabla \times \boldsymbol{H}(\boldsymbol{B})) = \nabla \times (\gamma(u)\boldsymbol{F}) \quad \text{in } Q_T \tag{3.1}$$

with the initial condition

$$\boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x}) \quad \text{in} \quad \Omega. \tag{3.2}$$

Moreover, because of (1.4), we also assume that

$$\nabla \cdot \boldsymbol{B}_0(\boldsymbol{x}) = 0 \quad \text{in} \quad \Omega.$$

The boundary $\partial\Omega$ is assumed to be a perfect conductor. Hence,

$$\boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega. \tag{3.3}$$

The heat transfer part of our model is determined with the same nonlinear PDE with a truncated r.h.s. as in Chapter 2, one exception being that the *Joule heat* is expressed as

$$Q = \gamma \left| \boldsymbol{J} \right|^2 = \gamma \left| \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right|^2.$$

Then, the temperature function $u(\boldsymbol{x}, t)$ is determined by the solution of the following boundary value problem:

$$\begin{array}{rcll} \partial_t \theta(u) - \nabla \cdot (\lambda_0 \nabla u) & = & \mathcal{R}_r \left(\gamma(u) \left| \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right|^2 \right) & \text{in} \quad Q_T, \\ u(\boldsymbol{x}, 0) & = & u_0(\boldsymbol{x}) & \text{in} \quad \Omega, \\ u(\boldsymbol{x}) & = & 0 & \text{on} \quad \partial\Omega. \end{array} \tag{3.4}$$

3.1.1 Weak formulation

The weak formulation of (3.1) coupled with (3.4) reads as:
Find $\boldsymbol{B}$ from the space $L^2((0,T); \mathbf{L}^2(\Omega))$ with $\partial_t \boldsymbol{B} \in L^2((0,T); \mathbf{H}^{-1}(\mathbf{curl}\,; \Omega))$ and $\boldsymbol{H}(\boldsymbol{B}) \in L^2((0,T); \mathbf{H}_0(\mathbf{curl}\,; \Omega))$ and $u \in C([0,T]; \mathrm{L}^2(\Omega)) \cap L^\infty((0,T); H_0^1(\Omega))$ with $\partial_t u \in L^2((0,T); L^2(\Omega))$ such that

$$(\partial_t \boldsymbol{B}, \boldsymbol{\varphi}) + (\gamma(u) \nabla \times \boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{\varphi}) = (\gamma(u)\boldsymbol{F}, \nabla \times \boldsymbol{\varphi}), \tag{3.5}$$

$$\boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x}) \quad \text{in} \quad \Omega,$$

$$(\partial_t \theta(u), \psi) + (\lambda_0 \nabla u, \nabla \psi) = \left(\mathcal{R}_r \left(\gamma(u) \left| \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right|^2 \right), \psi \right), \tag{3.6}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \text{in} \quad \Omega,$$

is satisfied for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,; \Omega)$ and $\psi \in H_0^1(\Omega)$.

The main difference between the model introduced in Chapter 2 and the model (3.1), (3.2), (3.3), and (3.4) is in the position of the nonlinear term. To prove the existence of a weak solution of (3.5) and (3.6), we require strong convergence of the argument of the truncation function in (3.6). Hence, the strong convergence of $\nabla \times \boldsymbol{H}$ was required in Chapter 2. However, in this chapter, we need to show the strong convergence of the nonlinear vector field $\nabla \times \boldsymbol{H}(\boldsymbol{B})$.

3.1.2 Assumptions

Some necessary assumptions on the vector field $\boldsymbol{H}(\boldsymbol{x})$ have to be made. Namely, $\boldsymbol{H}(\boldsymbol{x})$ is supposed to be strongly monotone, Lipschitz continuous, hemicontinuous and potential, i.e.

$$\begin{aligned} (\boldsymbol{H}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq \omega_H \left|\boldsymbol{x} - \boldsymbol{y}\right|^2 &\qquad \omega_H > 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, \\ \left|\boldsymbol{H}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{y})\right| \leq C_L \left|\boldsymbol{x} - \boldsymbol{y}\right| &\qquad C_L > 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, \\ \boldsymbol{H}(\boldsymbol{0}) = \boldsymbol{0}. & \end{aligned} \tag{3.7}$$

The strong monotonicity of $\boldsymbol{H}(\boldsymbol{x})$ implies the existence of the inverse field $\boldsymbol{H}^{-1}(\boldsymbol{x})$ which is also strongly monotone and Lipschitz continuous.

Similarly, as in the previous chapter, we need to make a few calculations regarding the potential $\Phi_{\boldsymbol{H}}$ of the vector field $\boldsymbol{H}$

$$\begin{aligned} \Phi_{\boldsymbol{H}}(\boldsymbol{x}) &= \int_0^1 \boldsymbol{H}(s\boldsymbol{x}) \cdot \boldsymbol{x} \, \mathrm{d}s = \int_0^1 s^{-1} \boldsymbol{H}(s\boldsymbol{x}) \cdot s\boldsymbol{x} \, \mathrm{d}s \\ &\overset{(3.7)a}{\geq} \int_0^1 s^{-1} \omega_H \left|s\boldsymbol{x}\right|^2 \, \mathrm{d}s = \frac{\omega_H}{2} \left|\boldsymbol{x}\right|^2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \Phi_{\boldsymbol{H}}(\boldsymbol{x}) &= \int_0^1 \boldsymbol{H}(s\boldsymbol{x}) \cdot \boldsymbol{x} \, \mathrm{d}s \leq \int_0^1 \left|\boldsymbol{H}(s\boldsymbol{x})\right| \left|\boldsymbol{x}\right| \, \mathrm{d}s \\ &\overset{(3.7)b}{\leq} C_L \int_0^1 \left|s\boldsymbol{x}\right| \left|\boldsymbol{x}\right| \, \mathrm{d}s = \frac{C_L}{2} \left|\boldsymbol{x}\right|^2. \end{aligned} \tag{3.9}$$

3.2 Time discretization

Using the same technique and notations as in Section 2.3, we introduce a time-discretized decoupled version of the variational formulations (3.5) and (3.6)

$$(\delta \boldsymbol{b}_i, \boldsymbol{\varphi}) + (\gamma(u_{i-1}) \nabla \times \boldsymbol{H}(\boldsymbol{b}_i), \nabla \times \boldsymbol{\varphi}) = (\gamma(u_{i-1}) \boldsymbol{F}_i, \nabla \times \boldsymbol{\varphi}), \tag{3.10}$$

$$(\delta\theta_i, \psi) + (\lambda_0 \nabla u_i, \nabla\psi) = \left(\mathcal{R}_r\left(\gamma(u_{i-1})\left|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\right|^2\right), \psi\right) \tag{3.11}$$

at each time step t_i for $i = 1, \ldots, n$, $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and $\psi \in H_0^1(\Omega)$.

Existence of a unique solution of the variational formulation above at each time step t_i, $i = 1, \ldots, n$ is covered by the following lemma.

Lemma 3.1. *Let* (2.13) *and* (3.7) *hold true. Moreover, assume that* $\boldsymbol{B}_0 \in \mathbf{L}^2(\Omega)$, $\nabla\cdot\boldsymbol{B}_0 = 0$, $u_0 \in \mathrm{L}^2(\Omega)$, $\boldsymbol{F} : [0,T] \to \mathbf{L}^2(\Omega)$ *and* $\boldsymbol{F} \in L^2((0,T);\mathbf{L}^2(\Omega))$. *Then there exist uniquely determined* $\boldsymbol{b}_i \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}(\boldsymbol{b}_i) \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ *and* $u_i \in H_0^1(\Omega)$ *solving (3.10) and (3.11) for any* $i = 1, \ldots, n$.

Proof. Let us define the operators M_γ and N in the following manner

$$\langle M_\gamma(\boldsymbol{v}), \boldsymbol{\varphi}\rangle := \left(\frac{\boldsymbol{H}^{-1}(\boldsymbol{v})}{\tau}, \boldsymbol{\varphi}\right) + (\gamma\nabla\times\boldsymbol{v}, \nabla\times\boldsymbol{\varphi}),$$

$$\langle N(u), \psi\rangle := \left(\frac{\theta(u)}{\tau}, \psi\right) + (\lambda_0\nabla u, \nabla\psi).$$

We need to show that operators defined above are hemicontinuous, strictly monotone and coercive (for $0 < \tau < 1$). Hemicontinuity follows from the Lipschitz continuity of $\boldsymbol{H}^{-1}(\boldsymbol{v})$ and the continuity of $\theta(u)$, cf.[79, Definition 1.8]. The next step is to show strict monotonicity. We start with the operator M_γ (we use strong monotonicity of the vector field $\boldsymbol{H}^{-1}(\boldsymbol{v})$):

$$\begin{aligned}
&\langle M_\gamma(\boldsymbol{v}_1) - M_\gamma(\boldsymbol{v}_2), \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle \\
&= \left(\frac{\boldsymbol{H}^{-1}(\boldsymbol{v}_1) - \boldsymbol{H}^{-1}(\boldsymbol{v}_2)}{\tau}, \boldsymbol{v}_1 - \boldsymbol{v}_2\right) + (\gamma(\nabla\times\boldsymbol{v}_1 - \nabla\times\boldsymbol{v}_2), \nabla\times\boldsymbol{v}_1 - \nabla\times\boldsymbol{v}_2) \\
&\geq \omega_{H^{-1}}\left\|\boldsymbol{v}_1 - \boldsymbol{v}_2\right\|^2 + \gamma_*\left\|\nabla\times\boldsymbol{v}_1 - \nabla\times\boldsymbol{v}_2\right\|^2 \\
&\geq \min\{\omega_{H^{-1}}, \gamma_*\}\left\|\boldsymbol{v}_1 - \boldsymbol{v}_2\right\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \\
&> 0
\end{aligned}$$

for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$, $\boldsymbol{v}_1 \neq \boldsymbol{v}_2$. Hence, the operator M_γ is strongly monotone. Strong monotonicity of the operator N is shown correspondingly

$$\begin{aligned}
&\langle N(u_1) - N(u_2), u_1 - u_2\rangle \\
&= \left(\frac{\theta(u_1) - \theta(u_2)}{\tau}, u_1 - u_2\right) + (\lambda_0(\nabla u_1 - \nabla u_2), \nabla u_1 - \nabla u_2) \\
&= \left(\theta'(\xi)\frac{u_1 - u_2}{\tau}, u_1 - u_2\right) + (\lambda_0(\nabla u_1 - \nabla u_2), \nabla u_1 - \nabla u_2)
\end{aligned}$$

$$\ge \theta_* \|u_1 - u_2\|^2 + \lambda_* \|\nabla u_1 - \nabla u_2\|^2$$
$$\ge \min\{\theta_*, \lambda_*\} \|u_1 - u_2\|^2_{H^1(\Omega)}$$
$$> 0$$

for any $u_1, u_2 \in H_0^1(\Omega)$, $u_1 \neq u_2$. Coercivity of both operators follows from the fact that $\boldsymbol{H}^{-1}(\mathbf{0}) = \mathbf{0}$ and $\theta(0) = 0$. Indeed,

$$\begin{aligned}\langle M_\gamma(\boldsymbol{v}), \boldsymbol{v}\rangle &= \left(\frac{\boldsymbol{H}^{-1}(\boldsymbol{v}) - \boldsymbol{H}^{-1}(\mathbf{0})}{\tau}, \boldsymbol{v} - \mathbf{0}\right) + (\gamma \nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{v}) \\ &\ge \omega_{H^{-1}} \|\boldsymbol{v}\|^2 + \gamma_* \|\nabla \times \boldsymbol{v}\|^2 \\ &\ge \min\{\omega_{H^{-1}}, \gamma_*\} \|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)}\end{aligned}$$

and

$$\begin{aligned}\langle N(u), u\rangle &= \left(\frac{\theta'(\xi)(u-0)}{\tau}, u - 0\right) + (\lambda_0 \nabla u, \nabla u) \\ &\ge \theta_* \|u\|^2 + \lambda_* \|\nabla u\|^2 \\ &\ge \min\{\theta_*, \lambda_*\} \|u\|^2_{H^1(\Omega)}.\end{aligned}$$

Thus,

$$\lim_{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \to \infty} \frac{\langle M_\gamma(\boldsymbol{v}), \boldsymbol{v}\rangle}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}} \ge +\infty$$

and

$$\lim_{\|u\|_{H^1(\Omega)} \to \infty} \frac{\langle N(u), u\rangle}{\|u\|_{H^1(\Omega)}} \ge +\infty.$$

To conclude the proof, we present a pseudoalgorithm which provides a unique solution pair at each time step:

Algorithm 3 Implicit Euler

Require: $\boldsymbol{v}_0 := \boldsymbol{H}(\boldsymbol{B}_0) \in \mathbf{H}_0(\mathbf{curl}\,;\Omega), u_0 \in H_0^1(\Omega), n \in \mathbb{N}$ ▷ $\boldsymbol{v}_i := \boldsymbol{H}(\boldsymbol{b}_i)$

```
1: for i = 1, i ≤ n do
2:     v_i ← Solve: ⟨M_{γ(u_{i-1})}(v_i), φ⟩ = (H^{-1}(v_{i-1})/τ, φ) + (γ(u_{i-1}) F_i, ∇ × φ)
3:     b_{i-1} ← H^{-1}(v_{i-1})                         ▷ b_{i-1} ∈ L²(Ω)
4:     b_i ← H^{-1}(v_i)                                 ▷ b_i ∈ L²(Ω), H(b_i) ∈ H_0(curl; Ω)
5:     u_i ← Solve: ⟨N(u_i), ψ⟩ = (θ(u_{i-1})/τ, ψ) + (R_r(γ(u_{i-1}) |∇ × H(b_i)|²), ψ)
6:     i ← i + 1
7: return {b_1, u_1}, ..., {b_n, u_n}
```

Line 2: $\boldsymbol{v}_i \leftarrow$ Solve: $\left\langle M_{\gamma(u_{i-1})}(\boldsymbol{v}_i), \boldsymbol{\varphi}\right\rangle = \left(\frac{\boldsymbol{H}^{-1}(\boldsymbol{v}_{i-1})}{\tau}, \boldsymbol{\varphi}\right) + (\gamma(u_{i-1})\boldsymbol{F}_i, \nabla \times \boldsymbol{\varphi})$

Line 5: $u_i \leftarrow$ Solve: $\langle N(u_i), \psi\rangle = \left(\frac{\theta(u_{i-1})}{\tau}, \psi\right) + \left(\mathcal{R}_r\left(\gamma(u_{i-1})\left|\nabla \times \boldsymbol{H}(\boldsymbol{b}_i)\right|^2\right), \psi\right)$

□

3.2.1 A priori energy estimates

Since the evolution of temperature in Ω is modeled with the same nonlinear PDE as in Chapter 2, the energy estimates for the functions u_i remain the same. For this reason, we find it unnecessary to state the same lemma again and we refer the reader to Lemma 2.4.

The basic energy estimates and stability results for the functions $\boldsymbol{b}_i$ and $\boldsymbol{H}(\boldsymbol{b}_i)$ are covered by the following lemma.

Lemma 3.2. *Let the assumptions of Lemma 3.1 be satisfied. Then there exists a positive constant C such that*

$$
\begin{aligned}
(i)&\quad \max_{1\le j\le n}\|\boldsymbol{b}_j\|^2+\sum_{i=1}^n\|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\|^2\tau\le C,\\
(ii)&\quad \sum_{i=1}^n\|\delta\boldsymbol{b}_i\|^2_{\mathbf{H}^{-1}(\mathbf{curl};\Omega)}\tau\le C,\\
(iii)&\quad \nabla\cdot\boldsymbol{b}_i=0\qquad for\qquad i=1,\dots,n.
\end{aligned}
$$

Proof. (i) Set $\boldsymbol{\varphi}=\boldsymbol{H}(\boldsymbol{b}_i)\tau$ in (3.10) and sum it up for $i=1,\dots,j$, to obtain

$$
\sum_{i=1}^j(\boldsymbol{b}_i-\boldsymbol{b}_{i-1},\boldsymbol{H}(\boldsymbol{b}_i))+\sum_{i=1}^j(\gamma(u_{i-1})\nabla\times\boldsymbol{H}(\boldsymbol{b}_i),\nabla\times\boldsymbol{H}(\boldsymbol{b}_i))\,\tau
=\sum_{i=1}^j(\gamma(u_{i-1})\boldsymbol{F}_i,\nabla\times\boldsymbol{H}(\boldsymbol{b}_i))\,\tau. \tag{3.12}
$$

With the help of Cauchy-Schwarz's and Young's inequalities, we are able to handle the r.h.s. in (3.12)

$$
\sum_{i=1}^j(\gamma(u_{i-1})\boldsymbol{F}_i,\nabla\times\boldsymbol{H}(\boldsymbol{b}_i))\,\tau\le C_\varepsilon\sum_{i=1}^j\|\boldsymbol{F}_i\|^2\tau+\varepsilon\sum_{i=1}^j\|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\|^2\tau.
$$

The second term on the l.h.s. in (3.12) can be bounded from below since the function γ is bounded and positive

$$
\sum_{i=1}^j(\gamma(u_{i-1})\nabla\times\boldsymbol{H}(\boldsymbol{b}_i),\nabla\times\boldsymbol{H}(\boldsymbol{b}_i))\ge\gamma_*\sum_{i=1}^j\|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\|^2\tau.
$$

The trickiest term in (3.12) is the first one. First, we use the fact that the potential $\Phi_{\boldsymbol{H}}$ is convex, cf. [79, Theorem 1.6] and then the monotonicity and Lipschitz continuity of the vector field $\boldsymbol{H}$ to obtain that

$$\begin{aligned}\sum_{i=1}^{j}(\boldsymbol{b}_i-\boldsymbol{b}_{i-1},\boldsymbol{H}(\boldsymbol{b}_i)) &= \sum_{i=1}^{j}\int_\Omega(\boldsymbol{b}_i-\boldsymbol{b}_{i-1})\cdot\boldsymbol{H}(\boldsymbol{b}_i)\,\mathrm{d}\boldsymbol{x}\\ &\overset{Theorem\ 1.11}{\geq} \sum_{i=1}^{j}\int_\Omega \Phi_{\boldsymbol{H}}(\boldsymbol{b}_i)-\Phi_{\boldsymbol{H}}(\boldsymbol{b}_{i-1})\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega \Phi_{\boldsymbol{H}}(\boldsymbol{b}_j)\,\mathrm{d}\boldsymbol{x}-\int_\Omega \Phi_{\boldsymbol{H}}(\boldsymbol{B}_0)\,\mathrm{d}\boldsymbol{x}\\ &\overset{(3.8),(3.9)}{\geq} \frac{\omega_H}{2}\left\|\boldsymbol{b}_j\right\|^2-\frac{C_L}{2}\left\|\boldsymbol{B}_0\right\|^2.\end{aligned}$$

Collecting all estimates, we get

$$\frac{\omega_H}{2}\left\|\boldsymbol{b}_j\right\|^2+(\gamma_*-\varepsilon)\sum_{i=1}^{j}\left\|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\right\|^2\tau \leq \frac{C_L}{2}\left\|\boldsymbol{B}_0\right\|^2+C_\varepsilon\sum_{i=1}^{j}\left\|\boldsymbol{F}_i\right\|^2\tau.$$

Taking $\varepsilon\in(0,\gamma_*)$, we arrive at

$$\left\|\boldsymbol{b}_j\right\|^2+\sum_{i=1}^{j}\left\|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\right\|^2\tau\leq C.$$

(ii) From (3.10), we have for any $\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl}\,;\Omega)$ that

$$\begin{aligned}(\delta\boldsymbol{b}_i,\boldsymbol{\varphi}) &= (\gamma(u_{i-1})\boldsymbol{F}_i,\nabla\times\boldsymbol{\varphi})-(\gamma(u_{i-1})\nabla\times\boldsymbol{H}(\boldsymbol{b}_i),\nabla\times\boldsymbol{\varphi})\\ &\leq C(\|\boldsymbol{F}_i\|+\|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\|)\left\|\boldsymbol{\varphi}\right\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}.\end{aligned}$$

This, together with the definition of the norm in the space $\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)$, gives us

$$\|\delta\boldsymbol{b}_i\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\leq C(\|\boldsymbol{F}_i\|+\|\nabla\times\boldsymbol{H}(\boldsymbol{b}_i)\|).$$

Estimates obtained in (i) allow us to write

$$\sum_{i=1}^{n}\|\delta\boldsymbol{b}_i\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\tau\leq C.$$

(iii) Take any $\Phi\in H^1_0(\Omega)$. Put $\boldsymbol{\varphi}=\nabla\Phi$ in (3.10) to find out that

$$0=(\delta\boldsymbol{b}_i,\nabla\Phi)=-\left(\nabla\cdot\delta\boldsymbol{b}_i,\Phi\right)=-\left(\delta\nabla\cdot\boldsymbol{b}_i,\Phi\right).$$

Hence,

$$(\nabla\cdot\boldsymbol{b}_i,\Phi)=(\nabla\cdot\boldsymbol{b}_{i-1},\Phi)=\cdots=(\nabla\cdot\boldsymbol{b}_0,\Phi)=0.$$

From this we conclude that $\nabla\cdot\boldsymbol{b}_i=0$ for $i=1,\ldots,n$. □

3.3 The existence of a global solution

Following the same notations as in Section 2.4, we rewrite (3.10) and (3.11) on the whole time frame $[0,T]$ as

$$(\partial_t \boldsymbol{b}_n, \boldsymbol{\varphi}) + \left(\overline{\gamma_n}(t-\tau)\nabla\times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla\times\boldsymbol{\varphi}\right) = \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla\times\boldsymbol{\varphi}\right), \tag{3.13}$$

$$(\partial_t \theta_n, \psi) + (\lambda_0 \nabla \overline{u_n}, \nabla\psi) = \left(\mathcal{R}_r\left(\overline{\gamma_n}(t-\tau)\left|\nabla\times\boldsymbol{H}(\overline{\boldsymbol{b}_n})\right|^2\right), \psi\right) \tag{3.14}$$

for any $\boldsymbol{\varphi}\in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and $\psi\in H_0^1(\Omega)$.

Theorem 3.1. *Suppose that all assumptions of Lemma 3.1 are fulfilled. In addition, assume that γ is a global Lipschitz continuous function and $\boldsymbol{F}$ is Lipschitz continuous in time. Then there exist u from $C([0,T];L^2(\Omega))\cap L^\infty((0,T);H_0^1(\Omega))$ with $\partial_t u$ being in $L^2((0,T);L^2(\Omega))$ and $\boldsymbol{B}$ from $L^2((0,T);\mathbf{L}^2(\Omega))$ with $\partial_t\boldsymbol{B}$ being in $L^2((0,T);L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)))$ and $\boldsymbol{H}(\boldsymbol{B})$ being in $L^2((0,T);\mathbf{H}_0(\mathbf{curl}\,;\Omega))$ and subsequences of u_n and $\boldsymbol{b}_n$ (still denoted as n) such that*

$$(A)\begin{cases} u_n \to u & \text{in } C([0,T];L^2(\Omega)),\\ \overline{u_n}(t) \rightharpoonup u(t) & \text{in } H_0^1(\Omega), \forall t\in[0,T],\\ \overline{\gamma_n}(t-\tau)\to\gamma(u) & \text{in } L^2((0,T);L^2(\Omega)),\\ \overline{\theta_n}\to\theta(u) & \text{in } L^2((0,T);L^2(\Omega)),\end{cases}$$

$$(B)\begin{cases} \overline{\boldsymbol{b}_n}\to\boldsymbol{B} & \text{in } L^2((0,T);\mathbf{L}^2(\Omega)),\\ \partial_t\boldsymbol{b}_n\rightharpoonup\partial_t\boldsymbol{B} & \text{in } L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)),\\ \boldsymbol{H}(\overline{\boldsymbol{b}_n})\to\boldsymbol{H}(\boldsymbol{B}) & \text{in } L^2((0,T);\mathbf{H}(\mathbf{curl}\,;\Omega)),\end{cases}$$

(C) *u and $\boldsymbol{B}$ solve* (3.5) *and* (3.6).

Proof. We prove the theorem in three main parts. In the first part, we provide some convergence results for functions u_n and $\overline{u_n}$ and mark these partial results for further reference. Next, we prove the convergence results for functions $\boldsymbol{b}_n$ and $\overline{\boldsymbol{b}_n}$. Finally, in the last part, we prove the existence of the solution pair $\{u,\boldsymbol{b}\}$ of (3.5) and (3.6).

Part (A): We omit the proof of this part since it has been proven in Theorem 2.1.

Part (B): Using the second part of Lemma 3.2, we write

$$\left|\left(\overline{\boldsymbol{b}_n}-\boldsymbol{b}_n,\boldsymbol{\varphi}\right)\right| \le \tau\left|(\partial_t\boldsymbol{b}_n,\boldsymbol{\varphi})\right| \le \tau\left\|\partial_t\boldsymbol{b}_n\right\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\left\|\boldsymbol{\varphi}\right\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)}.$$

Therefore,

$$(i)\quad \left\|\overline{\boldsymbol{b}_n}-\boldsymbol{b}_n\right\|_{L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega))} \le \tau\int_0^T \left\|\partial_t\boldsymbol{b}_n\right\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}\,\mathrm{d}t \le \tau C \overset{n\to\infty}{\to} 0.$$

Due to the Lipschitz continuity of the vector field $\boldsymbol{H}$ and Lemma 3.2, we have

$$\int_0^T \left\|\boldsymbol{H}(\overline{\boldsymbol{b}_n})\right\|^2 \, \mathrm{d}t = \sum_{i=1}^n \tau \int_\Omega |\boldsymbol{H}(\boldsymbol{b}_i)|^2 \, \mathrm{d}\boldsymbol{x} \le C \sum_{i=1}^n \tau \int_\Omega |\boldsymbol{b}_i|^2 \, \mathrm{d}\boldsymbol{x}$$
$$= C \sum_{i=1}^n \|\boldsymbol{b}_i\|^2 \tau \le C$$

and

$$\begin{aligned}\int_0^T \|\boldsymbol{b}_n\|^2 \, \mathrm{d}t &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\boldsymbol{b}_{i-1} + (t - t_{i-1})\delta \boldsymbol{b}_i\|^2 \, \mathrm{d}t \\ &\le \sum_{i=1}^n \left(\|\boldsymbol{b}_{i-1}\|^2 + \|\boldsymbol{b}_i - \boldsymbol{b}_{i-1}\|^2\right) \tau \\ &\le \|\boldsymbol{B}_0\|^2 + C \sum_{i=1}^n \|\boldsymbol{b}_i\|^2 \tau \le C.\end{aligned}$$

Now, combining the estimates above with Lemma 3.2, we get

$$\int_0^T \left(\left\|\boldsymbol{H}(\overline{\boldsymbol{b}_n})\right\|^2 + \left\|\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right\|^2\right) \mathrm{d}t \le C,$$
$$\int_0^T \left(\|\boldsymbol{b}_n\|^2 + \|\nabla \cdot \boldsymbol{b}_n\|^2 + \|\partial_t \boldsymbol{b}_n\|^2_{\mathbf{H}^{-1}(\mathbf{curl};\Omega)}\right) \mathrm{d}t \le C.$$

Moreover, the space $L^2((0,T);\mathbf{L}^2(\Omega))$ is a reflexive Banach space, therefore, there exist vector fields $\boldsymbol{p}$ and $\boldsymbol{B}$ and subsequences of $\boldsymbol{H}(\overline{\boldsymbol{b}_n})$ and $\boldsymbol{b}_n$ (still denoted with n) such that $\boldsymbol{H}(\overline{\boldsymbol{b}_n}) \rightharpoonup \boldsymbol{p}$ and $\boldsymbol{b}_n \rightharpoonup \boldsymbol{B}$ in that space. With all this considered, we prompt Lemma 3.1 from [69] to prove that

$$\lim_{n\to\infty} \int_0^T \left(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \Phi \boldsymbol{b}_n\right) \mathrm{d}t = \int_0^T (\boldsymbol{p}, \Phi \boldsymbol{B}) \, \mathrm{d}t \tag{3.15}$$

for any $\Phi \in C_0^\infty(\overline{\Omega})$. Because of the monotonicity of $\boldsymbol{H}$, we have

$$\lim_{n\to\infty} \int_0^T \left(\boldsymbol{H}(\overline{\boldsymbol{b}_n}) - \boldsymbol{H}(\boldsymbol{q}), \Phi(\overline{\boldsymbol{b}_n} - \boldsymbol{q})\right) \mathrm{d}t \ge 0$$

where $\boldsymbol{q} \in L^2((0,T);\mathbf{L}^2(\Omega))$ is arbitrary and $\Phi \in C_0^\infty(\overline{\Omega})$ is non-negative. The basic idea to investigate the limit of the term above is to split it into four parts and examine them separately, i.e.

$$I_1 := \int_0^T \left(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \Phi \overline{\boldsymbol{b}_n}\right) \mathrm{d}t, \qquad I_2 := \int_0^T \left(\boldsymbol{H}(\boldsymbol{q}), \Phi \overline{\boldsymbol{b}_n}\right) \mathrm{d}t,$$

$$I_3 := \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \varPhi \boldsymbol{q}\big)\ \mathrm{d}t, \qquad I_4 := \int_0^T (\boldsymbol{H}(\boldsymbol{q}), \varPhi \boldsymbol{q})\ \mathrm{d}t.$$

We rewrite the first term as follows

$$I_1 = \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \varPhi(\overline{\boldsymbol{b}_n} - \boldsymbol{b}_n)\big)\ \mathrm{d}t + \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \varPhi \boldsymbol{b}_n\big)\ \mathrm{d}t.$$

From Lemma 3.2 and (i), we get

$$\begin{aligned}
\left| \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \varPhi(\overline{\boldsymbol{b}_n} - \boldsymbol{b}_n)\big) \right| &\le C \int_0^T \left\|\boldsymbol{H}(\overline{\boldsymbol{b}_n})\right\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \int_0^T \left\|\overline{\boldsymbol{b}_n} - \boldsymbol{b}_n\right\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \\
&\le C \left\|\overline{\boldsymbol{b}_n} - \boldsymbol{b}_n\right\|_{L^2((0,T);\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega))} \\
&\overset{n\to\infty}{\to} 0.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} I_1 = \lim_{n\to\infty} \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \varPhi \boldsymbol{b}_n\big)\ \mathrm{d}t \overset{(3.15)}{=} \int_0^T (\boldsymbol{p}, \varPhi \boldsymbol{B})\ \mathrm{d}t.$$

We can pass to the limit for $n \to \infty$ in the remaining terms and see that

$$\begin{aligned}
&\lim_{n\to\infty} I_2 = \int_0^T (\boldsymbol{H}(\boldsymbol{q}), \varPhi \boldsymbol{B})\ \mathrm{d}t, \quad \lim_{n\to\infty} I_3 = \int_0^T (\boldsymbol{p}, \varPhi \boldsymbol{q})\ \mathrm{d}t, \\
&\lim_{n\to\infty} I_4 = \int_0^T (\boldsymbol{H}(\boldsymbol{q}), \varPhi \boldsymbol{q})\ \mathrm{d}t.
\end{aligned}$$

Therefore, gathering all partial results, we get

$$\lim_{n\to\infty} \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}) - \boldsymbol{H}(\boldsymbol{q}), \varPhi(\overline{\boldsymbol{b}_n} - \boldsymbol{q})\big)\ \mathrm{d}t = \int_0^T (\boldsymbol{p} - \boldsymbol{H}(\boldsymbol{q}), \varPhi(\boldsymbol{B} - \boldsymbol{q}))\ \mathrm{d}t \ge 0.$$

Now, let us set $\boldsymbol{q} = \boldsymbol{B} + \varepsilon \boldsymbol{v}$, where $\boldsymbol{v} \in L^2((0,T);\mathbf{L}^2(\Omega))$ and $\varepsilon > 0$, to obtain

$$\int_0^T (\boldsymbol{p} - \boldsymbol{H}(\boldsymbol{B} + \varepsilon \boldsymbol{v}), \varPhi \boldsymbol{v})\ \mathrm{d}t \le 0.$$

Passing to $\varepsilon \to 0$ brings us to

$$\int_0^T (\boldsymbol{p} - \boldsymbol{H}(\boldsymbol{B}), \varPhi \boldsymbol{v})\ \mathrm{d}t \le 0.$$

Due to the fact that the inequality above is valid for any $\boldsymbol{v} \in L^2((0,T); \mathbf{L}^2(\Omega))$, we replace $\boldsymbol{v}$ with $-\boldsymbol{v}$ and see that the reversed inequality holds also. Hence, we get

$$\int_0^T (\boldsymbol{p} - \boldsymbol{H}(\boldsymbol{B}), \Phi \boldsymbol{v}) \ \mathrm{d}t = 0$$

that is true for any $\boldsymbol{v} \in L^2((0,T); \mathbf{L}^2(\Omega))$ and all non-negative $\Phi \in C_0^\infty(\overline{\Omega})$. Therefore, $\boldsymbol{p} = \boldsymbol{H}(\boldsymbol{B})$ a.e. in $\Omega \times (0,T)$, i.e.

$$(ii) \ \ \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \rightharpoonup \boldsymbol{H}(\boldsymbol{B}) \ \text{ in } \ L^2((0,T); \mathbf{L}^2(\Omega)).$$

Let $\Phi \in C_0^\infty(\overline{\Omega})$ be non-negative. From (3.7), we have

$$\int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}) - \boldsymbol{H}(\boldsymbol{q}), \Phi(\overline{\boldsymbol{b}_n} - \boldsymbol{q})\big) \ \mathrm{d}t \geq \omega_H \int_0^T \left(\Phi, \left|\overline{\boldsymbol{b}_n} - \boldsymbol{q}\right|^2\right) \ \mathrm{d}t \geq 0.$$

Setting $\boldsymbol{q} = \boldsymbol{B}$ and using the same reasoning as in (ii), we conclude

$$0 = \lim_{n\to\infty} \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}) - \boldsymbol{H}(\boldsymbol{B}), \Phi(\overline{\boldsymbol{b}_n} - \boldsymbol{B})\big) \geq \omega_H \lim_{n\to\infty} \int_0^T \left(\Phi, \left|\overline{\boldsymbol{b}_n} - \boldsymbol{B}\right|^2\right) \geq 0.$$

This inequality is valid for any non-negative $\Phi \in C_0^\infty(\overline{\Omega})$, therefore

$$(iii) \ \ \overline{\boldsymbol{b}_n} \to \boldsymbol{B} \ \text{ in } \ L^2((0,T); \mathbf{L}^2(\Omega)).$$

Consider $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,; \Omega)$, then the following holds

$$\begin{aligned} (iv) \ \ \lim_{n\to\infty} \int_0^T \big(\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \boldsymbol{\varphi}\big) &= \lim_{n\to\infty} \int_0^T \big(\boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla \times \boldsymbol{\varphi}\big) \\ &= \int_0^T (\boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{\varphi}) = \int_0^T (\nabla \times \boldsymbol{H}(\boldsymbol{B}), \boldsymbol{\varphi}). \end{aligned}$$

The space $L^2((0,T); \mathbf{H}(\mathbf{curl}\,; \Omega))$ is reflexive, this combined with Lemma 3.2 guarantees the existence of $\boldsymbol{z} \in L^2((0,T); \mathbf{H}(\mathbf{curl}\,; \Omega))$ such that

$$\int_0^t (\partial_s \boldsymbol{B}_n, \boldsymbol{\varphi}) \ \mathrm{d}s \overset{n\to\infty}{\to} \int_0^t (\boldsymbol{z}, \boldsymbol{\varphi}) \ \mathrm{d}s.$$

The sequence $\boldsymbol{b}_n(t)$ is equibounded in $\mathbf{H}^{-1}(\mathbf{curl}\,; \Omega)$, i.e. for any $n \in \mathbb{N}$ it holds that

$$(\boldsymbol{b}_n(t), \boldsymbol{\varphi}) - (\boldsymbol{b}_n(0), \boldsymbol{\varphi}) = \int_0^t (\partial_s \boldsymbol{b}_n, \boldsymbol{\varphi}) \ \mathrm{d}s \tag{3.16}$$

$$\begin{aligned} &\leq \int_0^t \|\partial_s \boldsymbol{b}_n\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \,\mathrm{d}s \\ &\leq C \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} . \end{aligned} \tag{3.17}$$

Hence, we have

$$(\boldsymbol{b}_n(t), \boldsymbol{\varphi}) \leq C \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} + (\boldsymbol{B}_0, \boldsymbol{\varphi}) \leq C \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \tag{3.18}$$

which brings us to

$$\|\boldsymbol{b}_n(t)\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \leq C.$$

The same sequence is also equicontinuous. Indeed, for any $t_1, t_2 \in [0, T]$, the following holds

$$\begin{aligned} |(\boldsymbol{b}_n(t_2) - \boldsymbol{b}_n(t_1), \boldsymbol{\varphi})| = \left| \int_{t_1}^{t_2} (\partial_s \boldsymbol{b}_n, \boldsymbol{\varphi}) \right| &\leq \int_{t_1}^{t_2} \|\partial_s \boldsymbol{b}_n\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \\ &\leq \sqrt{\int_{t_1}^{t_2} 1^2} \sqrt{\int_{t_1}^{t_2} \|\partial_t \boldsymbol{b}_n\|^2_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)}} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} \\ &\leq C\sqrt{|t_2 - t_1|} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega)} . \end{aligned} \tag{3.19}$$

Therefore, we get

$$\|\boldsymbol{b}_n(t_2) - \boldsymbol{b}_n(t_1)\|_{\mathbf{H}^{-1}(\mathbf{curl}\,;\Omega)} \leq C |t_2 - t_1|^{\frac{1}{2}} .$$

Using Theorem 1.5-(i) yields

$$\lim_{n\to\infty} (\boldsymbol{b}_n(t), \boldsymbol{\varphi}) = (\boldsymbol{B}(t), \boldsymbol{\varphi}) \qquad \text{for any } \boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega) \text{ and for any } t \in [0, T].$$

Now, we conclude that $\boldsymbol{z} = \partial_t \boldsymbol{B}$ a.e. in $\Omega \times (0, T)$

$$\begin{aligned} (v) \quad \int_0^t (\partial_s \boldsymbol{B}, \boldsymbol{\varphi}) \,\mathrm{d}s = (\boldsymbol{B}(t) - \boldsymbol{B}_0, \boldsymbol{\varphi}) &= \lim_{n\to\infty} (\boldsymbol{b}_n(t) - \boldsymbol{b}_n(0), \boldsymbol{\varphi}) \\ &= \lim_{n\to\infty} \int_0^t (\partial_s \boldsymbol{b}_n, \boldsymbol{\varphi}) \,\mathrm{d}s = \int_0^t (\boldsymbol{z}, \boldsymbol{\varphi}) \,\mathrm{d}s. \end{aligned}$$

The Lipschitz continuity of $\boldsymbol{F}$ implies that

$$(vi) \ \ \overline{\boldsymbol{F}_n} \to \boldsymbol{F} \quad \text{in } L^2((0,T); L^2(\Omega)).$$

From (iii), we see that $\overline{\boldsymbol{b}_n} \to \boldsymbol{B}$ in $L^2((0,T); \mathbf{L}^2(\Omega))$. The continuity of $\boldsymbol{H}$ and the Lebesgue dominated convergence theorem imply that $\boldsymbol{H}(\overline{\boldsymbol{b}_n}) \to \boldsymbol{H}(\boldsymbol{B})$ in the space $L^2((0,T); \mathbf{L}^2(\Omega))$.

Now, we have to show that $\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \to \nabla \times \boldsymbol{H}(\boldsymbol{B})$ in $L^2((0,T);\mathbf{L}^2(\Omega))$. We know that $\overline{\boldsymbol{b}_n} \to \boldsymbol{B}$ in $L^2((0,T);\mathbf{L}^2(\Omega))$. Let us take any $\xi \in [0,T]$ such that $\overline{\boldsymbol{b}_n}(\xi) \to \boldsymbol{B}(\xi)$ in $\mathbf{L}^2(\Omega)$. The set of such ξ is dense in $[0,T]$. The positiveness of the function γ allows us to write

$$
\begin{aligned}
0 \le \gamma_* \int_0^\xi \int_\Omega & \left|\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) - \nabla \times \boldsymbol{H}(\boldsymbol{B})\right|^2 \le \\
\le \int_0^\xi \int_\Omega & \overline{\gamma_n}(t-\tau) \left|\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) - \nabla \times \boldsymbol{H}(\boldsymbol{B})\right|^2 \\
= \int_0^\xi & \left(\overline{\gamma_n}(t-\tau)\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right) \\
& + \int_0^\xi \left(\overline{\gamma_n}(t-\tau)\nabla \times \boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{H}(\boldsymbol{B})\right) \\
& - 2\int_0^\xi \left(\overline{\gamma_n}(t-\tau)\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla \times \boldsymbol{H}(\boldsymbol{B})\right).
\end{aligned}
\tag{3.20}
$$

Using the results from **Part** (A), (iv), and Lebesgue's dominated convergence theorem, we pass to the limit for $n \to \infty$ in the second and the third term on the r.h.s. in (3.20)

$$
\begin{aligned}
\lim_{n\to\infty} \int_0^\xi \left(\overline{\gamma_n}(t-\tau)\nabla \times \boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{H}(\boldsymbol{B})\right) &= \int_0^\xi \left(\gamma(u)\nabla \times \boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{H}(\boldsymbol{B})\right), \\
\lim_{n\to\infty} \int_0^\xi \left(\overline{\gamma_n}(t-\tau)\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla \times \boldsymbol{H}(\boldsymbol{B})\right) &= \int_0^\xi \left(\gamma(u)\nabla \times \boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{H}(\boldsymbol{B})\right).
\end{aligned}
$$

We can say that $\xi \in (t_{j-1}, t_j]$. Now, let us rewrite the first term on the r.h.s. in (3.20) using (3.13) as follows

$$
\begin{aligned}
\int_0^\xi & \left(\overline{\gamma_n}(t-\tau)\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right) = \\
&= -\int_0^\xi \left(\partial_t \boldsymbol{b}_n, \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right) + \int_0^\xi \left(\overline{\gamma_n}(t-\tau)\nabla \times \overline{\boldsymbol{F}}_n, \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right) \\
&= -\int_0^{t_j} \left(\partial_t \boldsymbol{b}_n, \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right) + \int_\xi^{t_j} \left(\partial_t \boldsymbol{b}_n, \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right) \\
&\qquad\qquad + \int_0^\xi \left(\overline{\gamma_n}(t-\tau)\nabla \times \overline{\boldsymbol{F}}_n, \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right) \\
&= -\sum_{i=1}^{j} \int_\Omega (\boldsymbol{b}_i - \boldsymbol{b}_{i-1}) \cdot \boldsymbol{H}(\boldsymbol{b}_i) + \int_\xi^{t_j} \left(\partial_t \boldsymbol{b}_n, \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \int_0^{\xi} \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{F}}_n, \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right) \\
&\overset{Thm.\ 1.11}{\leq} -\sum_{i=1}^{j} \int_\Omega \left[\Phi_{\boldsymbol{H}}(\boldsymbol{b}_i) - \Phi_{\boldsymbol{H}}(\boldsymbol{b}_{i-1}) \right] + \int_\xi^{t_j} \left(\partial_t \boldsymbol{b}_n, \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right) \\
&\qquad + \int_0^{\xi} \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{F}}_n, \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right) \\
&= -\int_\Omega \left[\Phi_{\boldsymbol{H}}(\boldsymbol{b}_j) - \Phi_{\boldsymbol{H}}(\boldsymbol{B}_0) \right] + \int_\xi^{t_j} \left(\partial_t \boldsymbol{b}_n, \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right) \\
&\qquad + \int_0^{\xi} \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{F}}_n, \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right) \\
&= -\int_\Omega \left[\Phi_{\boldsymbol{H}}(\overline{\boldsymbol{b}_n}(\xi)) - \Phi_{\boldsymbol{H}}(\boldsymbol{B}_0) \right] + \int_\xi^{t_j} \left(\partial_t \boldsymbol{b}_n, \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right) \\
&\qquad + \int_0^{\xi} \left(\overline{\gamma_n}(t-\tau) \nabla \times \overline{\boldsymbol{F}}_n, \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right).
\end{aligned}
$$

Now, using (A), (iv), and (vi), we pass to the limit for $n \to \infty$ to obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^{\xi} \left(\overline{\gamma_n}(t-\tau) \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \right) \, \mathrm{d}s \\
&\quad \leq -\int_\Omega \left[\Phi_{\boldsymbol{H}}(\boldsymbol{B}(\xi)) - \Phi_{\boldsymbol{H}}(\boldsymbol{B}_0) \right] \, \mathrm{d}\boldsymbol{x} + \int_0^{\xi} \left(\gamma(u) \nabla \times \boldsymbol{F}, \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right) \, \mathrm{d}s \\
&\quad = -\int_0^{\xi} \int_\Omega \frac{d\Phi_{\boldsymbol{H}}(\boldsymbol{B})}{dt} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s + \int_0^{\xi} \left(\gamma(u) \nabla \times \boldsymbol{F}, \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right) \, \mathrm{d}s \\
&\quad = -\int_0^{\xi} \left(\partial_t \boldsymbol{B}, \boldsymbol{H}(\boldsymbol{B}) \right) \, \mathrm{d}s + \int_0^{\xi} \left(\gamma(u) \nabla \times \boldsymbol{F}, \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right) \, \mathrm{d}s \\
&\quad \overset{(3.5)}{=} \int_0^{\xi} \left(\gamma(u) \nabla \times \boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right) \, \mathrm{d}s.
\end{aligned}
$$

Gathering all partial results above, we conclude

$$
0 \leq \lim_{n\to\infty} \gamma_* \int_0^{\xi} \int_\Omega \left(\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) - \nabla \times \boldsymbol{H}(\boldsymbol{B}) \right)^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s \leq 0.
$$

This is valid for any $\xi \in [0, T]$ for which $\overline{\boldsymbol{b}_n}(\xi) \to \boldsymbol{B}(\xi)$ in $\mathbf{L}^2(\Omega)$. Density of the set of such ξ in $[0, T]$ yields

$$
(xii) \ \ \nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}) \to \nabla \times \boldsymbol{H}(\boldsymbol{B}) \ \text{ in } \ L^2((0,T); \mathbf{L}^2(\Omega)).
$$

Part (C)**:** Take $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\overline{\Omega})$ in (3.13) and integrate it in time to obtain

$$\int_0^\xi (\partial_t \boldsymbol{b}_n, \boldsymbol{\varphi}) + \int_0^\xi \left(\overline{\gamma_n}(t-\tau)\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n}), \nabla \times \boldsymbol{\varphi}\right) = \int_0^\xi \left(\overline{\gamma_n}(t-\tau)\overline{\boldsymbol{F}_n}, \nabla \times \boldsymbol{\varphi}\right).$$

Thanks to $(A), (iv), (v)$, and (vi) we pass to the limit for $n \to \infty$

$$\int_0^\xi (\partial_t \boldsymbol{B}, \boldsymbol{\varphi}) + \int_0^\xi (\gamma(u)\nabla \times \boldsymbol{H}(\boldsymbol{B}), \nabla \times \boldsymbol{\varphi}) = \int_0^\xi (\gamma(u)\boldsymbol{F}, \nabla \times \boldsymbol{\varphi}).$$

Now, considering the fact that $\mathbf{C}_0^\infty(\overline{\Omega})$ is dense in $\mathbf{H}_0(\mathbf{curl}\,; \Omega)$ and differentiating with respect to the time variable we see that $\boldsymbol{B}$ and u solve (3.5).

Let us integrate (3.14) in time

$$\left(\overline{\theta_n}(t), \psi\right) - (\theta_n(0), \psi) + \left(\theta_n(t) - \overline{\theta_n}(t), \psi\right) + \int_0^t (\lambda_0 \nabla \overline{u_n}, \nabla \psi)\ \mathrm{d}s \qquad (3.21)$$
$$= \int_0^t \left(\mathcal{R}_r\left(\overline{\gamma_n}(s-\tau)\left|\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right|^2\right), \psi\right)\ \mathrm{d}s.$$

The fact that $\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})$ converges strongly to $\nabla \times \boldsymbol{H}(\boldsymbol{B})$ in $L^2((0,T); \mathbf{L}^2(\Omega))$ combined with $(A), (vi)$ and Lebesgue's dominated convergence theorem let us pass to the limit for $n \to \infty$ in the term on the r.h.s. of the equation above. Hence,

$$\lim_{n\to\infty} \int_0^t \left(\mathcal{R}_r\left(\overline{\gamma_n}(s-\tau)\left|\nabla \times \boldsymbol{H}(\overline{\boldsymbol{b}_n})\right|^2\right), \psi\right)\ \mathrm{d}s$$
$$= \int_0^t \left(\mathcal{R}_r\left(\gamma(u)\left|\nabla \times \boldsymbol{H}(\boldsymbol{B})\right|^2\right), \psi\right)\ \mathrm{d}s.$$

Due to (A), we also see that

$$\lim_{n\to\infty} \left(\theta_n(t) - \overline{\theta_n}(t), \psi\right) = 0 \qquad \text{for every } t \in [0,T].$$

Collecting all results above and using (A), we finally pass to the limit for $n \to \infty$ in (3.21) to obtain

$$(\theta(u(t)), \psi) - (\theta(u(0)), \psi) + \int_0^t (\lambda_0 \nabla u, \nabla \psi)\ \mathrm{d}s$$
$$= \int_0^t \left(\mathcal{R}_r\left(\gamma(u)\left|\nabla \times \boldsymbol{H}(\boldsymbol{B})\right|^2\right), \psi\right)\ \mathrm{d}s.$$

Now, differentiation in time shows that u and $\boldsymbol{B}$ solve (3.6). □

Chapter 4

A vector-scalar potential formulation of the induction hardening model considering a nonlinear law for the magnetic field

This chapter is based on an article that has been published in the journal of *Computer Methods in Applied Mechanics and Engineering.*

4.1 Introduction

Induction hardening is a form of a heat treatment in which a metal part of the workpiece is heated by the induction heating and then quenched by cool water. The quenched metal undergoes a martensitic transformation, increasing the hardness, strength and fatigue resistance of the metal part. The advantage of the induction hardening is that it can be applied only on a part of the workpiece (surface, layers, pins, gears) to sustain the properties of the remaining parts.

The most extensive mathematical study on the induction hardening topic has been done by Dietmar Hömberg who provided very detailed models of the induction hardening in [38] and [39]. However, these models did assume neither the monotone

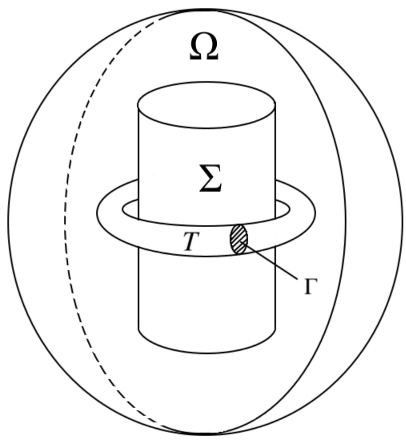

Figure 4.1: Illustration of the domain.

character of the fields $\boldsymbol{H}$ and $\boldsymbol{B}$ nor a temperature dependency of the electric conductivity function in Ohm's law (2.1).

Mathematical models introduced in Chapters 2 and 3 were general models of the induction heating process. We assumed that the domain Ω was occupied by a conducting magnetic material. In this chapter we present a vector-scalar potential formulation of a mathematical model of the induction hardening. Again, the non-linear relation between $\boldsymbol{H}$ and $\boldsymbol{B}$ is assumed and so is the temperature sensitivity of electric conductivity. We also assume conductive and non-conductive parts in our domain Ω. This means that material coefficients may have jumps across the interfaces.

4.2 Derivation of the mathematical model

Our working domain is illustrated in Fig. 4.1. The time frame is denoted by $[0, \mathcal{T}]$. The area occupied by the electromagnetic field is denoted by Ω. It is supposed to be a bounded sphere in $\mathbb{R}^3$. The workpiece and the coil are represented by Σ and T, respectively. Both Σ and T are closed subsets of Ω and the following holds

$$\Sigma \cap T = \varnothing, \text{ and } \partial\Sigma, \partial T, \partial\Omega \text{ are of class } C^{1,1}. \tag{4.1}$$

Conductors are affected by temperature, therefore, we separate them from the rest of the domain Ω by denoting $\pi = \Sigma \cup T$. Current in the coil is supplied through the area denoted as Γ. This is modeled via an interface condition on Γ. By $\boldsymbol{n}$ we denote the standard outer normal unit vector associated with the surfaces of the materials under consideration.

We present a nonlinear relation between $\boldsymbol{H}$ and $\boldsymbol{B}$ in the following form:

$$\boldsymbol{H} := \mu \boldsymbol{M}(\boldsymbol{B}) = \frac{1}{\mu^*} m(|\boldsymbol{B}|)\boldsymbol{B}. \tag{4.2}$$

Remark 4.1. *The vector field $\boldsymbol{H}$ has a monotone character in the conductive parts of our domain. Hence, in those parts it is expressed as* (4.2)*. However, in non-conductive parts the relation between $\boldsymbol{H}$ and $\boldsymbol{B}$ is linear. Therefore,* (4.2) *simplifies to $\boldsymbol{H} = \mu\boldsymbol{B}$.*

The magnetic permeability $\mu = \frac{1}{\mu^*}$ might behave differently in the workpiece and in the air, therefore, we need to specify it as a split function

$$\mu(\boldsymbol{x}) = \begin{cases} \mu_\pi(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \overline{\pi}, \\ \mu_A(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \Omega \setminus \overline{\pi}. \end{cases} \tag{4.3}$$

Both μ_π and μ_A are strictly positive and bounded. There is no jump in the tangential component of $\boldsymbol{H}$ along the boundaries between different materials, i.e.

$$[\boldsymbol{n} \times \mu\boldsymbol{M}(\boldsymbol{B})]_{\partial\pi} = \boldsymbol{0}.$$

The boundary $\partial\Omega$ is assumed to be a perfect conductor. Hence, the tangential component of $\boldsymbol{B}$ vanishes across the boundary, i.e.

$$\boldsymbol{n} \times \boldsymbol{B} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega.$$

The vector field $\boldsymbol{M}$ is supposed to be potential, strongly monotone and Lipschitz continuous. The Ohm law is adopted in the following form:

$$\boldsymbol{J} = \sigma\boldsymbol{E}. \tag{4.4}$$

The function σ represents the electric conductivity and it is defined as follows

$$\sigma(u(\boldsymbol{x},t)) = \begin{cases} \sigma_\pi(u(\boldsymbol{x},t)), & \text{if } \boldsymbol{x} \in \overline{\pi},\ t \in (0,\mathcal{T}), \\ 0, & \text{if } \boldsymbol{x} \in \Omega \setminus \overline{\pi},\ t \in (0,\mathcal{T}) \end{cases} \tag{4.5}$$

where $u(\boldsymbol{x},t)$ is the function of temperature in the workpiece and in the coil. We consider σ to be continuous, bounded and strictly positive in $\overline{\pi}$. Since Ω is a simply-connected domain and (1.4) is true in the whole domain Ω, we use Theorem 1.16 to obtain exactly one magnetic vector potential $\boldsymbol{A} \in \mathbf{H}(\mathbf{curl}\,;\Omega)$ with the following properties:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \nabla \cdot \boldsymbol{A} = 0, \qquad \boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega. \tag{4.6}$$

Substituting (4.6) into (1.5), we get

$$\nabla \times (\boldsymbol{E} + \partial_t \boldsymbol{A}) = \boldsymbol{0} \quad \text{in} \quad \Omega. \tag{4.7}$$

Using (4.7), we apply Theorem 1.17 to acquire a unique scalar potential $\phi \in H^1(\Omega)/\mathbb{R}$ such that:

$$\boldsymbol{E} + \partial_t \boldsymbol{A} = -\nabla \phi. \tag{4.8}$$

Combining (4.8), (4.6), (4.4), (4.2), and (1.6), we arrive at the following boundary value problem for the vector potential $\boldsymbol{A}$:

$$\begin{aligned} \sigma(u)\partial_t \boldsymbol{A} + \nabla \times \mu \boldsymbol{M}(\nabla \times \boldsymbol{A}) + \sigma(u)\chi_T \nabla \phi = \boldsymbol{0} \quad &\text{a.e. in} \quad \Omega \times (0,\mathcal{T}) = Q_{\mathcal{T}}, \\ \boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0} \quad &\text{on} \quad \partial\Omega, \\ \boldsymbol{A}(\boldsymbol{x},0) = \boldsymbol{A}_0(\boldsymbol{x}) \quad &\text{in} \quad \Omega. \end{aligned} \tag{4.9}$$

The characteristic function χ_T has value 1, if $\boldsymbol{x} \in T$ and 0 otherwise. We use it because the external source of the current which is defined by the gradient of the scalar potential is present only in the coil T (see Fig. 4.1). Combination of (4.4) and (4.8) gives us an expression for the total current density $\boldsymbol{J}$

$$\boldsymbol{J} = -\sigma(u)\partial_t \boldsymbol{A} - \sigma(u)\chi_T \nabla \phi.$$

The impressed part $\boldsymbol{J}_{source} = -\sigma(u)\chi_T \nabla \phi$ is caused by the external source and the induced part $\boldsymbol{J}_{induced} = -\sigma(u)\partial_t \boldsymbol{A}$ is caused by the magnetic induction field $\boldsymbol{B}$ in the coil. Demanding that the continuity equation holds for the source current $\boldsymbol{J}_{source}$, i.e.

$$\nabla \cdot \boldsymbol{J}_{source} = 0,$$

we define the scalar potential ϕ by the following elliptic equation with homogeneous Neumann boundary condition on ∂T and interface condition on Γ, cf. [38]:

$$\begin{aligned} -\nabla \cdot (\sigma_\pi(u)\nabla\phi) = 0 \qquad &\text{a.e. in} \quad T, \\ -\sigma_\pi(u)\frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \qquad &\text{a.e. on} \quad \partial T, \\ \left[-\sigma_\pi(u)\frac{\partial \phi}{\partial \boldsymbol{n}}\right]_\Gamma = j \qquad &\text{a.e. in} \quad \Gamma. \end{aligned} \tag{4.10}$$

External source current density is represented by the function $j(\boldsymbol{x},t)$ which is assumed to be Lipschitz continuous in time. Jump across the interface Γ is indicated by $[\cdot]_\Gamma$.

Evolution of temperature in $\mathcal{T}$ is modeled similarly as in Chapters 2 and 3. The *Joule heat* term is expressed as

$$\sigma_\pi(u)\left|\boldsymbol{E}\right|^2 \overset{(4.8)}{=} \sigma_\pi(u)\left|\partial_t \boldsymbol{A} + \chi_T \nabla\phi\right|^2. \tag{4.11}$$

To simulate the cooling effect on the boundary $\partial\Sigma$, we need to involve a Robin boundary condition for the temperature evolution function $u(\boldsymbol{x}, t)$, i.e.

$$\lambda_0 \frac{\partial u}{\partial \boldsymbol{n}} = \alpha(u - u_c) \quad \text{on} \quad \partial\Pi \tag{4.12}$$

where u_c is temperature of cooling water, λ_0 is same as in (2.12) and α is the heat transfer coefficient defined as

$$\alpha(\boldsymbol{x}, t) := \begin{cases} 0 & \text{on} \quad \partial T \times (0, \mathcal{T}), \\ \alpha_\Sigma(t) & \text{on} \quad \partial\Sigma \times (0, \mathcal{T}). \end{cases}$$

The coefficient $\alpha_\Sigma(t)$ is 0 during the heating time. After the current is switched-off and cooling water is applied $\alpha_\Sigma(t)$ becomes positive. We restrict ourselves to model only the heating part of the induction hardening process and omit the cooling effects. Hence, the boundary value problem for the temperature evolution function $u(\boldsymbol{x}, t)$ reads as:

$$\begin{aligned} \partial_t \theta(u) - \nabla \cdot (\lambda_0 \nabla u) &= \mathcal{R}_r \left(\sigma_\pi(u) \left| \partial_t \boldsymbol{A} + \chi_T \nabla\phi \right|^2 \right) && \text{in} \quad \Pi \times (0, \mathcal{T}), \\ -\lambda_0 \frac{\partial u}{\partial \boldsymbol{\nu}} &= 0 && \text{on} \quad \partial\Pi, \\ u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) && \text{in} \quad \Pi \end{aligned} \tag{4.13}$$

where $\mathcal{R}_r$ and θ are same as in (2.14) and (2.15), respectively.

Equations (4.9), (4.10) and (4.13) model the process of induction hardening in our simplified domain Ω.

4.2.1 Weak formulation

We recall the space $\boldsymbol{X}_{N,0}$, which has been properly defined in Section 1.3

$$\boldsymbol{X}_{N,0} = \{\boldsymbol{\varphi} \in \mathbf{H}(\mathbf{curl}\,; \Omega);\ \nabla \cdot \boldsymbol{\varphi} = 0, \text{ and } \boldsymbol{\varphi} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \ \partial\Omega\}.$$

The norm in this space is defined as

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_{N,0}} = \|\nabla \times \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}\,.$$

Taking into account (4.1), we use Theorem 1.8 to conclude that $\boldsymbol{X}_{N,0}$ is a closed subspace of $\mathbf{H}^1(\Omega)^*$.

*This is a very important embedding, since $\mathbf{H}^1(\Omega)$ is compactly embedded in $\mathbf{L}^2(\Omega)$, cf. Theorem 1.9. The inclusion $\boldsymbol{X}_{N,0} \subset \mathbf{H}^1(\Omega)$ is also valid for convex domains (with non-smooth boundary). In such case we rely on [3, Theorem 2.17]

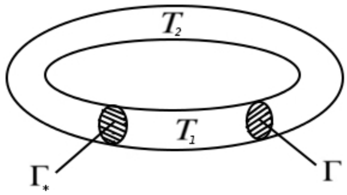

Figure 4.2: Dissection of T.

Multiplying (4.9) by a test function $\boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}$, integrating over Ω, using Green's Theorem 1.12, and taking into account the boundary conditions, we obtain the variational formulation for the vector potential $\boldsymbol{A}$:

$$(\sigma_\pi \partial_t \boldsymbol{A}, \boldsymbol{\varphi})_\pi + (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}), \nabla \times \boldsymbol{\varphi})_\Omega + (\sigma_\pi \nabla \phi, \boldsymbol{\varphi})_T = 0, \tag{4.14}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}$. Please note that the scalar product of any two given functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in the domain $\heartsuit$ is defined as

$$(\boldsymbol{f}, \boldsymbol{g})_\heartsuit = \int_\heartsuit \boldsymbol{f} \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{x}.$$

To obtain the variational formulation for (4.10), we split T in two separate parts T_1 and T_2. Flux of the scalar potential on the new interface Γ_* is supposed to be continuous. Moreover, $\Gamma_* \cap \Gamma = \emptyset$ and $T_1 \cap T_2 = \Gamma_* \cup \Gamma$ (see Fig. 4.2). Now, we multiply (4.10) by a test function $\xi \in H^1(T)/\mathbb{R}$ and integrate in T_1 and T_2. Using Green's theorem and continuous condition on Γ_*, we get

$$\begin{aligned} &\int_{T_1} \sigma_\pi \nabla \phi \cdot \nabla \xi + \int_{T_2} \sigma_\pi \nabla \phi \cdot \nabla \xi - \int_\Gamma \sigma_\pi \frac{\partial \nabla \phi}{\partial \boldsymbol{n}} - \int_\Gamma \sigma_\pi \frac{\partial \nabla \phi}{\partial \boldsymbol{n}^-} \\ &= \int_T \sigma_\pi \nabla \phi \cdot \nabla \xi - \int_\Gamma \left[\sigma_\pi \frac{\partial \nabla \phi}{\partial \boldsymbol{n}} \right]_\Gamma . \end{aligned}$$

Here, $\boldsymbol{n}^-$ denotes a unit outward normal vector with opposite orientation as $\boldsymbol{n}$. Taking into account the interface condition (4.10) we arrive at the following variational formulation for scalar potential ϕ:

$$(\sigma_\pi \nabla \phi, \nabla \xi)_T + (j, \xi)_\Gamma = 0 \tag{4.15}$$

for any $\xi \in H^1(T)/\mathbb{R}^\dagger$. From now on the set of all functions $\phi + c$, where $\phi \in H^1(T)$ and c is a constant is marked as ϕ_c

† The choice of the test space $H^1(T)/\mathbb{R}$ is just to obtain a unique solvability.

Lemma 4.1. *There are positive constants c_1 and c_2 such that:*

$$c_1 \|\phi_c\|^2_{H^1(T)/\mathbb{R}} \leq \|\nabla\phi\|^2_{\mathbf{L}^2(T)} \leq c_2 \|\phi_c\|^2_{H^1(T)/\mathbb{R}}.$$

Proof. The norm in $H^1(T)/\mathbb{R}$ is defined as $\|\phi_c\|_{H^1(T)/\mathbb{R}} := \inf_{\phi\in\phi_c} \|\phi\|_{H^1(T)}$. This norm is minimal for $c = -\frac{1}{|T|}\int_T \phi \, \mathrm{d}\boldsymbol{x}$, indeed, let us take a closer look.

$$\begin{aligned} 0 &= \frac{d}{dc}\left(\int_T (\phi + c)^2 + |\nabla\phi|^2\right) \mathrm{d}\boldsymbol{x} \\ &= 2\int_T \phi \, \mathrm{d}\boldsymbol{x} + 2\int_T c \, \mathrm{d}\boldsymbol{x} \\ &\Longrightarrow c = -\frac{1}{|T|}\int_T \phi \, \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Now, we write $\|\phi_c\|_{H^1(T)/\mathbb{R}} = \left\|\phi - \frac{1}{|T|}\int_T \phi \, \mathrm{d}\boldsymbol{x}\right\|_{H^1(T)}$. Employing the Poincaré-Wirtinger inequality, cf. Theorem 1.13, we conclude the following:

$$\begin{aligned} \|\phi_c\|^2_{H^1(T)/\mathbb{R}} &= \left\|\phi - \frac{1}{|T|}\int_T \phi \, \mathrm{d}\boldsymbol{x}\right\|^2_{L^2(T)} + \|\nabla\phi\|^2_{\mathbf{L}^2(T)} \\ &\leq c_{PW} \|\nabla\phi\|^2_{\mathbf{L}^2(T)} + \|\nabla\phi\|^2_{\mathbf{L}^2(T)} \\ &= (c_{PW} + 1) \|\nabla\phi\|^2_{\mathbf{L}^2(T)} \end{aligned}$$

where c_{PW} is a positive constant. Taking $c_2 = 1$ and $c_1 = \frac{1}{1+c_{PW}}$, the proof is completed. □

For equation (4.13) we follow identical steps as above, using $\psi \in H^1(\pi)$ as a test function, brings us to the variational formulation for the function u:

$$(\partial_t\theta(u), \psi)_\pi + (\lambda_0\nabla u, \nabla\psi)_\pi = \left(\mathcal{R}_r\left(\sigma_\pi\left|\partial_t\boldsymbol{A} + \chi_T\nabla\phi\right|^2\right), \psi\right)_\pi \tag{4.16}$$

for any $\psi \in H^1(\pi)$.

The weak formulation of our model (4.9), (4.10), and (4.13) then reads as: Find $\boldsymbol{A} \in L^2((0,\mathcal{T}); \boldsymbol{X}_{N,0})$ with $\partial_t\boldsymbol{A} \in L^2((0,\mathcal{T}); \mathbf{L}^2(\pi))$, $u \in C([0,T]; \mathrm{L}^2(\Omega)) \cap L^\infty((0,T); H^1_0(\Omega))$ with $\partial_t u \in L^2((0,T); L^2(\Omega))$ and $\phi \in L^2((0,\mathcal{T}); H^1(T)/\mathbb{R})$ such that they satisfy (4.14), (4.16), and (4.15) for any $\boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}$, $\psi \in H^1(\pi)$ and $\xi \in H^1(T)/\mathbb{R}$.

4.2.2 Assumptions

To achieve better clarity and readability of this chapter, we list all assumptions together:

$$
\begin{aligned}
&(a1) \quad 0 < \mu_{\pi *} \le \mu_\pi(\boldsymbol{x}) \le \mu_\pi^* < \infty && \forall \boldsymbol{x} \in \overline{\Sigma},\\
&(a2) \quad 0 < \mu_{A *} \le \mu_A(\boldsymbol{x}) \le \mu_A^* < \infty && \forall \boldsymbol{x} \in \Omega \setminus \overline{\Sigma},\\
&(b) \quad \mu_* = \min\{\mu_{\pi *}, \mu_{A *}\},\ \mu^* = \max\{\mu_\pi^*, \mu_A^*\}, &&\\
&(c1) \quad \mu \in H^1(\mathcal{T}), &&\\
&(c2) \quad \mu \in H^1(\Omega \setminus \mathcal{T}), &&\\
&(d) \quad 0 < \sigma_* \le \sigma(u(\boldsymbol{x}, t)) \le \sigma^* < \infty && \forall (\boldsymbol{x}, t) \in \mathcal{T} \times (0, \mathcal{T}),\\
&(e) \quad |j(\boldsymbol{x}, t_2) - j(\boldsymbol{x}, t_1)| \le C_j\,|t_2 - t_1| && C_j > 0, \forall \boldsymbol{x} \in \Gamma, \forall t_2, t_1 \in (0, \mathcal{T}),\\
&(f) \quad j \in L^2((0, \mathcal{T}); H^{-1/2}(\Gamma)), \quad \int_\Gamma j(t)\ \mathrm{d}\gamma = 0 && \forall t \in (0, \mathcal{T}),\\
&(g) \quad u_0 \in H_0^1(\mathcal{T}), &&\\
&(h) \quad \boldsymbol{A}_0 \in \boldsymbol{X}_{N,0}, &&\\
&(i) \quad \theta \text{ is continuous}, \theta(0) = 0, && (4.17)\\
&\qquad |\theta(x)| \le C_\theta(1 + |x|), \quad 0 < \theta_* \le \theta'(x) && C_\theta > 0, \forall x \in \mathbb{R},\\
&(j_1) \quad (\boldsymbol{M}(\boldsymbol{x}) - \boldsymbol{M}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \ge c_M\,|\boldsymbol{x} - \boldsymbol{y}|^2 && c_M > 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3,\\
&(j_2) \quad |\boldsymbol{M}(\boldsymbol{x}) - \boldsymbol{M}(\boldsymbol{y})| \le C_M\,|\boldsymbol{x} - \boldsymbol{y}| && C_M > 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3,\\
&(j_3) \quad \boldsymbol{M}(\boldsymbol{0}) = \boldsymbol{0}. &&
\end{aligned}
$$

Following [79, Theorem 5.1], we see that the potential $\Phi_{\boldsymbol{M}}$ of the vector field $\boldsymbol{M}$ with properties $(j_1) - (j_3)$ is strictly convex. Applying Theorem 1.11, we get

$$\boldsymbol{M}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \ge \Phi_{\boldsymbol{M}}(\boldsymbol{x}) - \Phi_{\boldsymbol{M}}(\boldsymbol{y}) \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3. \tag{4.18}$$

Thanks to (j_1) and (j_2), we bound $\Phi_{\boldsymbol{M}}$ from below

$$
\begin{aligned}
\Phi_{\boldsymbol{M}}(\boldsymbol{x}) &= \int_0^1 \boldsymbol{M}(\boldsymbol{x}p) \cdot \boldsymbol{x}\ \mathrm{d}p = \int_0^1 \boldsymbol{M}(\boldsymbol{x}p) \cdot (\boldsymbol{x}p)p^{-1}\ \mathrm{d}p\\
&\ge \int_0^1 c_M\,|\boldsymbol{x}p|^2\,p^{-1}\ \mathrm{d}p = \frac{c_M}{2}\,|\boldsymbol{x}|^2
\end{aligned} \tag{4.19}
$$

and from above

$$\Phi_{\boldsymbol{M}}(\boldsymbol{x}) = \int_0^1 \boldsymbol{M}(\boldsymbol{x}p) \cdot \boldsymbol{x}\ \mathrm{d}p \le \int_0^1 |\boldsymbol{M}(\boldsymbol{x}p)|\,|\boldsymbol{x}|\ \mathrm{d}p$$

$$\leq C_M \int_0^1 |\boldsymbol{x}p|\,|\boldsymbol{x}| \ \mathrm{d}p = \frac{C_M}{2}\,|\boldsymbol{x}|^2\,. \tag{4.20}$$

4.3 Time discretization

In this section we discretize the time interval $[0,\mathcal{T}]$ and solve a system of steady-state differential equations at each time step similarly as in Chapter 2. Using the same notation for any function f

$$f_i = f(t_i), \quad \delta f_i = \frac{f_i - f_{i-1}}{\tau},$$

we split the time interval in n equidistant parts, i.e. $n\tau = \mathcal{T}$, where $n \in \mathbb{N}$. Now, we approximate the system (4.14)-(4.16) at every time step $t_i = \tau i$ for $i = 1,\dots,n$

$$(\sigma_\pi(u_{i-1})\nabla\phi_{c_i}, \nabla\xi)_T + (j_i,\xi)_\Gamma = 0, \tag{4.21}$$

$$(\sigma_\pi(u_{i-1})\delta\boldsymbol{A}_i, \boldsymbol{\varphi})_\pi + (\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i), \nabla\times\boldsymbol{\varphi})_\Omega + (\sigma_\pi(u_{i-1})\nabla\phi_{c_i}, \boldsymbol{\varphi})_T = 0, \tag{4.22}$$

$$(\delta\theta(u_i),\psi)_\pi + (\lambda_0\nabla u_i,\psi)_\pi = \left(\mathcal{R}_r\left(\sigma_\pi(u_{i-1})\,|\delta\boldsymbol{A}_i + \chi_T\nabla\phi_{c_i}|^2\right),\psi\right)_\pi \tag{4.23}$$

for any $\xi \in H^1(T)/\mathbb{R}, \boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}$ and $\psi \in H^1(\pi)$.

Remark 4.2. *In system* (4.21)-(4.23) *we use* u_{i-1} *as an argument for the function* σ*. The reason to take this action is to be able to decouple the whole system. As we will see in the sequel, this small adjustment does not affect the convergence results.*

The solvability at each time step is proven in the following lemma.

Lemma 4.2. *Assume that* (4.17) *holds. Then for any* $i = 1\dots n$*, there exists a uniquely determined triplet* $\phi_{c_i} \in H^1(T)/\mathbb{R}$*,* $\boldsymbol{A}_i \in \boldsymbol{X}_{N,0}$ *and* $u_i \in H^1(\pi)$ *solving system* (4.21)-(4.23).

Proof. Let us define operators: $\mathcal{F}_\sigma : \boldsymbol{X}_{N,0} \to (\boldsymbol{X}_{N,0})^*$ and $\mathcal{G} : H^1(\pi) \to (H^1(\pi))^*$

$$\langle\mathcal{F}_\sigma(\boldsymbol{A}),\boldsymbol{\varphi}\rangle := \left(\sigma\frac{\boldsymbol{A}}{\tau},\boldsymbol{\varphi}\right)_\pi + (\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}),\nabla\times\boldsymbol{\varphi})_\Omega\,,$$

$$\langle\mathcal{G}(u),\psi\rangle := \left(\frac{\theta(u)}{\tau},\psi\right)_\pi + (\lambda_0\nabla u,\nabla\psi)_\pi\,.$$

Assuming that τ is small enough, i.e. $0 < \tau < 1$, we show strict monotonicity, coercivity, and hemicontinuity of these operators in the same way as in Lemmas

2.1 and 3.1. The rest of the proof serves as a guideline for obtaining a solution-triplet at every time step $t = t_i$, for $i = 1, \ldots, n$. Applying the Lax-Milgram Lemma 1.1 to (4.21), we obtain a unique solution $\phi_{c_i} \in H^1(\mathcal{T})/\mathbb{R}$ at a time step $t = t_i$ (u_{i-1} is known on this time step).

To obtain a unique solution $\boldsymbol{A}_i$ at a time step t_i, we have to solve the following identity:

$$\left\langle \mathcal{F}_{\sigma_\pi(u_{i-1})}(\boldsymbol{A}_i), \boldsymbol{\varphi} \right\rangle = \left(\sigma_\pi(u_{i-1}) \frac{\boldsymbol{A}_{i-1}}{\tau}, \boldsymbol{\varphi} \right)_\pi - (\sigma_\pi(u_{i-1}) \nabla \phi_{c_i}, \boldsymbol{\varphi})_T .$$

Since the r.h.s. is known, we use Theorem 1.10 to provide the solution. Now, we involve the same theorem again to acquire a unique solution $u_i \in H^1(\mathcal{T})$ of the setting below (taking into account that the r.h.s. is known)

$$\langle \mathcal{G}(u_i), \psi \rangle = \left(\frac{\theta(u_{i-1})}{\tau}, \psi \right)_\pi + \left(\mathcal{R}_r \left(\sigma_\pi(u_{i-1}) \left| \delta \boldsymbol{A}_i + \chi_T \nabla \phi_{c_i} \right|^2 \right), \psi \right)_\pi .$$

This provides us with the solution-triplet $\{\phi_{c_i}, \boldsymbol{A}_i, u_i\}$ at a time step $t = t_i$, for $i = 1 \ldots n$. □

To wrap everything together, we introduce a pseudoalgorithm for obtaining the solution-triplet $\{\phi_{c_i}, \boldsymbol{A}_i, u_i\}$ at every time step $t = t_i$:

Algorithm 4 Implicit Euler

Require: $\boldsymbol{A}_0 \in \boldsymbol{X}_{N,0}, u_0 \in H_0^1(\mathcal{T}), j \in L^2((0,\mathcal{T}); H^{-1/2}(\Gamma)), n \in \mathbb{N}$

1: **for** $i = 1, i \leq n$ **do**

2: $\nabla \phi_{c_i} \leftarrow$ Solve: $(\sigma_\pi(u_{i-1}) \nabla \phi_{c_i}, \nabla \xi)_T + (j_i, \xi)_\Gamma = 0$

3: $\boldsymbol{A}_i \leftarrow$ Solve: $\left(\sigma_\pi(u_{i-1}) \frac{\boldsymbol{A}_i}{\tau}, \boldsymbol{\varphi} \right)_\pi + (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i), \nabla \times \boldsymbol{\varphi})_\Omega = \left(\sigma_\pi(u_{i-1}) \frac{\boldsymbol{A}_{i-1}}{\tau}, \boldsymbol{\varphi} \right)_\pi - (\sigma_\pi(u_{i-1}) \nabla \phi_{c_i}, \boldsymbol{\varphi})_T$

4: $u_i \leftarrow$ Solve: $\left(\frac{\theta(u_i)}{\tau}, \psi \right)_\pi + (\lambda_0 \nabla u_i, \nabla \psi)_\pi = \left(\frac{\theta(u_{i-1})}{\tau}, \psi \right)_\pi + \left(\mathcal{R}_r \left(\sigma_\pi(u_{i-1}) \left| \delta \boldsymbol{A}_i + \chi_T \nabla \phi_{c_i} \right|^2 \right), \psi \right)_\pi$

5: $i \leftarrow i + 1$

6: **return** $\{\nabla \phi_{c_1}, \boldsymbol{A}_1, u_1\}, \ldots, \{\nabla \phi_{c_n}, \boldsymbol{A}_n, u_n\}$

4.3.1 A priori energy estimates

Before we proceed to the convergence part, we have to derive some basic energy estimates for $\phi_{c_i}, \boldsymbol{A}_i$ and u_i. They are covered by the following lemmas.

Lemma 4.3. *Assume* (4.17). *Then there exists a positive constant* C *such that*

$$\sum_{i=1}^{n} \|\nabla \phi_{c_i}\|^2_{\mathbf{L}^2(T)} \tau \leq C.$$

Proof. Take $\xi = \phi_{c_i}\tau$ in (4.21) and sum it up for $i = 1, \ldots, l \leq n$ to get

$$\sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\nabla\phi_{c_i}, \nabla\phi_{c_i})_T \, \tau = -\sum_{i=1}^{l} (j_i, \phi_{c_i})_\Gamma \, \tau.$$

We can bound the l.h.s. from below

$$\sigma_* \sum_{i=1}^{l} \|\nabla \phi_{c_i}\|^2_{\mathbf{L}^2(T)} \tau \leq \sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\nabla\phi_{c_i}, \nabla\phi_{c_i})_T \, \tau.$$

Using Cauchy-Schwarz's and Young's inequalities, we bound the r.h.s. by

$$\begin{aligned}\sum_{i=1}^{l} (j_i, \phi_{c_i})_\Gamma \, \tau &\leq \frac{1}{2\varepsilon} \sum_{i=1}^{l} \|j_i\|^2_{H^{-1/2}(\Gamma)} \tau + \frac{\varepsilon}{2} \sum_{i=1}^{l} \|\phi_{c_i}\|^2_{H^{1/2}(\Gamma)} \tau \\ &\leq C_\varepsilon + \varepsilon \sum_{i=1}^{l} \|\phi_{c_i}\|^2_{H^{1/2}(\Gamma)} \tau\end{aligned}$$

where $\varepsilon > 0$. Since $H^1(T)/\mathbb{R} \subset H^{1/2}(\Gamma)$, we use Lemma 4.1 to write

$$\sum_{i=1}^{l} \|\phi_{c_i}\|^2_{H^{1/2}(\Gamma)} \tau \leq C \sum_{i=1}^{l} \|\nabla \phi_{c_i}\|^2_{\mathbf{L}^2(T)} \tau.$$

Now, fixing a sufficiently small ε, we conclude the proof. □

Lemma 4.4. *Assume* (4.17). *Then there exists a positive constant* C *such that*

(i) $\displaystyle\sum_{i=1}^{n} \|\delta \boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)} \tau + \max_{1\leq l\leq n} \|\nabla \times \boldsymbol{A}_l\|^2_{\mathbf{L}^2(\Omega)} \leq C$

(ii) $\displaystyle\sum_{i=1}^{n} \|\nabla \times (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i))\|^2_{\mathbf{L}^2(\pi)} \tau \leq C.$

Proof. (i) Taking $\boldsymbol{\varphi} = \delta \boldsymbol{A}_i \tau$ in (4.22) and summing up for $i = 1, \ldots, l \leq n$ yields

$$\sum_{i=1}^{l} \left(\sigma_\pi(u_{i-1})\delta\boldsymbol{A}_i, \delta\boldsymbol{A}_i\right)_\pi \tau + \sum_{i=1}^{l} \left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i), \nabla\times\boldsymbol{A}_i - \nabla\times\boldsymbol{A}_{i-1}\right)_\Omega$$
$$= -\sum_{i=1}^{l} \left(\sigma_\pi(u_{i-1})\nabla\phi_{c_i}, \delta\boldsymbol{A}_i\right)_T \tau.$$

Using Lemma 4.3, Cauchy-Schwarz's, and Young's inequalities, we bound the first term on the l.h.s. and the term on the r.h.s. as follows

$$\sigma_* \sum_{i=1}^{l} \|\delta\boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)} \tau \leq \sum_{i=1}^{l} \left(\sigma_\pi(u_{i-1})\delta\boldsymbol{A}_i, \delta\boldsymbol{A}_i\right)_\pi \tau$$

and

$$-\sum_{i=1}^{l} \left(\sigma_\pi(u_{i-1})\nabla\phi_{c_i}, \delta\boldsymbol{A}_i\right)_T \tau \leq \frac{\sigma^*}{2\varepsilon} \sum_{i=1}^{l} \|\nabla\phi_{c_i}\|^2_{\mathbf{L}^2(T)} \tau + \frac{\varepsilon\sigma^* C_\pi}{2} \sum_{i=1}^{l} \|\delta\boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)} \tau$$
$$\leq C\frac{\sigma^*}{2\varepsilon} + \frac{\varepsilon\sigma^* C_\pi}{2} \sum_{i=1}^{l} \|\delta\boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)} \tau.$$

To estimate the second term on the l.h.s., we take into account (4.19) and (4.20)

$$\sum_{i=1}^{l} \int_\Omega \mu\left\{\boldsymbol{M}(\nabla\times\boldsymbol{A}_i)\cdot(\nabla\times\boldsymbol{A}_i - \nabla\times\boldsymbol{A}_{i-1})\right\}\,\mathrm{d}\boldsymbol{x}$$
$$\geq \sum_{i=1}^{l} \int_\Omega \mu(\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_i) - \Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_{i-1}))\,\mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega \mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_l)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_0)\,\mathrm{d}\boldsymbol{x}$$
$$\geq \frac{c_M\mu_*}{2}\|\nabla\times\boldsymbol{A}_l\|^2_{\mathbf{L}^2(\Omega)} - \frac{C_M\mu^*}{2}\|\nabla\times\boldsymbol{A}_0\|^2_{\mathbf{L}^2(\Omega)}.$$

We relocate the terms to get

$$\left(\sigma_* - \frac{\varepsilon}{2}\sigma^* C_\pi\right)\sum_{i=1}^{l}\|\delta\boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)}\tau + \frac{c_M\mu_*}{2}\|\nabla\times\boldsymbol{A}_l\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq C\frac{\sigma^*}{2\varepsilon} + \frac{C_M\mu^*}{2}\|\nabla\times\boldsymbol{A}_0\|^2_{\mathbf{L}^2(\Omega)}.$$

Fixing $\varepsilon \in \left(0, \frac{2\sigma_*}{\sigma^* C_\pi}\right)$ and assuming that $\boldsymbol{A}_0 \in \boldsymbol{X}_{N,0}$, we obtain that

$$\sum_{i=1}^{l} \|\delta \boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)} \tau + \|\nabla \times \boldsymbol{A}_l\|^2_{\mathbf{L}^2(\Omega)} \leq C.$$

This is valid for any $1 \leq l \leq n$ which concludes the proof of (i).

(ii) Take $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\mathcal{T})$. It holds that

$$\begin{aligned}(\sigma_\pi(u_{i-1})\delta \boldsymbol{A}_i, \boldsymbol{\varphi})_\pi + (\sigma_\pi(u_{i-1})\nabla\phi_{c_i}, \boldsymbol{\varphi})_T &= -\left(\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i), \nabla \times \boldsymbol{\varphi}\right)_\Omega \\ &\overset{Green's\, theorem}{=} -\left(\nabla \times \left(\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i)\right), \boldsymbol{\varphi}\right)_\Omega .\end{aligned}$$

Based on Lemma 4.3 and Lemma 4.4 (i), we see that the l.h.s. can be seen as a linear bounded functional in $L^2((0,\mathcal{T}); \mathbf{L}^2(\mathcal{T}))$. According to the Hahn-Banach Theorem 1.1, the same holds true for the r.h.s., i.e.

$$\sum_{i=1}^{n} \|\nabla \times (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i))\|^2_{\mathbf{L}^2(\pi)} \tau \leq C.$$

□

Lemma 4.5. *Let* (4.17) *be fulfilled. Then there exists a positive constant C_r, depending only on parameter r of truncation function $\mathcal{R}_r$ such that*

$$(i) \quad \sum_{i=1}^{n} \|\delta u_i\|^2_{L^2(\pi)} \tau + \sum_{i=1}^{n} \|\nabla u_i - \nabla u_{i-1}\|^2_{\mathbf{L}^2(\pi)} + \max_{1\leq i\leq n} \|\nabla u_i\|_{\mathbf{L}^2(\pi)} \leq C_r,$$

$$(ii) \quad \max_{1\leq i\leq n} \|u_i\|^2_{L^2(\pi)} \leq C_r,$$

$$(iii) \quad \max_{1\leq i\leq n} \|\delta\theta(u_i)\|^2_{(H^1(\pi))^*} \leq C_r.$$

Proof. We omit the proof since it follows the same pattern as the proof of Lemma 2.4. □

4.4 The existence of a global solution

We construct piece-wise constant and piece-wise linear in time functions in the same manner as in Chapter 2, i.e.

$$\begin{aligned}\overline{f_n}(t) &= f_i && \text{for } t \in (t_{i-1}, t_i], \\ f_n(t) &= f_{i-1} + (t - t_{i-1})\delta f_i && \text{for } t \in (t_{i-1}, t_i], \\ \overline{f_n}(0) &= f_n(0) = f_0. \end{aligned}$$

Using this notation, we rewrite (4.21), (4.22), and (4.23) for $t \in [0, \mathcal{T}]$ as follows

$$\left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \nabla\xi\right)_T + \left(\overline{j_n}, \xi\right)_\Gamma = 0, \tag{4.24}$$

$$\begin{aligned}\left(\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}_n, \boldsymbol{\varphi}\right)_\pi &+ \left(\mu \boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}), \nabla\times\boldsymbol{\varphi}\right)_\Omega \\ &+ \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \boldsymbol{\varphi}\right)_T = 0,\end{aligned} \tag{4.25}$$

$$(\partial_t\theta_n, \psi)_\pi + (\lambda_0\nabla\overline{u_n}, \psi)_\pi = \left(\mathcal{R}_r\left(\overline{\sigma_{\pi_n}}(t-\tau)\left|\partial_t\boldsymbol{A}_n + \chi_T\nabla\overline{\phi_n}\right|^2\right), \psi\right)_\pi \tag{4.26}$$

for any $\psi \in H^1(\mathcal{\pi}), \boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}$ and $\xi \in H^1(T)/\mathbb{R}$.

The proof of existence of a solution to (4.14)-(4.16) is very long from the technical point of view, therefore, we split it into three parts.

Proposition 4.1. *Suppose* (4.17). *Moreover, assume that σ is globally Lipschitz continuous. Then there exists $u \in C([0,\mathcal{T}]; L^2(\mathcal{\pi})) \cap L^\infty((0,\mathcal{T}); H^1_0(\mathcal{\pi}))$ with $\partial_t u \in L^2((0,\mathcal{T}); L^2(\mathcal{\pi}))$ and a subsequence of u_n (denoted by the same symbol again) such that*

$$\begin{array}{lll}
(i) & u_n \to u & \text{in } C\left([0,\mathcal{T}]; L^2(\mathcal{\pi})\right), \\
 & \overline{u_n}(t) \rightharpoonup u(t) & \text{in } H^1(\mathcal{\pi}),\ \forall t \in [0,\mathcal{T}], \\
 & \overline{u_n} \to u & \text{in } L^2((0,\mathcal{T}); L^2(\mathcal{\pi})), \\
(ii) & \overline{\sigma_{\pi_n}} \to \sigma_\pi(u),\ \overline{\sigma_{\pi_n}}(t-\tau) \to \sigma_\pi(u) & \text{in } L^2((0,\mathcal{T}); L^2(\mathcal{\pi})), \\
(iii) & \overline{\theta_n} - \theta_n \to 0 & \text{in } C\left([0,\mathcal{T}]; (H^1(\mathcal{\pi}))^*\right), \\
(iv) & \overline{\theta_n} \to \theta(u) & \text{in } L^2((0,\mathcal{T}); L^2(\mathcal{\pi})), \\
(v) & \overline{j_n} \to j & \text{in } L^2((0,\mathcal{T}); H^{-1/2}(\Gamma)).
\end{array}$$

Proof. Statements $(i)-(iv)$ of this proposition are proved in the same fashion as in the Part (A) in Theorem 3.1.

(v) Assuming that j is Lipschitz continuous in time, we are allowed to write

$$\int_0^{\mathcal{T}} \left\|\overline{j_n} - j\right\|^2_{H^{-1/2}(\Gamma)}\, \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|j(t_i) - j(t)\|^2_{H^{-1/2}(\Gamma)}\, \mathrm{d}t \leq C\tau^2 \overset{n\to\infty}{\to} 0.$$

□

Proposition 4.2. *Suppose that all assumptions of Proposition 4.1 are satisfied. Then there exists $\boldsymbol{A} \in L^2((0,\mathcal{T}); \boldsymbol{X}_{N,0})$ with $\partial_t\boldsymbol{A} \in L^2((0,\mathcal{T}); \mathbf{L}^2(\pi))$ and a subse-*

quence of $\boldsymbol{A}_n$ *(denoted by the same symbol again) such that*

$$
\begin{array}{llll}
(i) & \overline{\boldsymbol{A}_n} \rightharpoonup \boldsymbol{A},\ \nabla\times\overline{\boldsymbol{A}_n} \rightharpoonup \nabla\times\boldsymbol{A} & in & L^2((0,\mathcal{T});\mathbf{L}^2(\Omega)),\\
& \mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) \rightharpoonup \mu\boldsymbol{M}(\nabla\times\boldsymbol{A}) & in & L^2((0,\mathcal{T});\mathbf{L}^2(\Omega\setminus\pi)),\\
& \boldsymbol{A}_n \to \boldsymbol{A} & in & C\left([0,\mathcal{T}];\mathbf{L}^2(\pi)\right),\\
& \boldsymbol{A}_n(t) \rightharpoonup \boldsymbol{A}(t),\ \overline{\boldsymbol{A}_n}(t) \rightharpoonup \boldsymbol{A}(t) & in & \mathbf{H}^1(\pi),\ \forall t,\\
& \partial_t\boldsymbol{A}_n \rightharpoonup \partial_t\boldsymbol{A} & in & L^2((0,\mathcal{T});\mathbf{L}^2(\pi)),\\
(ii) & \boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) \rightharpoonup \boldsymbol{M}(\nabla\times\boldsymbol{A}) & in & L^2((0,\mathcal{T});\mathbf{L}^2(\pi)),\\
(iii) & \nabla\times\overline{\boldsymbol{A}_n} \to \nabla\times\boldsymbol{A} & in & L^2((0,\mathcal{T});\mathbf{L}^2(\pi)),\\
& \boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) \to \boldsymbol{M}(\nabla\times\boldsymbol{A}) & in & L^2((0,\mathcal{T});\mathbf{L}^2(\pi)).
\end{array}
$$

Proof. (i) Lemma 4.4 yields

$$\int_0^{\mathcal{T}} \left\|\overline{\boldsymbol{A}_n}\right\|^2_{\boldsymbol{X}_{N,0}}\ \mathrm{d}t \le C.$$

Since $L^2((0,\mathcal{T});\boldsymbol{X}_{N,0})$ is reflexive, we use Theorem 1.2 to obtain a subsequence such that $\overline{\boldsymbol{A}_n} \rightharpoonup \boldsymbol{A}$ in that space. One can easily see that

$$\overline{\boldsymbol{A}_n} \rightharpoonup \boldsymbol{A}, \qquad \nabla\times\overline{\boldsymbol{A}_n} \rightharpoonup \nabla\times\boldsymbol{A} \qquad \text{in } L^2((0,\mathcal{T});\mathbf{L}^2(\Omega)),$$

due to the density of $\boldsymbol{C}_0^\infty(\overline{\Omega})$ in $\mathbf{L}^2(\Omega)$, see [49, Theorem 2.6.1]. Take now $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\Omega\setminus\pi)$. Using $\mu \in H^1(\Omega\setminus\pi)$ and taking into account Remark 4.1, we have

$$\int_0^{\mathcal{T}} \left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}),\boldsymbol{\varphi}\right)_{\Omega\setminus\pi} = \int_0^{\mathcal{T}} \left(\mu\nabla\times\overline{\boldsymbol{A}_n},\boldsymbol{\varphi}\right)_{\Omega\setminus\pi} = \int_0^{\mathcal{T}} \left(\overline{\boldsymbol{A}_n},\nabla\times(\mu\boldsymbol{\varphi})\right)_{\Omega\setminus\pi}.$$

Passing to the limit for $n\to\infty$, we get

$$\lim_{n\to\infty}\int_0^{\mathcal{T}} \left(\mu\nabla\times\overline{\boldsymbol{A}_n},\boldsymbol{\varphi}\right)_{\Omega\setminus\pi} = \int_0^{\mathcal{T}} \left(\boldsymbol{A},\nabla\times(\mu\boldsymbol{\varphi})\right)_{\Omega\setminus\pi} = \int_0^{\mathcal{T}} \left(\mu\nabla\times\boldsymbol{A},\boldsymbol{\varphi}\right)_{\Omega\setminus\pi}.$$

Using the density argument of $\mathbf{C}_0^\infty(\Omega\setminus\pi)$ in $\mathbf{L}^2(\Omega\setminus\pi)$, we have that

$$\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) = \mu\nabla\times\overline{\boldsymbol{A}_n} \rightharpoonup \mu\nabla\times\boldsymbol{A} = \mu\boldsymbol{M}(\nabla\times\boldsymbol{A}) \quad \text{in } L^2((0,\mathcal{T});\mathbf{L}^2(\Omega\setminus\pi)).$$

Lemma 4.4 together with $\boldsymbol{X}_{N,0}\subset\mathbf{H}^1(\Omega)$ implies that

$$\int_0^{\mathcal{T}} \left\|\partial_t\boldsymbol{A}_n\right\|^2_{L^2(\pi)}\ \mathrm{d}t \le C, \qquad \left\|\overline{\boldsymbol{A}_n}\right\|_{\mathbf{H}^1(\pi)} \le \left\|\overline{\boldsymbol{A}_n}\right\|_{\mathbf{H}^1(\Omega)} \le C.$$

Employing Lemma 2.5 for $V = \mathbf{H}^1(\mathcal{T})$ and $Y = \mathbf{L}^2(\mathcal{T})$, we get for a subsequence that

$$\begin{array}{lll} \boldsymbol{A}_n \to \boldsymbol{A} & \text{in} & C\left([0,\mathcal{T}];\mathbf{L}^2(\mathcal{T})\right); \\ \boldsymbol{A}_n(t) \rightharpoonup \boldsymbol{A}(t),\ \overline{\boldsymbol{A}_n}(t) \rightharpoonup \boldsymbol{A}(t) & \text{in} & \mathbf{H}^1(\mathcal{T}),\ \forall t; \\ \partial_t \boldsymbol{A}_n \rightharpoonup \partial_t \boldsymbol{A} & \text{in} & L^2((0,\mathcal{T});\mathbf{L}^2(\mathcal{T})). \end{array}$$

(ii) The sequence $\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n})$ is bounded in $L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$. Therefore, there exists $\boldsymbol{p}$ from $L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$ such that $\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}) \rightharpoonup \boldsymbol{p}$ in that space (for a subsequence). Now let us investigate the following inequality

$$0 \leq \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}) - \boldsymbol{M}(\boldsymbol{b}), \psi\mu\left(\nabla \times \overline{\boldsymbol{A}_n} - \boldsymbol{b}\right)\right)_\Omega \,\mathrm{d}t = I_1 + I_2 + I_3 + I_4 \quad (4.27)$$

where

$$\begin{array}{ll} I_1 := \displaystyle\int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}), \psi\mu\nabla \times \overline{\boldsymbol{A}_n}\right)_\Omega \,\mathrm{d}t, & I_2 := \displaystyle\int_0^{\mathcal{T}} \left(\boldsymbol{M}(\boldsymbol{b}), \psi\mu\nabla \times \overline{\boldsymbol{A}_n}\right)_\Omega \,\mathrm{d}t, \\ I_3 := \displaystyle\int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}), \psi\mu\boldsymbol{b}\right)_\Omega \,\mathrm{d}t, & I_4 := \displaystyle\int_0^{\mathcal{T}} \left(\boldsymbol{M}(\boldsymbol{b}), \psi\mu\boldsymbol{b}\right)_\Omega \,\mathrm{d}t. \end{array}$$

This inequality holds true for any $\boldsymbol{b} \in L^2((0,T);\mathbf{L}^2(\Omega))$ and any non-negative $\psi \in C_0^\infty(\mathcal{T})$. We want to pass to the limit for $n \to \infty$ in (4.27). We do it for each term in (4.27) separately.

It holds that

$$\begin{aligned} I_1 &= \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}), \psi\mu\nabla \times \overline{\boldsymbol{A}_n}\right)_\Omega \,\mathrm{d}t \\ &= \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}), \psi\mu\nabla \times \left(\overline{\boldsymbol{A}_n} - \boldsymbol{A}\right)\right)_\Omega \,\mathrm{d}t + \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}), \psi\mu\nabla \times \boldsymbol{A}\right)_\Omega \,\mathrm{d}t \\ &= \int_0^{\mathcal{T}} \left(\nabla \times \left[\psi\mu\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n})\right], \overline{\boldsymbol{A}_n} - \boldsymbol{A}\right)_\Omega \,\mathrm{d}t + \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}), \psi\mu\nabla \times \boldsymbol{A}\right)_\Omega \,\mathrm{d}t \\ &= \int_0^{\mathcal{T}} \left(\psi\nabla \times \left[\mu\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n})\right], \overline{\boldsymbol{A}_n} - \boldsymbol{A}\right)_\Omega \,\mathrm{d}t \\ &+ \int_0^{\mathcal{T}} \left(\nabla\psi \times \left[\mu\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n})\right], \overline{\boldsymbol{A}_n} - \boldsymbol{A}\right)_\Omega \,\mathrm{d}t + \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}_n}), \psi\mu\nabla \times \boldsymbol{A}\right)_\Omega \,\mathrm{d}t. \end{aligned}$$

Here, we used the Green Theorem 1.12 and the Curl identity from Section 1.6. We know that $\boldsymbol{A}_n \to \boldsymbol{A}$ in the space $C\left([0,\mathcal{T}];\mathbf{L}^2(\mathcal{T})\right)$ and $\partial_t \boldsymbol{A}_n$ is bounded in the space $L^2((0,\mathcal{T});\mathbf{L}^2(\mathcal{T}))$. Therefore, also $\overline{\boldsymbol{A}_n} \to \boldsymbol{A}$ in $C\left([0,\mathcal{T}];\mathbf{L}^2(\mathcal{T})\right)$. Thus, using $\mu \in H^1(\mathcal{T})$, it is not difficult to see that

$$\lim_{n\to\infty} I_1 = \int_0^{\mathcal{T}} \left(\boldsymbol{p}, \psi\mu\nabla \times \boldsymbol{A}\right)_\Omega \,\mathrm{d}t.$$

Clearly,

$$\begin{aligned}\lim_{n\to\infty} I_2 &= \int_0^{\mathcal{T}} (\boldsymbol{M}(\boldsymbol{b}), \psi\mu\nabla\times\boldsymbol{A})_\Omega \ \mathrm{d}t,\\ \lim_{n\to\infty} I_3 &= \int_0^{\mathcal{T}} (\boldsymbol{p}, \psi\mu\boldsymbol{b})_\Omega \ \mathrm{d}t,\\ \lim_{n\to\infty} I_4 &= \int_0^{\mathcal{T}} (\boldsymbol{M}(\boldsymbol{b}), \psi\mu\boldsymbol{b})_\Omega \ \mathrm{d}t.\end{aligned}$$

Assembling these auxiliary results, we arrive at

$$\begin{aligned}&\lim_{n\to\infty}\int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) - \boldsymbol{M}(\boldsymbol{b}), \psi\mu\left(\nabla\times\overline{\boldsymbol{A}_n}-\boldsymbol{b}\right)\right)_\Omega \ \mathrm{d}t\\ &\quad= \int_0^{\mathcal{T}} (\boldsymbol{p}-\boldsymbol{M}(\boldsymbol{b}), \psi\mu\left(\nabla\times\boldsymbol{A}-\boldsymbol{b}\right))_\Omega \ \mathrm{d}t \geq 0.\end{aligned}$$

Since $\boldsymbol{b}$ was taken as an arbitrary element of $L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$, we choose it as $\boldsymbol{b} = \omega\boldsymbol{q} + \nabla\times\boldsymbol{A}$ where $\boldsymbol{q}\in L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$ is again arbitrary and $\omega > 0$. Using this substitution in the equation above, we obtain that

$$\begin{aligned}&\int_0^{\mathcal{T}} (\boldsymbol{p}-\boldsymbol{M}(\nabla\times\boldsymbol{A}+\omega\boldsymbol{q}), \mu\psi\left(-\omega\boldsymbol{q}\right))_\Omega \ \mathrm{d}t \geq 0 \quad / \text{ multiply by } \frac{1}{\omega},\\ &\int_0^{\mathcal{T}} (\boldsymbol{p}-\boldsymbol{M}(\nabla\times\boldsymbol{A}+\omega\boldsymbol{q}), \mu\psi\left(-\boldsymbol{q}\right))_\Omega \ \mathrm{d}t \geq 0 \quad / \text{ pass to the limit } \omega\to 0,\\ &\int_0^{\mathcal{T}} (\boldsymbol{p}-\boldsymbol{M}(\nabla\times\boldsymbol{A}), \mu\psi\left(-\boldsymbol{q}\right))_\Omega \ \mathrm{d}t \geq 0 \quad / \text{ since } \boldsymbol{q} \text{ is arbitrary}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{we choose it as } \boldsymbol{q} = -\boldsymbol{q}\\ &\int_0^{\mathcal{T}} (\boldsymbol{p}-\boldsymbol{M}(\nabla\times\boldsymbol{A}), \mu\psi\left(-\boldsymbol{q}\right))_\Omega \ \mathrm{d}t \leq 0.\end{aligned}$$

The conclusion is that $\int_0^{\mathcal{T}} (\boldsymbol{p}-\boldsymbol{M}(\nabla\times\boldsymbol{A}), \mu\psi\boldsymbol{q})_\Omega \ \mathrm{d}t = 0$ for any non-negative $\psi\in C_0^\infty(\pi)$ and every $\boldsymbol{q}\in L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$. Hence, $\boldsymbol{p} = \boldsymbol{M}(\nabla\times\boldsymbol{A})$ a.e. in $(0,\mathcal{T})\times\pi$ and $\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) \rightharpoonup \boldsymbol{M}(\nabla\times\boldsymbol{A})$ in $L^2((0,\mathcal{T});\mathbf{L}^2(\pi))$.

(*iii*) Analogously as in (*ii*), using the strong monotonicity of $\boldsymbol{M}$ (j_1), we conclude

$$\begin{aligned}0 &= \lim_{n\to\infty}\int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) - \boldsymbol{M}(\nabla\times\boldsymbol{A}), \mu\psi\left(\nabla\times\overline{\boldsymbol{A}_n}-\nabla\times\boldsymbol{A}\right)\right)_\Omega \ \mathrm{d}t\\ &\geq \lim_{n\to\infty} c_M\int_0^{\mathcal{T}} \left(\mu\psi, \left|\nabla\times\overline{\boldsymbol{A}_n}-\nabla\times\boldsymbol{A}\right|^2\right)_\Omega \ \mathrm{d}t \geq 0.\end{aligned}$$

Therefore, $\lim_{n\to\infty}\int_0^{\mathcal{T}} \left(\mu\psi, \left|\nabla\times\overline{\boldsymbol{A}_n} - \nabla\times\boldsymbol{A}\right|^2\right)_\Omega \,\mathrm{d}t = 0$ for every $0 \le \psi \in C_0^\infty(\Pi)$ which implies $\nabla\times\overline{\boldsymbol{A}_n} \to \nabla\times\boldsymbol{A}$ in $L^2((0,\mathcal{T});\mathbf{L}^2(\Pi))$. The vector field $\boldsymbol{M}$ is also Lipschitz continuous, hence, $\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}) \to \boldsymbol{M}(\nabla\times\boldsymbol{A})$ in $L^2((0,\mathcal{T});\mathbf{L}^2(\Pi))$ as well. □

Now, we are in the position to state our convergence theorem.

Theorem 4.1. *Suppose that Proposition 4.1 holds. Then there exists a solution-triplet* $\{\phi, \boldsymbol{A}, u\}$ *with* ϕ *being from the space* $L^2((0,\mathcal{T}); H^1(T)/\mathbb{R})$, $\boldsymbol{A}$ *being from the space* $L^2((0,\mathcal{T}); \boldsymbol{X}_{N,0})$ *with* $\partial_t\boldsymbol{A} \in L^2((0,\mathcal{T});\mathbf{L}^2(\Pi))$ *and* u *being from the space* $C([0,\mathcal{T}]; L^2(\Pi)) \cap L^\infty((0,\mathcal{T}); H_0^1(\Pi))$ *with* $\partial_t u \in L^2((0,\mathcal{T}); L^2(\Pi))$ *and subsequences of* $\overline{\phi_n}$, $\boldsymbol{A}_n$ *and* u_n *(denoted by the same symbol again) such that*

(i) ϕ *and* u *solve* (4.15),

(ii) $\nabla\overline{\phi_n} \to \nabla\phi$ *in* $L^2((0,\mathcal{T});\mathbf{L}^2(T))$,

(iii) ϕ, u *and* $\boldsymbol{A}$ *solve* (4.14),

(iv) $\partial_t\boldsymbol{A}_n \to \partial_t\boldsymbol{A}$ *in* $L^2((0,\mathcal{T});\mathbf{L}^2(\Pi))$,

(v) ϕ, u *and* $\boldsymbol{A}$ *solve* (4.16).

Proof. (i) Existence of a potential $\phi \in H^1(T)/\mathbb{R}$ such that $\nabla\overline{\phi_n} \rightharpoonup \nabla\phi$ in the space $L^2((0,\mathcal{T});\mathbf{L}^2(T))$ follows from the reflexivity of $L^2((0,\mathcal{T});\mathbf{L}^2(T))$ and Theorem 1.2. The function ϕ has in fact a zero mean over T, cf. proof of Lemma 4.1.

Take $\xi \in H^1(T)/\mathbb{R}$ in (4.24) and integrate in time

$$\int_0^\zeta \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \xi\right)_T \,\mathrm{d}s + \int_0^\zeta \left(\overline{j_n}, \xi\right)_\Gamma \,\mathrm{d}s = 0.$$

Thanks to Proposition 4.1 (ii), and (v), we pass to the limit for $n \to \infty$ to get

$$\int_0^\zeta (\sigma_\pi(u)\nabla\phi, \xi)_T \,\mathrm{d}s + \int_0^\zeta (j, \xi)_\Gamma \,\mathrm{d}s = 0.$$

Now, differentiating with respect to time, we see that ϕ and u solve (4.15).

(ii) It holds that

$$0 \le \sigma_* \int_0^{\mathcal{T}} \left\|\nabla\left[\overline{\phi_n} - \phi\right]\right\|_{L^2(T)}^2 \,\mathrm{d}t$$

$$
\begin{aligned}
&\leq \int_0^{\mathcal{T}} \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\left[\overline{\phi_n}-\phi\right], \nabla\left[\overline{\phi_n}-\phi\right]\right)_T \;\mathrm{d}t \\
&= \int_0^{\mathcal{T}} \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\phi, \nabla\phi\right)_T \;\mathrm{d}t + \int_0^{\mathcal{T}} \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \nabla\overline{\phi_n}\right)_T \;\mathrm{d}t \\
&- 2\int_0^{\mathcal{T}} \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \nabla\phi\right)_T \;\mathrm{d}t \\
&\overset{(4.24)}{=} \int_0^{\mathcal{T}} \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\phi, \nabla\phi\right)_T \;\mathrm{d}t - \int_0^{\mathcal{T}} \left(\overline{j_n}, \overline{\phi_n}\right)_\Gamma \;\mathrm{d}t \\
&- 2\int_0^{\mathcal{T}} \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \nabla\phi\right)_T \;\mathrm{d}t.
\end{aligned}
$$

Passing to the limit, we conclude that

$$
0 \leq \lim_{n\to\infty} \sigma_* \int_0^{\mathcal{T}} \left\|\nabla\left[\overline{\phi_n}-\phi\right]\right\|^2_{L^2(T)} \leq -\int_0^{\mathcal{T}} \left(\sigma_\pi(u)\nabla\phi, \nabla\phi\right)_T - \int_0^{\mathcal{T}} \left(j,\phi\right)_\Gamma \overset{(i)}{=} 0.
$$

Therefore, $\nabla\overline{\phi_n} \to \nabla\phi$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(T))$.

(iii) We integrate (4.25) in time to get

$$
\begin{aligned}
\int_0^{\zeta} \left(\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}_n, \boldsymbol{\varphi}\right)_\pi \;\mathrm{d}s + \int_0^{\zeta} \left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}), \nabla\times\boldsymbol{\varphi}\right)_\Omega \;\mathrm{d}s& \\
+ \int_0^{\zeta} \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \boldsymbol{\varphi}\right)_T \;\mathrm{d}s = 0.&
\end{aligned}
$$

Using Proposition 4.1 (ii), Proposition 4.2, and Theorem 4.1 (ii), we pass to the limit for $n\to\infty$ to see

$$
\int_0^{\zeta} \left(\sigma_\pi(u)\partial_t \boldsymbol{A}, \boldsymbol{\varphi}\right)_\pi + \int_0^{\zeta} \left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}), \nabla\times\boldsymbol{\varphi}\right)_\Omega + \int_0^{\zeta} \left(\sigma_\pi(u)\nabla\phi, \boldsymbol{\varphi}\right)_T = 0.
$$

Thus, ϕ, u and $\boldsymbol{A}$ solve (4.14).

(iv) The strong convergence of $\nabla\times\overline{\boldsymbol{A}_n} \to \nabla\times\boldsymbol{A}$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\pi))$ is guaranteed by Proposition 4.2 (iii). Let us take any $\zeta \in [0,\mathcal{T}]$ such that $\nabla\times\overline{\boldsymbol{A}_n}(\zeta) \to \nabla\times\boldsymbol{A}(\zeta)$ in $\mathbf{L}^2(\pi)$. This set is dense in $[0,\mathcal{T}]$. Take any non-negative $\psi \in C_0^\infty(\pi)$. We use the positiveness of σ to estimate the following

$$
\begin{aligned}
0 &\leq \sigma_* \int_0^{\zeta}\int_\pi \psi\left|\partial_t\boldsymbol{A}_n - \partial_t\boldsymbol{A}\right|^2 \;\mathrm{d}\boldsymbol{x}\;\mathrm{d}s \\
&\leq \int_0^{\zeta}\int_\pi \psi\overline{\sigma_{\pi_n}}(t-\tau)\left|\partial_t\boldsymbol{A}_n - \partial_t\boldsymbol{A}\right|^2 \;\mathrm{d}\boldsymbol{x}\;\mathrm{d}s
\end{aligned}
$$

$$= -2\int_0^\zeta (\psi\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}_n, \partial_t \boldsymbol{A})_\pi \ \mathrm{d}s + \int_0^\zeta (\psi\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}, \partial_t \boldsymbol{A})_\pi \ \mathrm{d}s$$
$$+ \int_0^\zeta (\psi\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}_n, \partial_t \boldsymbol{A}_n)_\pi \ \mathrm{d}s.$$

We use Lebesgue's dominated convergence Theorem 1.3 combined with Proposition 4.1 (*ii*) and Proposition 4.2 (*i*) to pass to the limit for $n \to \infty$ in the first two terms

$$\lim_{n\to\infty} -2\int_0^\zeta (\psi\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}_n, \partial_t \boldsymbol{A})_\pi \ \mathrm{d}s = -2\int_0^\zeta (\psi\sigma_\pi(u)\partial_t \boldsymbol{A}, \partial_t \boldsymbol{A})_\pi \ \mathrm{d}s,$$
$$\lim_{n\to\infty} \int_0^\zeta (\psi\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}, \partial_t \boldsymbol{A})_\pi \ \mathrm{d}s = \int_0^\zeta (\psi\sigma_\pi(u)\partial_t \boldsymbol{A}, \partial_t \boldsymbol{A})_\pi \ \mathrm{d}s.$$

We can assume that $\zeta \in (t_{j-1}, t_j]$ and use variational formulation (4.25) to rewrite the third term as

$$\int_0^\zeta (\psi\overline{\sigma_{\pi_n}}(t-\tau)\partial_t \boldsymbol{A}_n, \partial_t \boldsymbol{A}_n)_\pi \ \mathrm{d}s =$$
$$= -\int_0^\zeta \left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}), \nabla\times(\psi\partial_t\boldsymbol{A}_n)\right)_\Omega \ \mathrm{d}s - \int_0^\zeta \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \psi\partial_t\boldsymbol{A}_n\right)_T \ \mathrm{d}s$$
$$= -\int_0^\zeta \left(\psi\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}), \nabla\times\partial_t\boldsymbol{A}_n\right)_\Omega \ \mathrm{d}s - \int_0^\zeta \left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}), \nabla\psi\times\partial_t\boldsymbol{A}_n\right)_\Omega \ \mathrm{d}s$$
$$- \int_0^\zeta \left(\overline{\sigma_{\pi_n}}(t-\tau)\nabla\overline{\phi_n}, \psi\partial_t\boldsymbol{A}_n\right)_T \ \mathrm{d}s$$
$$=: R_1 + R_2 + R_3.$$

Let us rewrite the first term on the r.h.s. and examine it closely

$$R_1 = -\int_0^{t_j} \left(\psi\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}), \nabla\times\partial_t\boldsymbol{A}_n\right)_\Omega + \int_\zeta^{t_j} \left(\psi\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}), \nabla\times\partial_t\boldsymbol{A}_n\right)_\Omega$$
$$= -\sum_{i=1}^j \int_\Omega \psi\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i)\cdot(\nabla\times\boldsymbol{A}_i - \nabla\times\boldsymbol{A}_{i-1}))$$
$$+ \int_\zeta^{t_j} \left(\nabla\times\left(\psi\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n})\right), \partial_t\boldsymbol{A}_n\right)_\Omega$$
$$\overset{(4.18)}{\leq} -\sum_{i=1}^j \int_\Omega \psi\mu\left(\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_i) - \Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_{i-1})\right)$$
$$+ \int_\zeta^{t_j} \left(\nabla\psi\times\left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n})\right), \partial_t\boldsymbol{A}_n\right)_\Omega$$

$$
\begin{aligned}
&+\int_{\zeta}^{t_j}\left(\psi\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n})\right),\partial_t\boldsymbol{A}_n\right)_\Omega\\
=&-\int_\Omega\psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_j)+\int_\Omega\psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_0)\\
&+\int_{\zeta}^{t_j}\left(\nabla\psi\times\left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n})\right),\partial_t\boldsymbol{A}_n\right)_\Omega\\
&+\int_{\zeta}^{t_j}\left(\psi\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n})\right),\partial_t\boldsymbol{A}_n\right)_\Omega\\
=&-\int_\Omega\psi\mu\Phi_{\boldsymbol{M}}(\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n}(\zeta))+\int_\Omega\psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_0)\\
&+\int_{\zeta}^{t_j}\left(\nabla\psi\times\left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n})\right),\partial_t\boldsymbol{A}_n\right)_\Omega\\
&+\int_{\zeta}^{t_j}\left(\psi\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\overline{\boldsymbol{A}_n})\right),\partial_t\boldsymbol{A}_n\right)_\Omega.
\end{aligned}
$$

Now, we are able to pass to the limit for $n\to\infty$ to find that

$$
\lim_{n\to\infty} R_2=-\int_0^{\zeta}(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}),\nabla\psi\times\partial_t\boldsymbol{A})_\Omega\ \mathrm{d}s,
$$

$$
\lim_{n\to\infty} R_3=-\int_0^{\zeta}(\sigma_\pi(u)\nabla\phi,\psi\partial_t\boldsymbol{A})_T\ \mathrm{d}s,
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} R_1&\le-\int_\Omega\psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}(\zeta))\ \mathrm{d}\boldsymbol{x}+\int_\Omega\psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}(0))\ \mathrm{d}\boldsymbol{x}\\
&=-\int_0^{\zeta}\int_\Omega\psi\mu\frac{\mathrm{d}\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A})}{\mathrm{d}t}\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}s\\
&=-\int_0^{\zeta}\int_\Omega\psi\mu\boldsymbol{M}(\nabla\times\boldsymbol{A})\cdot\partial_t\left(\nabla\times\boldsymbol{A}\right)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}s\\
&=-\int_0^{\zeta}(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}),\psi\nabla\times(\partial_t\boldsymbol{A}))_\Omega\ \mathrm{d}s.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_{n\to\infty} R_1+R_2+R_3\le&-\int_0^{\zeta}(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}),\nabla\times(\psi\partial_t\boldsymbol{A}))_\Omega\ \mathrm{d}s\\
&-\int_0^{\zeta}(\sigma_\pi(u)\nabla\phi,\psi\partial_t\boldsymbol{A})_T\ \mathrm{d}s
\end{aligned}
$$

$$\overset{(4.14)}{=} \int_0^{\zeta} (\psi\sigma_\pi(u)\partial_t \boldsymbol{A}, \partial_t \boldsymbol{A})_\pi \ \mathrm{d}s.$$

Thus, collecting all estimates above, we see that

$$0 \leq \lim_{n\to\infty} \int_0^{\zeta} \int_\pi \psi \left|\partial_t \boldsymbol{A}_n - \partial_t \boldsymbol{A}\right|^2 \ \mathrm{d}\boldsymbol{x} \ \mathrm{d}s \leq 0.$$

Please note that this is valid for any non-negative $\psi \in C_0^\infty(\mathcal{T})$. Since the set of $\zeta \in [0, \mathcal{T}]$ for which $\nabla \times \overline{\boldsymbol{A}_n}(\zeta) \to \nabla \times \boldsymbol{A}(\zeta)$ in $\mathbf{L}^2(\Omega)$ is dense in $[0, \mathcal{T}]$, we achieve a strong convergence of $\partial_t \boldsymbol{A}_n$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\mathcal{T}))$, i.e. $\partial_t \boldsymbol{A}_n \to \partial_t \boldsymbol{A}$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\mathcal{T}))$.

(v) Take $\psi \in H^1(\mathcal{T})$ in (4.26) and integrate in time to get

$$\left(\overline{\theta_n}(t) - \theta_n(0), \psi\right)_\pi + \left(\theta_n(t) - \overline{\theta_n}(t), \psi\right)_\pi + \int_0^t (\lambda_0 \nabla \overline{u_n}, \nabla\psi)_\pi \ \mathrm{d}s$$
$$= \int_0^t \left(\mathcal{R}_r\left(\overline{\sigma_{\pi_n}}(s-\tau)\left|\partial_t \boldsymbol{A}_n + \chi_T \nabla\overline{\phi_n}\right|^2\right), \psi\right)_\pi \ \mathrm{d}s.$$

Lebesgue's dominated convergence Theorem 1.3 together with Proposition 4.1-(ii), Theorem 4.1-(ii), and (iv) let us pass to the limit for $n \to \infty$ on the r.h.s. of the equation above

$$\lim_{n\to\infty} \int_0^t \left(\mathcal{R}_r\left(\overline{\sigma_{\pi_n}}(s-\tau)\left|\partial_t \boldsymbol{A}_n + \chi_T \nabla\overline{\phi_n}\right|^2\right), \psi\right)_\pi \ \mathrm{d}s =$$
$$= \int_0^t \left(\mathcal{R}_r\left(\sigma_\pi(u)\left|\partial_t \boldsymbol{A} + \chi_T \nabla\phi\right|^2\right), \psi\right)_\pi \ \mathrm{d}s.$$

Proposition 4.1 lets us pass to the limit for $n \to \infty$ on the l.h.s. of the equation above. Please note that the term $\left(\theta_n(t) - \overline{\theta_n}(t), \psi\right)_\pi$ vanishes since $\lim_{n\to\infty} \left(\theta_n(t) - \overline{\theta_n}(t), \psi\right)_\pi = 0$ for every $t \in [0, \mathcal{T}]$. Therefore, gathering all results above brings us to

$$(\theta(u(t)) - \theta(u(0)), \psi)_\pi + \int_0^t (\lambda_0 \nabla u, \nabla\psi)_\pi \ \mathrm{d}s$$
$$= \int_0^t \left(\mathcal{R}_r\left(\sigma_\pi(u)\left|\partial_t \boldsymbol{A} + \chi_T \nabla\phi\right|^2\right), \psi\right)_\pi \ \mathrm{d}s.$$

After differentiation with respect to the time variable, we see that ϕ, u and $\boldsymbol{A}$ solve (4.16). □

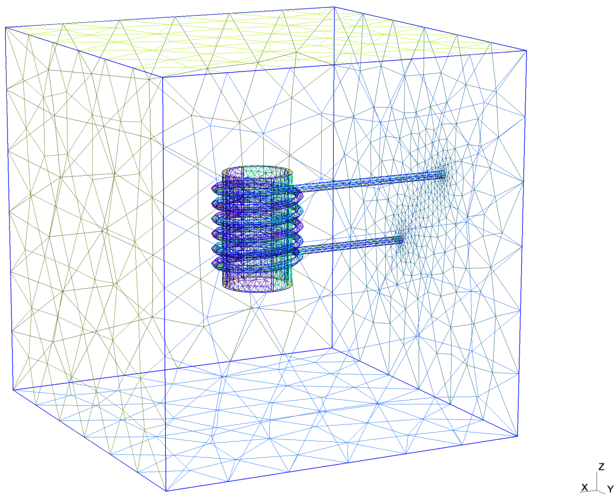

Figure 4.3: Meshed domain.

4.5 Numerical simulation

To support our proposed numerical scheme obtained from the variational formulation (4.23) - (4.21), we provide a numerical simulation of the induction hardening process. The domain used in the simulation is reported in Fig. 4.3. This domain is more complex than its simplified version in Fig. 4.1, but our theoretical results for this type hold regardless, because the inclusion $\boldsymbol{X}_{N,0} \subset \mathbf{H}^1(\Omega)$ holds true also for convex domains (without a smooth boundary). Since we want our simulation to be realistic, we use physical constants:

$c_{\text{fe}} = 502.4 \ \ [\text{J/kg}]$	specific heat of steel,
$\rho_{\text{fe}} = 7850 \ \ [\text{kg/m}^3]$	density of steel,
$\lambda_{\text{fe}} = 43 \ \ [\text{W/m}]$	thermal conductivity of steel,
$\sigma_{\text{fe}} = 6.21 \times 10^6 \ \ [\text{S/m}]$	electric conductivity of steel,
$\mu_{\text{fe}} = 1.26 \times 10^{-4} \ \ [\text{H/m}]$	magnetic permeability of steel,
$\mu_{\text{air}} = 1.256 \times 10^{-6} \ \ [\text{H/m}]$	magnetic permeability of air.

Unknown functions representing nonlinearities are chosen accordingly to satisfy (4.17):

$$\sigma_\pi(u) = 2\sigma_{\text{fe}} + \sigma_{\text{fe}} \left(2 - \left(1 + \frac{1}{1+u} \right)^{1+u} \right),$$

$$\theta(u) = \rho_{\text{fe}} c_{\text{fe}} \sqrt{u},$$
$$\boldsymbol{M}(\nabla \times \boldsymbol{A}) = m(|\nabla \times \boldsymbol{A}|)\nabla \times \boldsymbol{A} = \left(1 + e^{-|\nabla \times \boldsymbol{A}|}\right) \nabla \times \boldsymbol{A}.$$

The range of the electric conductivity function $\sigma_\pi(u)$ is

$$0 \leq \sigma_c(4 - e) < \sigma_\pi(u) \leq 2\sigma_c \quad \text{for } u \geq 0.$$

Thus, $\sigma_\pi(u)$ satisfies the assumption (d) (it is positive and bounded) from Section 4.2.2. The function $\theta(u)$ defined above is of a linear growth, its derivative is always positive and it also satisfies the condition $\theta(0) = 0$. We need the vector field $\boldsymbol{M}$ to satisfy the condition $\boldsymbol{M}(\boldsymbol{0}) = \boldsymbol{0}$, to be Lipschitz continuous and also strongly monotone. Clearly, the first condition is met. To check the second condition, we need to compute the gradient of $\boldsymbol{M}(\boldsymbol{x})$ in the $\boldsymbol{h}$ direction, i.e.

$$\langle grad\ \boldsymbol{M}(\boldsymbol{x}), \boldsymbol{h}\rangle = \langle grad\ m(|\boldsymbol{x}|)\boldsymbol{x}, \boldsymbol{h}\rangle = m'(|\boldsymbol{x}|)\frac{\boldsymbol{h}\cdot\boldsymbol{x}}{|\boldsymbol{x}|}\boldsymbol{x} + m(|\boldsymbol{x}|)\boldsymbol{h}.$$

Using the mean value theorem for vector functions, we obtain for some $\xi \in (0,1)$

$$\begin{aligned}
|\boldsymbol{M}(\boldsymbol{x}) - \boldsymbol{M}(\boldsymbol{y})| &= |\langle grad\ \boldsymbol{M}(\boldsymbol{x} + \xi(\boldsymbol{y}-\boldsymbol{x})), \boldsymbol{x}-\boldsymbol{y}\rangle| \\
&= \Big| m'\left(|\boldsymbol{x} + \xi(\boldsymbol{y}-\boldsymbol{x})|\right) \frac{(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x}))}{|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|}(\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})) \\
&\qquad + m(|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|)(\boldsymbol{x}-\boldsymbol{y})| \\
&\leq |\boldsymbol{x}-\boldsymbol{y}| \left\{|m(|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|)|\right. \\
&\qquad\qquad \left. + |m'(|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|)|\,|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|\right\} \\
&= |\boldsymbol{x}-\boldsymbol{y}| \left\{1 + e^{-|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|} + e^{-|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|}\,|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|\right\} \\
&\leq 2\,|\boldsymbol{y}-\boldsymbol{x}| \quad \forall\, \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3\ \boldsymbol{x} \neq \boldsymbol{y}
\end{aligned}$$

since the function $f(s) = 1 + e^{-s} + e^{-s}s$ reaches its global maximum at $s = 0$ and it is equal to 2. It only remains to show the strongly monotone character of $\boldsymbol{M}(\boldsymbol{x})$. Again, using the mean value theorem, we have for some $\xi \in (0,1)$

$$\begin{aligned}
[\boldsymbol{M}(\boldsymbol{x}) - \boldsymbol{M}(\boldsymbol{y})]\cdot(\boldsymbol{x}-\boldsymbol{y}) &= \langle grad\ \boldsymbol{M}(\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})), \boldsymbol{x}-\boldsymbol{y}\rangle \cdot (\boldsymbol{x}-\boldsymbol{y}) \\
&= m'(|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|)\frac{\left((\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x}))\right)^2}{|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|} \\
&\quad + m(|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|)\,|\boldsymbol{x}-\boldsymbol{y}|^2 \\
&\geq |\boldsymbol{x}-\boldsymbol{y}|^2 \left\{m(|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|)\right. \\
&\qquad\qquad \left. - |m'(|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|)|\,|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|\right\}
\end{aligned}$$

$$= |\boldsymbol{x}-\boldsymbol{y}|^2 \Big\{1 + e^{-|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|} \\ - e^{-|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|}\,|\boldsymbol{x}+\xi(\boldsymbol{y}-\boldsymbol{x})|\Big\} \\ \geq \left(1-\frac{1}{e^2}\right)|\boldsymbol{x}-\boldsymbol{y}|^2 \quad \forall\, \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3,\ \boldsymbol{x}\neq\boldsymbol{y},$$

because the function $p(s) = 1 + e^{-s} - e^{-s}s$ reaches its global minimum at $s = 2$ and it is equal to $\left(1-\frac{1}{e^2}\right)$.

We split the time interval $[0, \mathcal{T}] = [0, 0.02]$ in 1280 equidistant parts ($\tau = 1.5625\text{e}10^{-5}$) and use the open source finite element environment Gmsh/GetDP [25, 33], available online on `http://www.onelab.info`, to solve the system (4.23) - (4.21) at each time step after spatial discretization using Whitney finite elements on tetrahedra (edge elements for the magnetic vector potential, nodal elements for the electric scalar potential and the temperature) [15]. The mesh contained 26765 tetrahedra, leading to a total of 29714 unknowns. We denote the obtained solutions for the magnetic induction field and the temperature function as reference solutions $\boldsymbol{b}_{ref}$ and u_{ref}, respectively. Typical solutions are plotted in Fig. 4.4 and Fig. 4.5.

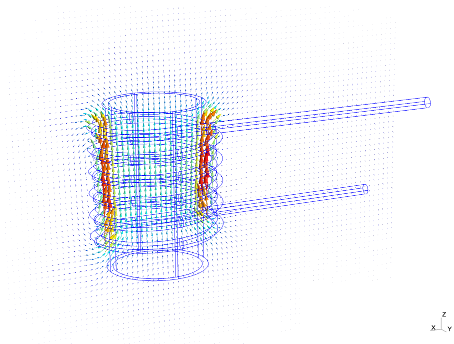

Figure 4.4: Magnetic induction field.

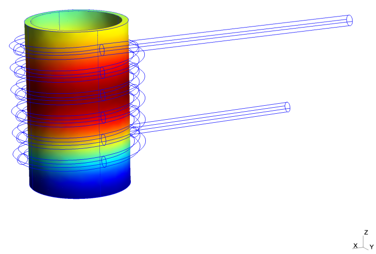

Figure 4.5: Temperature.

Figure 4.6: Reference solutions in time $t = 0.015$.

To show that our scheme is converging to $\boldsymbol{b}_{ref}$ and u_{ref}, we compute other numerical solutions for number of time steps $10, 20, 40, 80, 160, 320$ and 640 and compare them with $\boldsymbol{b}_{ref}$ and u_{ref}. We analyze these solutions in certain measurement points of our domain (see Fig. 4.7) and at certain time steps, namely $t_i = 0.002i$, where $i = 1, \ldots, 10$. Relative errors of a given numerical solution $\boldsymbol{b}_n$ from the reference solution $\boldsymbol{b}_{ref}$ and u_n from u_{ref} are then calculated in the following manner

$$
\begin{aligned}
|\boldsymbol{b}_{ref}| &= \sum_{P_j \in P} \sum_{i=1}^{10} |\boldsymbol{b}_{ref}(P_j, t_i)|, \\
|u_{ref}| &= \sum_{P_j \in P} \sum_{i=1}^{10} |u_{ref}(P_j, t_i)|, \\
|\boldsymbol{b}_{ref} - \boldsymbol{b}_n| &= \sum_{P_j \in P} \sum_{i=1}^{10} |\boldsymbol{b}_{ref}(P_j, t_i) - \boldsymbol{b}_n(P_j, t_i)|, \\
|u_{ref} - u_n| &= \sum_{P_j \in P} \sum_{i=1}^{10} |u_{ref}(P_j, t_i) - u_n(P_j, t_i)|,
\end{aligned}
$$

$$
\text{rel } \boldsymbol{b}_n = \frac{|\boldsymbol{b}_{ref} - \boldsymbol{b}_n|}{|\boldsymbol{b}_{ref}|} \quad \text{and} \quad \text{rel } u_n = \frac{|u_{ref} - u_n|}{|u_{ref}|}
$$

where P is the set of measurement points.

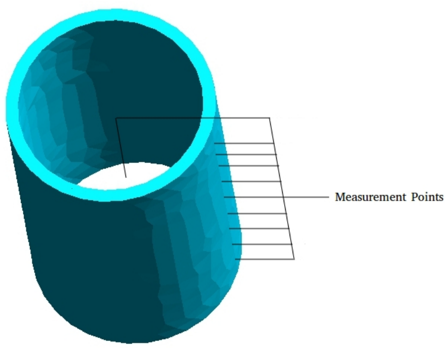

Figure 4.7: Measurement points.

Please bear in mind that the index n refers to the numerical solution computed on a mesh with $2^{n-1} \cdot 10$ time steps. The evolution of these errors with increasing number of time steps is shown in Fig. 4.8 and Fig. 4.9.

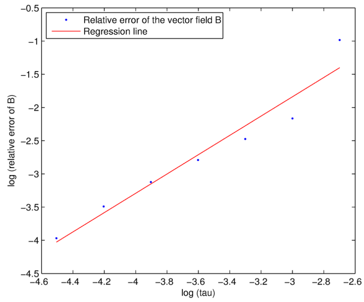

Figure 4.8: Relative error of the magnetic induction field $\boldsymbol{B}$ with respect to a decreasing time step τ.

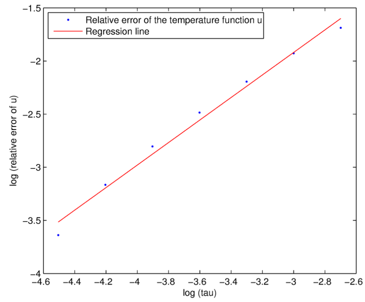

Figure 4.9: Relative error of the temperature function u with respect to a decreasing time step τ.

If the error of a given numerical solution f_τ from the exact solution f depends smoothly on a time step τ then there exist an error coefficient D such that

$$f_\tau - f = D\tau^p + \mathcal{O}(\tau^{p+1})$$

where p represents the order of convergence. Using the fact that the difference of $f_\tau - f_{\tau/2}$ decays to zero with the same speed as $f_\tau - f$, we can estimate the order of convergence without knowing the exact solution f, i.e.

$$\frac{f_\tau - f_{\tau/2}}{f_{\tau/2} - f_{\tau/4}} = \frac{D\tau^p - D(\tau/2)^p + \mathcal{O}(\tau^{p+1})}{D(\tau/2)^p - D(\tau/4)^p + \mathcal{O}(\tau^{p+1})} = 2^p + \mathcal{O}(\tau).$$

Which gives us

$$\log_2 \left(\frac{f_\tau - f_{\tau/2}}{f_{\tau/2} - f_{\tau/4}} \right) = p + \mathcal{O}(\tau).$$

Applying the formula above to our numerical solutions, we obtain an estimation for the order of convergence of $\{u_n\}$ and $\{\boldsymbol{b}_n\}$

$$p_u \approx 0.9830 \qquad \text{and} \qquad p_{\boldsymbol{b}} \approx 1.0010.$$

This provides a strong indication that the convergence of our numerical scheme is linear.

Part II

On an inverse source problem in Maxwell's equations

Chapter 5

Reconstruction of a time dependent source term from a single boundary measurement considering nonlinear generalized Ohm's law

5.1 Introduction to inverse (source) problems

An inverse problem assumes a direct problem that is a well-posed problem of mathematical physics. In other words, if we know completely a physical device, we have a mathematical description of this device including uniqueness, stability, and the existence of a solution of the corresponding mathematical problem. But, if one of the (functional) parameters describing this device is to be found from additional boundary (or experimental) data then we arrive at an inverse problem.

Many inverse problems arise naturally and have important applications. As a rule, these problems are rather difficult to solve for two reasons: they are nonlinear and they are improperly posed.

Most direct problems can be reduced to finding values $y = Ax$ of a continuous (not necessarily linear) operator A operating from a Banach space X onto a Banach space Y. The inverse problem is then connected with the inverse operator or with solving the equation

$$Ax = y. \tag{5.1}$$

Many direct problems are equivalent to such an equation. A related problem is said to be well-posed in the sense of Hadamard if the following conditions are satisfied:

(i) for any $y \in Y$ there is no more than one $x \in X$ satisfying (5.1) (uniqueness),

(ii) for any $y \in Y$ there exists a solution $x \in X$ (existence),

(iii) a solution x to Equation (5.1) is stable, i.e., if $\bar{y} \to y$ in Y, then related solutions $\bar{x} \to x$ in X (continuous dependence on data).

If one of the conditions $(i) - (iii)$ is not satisfied, the problem (5.1) is called ill-posed.

The conditions above are of different degrees of importance. For instance, if we cannot guarantee the uniqueness of a solution under any reasonable choice of X, then the inverse problem does not make much sense. On the other hand, the condition (ii) does not appear as restrictive since it only shows that we cannot describe conditions that guarantee the existence of a solution. Even without these conditions we can produce a stable numerical algorithm for finding x for given y. Hadamard especially stressed the meaning of the stability condition. In practice it is important because of the inevitable errors when calculating or measuring something.

A problem described by (5.1) is said to be conditionally correct (according to Tikhonov [78]) in a correctness class W if the following conditions are satisfied:

$(i)^*$ a solution x to Equation (5.1) is unique in W, i.e., if $Ax_1 = Ax_2$, $x_1, x_2 \in W$, then $x_1 = x_2$ (uniqueness in W),

$(ii)^*$ a solution is stable on W, i.e., $\bar{x} \to x$ in X if $\bar{x}, x \in W$ and $A\bar{x} \to Ax$ in Y (conditional stability).

Uniqueness questions are central in the theory of conditionally correct problems, nevertheless, the existence theorem are of importance as well since they guarantee that we do not use extra data.

Inverse source problems (ISP) are a special subclass of Inverse problems. In ISP we have to find the right-hand side (source). Identification problems can often be reduced to an ISP.

There is a broad range of applications, for instance, geophysics, physics, chemistry, medicine, optics, machine learning, signal processing, astronomy and medical imaging. For more details, we refer the reader to the books and articles of [7, 14, 19, 46, 58, 59, 76] and [77].

Many mathematical papers are dealing with ISP in hyperbolic PDEs. There are different techniques used to reconstruct the source term, for instance Carleman estimates have been used in [20, 47]. If the source is also space dependent, then an extra measurement in space (e.g. solution at the final time) is needed, cf. [42, 71]. Linear problems have been addressed in [37, 43, 82]. Source reconstructions from additional boundary measurements were studied in [62, 63, 64].

In some applications (geophysics), we can reach only certain parts of the considered domain and the rest remains unreachable. Hence, the electric and magnetic fields can be measured only on the reachable part. It can be a part of the considered domain or boundary (for better interpretation see Fig. 5.1). In this chapter we study an ISP in a nonlinear hyperbolic setting derived from the Maxwell equations (1.3)-(1.6). The additional measurement consists of a single boundary measurement on a part of the considered boundary.

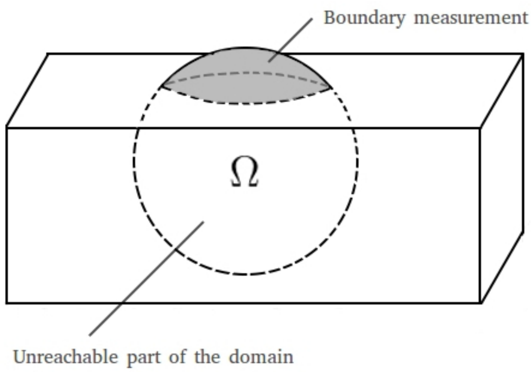

Figure 5.1: Example of a boundary measurement on a part of the boundary.

5.2 The mathematical model and ISP formulation

We assume that our domain $\Omega \subset \mathbb{R}^3$ is either smooth, i.e. $\Omega \in C^{1,1}$ or convex. The boundary of Ω is denoted by $\partial\Omega = \Gamma$ and the symbol $\boldsymbol{n}$ stands for the unit outward normal vector on the boundary Γ. We work in the time frame $[0, T]$.

The constitutive relations between the four vector fields $\boldsymbol{B}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{E}$ were described in Section 1.7.1. Their exact form depends on the particular physical phenomenon we are modelling. In many cases the present values of solutions depend on their previous values. These dependencies are expressed with a memory term. There are plenty of applications, for instance in chiral media [74], meta-materials [40, 41], nonlinear optics [8, 9, 10] or geophysics [23, 24, 72, 88]. The physical phenomenon which is often observed in geophysics is charge accumulation in rocks that serve as capacitors. The charge then decays, and this introduces an effective change to the traditional Ohm law that does not assume capacitance. Hence, it becomes a convolution in time, cf. [36].

Let us again consider the Maxwell equations (1.3)-(1.6). They form the cornerstone of our mathematical model. We adopt the generalized Ohm law in the following nonlinear form

$$\boldsymbol{J}(t) = (\sigma * \boldsymbol{E})(t) - (1 * \boldsymbol{N}(\boldsymbol{E}))(t) + \boldsymbol{F}(\boldsymbol{x}, t)$$

where the symbol $*$ stands for the usual convolution in time, e.g.

$$(f * g(\boldsymbol{x}))(t) = \int_0^t f(t-s) g(\boldsymbol{x}, s) \, \mathrm{d}s.$$

The vector function $\boldsymbol{F}$ describes the source current and $\boldsymbol{N}(\boldsymbol{E})$ is a nonlinear vector function of $\boldsymbol{E}$.

The electric conductivity term σ is assumed to be separable, i.e.

$$\sigma(\boldsymbol{x}, t) = \hat{\sigma}(\boldsymbol{x}) \sigma(t).$$

Both $\hat{\sigma}(\boldsymbol{x})$ and $\sigma(t)$ are known. We assume $\hat{\sigma}(\boldsymbol{x})$ to be a positive constant. Time dependent part $\sigma(t)$ is assumed to be Lipschitz continuous and bounded, e.g. $0 < \sigma_* \leq \sigma \leq \sigma^* < \infty$. The nonlinear function $\boldsymbol{N}$ is supposed to be globally Lipschitz continuous and it also fulfills the following boundary condition

$$\boldsymbol{N}(\boldsymbol{E}(t)) \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \ \forall t \in (0, T). \tag{5.2}$$

The domain Ω is occupied by a homogeneous dielectric material. Hence, the constitutive relations become

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}$$

where μ and ε are positive constants. Elimination of $\boldsymbol{H}$ in Maxwell's equations then yields

$$\varepsilon \partial_t^2 \boldsymbol{E} + \hat{\sigma} \partial_t (\sigma * \boldsymbol{E}) + \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) = \boldsymbol{N}(\boldsymbol{E}) - \partial_t \boldsymbol{F} \quad \text{in} \quad \Omega.$$

We assume that the tangential component of $\boldsymbol{E}$ is continuous across the boundary, i.e.

$$\boldsymbol{n} \times \boldsymbol{E}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma,\ \forall t \in (0, T). \tag{5.3}$$

And the initial data are prescribed as follows

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{W}_0(\boldsymbol{x}) \quad \text{in} \quad \Omega. \tag{5.4}$$

Time derivative of the source term $\boldsymbol{F}$ is assumed to be separable, i.e.

$$-\partial_t \boldsymbol{F} = h(t)\boldsymbol{f}(\boldsymbol{x}).$$

Here, the function $\boldsymbol{f}(\boldsymbol{x})$ is given but $h(t)$ is unknown. We suppose that

$$\boldsymbol{f} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma.$$

For the sake of simplicity, let us assume that $\mu \equiv \varepsilon \equiv \hat{\sigma} \equiv 1$. Then the PDE becomes

$$\partial_t^2 \boldsymbol{E} + \partial_t(\sigma * \boldsymbol{E}) + \nabla \times \nabla \times \boldsymbol{E} = h(t)\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{N}(\boldsymbol{E}) \quad \text{in} \quad \Omega. \tag{5.5}$$

The inverse source problem reads as finding a couple $\{\boldsymbol{E}(\boldsymbol{x}, t), h(t)\}$. The measurement introduced below will be used to recover the time dependent part of the source term $h(t)$

$$\int_\Gamma \phi \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = m(t). \tag{5.6}$$

Here, ϕ is a function from $C^\infty(\overline{\Omega})$ with $meas\{\operatorname{supp}(\phi) \cap \Gamma\} > 0$. After applying the measurement operator to equation (5.5) and assuming that $\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi \, \mathrm{d}\Gamma \neq 0$, we eliminate $h(t)$ to obtain

$$h(t) = \frac{m''(t) + (\sigma * m)'(t) + \int_\Omega \nabla \times \nabla \times \boldsymbol{E} \cdot \nabla\phi \, \mathrm{d}\boldsymbol{x} - \int_\Gamma \boldsymbol{N}(\boldsymbol{E}) \cdot \boldsymbol{n}\phi \, \mathrm{d}\Gamma}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi \, \mathrm{d}\Gamma}. \tag{5.7}$$

We used the fact that $\partial_t(f * g)(t) = (f * \partial_t g)(t) + f(t)g(0)$ for any given functions f, g and applied the Green theorem in the following way

$$\begin{aligned} \int_\Gamma (\nabla \times \nabla \times \boldsymbol{E} \cdot \boldsymbol{n})\phi &= \int_\Omega \nabla \times \nabla \times \boldsymbol{E} \cdot \nabla\phi + \int_\Omega \nabla \cdot (\nabla \times \nabla \times \boldsymbol{E})\phi \\ &= \int_\Omega \nabla \times \nabla \times \boldsymbol{E} \cdot \nabla\phi. \end{aligned} \tag{5.8}$$

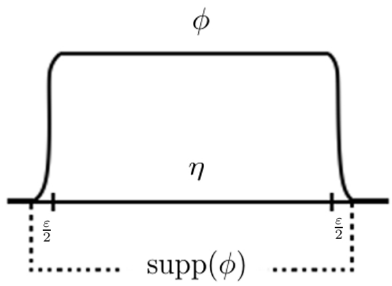

Figure 5.2: Vertical cut of the measured part of the boundary.

We measure only the normal component of $\boldsymbol{E}$ on a part of the boundary Γ which is modelled by the function ϕ. Let us explain the purpose of this function in more detail. Assume that the measurement is done on a part of the boundary denoted as η. Naturally, η is a subset of Γ, i.e. $\eta \subset \Gamma$. Then the function ϕ is defined as $\phi(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \eta$ and $meas\{\text{supp}(\phi) \cap \eta\} = |\overline{\eta}|$, moreover, $meas\{\text{supp}(\phi) \cap \Gamma\} = |\overline{\eta}| + \varepsilon$, for some small and positive ε. For better interpretation see Fig. 5.2. The implementation of ϕ in our measurement is solely due to mathematical reasons. With this addition, we can use the Green theorem as in equation (5.8).

5.2.1 Weak formulation

Let us recall the functional space $\boldsymbol{X}_N$* defined in Section 1.3

$$\boldsymbol{X}_N = \{\boldsymbol{\varphi} \in \mathbf{H}(\mathbf{curl}\,; \Omega) \cap \mathbf{H}(\text{div}; \Omega) : \boldsymbol{n} \times \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma\}.$$

With its norm defined as

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} = \|\boldsymbol{\varphi}\| + \|\nabla \times \boldsymbol{\varphi}\| + \|\nabla \cdot \boldsymbol{\varphi}\|.$$

Space $\boldsymbol{X}_N$ is associated with the solution of (5.5). To obtain the weak formulation of (5.9), we multiply it by a test function $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,; \Omega)$. Then integrate over Ω, take into account boundary condition (5.3) and use Green's theorem to obtain

$$\begin{aligned}\left(\partial_t^2 \boldsymbol{E}, \boldsymbol{\varphi}\right) + (\partial_t(\sigma * \boldsymbol{E}), \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\varphi}) = h(t)\,(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{\varphi}) \\ + (\boldsymbol{N}(\boldsymbol{E}), \boldsymbol{\varphi})\,, \qquad (5.9)\end{aligned}$$

*This space is embedded in $\mathbf{H}^1(\Omega)$, cf. Theorem 1.8. This embedding is very important since $\mathbf{H}^1(\Omega)$ is compactly embedded in $\mathbf{L}^2(\Omega)$. The embedding above also holds for a convex domain Ω (cf. [3, Theorem 2.17]).

for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$. Then, the weak formulation of problem (5.3), (5.4), (5.5) and (5.6) reads as:
Find a solution pair $\{h(t), \boldsymbol{E}(\boldsymbol{x},t)\}$ satisfying equation (5.7) and (5.9) such that $h(t) \in L^2((0,T)), \boldsymbol{E} \in C([0,T]; \mathbf{L}^2(\Omega)) \cap L^\infty((0,T); \boldsymbol{X}_N)$ with its first order time derivative $\partial_t \boldsymbol{E} \in L^2((0,T); \mathbf{L}^2(\Omega)) \cap C([0,T]; \boldsymbol{X}_N^*)$ and second order time derivative $\partial_t^2 \boldsymbol{E} \in L^2((0,T); (\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$.

5.3 Time discretization

To discretize our continuous formulation of equations (5.5) and (5.7), we start by splitting the time interval $[0,T]$ into $n \in \mathbb{N}$ equidistant parts with the time step $\tau = T/n$. Following the same notation as in the previous chapters, we write for any function w

$$t_i = i\tau, \qquad w_i = w(t_i), \qquad \delta w_i = \frac{w_i - w_{i-1}}{\tau}, \qquad \delta^2 w_i = \frac{\delta w_i - \delta w_{i-1}}{\tau^2}.$$

Please bear in mind that we approximate the second order time derivative in the following way

$$\partial_t^2 w = \partial_t(\partial_t w) \approx \delta(\delta w_i) = \delta^2 w_i.$$

The discretized convolution for any given functions f, g is then defined as

$$(f * g)_i = \sum_{k=0}^{i} f_{i-k} g_k \tau.$$

This also implies

$$\delta(f * g)_i = \frac{(f * g)_i - (f * g)_{i-1}}{\tau} = f_0 g_i + \sum_{k=0}^{i-1} \delta f_{i-k} g_k \tau, \text{ for } i \geq 1.$$

Now, we consider a system with unknown variables $\{\boldsymbol{e}_i, h_i\}$ and approximate our ISP at each time step t_i for $i = 1, \ldots, n$ as follows

$$\begin{aligned} \delta^2 \boldsymbol{e}_i + (\sigma * \delta \boldsymbol{e})_i + \nabla \times \nabla \times \boldsymbol{e}_i &= \boldsymbol{N}(\boldsymbol{e}_{i-1}) + h_i \boldsymbol{f} - \sigma_i \boldsymbol{E}_0 && \text{in } \Omega \\ \boldsymbol{e}_i \times \boldsymbol{n} &= \boldsymbol{0} && \text{on } \Gamma \\ \boldsymbol{e}_0 &= \boldsymbol{E}_0 \\ \delta \boldsymbol{e}_0 &= \boldsymbol{W}_0 \end{aligned} \tag{DPi}$$

and

$$h_i = \frac{m_i'' + (\sigma * m')_i + \sigma_i m_0 + (\nabla \times \nabla \times \boldsymbol{e}_{i-1}, \nabla \phi) - \int_\Gamma \boldsymbol{N}(\boldsymbol{e}_{i-1}) \cdot \boldsymbol{n} \phi}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n} \phi}. \tag{DMPi}$$

The scheme above is linear and decoupled. The pseudo-algorithm for obtaining the solution pair $\{\boldsymbol{e}_i, h_i\}$ at each time step t_i reads as

Algorithm 5 Implicit Euler

Require: $m, \sigma, \boldsymbol{f}, \delta\boldsymbol{e}_0 = \boldsymbol{E}_0, \delta\boldsymbol{e} = \boldsymbol{W}_0, n \in \mathbb{N}$

for $i = 1, i \leq n$ **do**
  $h_i \leftarrow$ Solve: $(DMPi)$
  $\boldsymbol{e}_i \leftarrow$ Solve: (DPi)
  $i \leftarrow i + 1$
return $\{h_1, \boldsymbol{e}_1\}, \ldots, \{h_n, \boldsymbol{e}_n\}$

We now proceed with a lemma that guarantees the existence of a unique solution pair $\{\boldsymbol{e}_i, h_i\}$ at each time step t_i for $i = 1, \ldots, n$.

Lemma 5.1. *Let $\Omega \in C^{1,1}$ or Ω be convex. Moreover, assume that $\boldsymbol{N}$ is globally Lipschitz continuous, $\phi \in C_0^\infty(\bar{\Omega})$ with meas$\{supp(\phi) \cap \Gamma\} > 0$, $\boldsymbol{f} \in \boldsymbol{X}_N, \boldsymbol{E}_0 \in \boldsymbol{X}_N, \nabla \times \nabla \times \boldsymbol{E}_0 \in \mathbf{L}^2(\Omega), \boldsymbol{W}_0 \in \boldsymbol{X}_N, m \in C^2([0,T])$, and $\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi \, d\Gamma \neq 0$ and also $0 < \sigma_* \leq \sigma(t) \leq \sigma^* < \infty$ for any $t \in [0,T]$. Then for any $i = 1, \ldots, n$ there exists a unique pair $\{\boldsymbol{e}_i, h_i\}$ solving* (DPi) *and* (DMPi). *Furthermore, $h_i \in \mathbb{R}, \boldsymbol{e}_i \in \boldsymbol{X}_N, \nabla \times \nabla \times \boldsymbol{e}_i \in \mathbf{L}^2(\Omega)$ and $\nabla \times \nabla \times \boldsymbol{e}_i \times \boldsymbol{n} = \mathbf{0}$ on Γ.*

Proof. For a given $\nabla \times \nabla \times \boldsymbol{e}_{i-1} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{e}_{i-1} \in \boldsymbol{X}_N$, we compute h_i from (DMPi). We also see that

$$\begin{aligned}
|h_i|^2 &\leq C\left(1 + \|\nabla\phi\|^2_{\mathbf{C}(\overline{\Omega})} \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\phi\|^2_{\mathbf{C}(\overline{\Omega})} \|\boldsymbol{N}(\boldsymbol{e}_{i-1})\|^2_{\mathbf{L}^2(\Gamma)}\right) \\
&\leq C\left(1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\mathbf{L}^2(\Gamma)}\right) \\
&\overset{\mathbf{H}^1(\Omega)\subset\mathbf{L}^2(\Gamma)}{\leq} C\left(1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\mathbf{H}^1(\Omega)}\right) \\
&\overset{\boldsymbol{X}_N\subset\mathbf{H}^1(\Omega)}{\leq} C\left(1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N}\right) \leq C_i.
\end{aligned}$$

Now, assume that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{i-1} \in \boldsymbol{X}_N$ and let us take a look at (DPi)

$$\begin{aligned}
\boldsymbol{e}_i\left(\sigma_0 + \frac{1}{\tau^2}\right) + \nabla \times \nabla \times \boldsymbol{e}_i = \frac{\delta\boldsymbol{e}_{i-1}}{\tau} &+ \boldsymbol{e}_{i-1}\left(\sigma_0 + \frac{1}{\tau^2}\right) + \boldsymbol{N}(\boldsymbol{e}_{i-1}) \\
&+ h_i\boldsymbol{f} - \sum_{k=0}^{i-1} \sigma_{i-k}\delta\boldsymbol{e}_k\tau - \sigma_i\boldsymbol{E}_0 \in \mathbf{L}^2(\Omega).
\end{aligned}$$

Thus, applying the Lax-Milgram Lemma 1.1, we obtain a unique $\boldsymbol{e}_i \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$. After applying the divergence operator to the equation above, we also see that

$$\begin{aligned}\nabla\cdot\boldsymbol{e}_i &= \nabla\cdot\boldsymbol{e}_{i-1} + \frac{\tau}{1+\sigma_0\tau^2}\nabla\cdot\delta\boldsymbol{e}_{i-1}\\ &\quad + \frac{\tau^2}{1+\sigma_0\tau^2}\left(\nabla\cdot\boldsymbol{N}(\boldsymbol{e}_{i-1}) + h_i\nabla\cdot\boldsymbol{f} - \sum_{k=0}^{i-1}\sigma_{i-1}\nabla\cdot\delta\boldsymbol{e}_k\tau - \sigma_i\nabla\cdot\boldsymbol{E}_0\right).\end{aligned}$$

Since $\left|\frac{\tau}{1+\sigma_0\tau^2}\right| \le C$ and $\left|\frac{\tau^2}{1+\sigma_0\tau^2}\right| \le C$, we obtain the following estimate for $\nabla\cdot\boldsymbol{e}_i$

$$\begin{aligned}\|\nabla\cdot\boldsymbol{e}_i\| &\le \|\nabla\cdot\boldsymbol{e}_{i-1}\| + C\left(1 + \|\nabla\cdot\delta\boldsymbol{e}_{i-1}\| + \|\boldsymbol{e}_{i-1}\|_{\mathbf{H}^1(\Omega)}\right)\\ &\quad + C\left(C_i\|\nabla\cdot\boldsymbol{f}\| + \sum_{k=0}^{i-1}\|\nabla\cdot\delta\boldsymbol{e}_k\|\,\tau + \|\nabla\cdot\boldsymbol{E}_0\|\right)\\ &\overset{\boldsymbol{X}_N\subset\mathbf{H}^1(\Omega)}{\le} C_i(1+\|\boldsymbol{e}_{i-1}\|_{\boldsymbol{X}_N}) \le C_i.\end{aligned}$$

Therefore, $\boldsymbol{e}_i \in \boldsymbol{X}_N$. Furthermore, we see that

$$\begin{aligned}\nabla\times\nabla\times\boldsymbol{e}_i &= \boldsymbol{N}(\boldsymbol{e}_{i-1}) + h_i\boldsymbol{f} - \sigma_i\boldsymbol{E}_0 - \delta^2\boldsymbol{e}_i - (\sigma*\delta\boldsymbol{e})_i \in \mathbf{L}^2(\Omega),\\ \nabla\times\nabla\times\boldsymbol{e}_i\times\boldsymbol{n} &= \boldsymbol{N}(\boldsymbol{e}_{i-1})\times\boldsymbol{n} + h_i\boldsymbol{f}\times\boldsymbol{n} - \sigma_i\boldsymbol{E}_0\times\boldsymbol{n}\\ &\quad - \delta^2\boldsymbol{e}_i\times\boldsymbol{n} - (\sigma*(\delta\boldsymbol{e}\times\boldsymbol{n}))_i\\ &= \mathbf{0} \text{ on } \Gamma\end{aligned}$$

which concludes our proof. □

5.3.1 A priori energy estimates

Several energy estimates for the functions $\boldsymbol{e}_i$ and h_i are provided in the following lemmas.

Lemma 5.2. *Let the assumptions of Lemma 5.1 be fulfilled. Moreover, assume that $\tau \le \tau^* < +\infty$. Then there exists a positive constant C such that*

$$\max_{1\le j\le n}\|\delta\boldsymbol{e}_j\|^2 + \sum_{i=1}^n\|\delta\boldsymbol{e}_i - \delta\boldsymbol{e}_{i-1}\|^2 \le C\left(1+\sum_{i=1}^n h_i^2\tau\right),$$

$$\max_{1\le j\le n}\|\nabla\times\boldsymbol{e}_j\|^2 + \sum_{i=1}^n\|\nabla\times\boldsymbol{e}_i - \nabla\times\boldsymbol{e}_{i-1}\|^2 \le C\left(1+\sum_{i=1}^n h_i^2\tau\right).$$

Proof. We multiply (scalar multiplication) (DPi) by $\delta \boldsymbol{e}_i \tau$, integrate over Ω, use the Green theorem and sum up for $i = 1, \ldots, j$ to get

$$\sum_{i=1}^{j} \left(\delta^2 \boldsymbol{e}_i, \delta \boldsymbol{e}_i\right) \tau + \sum_{i=1}^{j} \left((\sigma * \delta \boldsymbol{e})_i, \delta \boldsymbol{e}_i\right) \tau + \sum_{i=1}^{j} \left(\nabla \times \boldsymbol{e}_i, \delta \nabla \times \boldsymbol{e}_i\right) \tau$$
$$= \sum_{i=1}^{j} \left(\boldsymbol{N}(\boldsymbol{e}_{i-1}), \delta \boldsymbol{e}_i\right) \tau + \sum_{i=1}^{j} \left(h_i \boldsymbol{f}, \delta \boldsymbol{e}_i\right) \tau - \sum_{i=1}^{j} \sigma_i \left(\boldsymbol{E}_0, \delta \boldsymbol{e}_i\right) \tau.$$

The convolution term on the l.h.s. is bounded in the following way

$$\left| \sum_{i=1}^{j} \left((\sigma * \delta \boldsymbol{e})_i, \delta \boldsymbol{e}_i\right) \tau \right| \leq C + C \sum_{i=1}^{j} \left\| \delta \boldsymbol{e}_i \right\|^2 \tau.$$

Now, using Abel's summation formula, we rewrite the terms on the l.h.s. as follows

$$\sum_{i=1}^{j} \left(\delta^2 \boldsymbol{e}_i, \delta \boldsymbol{e}_i\right) \tau = \sum_{i=1}^{j} \left(\delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1}, \delta \boldsymbol{e}_i\right)$$
$$= \frac{\left\| \delta \boldsymbol{e}_j \right\|^2}{2} - \frac{\left\| \boldsymbol{W}_0 \right\|^2}{2} + \frac{1}{2} \sum_{i=1}^{j} \left\| \delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1} \right\|^2$$

and

$$\sum_{i=1}^{j} \left(\nabla \times \boldsymbol{e}_i, \delta \nabla \times \boldsymbol{e}_i\right) \tau = \frac{\left\| \nabla \times \boldsymbol{e}_j \right\|^2}{2} - \frac{\left\| \nabla \times \boldsymbol{E}_0 \right\|^2}{2}$$
$$+ \frac{1}{2} \sum_{i=1}^{j} \left\| \nabla \times \boldsymbol{e}_i - \nabla \times \boldsymbol{e}_{i-1} \right\|^2.$$

The first term on the r.h.s. is handled via Lipschitz continuity of $\boldsymbol{N}$, Young's and Cauchy's inequalities and the identity $\boldsymbol{e}_i = \boldsymbol{E}_0 + \sum_{k=1}^{i} \delta \boldsymbol{e}_k \tau$

$$\sum_{i=1}^{j} \left(\boldsymbol{N}(\boldsymbol{e}_{i-1}), \delta \boldsymbol{e}_i\right) \tau \leq C \sum_{i=1}^{j} (1 + \|\boldsymbol{e}_{i-1}\|) \|\delta \boldsymbol{e}_i\| \tau$$
$$= C \sum_{i=1}^{j} \left[1 + \left\| \boldsymbol{E}_0 + \sum_{k=1}^{i-1} \delta \boldsymbol{e}_k \tau \right\| \right] \|\delta \boldsymbol{e}_i\| \tau$$
$$\leq C \sum_{i=1}^{j} \left[1 + \|\boldsymbol{E}_0\|\right] \|\delta \boldsymbol{e}_i\| \tau + C \sum_{i=1}^{j} \sum_{k=1}^{i} \|\delta \boldsymbol{e}_k\| \|\delta \boldsymbol{e}_i\| \tau^2$$

$$\leq C + C \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i\|^2 \tau.$$

The rest of the r.h.s. is estimated using Young's and Cauchy's inequalities once again

$$\begin{aligned}
&\left| \sum_{i=1}^{j} (h_i \boldsymbol{f}, \delta \boldsymbol{e}_i) \tau - \sum_{i=1}^{j} \sigma_i (\boldsymbol{E}_0, \delta \boldsymbol{e}_i) \tau \right| \\
&\leq C \sum_{i=1}^{j} h_i^2 \|\boldsymbol{f}\|^2 \tau + \sigma^* C \|\boldsymbol{E}_0\|^2 + C \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i\|^2 \tau \\
&\leq C \left(1 + \sum_{i=1}^{j} h_i^2 \right) + C \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i\|^2 \tau.
\end{aligned}$$

Collecting all partial results above, we obtain

$$\begin{aligned}
&\|\delta \boldsymbol{e}_j\|^2 + \|\nabla \times \boldsymbol{e}_j\|^2 + \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1}\|^2 + \sum_{i=1}^{j} \|\nabla \times \boldsymbol{e}_i - \nabla \times \boldsymbol{e}_{i-1}\|^2 \\
&\leq C \left(1 + \sum_{i=1}^{j} h_i^2 \tau \right) + C \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i\|^2 \tau.
\end{aligned}$$

The rest of the proof follows from the application of the discrete version of Grönwall's Lemma 1.2. □

Remark 5.1. *Identity* $\boldsymbol{e}_j = \boldsymbol{E}_0 + \sum_{i=1}^{j} \delta \boldsymbol{e}_i \tau$ *and Lemma 5.2 above also imply*

$$\max_{1 \leq j \leq n} \|\boldsymbol{e}_j\|^2 \leq C \left(1 + \sum_{i=1}^{n} h_i^2 \tau \right).$$

Lemma 5.3. *Let the assumptions of Lemma 5.2 be fulfilled. Then there exists a positive constant C such that*

$$\max_{1 \leq jn} \|\nabla \cdot \delta \boldsymbol{e}_j\|^2 + \sum_{i=1}^{n} \|\nabla \cdot \delta \boldsymbol{e}_i - \nabla \cdot \delta \boldsymbol{e}_{i-1}\|^2 \leq C \left(1 + \sum_{i=1}^{n} h_i^2 \tau \right).$$

Proof. First, we apply a divergence operator on (DPi), then multiply by $\nabla \cdot \delta \boldsymbol{e}_i \tau$, integrate over Ω and sum up for $i = 1, \dots, j$. We obtain the following

$$\sum_{i=1}^{j} (\nabla \cdot \delta^2 \boldsymbol{e}_i, \nabla \cdot \delta \boldsymbol{e}_i) \tau = \sum_{i=1}^{j} (\nabla \cdot \boldsymbol{N}(\boldsymbol{e}_{i-1}), \nabla \cdot \delta \boldsymbol{e}_i) \tau + \sum_{i=1}^{j} h_i (\nabla \cdot \boldsymbol{f}, \nabla \cdot \delta \boldsymbol{e}_i) \tau$$

$$-\sum_{i=1}^{j}\left(\sigma_i \nabla\cdot \boldsymbol{E}_0, \nabla\cdot\delta\boldsymbol{e}_i\right)\tau-\sum_{i=1}^{j}\left((\sigma*\nabla\cdot\delta\boldsymbol{e})_i, \nabla\cdot\delta\boldsymbol{e}_i\right)\tau.$$

The l.h.s. can be rewritten using Abel's summation formula

$$\sum_{i=1}^{j}\left(\nabla\cdot\delta^2\boldsymbol{e}_i, \nabla\cdot\delta\boldsymbol{e}_i\right)\tau=\frac{\left\|\nabla\cdot\delta\boldsymbol{e}_j\right\|^2}{2}-\frac{\left\|\nabla\cdot\boldsymbol{W}_0\right\|^2}{2}+\frac{1}{2}\sum_{i=1}^{j}\left\|\nabla\cdot\delta\boldsymbol{e}_i-\nabla\cdot\delta\boldsymbol{e}_{i-1}\right\|^2.$$

The first term on the r.h.s. is estimated as

$$\begin{aligned}
&\sum_{i=1}^{j}\left(\nabla\cdot\boldsymbol{N}(\boldsymbol{e}_{i-1}), \nabla\cdot\delta\boldsymbol{e}_i\right)\tau\le\sum_{i=1}^{j}\left\|\nabla\cdot\boldsymbol{N}(\boldsymbol{e}_{i-1})\right\|\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|\tau\\
&\le C\sum_{i=1}^{j}\left\|\boldsymbol{e}_{i-1}\right\|_{\mathbf{H}^1(\Omega)}\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|\tau\le C\sum_{i=1}^{j}\left\|\boldsymbol{e}_{i-1}\right\|_{\boldsymbol{X}_N}\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|\tau\\
&=C\sum_{i=1}^{j}\left(\left\|\boldsymbol{e}_{i-1}\right\|+\left\|\nabla\times\boldsymbol{e}_{i-1}\right\|+\left\|\nabla\cdot\boldsymbol{e}_{i-1}\right\|\right)\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|\tau\\
&\overset{\text{Lemma 5.2}}{\le} C\left(1+\sum_{i=1}^{j}h_i^2\tau\right)+C\sum_{i=1}^{j}\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|^2\tau\\
&\qquad+C\sum_{i=1}^{j}\left\|\nabla\cdot\boldsymbol{e}_i\right\|\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|\tau\\
&\le C\left(1+\sum_{i=1}^{j}h_i^2\tau+\sum_{i=1}^{j}\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|^2\tau\right)\\
&\quad+C\sum_{i=1}^{j}\left\|\nabla\cdot\boldsymbol{E}_0+\sum_{k=1}^{i}\nabla\cdot\delta\boldsymbol{e}_k\tau\right\|\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|\tau\\
&\le C\left(1+\sum_{i=1}^{j}h_i^2\tau+\sum_{i=1}^{j}\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|^2\tau\right).
\end{aligned}$$

The other terms on the r.h.s. are estimated using Cauchy's and Young's inequalities. Therefore, gathering all partial results, we arrive at

$$\left\|\nabla\cdot\delta\boldsymbol{e}_j\right\|^2+\sum_{i=1}^{j}\left\|\nabla\cdot\delta\boldsymbol{e}_i-\nabla\cdot\delta\boldsymbol{e}_{i-1}\right\|^2\le C\left(1+\sum_{i=1}^{j}h_i^2\tau+\sum_{i=1}^{j}\left\|\nabla\cdot\delta\boldsymbol{e}_i\right\|^2\tau\right).$$

An application of discrete version of the Grönwall Lemma 1.2 and taking maximum over $1 \le j \le n$ yields to the desired result. $\square$

Remark 5.2. *Using identity* $\nabla \cdot \boldsymbol{e}_i = \nabla \cdot \boldsymbol{E}_0 + \sum_{j=1}^{i} \nabla \cdot \delta \boldsymbol{e}_j \tau$ *and Lemma 5.3 above we obtain an estimate for* $\nabla \cdot \boldsymbol{e}_i$*, i.e.*

$$\max_{1 \le j \le n} \|\nabla \cdot \boldsymbol{e}_j\|^2 \le C \left(1 + \sum_{i=1}^{n} h_i^2 \tau\right).$$

Lemma 5.4. *Suppose that all assumptions of Lemma 5.2 are met. Then there exists a positive constant* C *such that*

$$\max_{1 \le j \le n} \|\nabla \times \delta \boldsymbol{e}_j\|^2 + \sum_{i=1}^{n} \|\nabla \times \delta \boldsymbol{e}_i - \nabla \times \delta \boldsymbol{e}_{i-1}\|^2 \le C \left(1 + \sum_{i=1}^{n} h_i^2 \tau\right),$$

$$\max_{1 \le j \le n} \|\nabla \times \nabla \times \boldsymbol{e}_j\|^2 + \sum_{i=1}^{n} \|\nabla \times \nabla \times \boldsymbol{e}_i - \nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 \le C \left(1 + \sum_{i=1}^{n} h_i^2 \tau\right).$$

Proof. We start by applying the curl operator to (DPi), then we multiply it with $\nabla \times \delta \boldsymbol{e}_i \tau$, integrate over Ω, implement the Green Theorem (Lemma 5.1 guarantees $\nabla \times \nabla \times \boldsymbol{e}_i \times \boldsymbol{n} = \boldsymbol{0}$ on Γ) and sum up for $i = 1, \dots, j$ to obtain that

$$\begin{aligned}
&\sum_{i=1}^{j} \left(\nabla \times \delta^2 \boldsymbol{e}_i, \nabla \times \delta \boldsymbol{e}_i\right) \tau + \sum_{i=1}^{j} \left(\nabla \times \nabla \times \boldsymbol{e}_i, \nabla \times \nabla \times \delta \boldsymbol{e}_i\right) \tau \\
&\quad + \sum_{i=1}^{j} \left((\sigma * \nabla \times \delta \boldsymbol{e})_i, \nabla \times \delta \boldsymbol{e}_i\right) \tau \\
&= \sum_{i=1}^{j} \left(\nabla \times \boldsymbol{N}(\boldsymbol{e}_{i-1}), \nabla \times \delta \boldsymbol{e}_i\right) \tau + \sum_{i=1}^{j} h_i \left(\nabla \times \boldsymbol{f}, \nabla \times \delta \boldsymbol{e}_i\right) \tau \\
&\quad - \sum_{i=1}^{j} \left(\sigma_i \nabla \times \boldsymbol{E}_0, \nabla \times \delta \boldsymbol{e}_i\right) \tau.
\end{aligned}$$

To bound the first two terms on the l.h.s., we employ Abel's summation rule once again, i.e.

$$\begin{aligned}
\sum_{i=1}^{j} \left(\nabla \times \delta^2 \boldsymbol{e}_i, \nabla \times \delta \boldsymbol{e}_i\right) \tau = &\frac{\|\nabla \times \delta \boldsymbol{e}_j\|^2}{2} - \frac{\|\nabla \times \boldsymbol{W}_0\|^2}{2} \\
&+ \frac{1}{2} \sum_{i=1}^{j} \|\nabla \times \delta \boldsymbol{e}_i - \nabla \times \delta \boldsymbol{e}_{i-1}\|^2
\end{aligned}$$

and

$$\sum_{i=1}^{j} (\nabla \times \nabla \times \boldsymbol{e}_i, \nabla \times \nabla \times \delta \boldsymbol{e}_i)\,\tau = \frac{\left\|\nabla \times \nabla \times \boldsymbol{e}_j\right\|^2}{2} - \frac{\left\|\nabla \times \nabla \times \boldsymbol{E}_0\right\|^2}{2} + \frac{1}{2}\sum_{i=1}^{j} \left\|\nabla \times \nabla \times \boldsymbol{e}_i - \nabla \times \nabla \times \boldsymbol{e}_{i-1}\right\|^2.$$

The last term on the l.h.s. is bounded as follows

$$\left|\sum_{i=1}^{j} ((\sigma * \nabla \times \delta \boldsymbol{e})_i, \nabla \times \delta \boldsymbol{e}_i)\,\tau\right| \le C\left(1 + \sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau\right).$$

Afterwards, we continue with estimates for the r.h.s., starting with the first term, we obtain

$$\begin{aligned}
&\sum_{i=1}^{j} (\nabla \times \boldsymbol{N}(\boldsymbol{e}_{i-1}), \nabla \times \delta \boldsymbol{e}_i)\,\tau \\
&\le C\left(\|\boldsymbol{E}_0\|^2_{\mathbf{H}^1(\Omega)} + \sum_{i=1}^{j} \|\boldsymbol{e}_i\|^2_{\mathbf{H}^1(\Omega)}\,\tau\right) + C\sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau \\
&\le C\left(\|\boldsymbol{E}_0\|^2_{\boldsymbol{X}_N} + \sum_{i=1}^{j} \|\boldsymbol{e}_i\|^2_{\boldsymbol{X}_N}\,\tau\right) + C\sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau \\
&\overset{Lemma\ 5.2,5.3}{\le} C\left(1 + \sum_{i=1}^{j} h_i^2 \tau\right) + C\sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau.
\end{aligned}$$

The rest of the r.h.s. terms can be bounded via Cauchy's and Young's inequalities

$$\begin{aligned}
\sum_{i=1}^{j} h_i\,(\nabla \times \boldsymbol{f}, \nabla \times \delta \boldsymbol{e}_i)\,\tau &\le C\sum_{i=1}^{j} h_i^2\,\|\boldsymbol{f}\|^2\,\tau + C\sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau \\
&\le C\sum_{i=1}^{j} h_i^2 \tau + C\sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau
\end{aligned}$$

and

$$\begin{aligned}
\left|\sum_{i=1}^{j} (\sigma_i \nabla \times \boldsymbol{E}_0, \nabla \times \delta \boldsymbol{e}_i)\,\tau\right| &\le CT\sigma^* \left\|\nabla \times \boldsymbol{E}_0\right\|^2 + C\sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau \\
&\le C\left(1 + \sum_{i=1}^{j} \left\|\nabla \times \delta \boldsymbol{e}_i\right\|^2 \tau\right).
\end{aligned}$$

Now, we congregate all partial results above to see that

$$\begin{aligned}
&\|\nabla \times \delta \boldsymbol{e}_j\|^2 + \sum_{i=1}^{j} \|\nabla \times \delta \boldsymbol{e}_i - \nabla \times \delta \boldsymbol{e}_{i-1}\|^2 + \|\nabla \times \nabla \times \boldsymbol{e}_j\|^2 \\
&\quad + \sum_{i=1}^{j} \|\nabla \times \nabla \times \boldsymbol{e}_i - \nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 \\
&\leq C \left(1 + \sum_{i=1}^{j} h_i^2 \tau\right) + C \sum_{i=1}^{j} \|\nabla \times \delta \boldsymbol{e}_i\|^2 \tau.
\end{aligned}$$

Using Grönwall's argument and taking maximum over $1 \leq j \leq n$, we conclude the proof. □

Lemma 5.5. *Let all assumptions of Lemma 5.2 be fulfilled. Then there exists a positive constant C such that*

$$\begin{aligned}
(i) &\quad \max_{1\leq j\leq n} \|\boldsymbol{e}_j\|^2_{\boldsymbol{X}_N} + \max_{1\leq j\leq n} \|\nabla \times \nabla \times \boldsymbol{e}_j\|^2 \leq C \\
(ii) &\quad \max_{1\leq j\leq n} |h_j|^2 \leq C \\
(iii) &\quad \max_{1\leq j\leq n} \left\|\delta^2 \boldsymbol{e}_j\right\|_{(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*} \leq C.
\end{aligned}$$

Proof. (i) According to the proof of Lemma 5.1 and (DMPi), we have

$$\begin{aligned}
&h_i^2 \leq C(1 + \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 + \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N}) \\
&\quad \Longrightarrow \sum_{i=1}^{j} h_i^2 \tau \leq C \left(1 + \sum_{i=1}^{j} \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 \tau + \sum_{i=1}^{j} \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N} \tau\right).
\end{aligned}$$

Lemmas 5.2, 5.3 and 5.4 together with the bound above yield

$$\begin{aligned}
\|\boldsymbol{e}_j\|^2_{\boldsymbol{X}_N} + \|\nabla \times \nabla \times \boldsymbol{e}_j\|^2 &\leq C \left(1 + \sum_{i=1}^{j} h_i^2 \tau\right) \\
&\leq C \left(1 + \sum_{i=1}^{j} \|\boldsymbol{e}_{i-1}\|^2_{\boldsymbol{X}_N} \tau + \sum_{i=1}^{j} \|\nabla \times \nabla \times \boldsymbol{e}_{i-1}\|^2 \tau\right).
\end{aligned}$$

Thus, employing Grönwall's lemma again and taking maximum over $1 \leq j \leq n$, the first statement of Lemma 5.5 is proven.

(ii) The second statement is directly implied by (i).

(*iii*) We take $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and make a scalar multiplication with (DPi). Then, we integrate in Ω and use the Green theorem to observe that

$$\begin{aligned}\left(\delta^2 \boldsymbol{e}_i, \boldsymbol{\varphi}\right) &= (\boldsymbol{N}(\boldsymbol{e}_{i-1}), \boldsymbol{\varphi}) + h_i\,(\boldsymbol{f}, \boldsymbol{\varphi}) \\ &\quad - (\sigma_i \boldsymbol{E}_0, \boldsymbol{\varphi}) - ((\sigma * \delta \boldsymbol{e})_i, \boldsymbol{\varphi}) - (\nabla \times \boldsymbol{e}_i, \nabla \times \boldsymbol{\varphi})\,.\end{aligned}$$

Using statements $(i), (ii)$, and Lemma 5.2, we conclude

$$\left|\left(\delta^2 \boldsymbol{e}_i, \boldsymbol{\varphi}\right)\right| \le C\,\|\boldsymbol{\varphi}\| + C\,\|\nabla \times \boldsymbol{\varphi}\| \le C\,\|\boldsymbol{\varphi}\|_{\mathbf{H}_0(\mathbf{curl}\,;\Omega)}\,.$$

Therefore,

$$\left\|\delta^2 \boldsymbol{e}_j\right\|_{(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*} \le C.$$

□

5.4 The existence of a global solution

We construct piece-wise constant and piece-wise linear in time functions and show the convergence of subsequences of these functions towards a weak solution $\{\boldsymbol{E}, h\}$ which satisfies equation (5.9) and (5.7). They are created in the same fashion as in the previous chapters, i.e.

$$\begin{array}{ll}
\overline{\boldsymbol{E}_n}(t) = \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\boldsymbol{E}_n(t) = \boldsymbol{e}_{i-1} + (t - t_{i-1})\delta \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\overline{\boldsymbol{E}_n}(0) = \boldsymbol{E}_n(0) = \boldsymbol{E}_0, & \\
\overline{\boldsymbol{W}_n}(t) = \delta \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\boldsymbol{W}_n(t) = \delta \boldsymbol{e}_{i-1} + (t - t_{i-1})\delta \boldsymbol{e}_i & t \in (t_{i-1}, t], \\
\overline{\boldsymbol{W}_n}(0) = \boldsymbol{W}_n(0) = \boldsymbol{W}_0, & \\
\overline{h_n}(t) = h_i & t \in (t_{i-1}, t], \\
\overline{m_n}(t) = m_i,\ \overline{m_n'}(t) = m_i',\ \overline{m_n''}(t) = m_i'' & t \in (t_{i-1}, t], \\
\overline{\sigma_n}(t) = \sigma_i & t \in (t_{i-1}, t].
\end{array}$$

Now, we rewrite (DPi) and (DMPi) in a continuous form (for $t \in (t_{i-1}, t_i]$)

$$\begin{aligned}\partial_t \boldsymbol{W}_n(t) + (\overline{\sigma_n} * \overline{\boldsymbol{W}_n})(t_i) + \nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t) &= \boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau)) + \overline{h_n}(t)\boldsymbol{f} - \overline{\sigma_n}(t)\boldsymbol{E}_0 \\ \overline{\boldsymbol{E}_n}(t) \times \boldsymbol{n} &= 0 \\ \boldsymbol{E}_n(0) &= \boldsymbol{E}_0 \\ \boldsymbol{W}_n(0) &= \boldsymbol{W}_0,\end{aligned} \tag{DP}$$

$$\overline{h_n}(t) = \frac{1}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n}\phi} \left\{\overline{m_n''}(t) + (\overline{\sigma_n} * \overline{m_n'})(t_i) + \overline{\sigma_n} m(0)\right.$$

$$+\int_{\Omega} \nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t-\tau) \cdot \nabla\phi - \int_{\Gamma} \boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau)) \cdot \boldsymbol{n}\phi \Bigg\}. \tag{DMP}$$

Then the variational formulation of (DP) has the following structure for any $t \in (t_{i-1}, t_i]$ and $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$

$$\begin{aligned}&(\partial_t \boldsymbol{W}_n(t), \boldsymbol{\varphi}) + \big((\overline{\sigma_n} * \overline{\boldsymbol{W}_n})(t_i) + \overline{\sigma_n}(t)\boldsymbol{E}_0, \boldsymbol{\varphi}\big) + \big(\nabla \times \overline{\boldsymbol{E}_n}(t), \nabla \times \boldsymbol{\varphi}\big)\\ &= \big(\boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau)), \boldsymbol{\varphi}\big) + \overline{h_n}(t)\,(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{\varphi})\,.\end{aligned} \tag{5.10}$$

Theorem 5.1. *Let $\Omega \in C^{1,1}$ or Ω be convex. Assume that $\boldsymbol{N}$ and σ are global Lipschitz continuous functions and $\boldsymbol{f} \in \boldsymbol{X}_N, \boldsymbol{E}_0 \in \boldsymbol{X}_N, \boldsymbol{W}_0 \in \boldsymbol{X}_N, \nabla \times \nabla \times \boldsymbol{E}_0 \in \mathbf{L}^2(\Omega), m \in C^2([0,T]), \int_\Gamma \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{n}\phi \, d\Gamma \neq 0, \phi \in C^\infty(\bar{\Omega})$ with $meas\{supp(\phi)\cap\Gamma\} > 0$ and $0 < \sigma_* \leq \sigma \leq \sigma^* < \infty$. Then there exists a weak solution pair $\{\boldsymbol{E}, h\}$ that satisfies equation (5.9) and (5.7). Furthermore, we have $h \in L^2((0,T))$, $\boldsymbol{E} \in C([0,T];\mathbf{L}^2(\Omega)) \cap L^\infty((0,T);\boldsymbol{X}_N)$ with $\partial_t \boldsymbol{E} \in L^2((0,T);\mathbf{L}^2(\Omega)) \cap C([0,T];\boldsymbol{X}_N^*)$, $\partial_t^2 \boldsymbol{E} \in L^2((0,T);(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$ and $\nabla \times \nabla \times \boldsymbol{E} \in L^\infty((0,T);\mathbf{L}^2(\Omega))$.*

Proof. The Lipschitz continuity of σ implies

$$\overline{\sigma_n} \to \sigma \ \text{ in } \ L^2([0,T]). \tag{5.11}$$

Lemma 5.5 says that $\int_0^T \left|\overline{h_n}(s)\right|^2 \, \mathrm{d}s \leq C$. From reflexivity of the space $L^2((0,T))$, we have a subsequence of $\overline{h_n}$ which converges weakly to h in this space, i.e.

$$\overline{h_n}(t) \rightharpoonup h(t) \quad \text{in} \quad L^2((0,T)). \tag{5.12}$$

From Theorem 1.8, we have the following compact embedding for any Lipschitz domain Ω

$$\boldsymbol{X}_N \Subset \mathbf{L}^2(\Omega).$$

Deducing from Lemmas 5.2, 5.3, 5.4, and 5.5, we obtain

$$\int_0^T \|\partial_t \boldsymbol{E}_n(t)\|^2 \, \mathrm{d}t \leq C, \qquad \|\boldsymbol{E}_n(t)\|_{\boldsymbol{X}_N} \leq C \ \ \forall t \in [0,T].$$

Using Lemma 2.5 with $V = \boldsymbol{X}_N$ and $Y = \mathbf{L}^2(\Omega)$, we obtain the existence of a vector field $\boldsymbol{E}$ from $C([0,T];\mathbf{L}^2(\Omega)) \cap L^\infty((0,T);\boldsymbol{X}_N)$ with $\partial_t \boldsymbol{E} \in L^2((0,T);\mathbf{L}^2(\Omega))$ and a subsequence of $\boldsymbol{E}_n$ for which the following convergence results hold

$$\begin{array}{ll}\boldsymbol{E}_n \to \boldsymbol{E} & \text{in } \ C([0,T];\mathbf{L}^2(\Omega))\\ \overline{\boldsymbol{E}_n} \to \boldsymbol{E} & \text{in } \ L^2((0,T);\mathbf{L}^2(\Omega))\\ \boldsymbol{E}_n(t) \rightharpoonup \boldsymbol{E}(t) & \text{in } \ \boldsymbol{X}_N, \ \ \forall t \in [0,T]\\ \overline{\boldsymbol{E}_n}(t) \rightharpoonup \boldsymbol{E}(t) & \text{in } \ \boldsymbol{X}_N, \ \ \forall t \in [0,T]\\ \overline{\boldsymbol{W}_n} = \partial_t \boldsymbol{E}_n \rightharpoonup \partial_t \boldsymbol{E} & \text{in } \ L^2((0,T);\mathbf{L}^2(\Omega)).\end{array} \tag{5.13}$$

Lemma 5.5 together with $\boldsymbol{X}_N \hookrightarrow \mathbf{H}_0(\mathbf{curl}\,;\Omega) \hookrightarrow (\mathbf{H}_0(\mathbf{curl}\,;\Omega))^* \hookrightarrow \boldsymbol{X}_N^*$ implies

$$\|\partial_t \boldsymbol{W}_n\|_{\boldsymbol{X}_N^*} \le C\,\|\partial_t \boldsymbol{W}_n\|_{(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*} \le C.$$

Now, thanks to the embedding $\boldsymbol{X}_N \Subset \mathbf{L}^2(\Omega) \implies \mathbf{L}^2(\Omega) \Subset \boldsymbol{X}_N^*$, and Lemma 5.2, we also have for $t \in (t_{i-1}, t_i]$

$$\begin{aligned}\|\boldsymbol{W}_n\|_{\boldsymbol{X}_N^*} \le C\,\|\boldsymbol{W}_n\| &= C\,\left\|\delta \boldsymbol{e}_{i-1} + (t - t_{i-1})\delta^2 \boldsymbol{e}_i\right\| \\ &\le C(\|\delta \boldsymbol{e}_i\| + \|\delta \boldsymbol{e}_{i-1}\|) \le C.\end{aligned}$$

Hence, the sequence $\boldsymbol{W}_n$ is equibounded in $C([0,T];\boldsymbol{X}_N^*)$. Moreover, for any $t, s \in [0,T]$ with $t \neq s$ and any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$, we have

$$\begin{aligned}|(\boldsymbol{W}_n(t) - \boldsymbol{W}_n(s), \boldsymbol{\varphi})| &= \left|\int_s^t (\partial_t \boldsymbol{W}_n(z), \boldsymbol{\varphi})\ \mathrm{d}z\right| \\ &\le |t-s|\,\|\partial_t \boldsymbol{W}_n\|_{\boldsymbol{X}_N^*}\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} \\ &\le C\,|t-s|\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}.\end{aligned}$$

Thus, the sequence $\boldsymbol{W}_n$ is also equicontinuous in $C([0,T];\boldsymbol{X}_N^*)$ and so applying a modification of Arzelà-Ascoli Theorem 1.5 (*ii*), we conclude that the sequence is compact, i.e.

$$\boldsymbol{W}_n \to \boldsymbol{W} \ \text{ in the space } \ C([0,T];\boldsymbol{X}_N^*). \tag{5.14}$$

Now, for any $t \in (t_{i-1}, t_i]$ and any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$, we have

$$\begin{aligned}\left|\left(\boldsymbol{W}_n - \overline{\boldsymbol{W}_n}, \boldsymbol{\varphi}\right)\right| &= \left|\int_t^{t_i} (\partial_t \boldsymbol{W}_n(s), \boldsymbol{\varphi})\ \mathrm{d}s\right| \\ &\le \tau\,\|\partial_t \boldsymbol{W}_n\|_{\boldsymbol{X}_N^*}\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} \\ &\le C\tau\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} \\ &\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore,

$$\overline{\boldsymbol{W}_n} \to \boldsymbol{W} \ \text{ in } \ C([0,T];\boldsymbol{X}_N^*). \tag{5.15}$$

Using this and the convergence results of (5.13), we conclude the following for any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$

$$\int_0^T (\partial_t \boldsymbol{E}, \boldsymbol{\varphi})\ \mathrm{d}t = \lim_{n\to\infty} \int_0^T \left(\overline{\boldsymbol{W}_n}(t), \boldsymbol{\varphi}\right)\ \mathrm{d}t = \int_0^T (\boldsymbol{W}, \boldsymbol{\varphi})\ \mathrm{d}t \implies \partial_t \boldsymbol{E} = \boldsymbol{W}.$$

Lemma 5.5 implies $\partial_t \boldsymbol{W}_n \in L^2((0,T);(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$. Since this space is reflexive, there exists a function $\boldsymbol{z}$ from the space $\in L^2((0,T);(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$ such that $\partial_t \boldsymbol{W}_n \rightharpoonup \boldsymbol{z}$ in this space. Using the previous results for the sequence $\boldsymbol{W}_n$, we conclude for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$

$$
\begin{aligned}
\int_0^t \left(\partial_t^2 \boldsymbol{E}(s), \boldsymbol{\varphi}\right)\,\mathrm{d}s &= (\boldsymbol{W}(t) - \boldsymbol{W}(0), \boldsymbol{\varphi}) \\
&= \lim_{n\to\infty} (\boldsymbol{W}_n(t) - \boldsymbol{W}_n(0), \boldsymbol{\varphi}) \\
&= \lim_{n\to\infty} \int_0^t (\partial_t \boldsymbol{W}_n(s), \boldsymbol{\varphi})\,\mathrm{d}s \\
&= \int_0^t (\boldsymbol{z}(s), \boldsymbol{\varphi})\,\mathrm{d}s.
\end{aligned}
$$

Therefore, $\partial_t^2 \boldsymbol{E} = \boldsymbol{z}$, i.e. $\partial_t \boldsymbol{W}_n \rightharpoonup \partial_t^2 \boldsymbol{E}$ in $L^2((0,T);(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*)$. For the convolution term we have the following estimate for any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$ and $t \in (t_{i-1}, t_i]$

$$
\begin{aligned}
&\left|(\overline{\sigma_n} * (\overline{\boldsymbol{W}_n}, \boldsymbol{\varphi}))(t_i) - (\overline{\sigma_n} * (\overline{\boldsymbol{W}_n}, \boldsymbol{\varphi}))(t)\right| \\
&= \left|\int_0^{t_i} \left(\overline{\boldsymbol{W}_n}(t_i - s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\,\mathrm{d}s - \int_0^t \left(\overline{\boldsymbol{W}_n}(t-s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\,\mathrm{d}s\right| \\
&= \left|\int_0^t \left(\overline{\boldsymbol{W}_n}(t_i - s) - \overline{\boldsymbol{W}_n}(t-s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\,\mathrm{d}s - \int_t^{t_i} \left(\overline{\boldsymbol{W}_n}(t_i - s), \boldsymbol{\varphi}\right) \overline{\sigma_n}(s)\,\mathrm{d}s\right| \\
&\le C\tau \left\|\overline{\boldsymbol{W}_n}(t_i)\right\|_{\boldsymbol{X}_N^*} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N} + C\sigma^* \int_0^t \left\|\overline{\boldsymbol{W}_n}(t_i - s) - \overline{\boldsymbol{W}_n}(t-s)\right\|_{\boldsymbol{X}_N^*} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}\,\mathrm{d}s.
\end{aligned}
$$

First term on the r.h.s. is estimated as $C\tau \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}$ since $\left\|\overline{\boldsymbol{W}_n}\right\|_{\boldsymbol{X}_N^*} \le C$. Consider $t \in (t_{i-1}, t_i]$. Then, we bound the second term in the following manner

$$
\begin{aligned}
&\left\|\overline{\boldsymbol{W}_n}(t_i - s) - \overline{\boldsymbol{W}_n}(t-s)\right\|_{\boldsymbol{X}_N^*} \\
&= \left\|\overline{\boldsymbol{W}_n}(t_i - s) - \boldsymbol{W}_n(t-s) + \boldsymbol{W}_n(t-s) - \overline{\boldsymbol{W}_n}(t-s)\right\|_{\boldsymbol{X}_N^*} \\
&\le \left\|\overline{\boldsymbol{W}_n}(t_i - s) - \boldsymbol{W}_n(t-s)\right\|_{\boldsymbol{X}_N^*} + \left\|\boldsymbol{W}_n(t-s) - \overline{\boldsymbol{W}_n}(t-s)\right\|_{\boldsymbol{X}_N^*} \\
&\le C\tau \left\|\partial_t \boldsymbol{W}_n(t_i)\right\|_{\boldsymbol{X}_N^*} \\
&= C\tau \left\|\delta^2 \boldsymbol{e}_i\right\|_{\boldsymbol{X}_N^*} \\
&\le C\tau \left\|\delta^2 \boldsymbol{e}_i\right\|_{(\mathbf{H}_0(\mathbf{curl}\,;\Omega))^*} \\
&\le C\tau.
\end{aligned}
$$

This implies

$$
C\sigma^* \int_0^t \left\|\overline{\boldsymbol{W}_n}(t_i - s) - \overline{\boldsymbol{W}_n}(t-s)\right\|_{\boldsymbol{X}_N^*} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}\,\mathrm{d}s \le C\tau \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N}.
$$

Therefore,

$$\left|(\overline{\sigma_n} * (\overline{\boldsymbol{W}_n}, \boldsymbol{\varphi}))(t_i) - (\overline{\sigma_n} * (\overline{\boldsymbol{W}_n}, \boldsymbol{\varphi}))(t)\right| \le C\tau \left\|\boldsymbol{\varphi}\right\|_{\boldsymbol{X}_N} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Now, using the convergence results of (5.11), (5.15), and the Lebesgue dominated convergence Theorem 1.3, we conclude for any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$ and $t \in (t_{i-1}, t_i]$

$$\lim_{n\to\infty} (\overline{\sigma_n} * (\overline{\boldsymbol{W}_n}, \boldsymbol{\varphi}))(t_i) = \lim_{n\to\infty} (\overline{\sigma_n} * (\overline{\boldsymbol{W}_n}, \boldsymbol{\varphi}))(t) = (\sigma * (\partial_t \boldsymbol{E}, \boldsymbol{\varphi}))(t).$$

Thanks to Lemma 5.2 and the Lipschitz continuity of $\boldsymbol{N}$, we have for any $t \in (t_{i-1}, t_i]$ and $\boldsymbol{\varphi} \in \boldsymbol{X}_N$

$$\begin{aligned}\left|\left(\boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau)) - \boldsymbol{N}(\overline{\boldsymbol{E}_n}(t)), \boldsymbol{\varphi}\right)\right| &\le C \left\|\overline{\boldsymbol{E}_n}(t-\tau) - \overline{\boldsymbol{E}_n}(t)\right\| \left\|\boldsymbol{\varphi}\right\| \\ &= C \left\|\boldsymbol{e}_i - \boldsymbol{e}_{i-1}\right\| \left\|\boldsymbol{\varphi}\right\| \\ &= C \left\|\delta \boldsymbol{e}_i\right\| \left\|\boldsymbol{\varphi}\right\| \tau \le C\tau \left\|\boldsymbol{\varphi}\right\|.\end{aligned}$$

Since $\overline{\boldsymbol{E}_n} \to \boldsymbol{E}$ in $L^2((0,T); \mathbf{L}^2(\Omega))$ also $\boldsymbol{N}(\overline{\boldsymbol{E}_n}(t-\tau)) \to \boldsymbol{N}(\boldsymbol{E})$ in this space. Now, we integrate equation (5.10) in time over $t \in [0, \xi] \subset [0, T]$ and according to the results above, we pass to the limit for $n \to \infty$ and $\boldsymbol{\varphi} \in \boldsymbol{X}_N$ to obtain

$$\begin{aligned}&(\partial_t \boldsymbol{E}(\xi), \boldsymbol{\varphi}) - (\boldsymbol{W}_0, \boldsymbol{\varphi}) + \int_0^\xi ((\sigma * \partial_t \boldsymbol{E})(t) + \sigma(t)\boldsymbol{E}_0, \boldsymbol{\varphi}) \ \mathrm{d}t \\ &\quad + \int_0^\xi (\nabla \times \boldsymbol{E}(t), \nabla \times \boldsymbol{\varphi}) \ \mathrm{d}t \\ &= \int_0^\xi (\boldsymbol{N}(\boldsymbol{E}(t)), \boldsymbol{\varphi}) \ \mathrm{d}t + \int_0^\xi h(t) (\boldsymbol{f}, \boldsymbol{\varphi}) \ \mathrm{d}t.\end{aligned}$$

Then differentiation with respect to the time variable ξ yields

$$\begin{aligned}\left(\partial_t^2 \boldsymbol{E}, \boldsymbol{\varphi}\right) + ((\sigma * \partial_t \boldsymbol{E}) + \sigma \boldsymbol{E}_0, \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\varphi}) &= (\boldsymbol{N}(\boldsymbol{E}), \boldsymbol{\varphi}) \\ &\quad + h(t) (\boldsymbol{f}, \boldsymbol{\varphi}).\end{aligned}$$

The equation above is true a.e. in $[0, T]$ and for any $\boldsymbol{\varphi} \in \boldsymbol{X}_N$. Space $\boldsymbol{X}_N$ is dense in $\mathbf{H}_0(\mathbf{curl}; \Omega)$, therefore, equation (5.9) is valid for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ and $\partial_t^2 \boldsymbol{E} \in (\mathbf{H}_0(\mathbf{curl}; \Omega))^*$ a.e. in $[0, T]$.

The next step is to pass to the limit for $n \to \infty$ in (DMP). Since $m \in C^2([0, T])$ and σ is bounded, we deduce

$$\begin{aligned}&\left|(\overline{\sigma_n} * \overline{m_n'})(t_i) - (\overline{\sigma_n} * \overline{m_n'})(t)\right| \\ &= \left|\int_0^{t_i} \overline{m_n'}(t_i - s)\overline{\sigma_n}(s) \ \mathrm{d}s - \int_0^t \overline{m_n'}(t - s)\overline{\sigma_n}(s) \ \mathrm{d}s\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_0^t (\overline{m_n'}(t_i - s) - \overline{m_n'}(t - s))\overline{\sigma_n}(s)\ \mathrm{d}s - \int_t^{t_i} \overline{m_n'}(t_i - s)\overline{\sigma_n}(s)\ \mathrm{d}s \right| \\
&\leq \mathcal{O}(\tau) + \int_t^{t_i} \left|(\overline{m_n'}(t_i - s) - \overline{m_n'}(t - s))\overline{\sigma_n}(s)\right|\ \mathrm{d}s \\
&\stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Taking into account the convergence results of (5.11), we observe the following

$$
(\overline{\sigma_n} * \overline{m_n'})(t) \stackrel{n\to\infty}{\longrightarrow} (\sigma * m')(t).
$$

Thanks to Lemma 5.5 and the convergence results of (5.13), we have for any $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\Omega)$ and $t \in [0, T]$

$$
\begin{aligned}
\left(\nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t), \boldsymbol{\varphi}\right) &\stackrel{Green's\ theorem}{=} \left(\nabla \times \overline{\boldsymbol{E}_n}(t), \nabla \times \boldsymbol{\varphi}\right) \\
&\stackrel{Green's\ theorem}{=} \left(\overline{\boldsymbol{E}_n}(t), \nabla \times \nabla \times \boldsymbol{\varphi}\right) \\
&\stackrel{n\to\infty}{\longrightarrow} \quad (\boldsymbol{E}(t), \nabla \times \nabla \times \boldsymbol{\varphi}) = (\nabla \times \nabla \times \boldsymbol{E}(t), \boldsymbol{\varphi})\,.
\end{aligned}
$$

Since the space $\mathbf{C}_0^\infty(\Omega)$ is dense in the space $\mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and also in the space $\mathbf{L}^2(\Omega)$, we conclude that $\left(\nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t), \boldsymbol{\varphi}\right) \to (\nabla \times \nabla \times \boldsymbol{E}(t), \boldsymbol{\varphi})$ for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}\,;\Omega)$ and for any $t \in [0, T]$. Again, thanks to Lemma 5.4 and Lemma 5.5, we have

$$
\begin{aligned}
&\int_0^\xi \left\|\nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t) - \nabla \times \nabla \times \overline{\boldsymbol{E}_n}(t - \tau)\right\|^2\ \mathrm{d}t \\
&\leq \sum_{i=1}^n \left\|\nabla \times \nabla \times \boldsymbol{e}_i - \nabla \times \nabla \times \boldsymbol{e}_{i-1}\right\|^2 \tau \\
&\leq C\tau \\
&\stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Using this and the fact that $\phi \in H^1(\Omega)$ for any $\phi \in C_0^\infty(\overline{\Omega})$, we obtain the following convergence result

$$
\lim_{n\to\infty} \int_0^\xi \left(\nabla \times \nabla \times \bar{\boldsymbol{E}}_n(t - \tau), \nabla\phi\right)\ \mathrm{d}t = \int_0^\xi (\nabla \times \nabla \times \boldsymbol{E}(t), \nabla\phi)\ \mathrm{d}t.
$$

Lemma 5.2, Lemma 5.3 and the embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$ gives us an estimate for $\overline{\boldsymbol{E}_n}$, i.e.

$$
\left\|\overline{\boldsymbol{E}_n}(t)\right\|^2_{\mathbf{H}^1(\Omega)} \leq C.
$$

Strong convergence of $\overline{\boldsymbol{E}_n}(t)$ towards $\boldsymbol{E}(t)$ in $\mathbf{L}^2(\Omega)$ for any $t \in [0,T]$, cf. (5.13) and the Nečas inequality (1.14) imply that

$$\begin{aligned}\left\|\overline{\boldsymbol{E}_n}(t)-\boldsymbol{E}(t)\right\|^2_{\mathbf{L}^2(\Gamma)} &\le \varepsilon\left\|\overline{\boldsymbol{E}_n}(t)-\boldsymbol{E}(t)\right\|^2_{\mathbf{H}^1(\Omega)} + C_\varepsilon\left\|\overline{\boldsymbol{E}_n}(t)-\boldsymbol{E}(t)\right\|^2\\ &\le \varepsilon + C_\varepsilon\left\|\overline{\boldsymbol{E}_n}(t)-\boldsymbol{E}(t)\right\|^2.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\left\|\overline{\boldsymbol{E}_n}(t)-\boldsymbol{E}(t)\right\|^2_{\mathbf{L}^2(\Gamma)} \le \varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

Hence, $\overline{\boldsymbol{E}_n}(t)\to\boldsymbol{E}(t)$ in the space $\mathbf{L}^2(\Gamma)$ for any $t\in[0,T]$. Using Lemma 5.2 and the same technique as above, we conclude the following for $t\in[t_{i-1},t_i]$

$$\begin{aligned}&\left\|\overline{\boldsymbol{E}_n}(t)-\overline{\boldsymbol{E}_n}(t-\tau)\right\|^2_{\mathbf{L}^2(\Gamma)}\\ &\le \varepsilon\left\|\overline{\boldsymbol{E}_n}(t)-\overline{\boldsymbol{E}_n}(t-\tau)\right\|^2_{\mathbf{H}^1(\Omega)} + C_\varepsilon\left\|\overline{\boldsymbol{E}_n}(t)-\overline{\boldsymbol{E}_n}(t-\tau)\right\|^2\\ &\le \varepsilon + C_\varepsilon\left\|\delta\boldsymbol{e}_i\right\|^2\tau^2 \quad\stackrel{\tau\to 0}{\longrightarrow}\quad \varepsilon \quad\stackrel{\varepsilon\to 0}{\longrightarrow}\quad 0.\end{aligned}$$

Now, we integrate (DMP) in time over $t\in[0,\xi]\subset[0,T]$, consider all convergence results above, take into account that $m\in C^2([0,T])$ and the Lipschitz continuity of $\boldsymbol{N}$ and pass to the limit for $n\to\infty$ to obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^\xi \overline{h_n} = \frac{1}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{n}\phi\,\mathrm{d}\Gamma}&\left\{\int_0^\xi m''(t) + \int_0^\xi[(\sigma * m')(t)+\sigma(t)m(0)]\right.\\ &\left.+\int_0^\xi\int_\Omega \nabla\times\nabla\times\boldsymbol{E}\cdot\nabla\phi\,\mathrm{d}x - \int_0^\xi\int_\Gamma \boldsymbol{N}(\boldsymbol{E})\cdot\boldsymbol{n}\phi\,\mathrm{d}\Gamma\right\}.\end{aligned}$$

Differentiation with respect to the time variable ξ yields equation (5.7) that also concludes our proof. □

5.5 The uniqueness of a solution

Due to the nonlinear term $\boldsymbol{N}$ we are not able to provide the uniqueness proof without any further regularity assumptions on the solution $\boldsymbol{E}$. Thus, we assume $\boldsymbol{E}\in\mathbf{H}^{1,\infty}(\Omega)$. Taking this into account and also presume that $\boldsymbol{N}$ is supposedly smooth, i.e. $\boldsymbol{N}\in\mathbf{C^2}$, we conclude the following for any vector fields $\boldsymbol{u},\boldsymbol{v}:\mathbb{R}^3\to\mathbb{R}^3$ and some $\xi_1,\xi_2,\xi_3\in[0,1]$

$$\boldsymbol{N}(\boldsymbol{u})-\boldsymbol{N}(\boldsymbol{v}) = \begin{pmatrix}\boldsymbol{N}_1(\boldsymbol{u})-\boldsymbol{N}_1(\boldsymbol{v})\\ \boldsymbol{N}_2(\boldsymbol{u})-\boldsymbol{N}_2(\boldsymbol{v})\\ \boldsymbol{N}_3(\boldsymbol{u})-\boldsymbol{N}_3(\boldsymbol{v})\end{pmatrix} = \begin{pmatrix}\nabla\boldsymbol{N}_1(\boldsymbol{v}+\xi_1(\boldsymbol{u}-\boldsymbol{v}))\cdot(\boldsymbol{u}-\boldsymbol{v})\\ \nabla\boldsymbol{N}_2(\boldsymbol{v}+\xi_2(\boldsymbol{u}-\boldsymbol{v}))\cdot(\boldsymbol{u}-\boldsymbol{v})\\ \nabla\boldsymbol{N}_3(\boldsymbol{v}+\xi_3(\boldsymbol{u}-\boldsymbol{v}))\cdot(\boldsymbol{u}-\boldsymbol{v})\end{pmatrix}.$$

Assuming that $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}^{1,\infty}(\Omega)$, we use the Cauchy-Schwarz inequality to obtain an estimate for derivatives

$$\begin{aligned}|\partial x_j(\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}))| &\le \sum_{i=1}^{3} |\partial x_j \nabla \boldsymbol{N}_i(\boldsymbol{v} + \xi_i(\boldsymbol{u}-\boldsymbol{v}))|\, |\partial x_j(\boldsymbol{v} + \xi_i(\boldsymbol{u}-\boldsymbol{v}))|\, |\boldsymbol{u}-\boldsymbol{v}| \\ &\quad + |\nabla \boldsymbol{N}_i(\boldsymbol{v} + \xi_i(\boldsymbol{u}-\boldsymbol{v}))|\, |\partial x_j(\boldsymbol{u}-\boldsymbol{v})| \\ &\le C(|\boldsymbol{u}-\boldsymbol{v}| + |\partial x_j(\boldsymbol{u}-\boldsymbol{v})|)\end{aligned}$$

where the norm $|\cdot|$ is defined as $|\boldsymbol{u}| = \sqrt{\sum_{i=1}^{3} |u_i|^2}$. Now, we provide some further estimates that are obtained in the similar manner as the estimate above

$$\|\nabla \times (\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}))\| \le C\, \|\boldsymbol{u}-\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \quad \text{if } \boldsymbol{N} \in \mathbf{C^2}, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H^{1,\infty}(\Omega)} \tag{5.16}$$

and

$$\|\nabla \cdot (\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}))\| \le C\, \|\boldsymbol{u}-\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \quad \text{if } \boldsymbol{N} \in \mathbf{C^2}, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H^{1,\infty}(\Omega)}. \tag{5.17}$$

We pass on with the uniqueness theorem.

Theorem 5.2. *Let the assumptions of Theorem 5.1 be satisfied. Moreover, presume that $\boldsymbol{N} \in \mathbf{C^2}$. Then there exists at most one weak solution $\{\boldsymbol{E}, h\}$ to the problem* (5.5), (5.3), (5.4) *and* (5.7) *fulfilling $h \in L^2((0,T)), \boldsymbol{E} \in C([0,T]; \mathbf{L}^2(\Omega)) \cap L^\infty((0,T); \mathbf{H}^{1,\infty}(\Omega))$ with its first order time derivative $\partial_t \boldsymbol{E} \in L^2((0,T); \mathbf{L}^2(\Omega)) \cap C([0,T]; \boldsymbol{X}_N^*)$, second order time derivative $\partial_t^2 \boldsymbol{E} \in L^2((0,T); (\mathbf{H}_0(\mathbf{curl}\,; \Omega))^*)$ and $\nabla \times \nabla \times \boldsymbol{E} \in L^\infty((0,T); \mathbf{L}^2(\Omega))$.*

Proof. Let us have two solutions $\{\boldsymbol{E}, h\}$ and $\{\boldsymbol{G}, g\}$ to the problem (5.5), (5.3), (5.4), (5.7) and denote

$$\boldsymbol{E} - \boldsymbol{G} = \boldsymbol{P}, \qquad h(t) - g(t) = p(t).$$

Since $\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{G}(\boldsymbol{x}, 0) = \boldsymbol{E}_0$ and $\partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = \partial_t \boldsymbol{G}(\boldsymbol{x}, 0) = \boldsymbol{W}_0$, we have $\boldsymbol{P}(\boldsymbol{x}, 0) = \mathbf{0}$ and $\partial_t \boldsymbol{P}(\boldsymbol{x}, 0) = \mathbf{0}$. Moreover, $\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{G} \times \boldsymbol{n} = \mathbf{0}$ on Γ, therefore, $\boldsymbol{P} \times \boldsymbol{n} = \mathbf{0}$ on Γ as well. Our goal is to show that $\boldsymbol{P} = \mathbf{0}$ a.e. in $\Omega \times (0,T)$ and $p = 0$ a.e. in $(0,T)$. Measurements for both solutions are the same. Therefore, we have

$$p(t) = \frac{(\nabla \times \nabla \times \boldsymbol{P}(t), \nabla \phi) - \int_\Gamma (\boldsymbol{N}(\boldsymbol{E}(t)) - \boldsymbol{N}(\boldsymbol{G}(t))) \cdot \boldsymbol{n} \phi \,\mathrm{d}\Gamma}{\int_\Gamma \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n} \phi \,\mathrm{d}\Gamma}. \tag{5.18}$$

Subtracting equation (5.5) for $\boldsymbol{E}$ and $\boldsymbol{G}$ yields

$$\partial_t^2 \boldsymbol{P} + (\sigma * \partial_t \boldsymbol{P}) + \nabla \times \nabla \times \boldsymbol{P} = \boldsymbol{f} p + \boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}). \tag{5.19}$$

We now proceed with several energy estimates implied by equations (5.18) and (5.19).
Part (A) : Considering relation (5.18), taking into account the embedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Gamma)$ and the Lipschitz continuity of $\boldsymbol{N}$, we obtain

$$|p(\xi)|^2 \leq C \|\boldsymbol{P}(\xi)\|^2_{\mathbf{H}^1(\Omega)} + C \|\nabla \times \nabla \times \boldsymbol{P}(\xi)\|^2 . \tag{A}$$

Part (B) : We multiply equation (5.19) with $\partial_t \boldsymbol{P}$, integrate over Ω, use the Green theorem and then integrate in time to deduce

$$\begin{aligned} &\frac{1}{2} \|\partial_t \boldsymbol{P}(\xi)\|^2 + \frac{1}{2} \|\nabla \times \boldsymbol{P}(\xi)\|^2 \\ &\leq \int_0^\xi |p| \|\boldsymbol{f}\| \|\partial_t \boldsymbol{P}\| + \int_0^\xi \|\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G})\| \|\partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \partial_t \boldsymbol{P})\| \|\partial_t \boldsymbol{P}\| . \end{aligned}$$

We also have the following bounds for the terms on the r.h.s.

$$\begin{aligned} &\int_0^\xi |p| \|\boldsymbol{f}\| \|\partial_t \boldsymbol{P}\| \leq C \int_0^\xi |p|^2 + C \int_0^\xi \|\partial_t \boldsymbol{P}\|^2 , \\ &\int_0^\xi \|(\sigma * \partial_t \boldsymbol{P})\| \|\partial_t \boldsymbol{P}\| \leq C\sigma^* \int_0^\xi \|\partial_t \boldsymbol{P}\|^2 , \\ &\int_0^\xi \|\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G})\| \|\partial_t \boldsymbol{P}\| \leq C \int_0^\xi \|\boldsymbol{P}\| \|\partial_t \boldsymbol{P}\| \leq CT \int_0^\xi \|\partial_t \boldsymbol{P}\|^2 . \end{aligned}$$

Here, we used the traditional Cauchy and Young inequalities, boundedness of σ, the Lipschitz continuity of $\boldsymbol{N}$, and $\|\boldsymbol{P}(t)\| = \left\| \int_0^t \partial_t \boldsymbol{P}(s) \right\| \leq T \|\partial_t \boldsymbol{P}(t)\|$. Collecting all partial results and applying Grönwall's lemma, we conclude

$$\|\partial_t \boldsymbol{P}(\xi)\|^2 + \|\nabla \times \boldsymbol{P}(\xi)\|^2 \leq C \int_0^\xi |p|^2 . \tag{B}$$

Remark 5.3. *This result also implies* $\|\boldsymbol{P}(\xi)\|^2 \leq C \int_0^\xi |p|^2$.

Part (C) : We apply $\nabla \cdot$ operator to equation (5.19), then multiply it by $\nabla \cdot \partial_t \boldsymbol{P}$ and integrate in space and time to obtain

$$\begin{aligned} \frac{1}{2} \|\nabla \cdot \partial_t \boldsymbol{P}(\xi)\|^2 \leq &\int_0^\xi |p| \|\nabla \cdot \boldsymbol{f}\| \|\nabla \cdot \partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \nabla \cdot \partial_t \boldsymbol{P})\| \|\nabla \cdot \partial_t \boldsymbol{P}\| \\ &+ \int_0^\xi \|\nabla \cdot (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}))\| \|\nabla \cdot \partial_t \boldsymbol{P}\| . \end{aligned}$$

First two terms on the r.h.s. are estimated via the Young inequality, i.e.

$$\int_0^\xi |p| \left\| \nabla \cdot \boldsymbol{f} \right\| \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\| + \int_0^\xi \left\| (\sigma * \nabla \cdot \partial_t \boldsymbol{P}) \right\| \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|$$
$$\leq C \int_0^\xi |p|^2 + C \int_0^\xi \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|^2 .$$

For the last term, we use inequalities (5.17), (B), the embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$, and the identity $\left\| \nabla \cdot \boldsymbol{P}(t) \right\| = \left\| \int_0^t \nabla \cdot \partial_s \boldsymbol{P}(s) \, \mathrm{d}s \right\|$

$$\int_0^\xi \left\| \nabla \cdot (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G})) \right\| \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|$$
$$\leq C \int_0^\xi \left\| \boldsymbol{P} \right\|_{\mathbf{H}^1(\Omega)} \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|$$
$$\leq C \int_0^\xi \left\| \boldsymbol{P} \right\|_{\boldsymbol{X}_N} \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|$$
$$= C \int_0^\xi \left(\left\| \boldsymbol{P} \right\| + \left\| \nabla \times \boldsymbol{P} \right\| \right) \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\| + C \int_0^\xi \left\| \nabla \cdot \boldsymbol{P} \right\| \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|$$
$$\leq C \int_0^\xi \left\| \boldsymbol{P} \right\|^2 + \left\| \nabla \times \boldsymbol{P} \right\|^2 + C \int_0^\xi \left\| \nabla \cdot \boldsymbol{P} \right\|^2 + C \int_0^\xi \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|^2$$
$$\leq C \int_0^\xi |p|^2 + C \int_0^\xi \int_0^t \left\| \nabla \cdot \partial_s \boldsymbol{P} \right\|^2 + C \int_0^\xi \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|^2$$
$$\leq C \int_0^\xi |p|^2 + C \int_0^\xi \left\| \nabla \cdot \partial_t \boldsymbol{P} \right\|^2 .$$

We employ Grönwall's lemma to get

$$\left\| \nabla \cdot \partial_t \boldsymbol{P}(\xi) \right\|^2 \leq C \int_0^\xi |p|^2 \qquad \text{and} \qquad \left\| \nabla \cdot \boldsymbol{P}(\xi) \right\|^2 \leq C \int_0^\xi |p|^2 . \tag{C}$$

Part (D) : We apply $\nabla \times$ operator to equation (5.19), multiply it by $\nabla \times \partial_t \boldsymbol{P}$, then integrate in Ω and use the Green theorem and then integrate in time to attain

$$\frac{1}{2} \left\| \nabla \times \partial_t \boldsymbol{P}(\xi) \right\|^2 + \frac{1}{2} \left\| \nabla \times \nabla \times \boldsymbol{P}(\xi) \right\|^2$$
$$\leq \int_0^\xi |p| \left\| \nabla \times \boldsymbol{f} \right\| \left\| \nabla \times \partial_t \boldsymbol{P} \right\| + \int_0^\xi \left\| (\sigma * \nabla \times \partial_t \boldsymbol{P}) \right\| \left\| \nabla \times \partial_t \boldsymbol{P} \right\|$$
$$+ \int_0^\xi \left\| \nabla \times (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G})) \right\| \left\| \nabla \times \partial_t \boldsymbol{P} \right\| .$$

Again, we use Young's inequality to handle the first two terms on the r.h.s.

$$\int_0^\xi |p| \, \|\nabla \times \boldsymbol{f}\| \, \|\nabla \times \partial_t \boldsymbol{P}\| + \int_0^\xi \|(\sigma * \nabla \times \partial_t \boldsymbol{P})\| \, \|\nabla \times \partial_t \boldsymbol{P}\|$$
$$\leq C \int_0^\xi |p|^2 + C \int_0^\xi \|\nabla \times \partial_t \boldsymbol{P}\|^2 .$$

The last term on the r.h.s. is estimated with the help of inequalities (5.16), (B), (C), and the embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$

$$\int_0^\xi \|\nabla \times (\boldsymbol{N}(\boldsymbol{E}) - \boldsymbol{N}(\boldsymbol{G}))\| \, \|\nabla \times \partial_t \boldsymbol{P}\|$$
$$\leq C \int_0^\xi \|\boldsymbol{P}\|_{\mathbf{H}^1(\Omega)} \, \|\nabla \times \partial_t \boldsymbol{P}\|$$
$$\leq C \int_0^\xi \|\boldsymbol{P}\|_{\boldsymbol{X}_N} \, \|\nabla \times \partial_t \boldsymbol{P}\|$$
$$\leq C \int_0^\xi \|\boldsymbol{P}\|^2 + \|\nabla \cdot \boldsymbol{P}\|^2 + \|\nabla \times \boldsymbol{P}\|^2 + C \int_0^\xi \|\nabla \times \partial_t \boldsymbol{P}\|^2$$
$$\leq C \int_0^\xi |p|^2 + C \int_0^\xi \|\nabla \times \partial_t \boldsymbol{P}\|^2 .$$

Using the Grönwall lemma, we achieve the following estimate

$$\|\nabla \times \partial_t \boldsymbol{P}(\xi)\|^2 + \|\nabla \times \nabla \times \boldsymbol{P}(\xi)\|^2 \leq C \int_0^\xi |p|^2 . \qquad \text{(D)}$$

Summary : The embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$ together with the results from (A), (B), (C), and (D) gives us

$$\|\boldsymbol{P}(\xi)\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla \times \nabla \times \boldsymbol{P}(\xi)\|^2 \leq C \int_0^\xi |p|^2$$
$$\leq C \int_0^\xi \left(\|\boldsymbol{P}\|^2_{\mathbf{H}^1(\Omega)} + \|\nabla \times \nabla \times \boldsymbol{P}\|^2 \right) .$$

Thus, employing the Grönwall lemma one more time, we see that $\boldsymbol{P} = \mathbf{0}$ a.e. in $\Omega \times (0, T)$ and from inequality (A) we conclude that $p = 0$ a.e. in $(0, T)$. □

5.6 Numerical simulation

The main goal of this section is to support the theoretical results presented above. We want to demonstrate the convergence of the numerical scheme proposed in

Section 5.3. Since Rothe's method is semi-discrete, we only analyze the time dependent part of the error of the numerical solution. To show the convergence of our numerical scheme, we consider the following test problem:
Find the solution $\{\boldsymbol{E}(\boldsymbol{x},t), h(t)\}$ such that: †

$$\begin{aligned}\partial_t^2 \boldsymbol{E}(\boldsymbol{x},t) + (\sigma * \partial_t \boldsymbol{E})(t) + \sigma(t)\boldsymbol{E}(\boldsymbol{x},0) + \nabla\times\nabla\times \boldsymbol{E}(\boldsymbol{x},t) &\\ = \boldsymbol{f}(\boldsymbol{x})h(t) + \boldsymbol{N}(\boldsymbol{E}(\boldsymbol{x},t)) + \boldsymbol{U}(\boldsymbol{x},t) \ \text{ in } \ \Omega\times(0,T)&\\ \boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{n} = \boldsymbol{0} \ \text{ in } \ \Gamma\times(0,T)&\\ \boldsymbol{f}(\boldsymbol{x})\times\boldsymbol{n} = \boldsymbol{0} \ \text{ on } \ \Gamma.& \qquad (5.20)\end{aligned}$$

With initial data prescribed as

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \qquad \partial_t\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{W}_0(\boldsymbol{x})$$

and additional measurement in the form of (5.6).

5.6.1 Setting of the experiment

Let Ω be a sphere in $\mathbb{R}^3$ with radius $r=1$, i.e. the domain Ω can be expressed as $\Omega = \left\{\boldsymbol{x}=(x,y,z)\in\mathbb{R}^3 | x^2+y^2+z^2<1\right\}$ and $t\in(0,T)$ with $T=1$. To show the convergence of our scheme, we need an exact solution $\{\boldsymbol{E}(\boldsymbol{x},t), h(t)\}$, so, we compute the error of the numerical solution. For that reason, we define the exact solution as

$$\boldsymbol{E}(\boldsymbol{x},t) = e^t\begin{pmatrix} x\\ y\\ z\end{pmatrix}, \quad h(t) = e^t.$$

Remaining functions are determined accordingly to satisfy (5.20)

$$\begin{aligned}&\sigma(t) = 4t^3 + 8t^2 + 16t + 24,\\ &\boldsymbol{N}(\boldsymbol{E}(\boldsymbol{x},t)) = |\boldsymbol{E}(\boldsymbol{x},t)|^{-1/2}\,\boldsymbol{E}(\boldsymbol{x},t) + \boldsymbol{E}(\boldsymbol{x},t),\\ &\boldsymbol{f}(\boldsymbol{x}) = 80\begin{pmatrix} x\\ y\\ z\end{pmatrix}, \quad \boldsymbol{U}(\boldsymbol{x},t) = -\left(\frac{1}{(x^2+y^2+z^2)^{1/4}} + 12t^2 + 40t + 56\right)\begin{pmatrix} x\\ y\\ z\end{pmatrix}.\end{aligned}$$

In order to examine the nature of the error (whether it is diminishing with the decreasing time step) of our numerical solution $\{\boldsymbol{E}_{numerical}, h_{numerical}\}$, we

† If the vector field $\boldsymbol{U}(\boldsymbol{x},t)$ is sufficiently smooth then all theoretical results achieved in previous sections remain true. Hence, we add this term to the r.h.s. in (5.20).

compute multiple solutions for various time steps τ. In our experiment it is $\tau = [0.2, 0.1, 0.05, 0.025, 0.0125]$. The spatial part of our solution is then calculated in a free computing platform for PDE - FEniCS. The space-time domain is divided into 553 cells (tetrahedra) with diameters ranging from 0.38513 to 0.65231. We use Lagrange FEM (finite element method) of order 2 at each time step to provide a numerical solution. This leads to a system with 16590 DoF (degrees of freedom).

Part of the boundary where the measurement (5.6) was taken is displayed in Fig. 5.3.

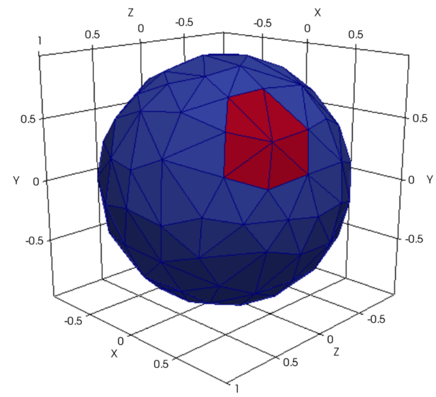

Figure 5.3: Boundary measurement.

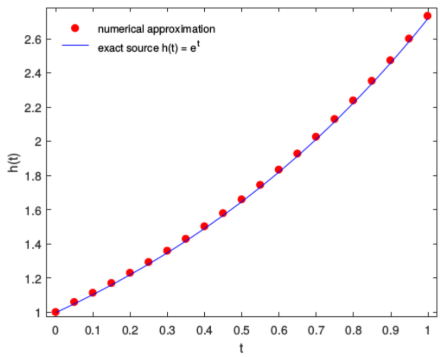

Figure 5.4: Reconstruction of the source term $h(t) = e^t$.

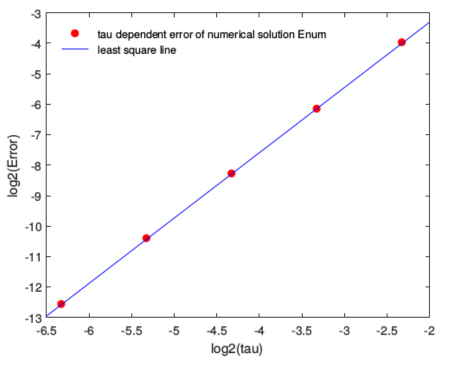

Figure 5.5: τ dependency of the error for $\boldsymbol{E}_{numerical}$.

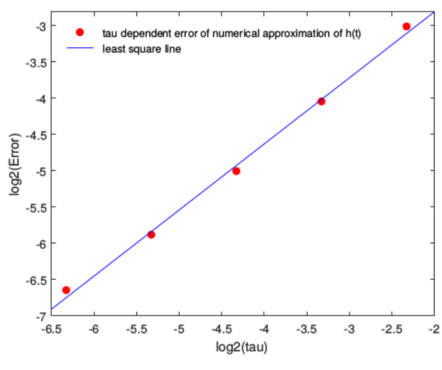

Figure 5.6: τ dependency of the error for $h_{numerical}$.

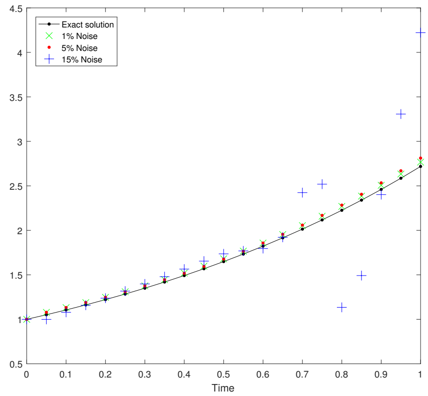

Figure 5.7: Source reconstruction using noisy data.

Errors for the numerical solutions are computed in the following manner

$$\boldsymbol{E}_{error} = \frac{\|\boldsymbol{E}_{numerical} - \boldsymbol{E}_{exact}\|^2_{L^2((0,T);\mathbf{L}^2(\Omega))}}{\|\boldsymbol{E}_{exact}\|^2_{L^2((0,T);\mathbf{L}^2(\Omega))}},$$

$$h_{error} = \sqrt{\int_0^T |h_{numerical} - h_{exact}|^2 \, \mathrm{d}t}.$$

The numerical reconstruction of the source term $h(t)$ is shown in Fig. 5.4. We can see the time step dependency of the errors for $\boldsymbol{E}_{numerical}$ and $h_{numerical}$ in Fig. 5.5 and Fig. 5.6 above. The performance of our algorithm with noise in the measurement is pictured in Fig. 5.7. As we can see, the reconstruction of the source term was quite good for 1% and 5% noise in the data. However, when 15% of noise was present, our reconstruction was slightly off. To reconstruct the source, we also need the information about the first and second order time derivatives of the function $m(t)$ (measurement). Therefore, if the noise in the data is too high (15%), the smoothness of $m(t)$ is not sufficient. This causes the errors in the reconstruction.

Chapter 6

Discussion and future research

The first part of the thesis presents a development of mathematical models for the induction heating phenomena. These physical events are described by the coupled system of nonlinear partial differential equations which is derived from Maxwell's equations and the heat transfer equation. In the second part an inverse source problem for Maxwell's equations is proposed and investigated. This chapter generates a discussion of obtained results in the previous chapters. Some ideas and improvements for the future research in the investigated areas are also proposed.

Part I

In Chapters 2 and 3 we develop and investigate similar mathematical models of the induction heating phenomena. To describe the electromagnetic part of the induction heating process, we use Maxwell's equations. The evolution of temperature in the considered domain is characterized by the nonlinear heat transfer equation. The difference between the mathematical model investigated in Chapter 2 and the model in Chapter 3 lies in the adopted constitutive relation between the magnetic field and the magnetic induction field. We propose two possible alternatives which are investigated separately in Chapter 2 and Chapter 3. We then define the weak formulation of the mathematical model and introduce a semi-discrete implicit Euler scheme (Rothe's method) which provides a unique solution at each time step. Lastly, the convergence of Rothe's functions towards a weak solution is shown, i.e. the existence of a global solution is proven.

The mathematical models investigated in Chapters 2 and 3 capture a broad range of the induction heating employment, such as induction cooking, induction brazing, induction sealing or induction hardening. In Chapter 4 we develop and investigate the mathematical model of the induction hardening process. Using a vector-scalar potential formulation to model the electromagnetic part and a nonlinear heat transfer equation to determine the temperature evolution in the workpiece and the coil, we obtain a coupled system of three partial differential equations. We put emphasis on the nonlinear nature of the constitutive relation between the magnetic field and the magnetic induction field in Maxwell's equations. Therefore, we obtain a strongly nonlinear system. We discretize the system in time (semi-discretization) and propose a decoupled numerical scheme to compute the solution at each time step. The existence of a global solution of the whole system is then demonstrated via Rothe's method. To supplement the theoretical results achieved in this chapter, we present a simple numerical simulation of the induction hardening process. We code the numerical scheme in the open source finite element environment Gmsh/GetDP and compute the solution using real physical constants.

The mathematical models introduced in Chapters 2, 3 and 4 share the same aspects of future improvements because of their similarities. For that reason, we tackle the questions regarding the future improvements in one bundle for all three mathematical models.

The uniqueness question

Although we provide the existence of a global solution for each mathematical model, we could not guarantee the existence of a unique solution. Terms containing either γ or σ seem to be the most difficult to handle when approaching this goal. The thermal dependency of γ/σ creates obstacles which prevent us from obtaining the desired energy estimates needed to prove the uniqueness of the solution.

Necessity of the truncation function

The next inherent question is whether we really need to apply the truncation function on the source term in the heat equation. If we did not apply the cut-off function we would not be able to control the source term. Let us demonstrate this statement on the mathematical model from Chapter 2. Better regularity of the solution would be needed to provide the same a priori estimates as in Lemma 2.4 These estimates are necessary to prove the existence of a weak solution. Let us consider (2.30) without the truncation, so it becomes

$$(\delta\theta(u_i), \psi) + (\lambda_0 \nabla u_i, \nabla\psi) = \left(\gamma(u_{i-1}) \left|\nabla \times \boldsymbol{h}_i\right|^2, \psi\right). \tag{6.1}$$

To obtain the same a priori estimates as in Lemma 2.4, we would need to set $\psi = u_i\tau$ in (6.1) and sum it for $i = 1, \dots, j$. Then, we can bound the l.h.s. in the same fashion as in Lemma 2.4. The r.h.s. can be estimated via the Cauchy and Young inequalities, i.e.

$$\sum_{i=1}^{j} \int_{\Omega} \gamma(u_{i-1}) \left|\nabla \times \boldsymbol{h}_i\right|^2 u_i \tau \leq \gamma^* \sum_{i=1}^{j} \sqrt{\int_{\Omega} \left|\nabla \times \boldsymbol{h}_i\right|^4} \sqrt{\left|u_i\right|^2} \tau$$
$$\leq \varepsilon \sum_{i=1}^{j} \left\|u_i\right\|^2 \tau + C_\varepsilon \sum_{i=1}^{j} \left\|\nabla \times \boldsymbol{h}_i\right\|_{\mathbf{L}^4(\Omega)}^4 \tau.$$

Here, we can see that in order to control the source term we would need $\nabla \times \boldsymbol{H}$ to be in the space $L^2((0,T);\mathbf{L}^4(\Omega))$ which we could not guarantee without any a priori estimates on the solution.

Moving conductor

In Chapter 4 we have considered a static conductor. In the case of a moving conductor, we need to take into account the Lorentz force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

The equation above describes the force $\boldsymbol{F}$ acting on a particle of electric charge q with instantaneous velocity $\boldsymbol{v}$. When the conductor is moving, the new electric field $\boldsymbol{E}'$ in Maxwell's equations becomes

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}.$$

Assuming the above relation in the vector-scalar potential formulation in Chapter 4, we would obtain the following nonlinear partial differential equation describing the electromagnetic phenomena:

$$\begin{aligned} \sigma \partial_t \boldsymbol{A} + \nabla \times \mu \boldsymbol{M}(\nabla \times \boldsymbol{A}) + \sigma \nabla \phi - \sigma \boldsymbol{v} \times \nabla \times \boldsymbol{A} &= \mathbf{0} \\ \boldsymbol{n} \times \boldsymbol{A} &= \mathbf{0} \\ \boldsymbol{A}(\boldsymbol{x}, 0) &= \boldsymbol{A}_0(\boldsymbol{x}). \end{aligned}$$

The implementation of a moving conductor in the mathematical model would make it even more realistic. Mathematical models of induction heating with moving conductor have been investigated in [81] and [5]. Authors of [81] has provided numerical simulations of transverse flux induction heating with moving-strips. Moreover, authors of [5] has presented a FEM approach to simulate the moving induction heating in weld-based additive manufacturing. However, they do not analyze the theoretical aspects of the problem, such as the existence or uniqueness of the solution. Therefore, it is a great direction to focus on in the further research of the induction hardening models.

Part II

In Chapter 5 we have investigated an inverse source problem for Maxwell's equations of hyperbolic type. We have considered a single boundary measurement as an additional information to reconstruct the missing time-dependent part of the source term. The main advantage of our measurement is that it is not necessary to measure the whole boundary but only a part of it. We have proposed a decoupled numerical scheme (backward Euler scheme) to compute the solution and the missing source. The existence of a weak solution has been proven by using Rothe's method and the important embedding $\boldsymbol{X}_N \subset \mathbf{H}^1(\Omega)$. The uniqueness has been shown in the case of a regular solution. In addition to the theoretical results, we have presented a numerical experiment showing convergence of the proposed numerical scheme.

(Non)linear media with memory

The material occupying the considered domain in Chapter 5 has been assumed to be homogeneous and dielectric. Because of that, the constitutive relations have taken the following form

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}.$$

However, in the case of a linear material with memory (for instance chiral media [74]) these relations become more complicated, i.e.

$$\begin{aligned} \boldsymbol{D} &= \varepsilon \boldsymbol{E} + \varepsilon_1 * \boldsymbol{E} + \xi * \boldsymbol{H}, \\ \boldsymbol{B} &= \mu \boldsymbol{H} + \mu_1 * \boldsymbol{H} + \eta * \boldsymbol{E}. \end{aligned}$$

where $*$ is defined as $\alpha * \boldsymbol{U} = \int_0^t \alpha(\boldsymbol{x}, s)\boldsymbol{U}(\boldsymbol{x}, t-s)\ \mathrm{d}s$. We could consider an even more general case. A nonlinear material with memory. Then the constitutive relations would become

$$\begin{aligned} \boldsymbol{D} &= \varepsilon \boldsymbol{E} + \varepsilon_1 * \boldsymbol{E} + \xi * \boldsymbol{H} + \varepsilon_2 f_1(|\boldsymbol{E}|^2)\boldsymbol{E}, \\ \boldsymbol{B} &= \mu \boldsymbol{H} + \mu_1 * \boldsymbol{H} + \eta * \boldsymbol{E} + \mu_2 f_2(|\boldsymbol{H}|^2)\boldsymbol{H}. \end{aligned}$$

An interesting subject for further research would be the implementation of the above constitutive relations in the mathematical model which has been investigated in Chapter 5. Consideration of the generalized Ohm law with the below expression

$$\boldsymbol{J} = \sigma * \boldsymbol{E} + \boldsymbol{F}$$

and assumption of a linear material with memory in Chapter 5 would lead us to the following coupled system:

$$\begin{aligned}\nabla \times \boldsymbol{H} &= \partial_t \left(\varepsilon \boldsymbol{E} + \varepsilon_1 * \boldsymbol{E} + \xi * \boldsymbol{H}\right) + \sigma * \boldsymbol{E} + \boldsymbol{f}(\boldsymbol{x})h(t),\\ \nabla \times \boldsymbol{E} &= -\partial_t \left(\mu \boldsymbol{H} + \mu_1 * \boldsymbol{H} + \eta * \boldsymbol{E}\right)\end{aligned}$$

where $h(t)$ is the missing time-dependent part of the source to be reconstructed.

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