Roy Oste

Laplace and Dirac operators, symmetry algebras, and their use in Fourier transforms and quantum oscillator models

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Laplace and Dirac operators, symmetry algebras, and their use in Fourier transforms and quantum oscillator models

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PREFACE

The thesis you are reading is a collection of research results that have been or will be published in scientific journals. For each chapter, the corresponding paper with the original place of publication and the correct citation details are listed below.

Chapter 1	De Bie H., Oste R., Van der Jeugt J.: Generalized Fourier Transforms Arising from the Enveloping Algebras of $\mathfrak{sl}(2)$ and $\mathfrak{osp}(1\|2)$. *Int. Math. Res. Not. IMRN* 2016 (2016), 4649–4705
Chapter 2	De Bie H., Oste R., Van der Jeugt J.: On the algebra of symmetries of Laplace and Dirac operators. arXiv: `1701.05760 [math-ph]`
Chapter 3	De Bie H., Oste R., Van der Jeugt J.: A Dirac equation on the two-sphere: the S_3 Dirac-Dunkl operator symmetry algebra. arXiv: `1705.08751 [math-ph]`
Chapter 4	Oste R., Van der Jeugt J.: Doubling (dual) Hahn polynomials: classification and applications. *SIGMA* 12 (2016), 003
Chapter 5	Oste R., Van der Jeugt J.: A finite oscillator model with equidistant position spectrum based on an extension of $\mathfrak{su}(2)$. *J. Phys. A: Math. Theor.* 49 (2016), 175204
Chapter 6	Oste R., Van der Jeugt J.: A finite quantum oscillator model related to special sets of Racah polynomials. *Phys. Atom. Nucl.* 80 (2017), 786–793
Chapter 7	Oste R., Van der Jeugt J.: Tridiagonal test matrices for eigenvalue computations: Two-parameter extensions of the Clement matrix. *J. Comput. Appl. Math.* 314 (2017), 30–39

The ideas and incentives from which these works originated, have arisen in discussion with the two supervisors of my doctoral research, professors Joris Van der Jeugt and Hendrik De Bie. For their guidance and support during this journey, I genuinely acknowledge my sincere appreciation and gratitude.

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Roy Oste
June 2017

CONTENTS

Part I

PROLOGUE

In this introductory part we describe our research problems and the region within the realms of mathematics and mathematical physics where they are situated.

INTRODUCTION

We first give an overview of the research questions that will be addressed and the context they appear in. After this follows an exposition of the approach followed in tackling these problems.

RESEARCH QUESTIONS

We begin our story by introducing a first protagonist in the form of the so-called Laplacian or Laplace operator. Named after Pierre-Simon de Laplace and denoted by Δ, on n-dimensional Euclidean space this second-order differential operator is defined as the divergence of the gradient. In a Cartesian coordinate system $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we thus have

$$\Delta = \langle \nabla, \nabla \rangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ contains the partial derivatives $\partial_{x_j} = \partial/\partial x_j$ and $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ is the standard inner product on $\mathbb{R}^n$.

The Laplace operator is used in differential equations modeling all sorts of phenomena in nature, and thus plays an important role in areas such as electromagnetism, wave theory, and (quantum) mechanics. Furthermore, the Laplacian is at the core of harmonic analysis, with the solid harmonics being the homogeneous polynomials in the kernel of Δ. The restriction of these polynomials to the surface of the sphere then yields the spherical harmonics, which will reappear later in our story. For the purposes at hand, the interaction of Δ with other operators is mainly of interest, so we assume the function space on which is being acted to be sufficiently well-behaved — at least twice-differentiable.

We elaborate upon the algebraic structure in which Δ is embedded. In this setting, a first research question will arise involving the well-known Fourier transform. Let $|x| = \sqrt{\langle x, x \rangle}$ denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. The squared norm $|x|^2 = \langle x, x \rangle$ is the natural partner for the Laplace operator Δ to generate a realization of the Lie algebra $\mathfrak{sl}(2)$. Indeed, the following commutation relations are readily verified to hold:

$$\left[|x|^2, -\Delta\right] = 4(\mathbb{E} + \tfrac{n}{2}), \quad \left[\mathbb{E} + \tfrac{n}{2}, |x|^2\right] = 2|x|^2, \quad \left[\mathbb{E} + \tfrac{n}{2}, -\Delta\right] = 2\Delta,$$

where the Lie bracket is the commutator $[X, Y] = XY - YX$, and $\mathbb{E} = \langle x, \nabla \rangle = \sum_{j=1}^{n} x_j \partial_{x_j}$ is the so-called Euler operator, which measures the degree of a homogeneous polynomial. This algebraic structure leads to a natural decomposition of the spherical harmonics, in the form of Howe duality [13, 14].

The starting point for Chapter 1 is the observation that the objects $-\Delta$ and $|x|^2$ are linked also as conjugate quantities in the sense that they are Fourier transform duals. Over $\mathbb{R}^n$, the Fourier transform $\mathcal{F}$ is defined as the integral transform

$$(\mathcal{F}f)(y)= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle x,y\rangle} f(x)\, dx\,.$$

The appropriate function space is now given by the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$. Related to the algebraic structure of Δ and $|x|^2$ is the following description of the Fourier transform as an operator exponential

$$\mathcal{F} = e^{i\frac{\pi}{2}H}\,, \qquad H = \frac{1}{2}\big(- \Delta_x + |x|^2 - n\big)\,.$$

This expression for $\mathcal{F}$ is an immediate consequence of the action on a basis of mutual eigenfunctions of both the Fourier transform and the operator H. These eigenfunctions can explicitly be given in terms of spherical harmonics and Laguerre polynomials. For the one-dimensional case, the Laguerre polynomials reduce to the well-known Hermite polynomials. Both Laguerre and Hermite polynomials are classical families of orthogonal polynomials classified in the Askey-scheme of hypergeometric orthogonal polynomials [18]. The latter is a way of organizing orthogonal polynomials of hypergeometric type which satisfy besides the standard three-terms recurrence relation, as all orthogonal polynomials do, also a second order differential (continuous) or difference (discrete) equation. Later on in our journey, we will encounter also polynomials in the discrete side of the Askey-scheme.

The duality of $-\Delta$ and $|x|^2$ is evident from $\mathcal{F}$ interchanging differentiation and coordinate multiplication as follows

$$\begin{aligned} \mathcal{F} \circ \partial_{x_j} &= -i\, y_j \circ \mathcal{F} \\ \mathcal{F} \circ x_j &= -i\, \partial_{y_j} \circ \mathcal{F} \end{aligned} \qquad (j = 1, \ldots, n).$$

This property uniquely characterizes the Fourier transform, though from the point of view of Δ, a natural follow-up question now arises: Is the Fourier transform unique in linking the naturally coupled quantities Δ and $|x|^2$? As the answer turns out to be negative, can we suitably restrict the set of all operators having this property to a select class of interesting ones? This forms the starting point of the matter covered in the first part of Chapter 1.

In a second part of Chapter 1, we address the same question for the related first-order Dirac operator. This square root of the Laplacian was first considered by the theoretical physicist Paul Dirac to make the Schrödinger equation of quantum mechanics consistent with Einstein's theory of relativity. This led to the formulation of the celebrated Dirac equation, which describes the behavior of spin-$\frac{1}{2}$ particles such as electrons and quarks that make up what is classically known as matter. The Dirac equation predicted the existence of the previously unsuspected and unobserved antimatter, which was later experimentally confirmed. We will get back to the Dirac equation in Chapter 3.

The algebraic structure governing the matrices which Dirac used to construct the equation named after him, is but one realization of what is more generally called a Clifford algebra [11]. In this context, the Fourier dual of the Dirac operator $\underline{D}$ is the vector variable $\underline{x}$ satisfying $\underline{x}^2 = |x|^2$. Together, $\underline{D}$ and $\underline{x}$ satisfy the defining relations of the Lie superalgebra $\mathfrak{osp}(1|2)$:

$$[\{\underline{D}, \underline{x}\}, \underline{D}] = -2\underline{D}, \qquad [\{\underline{D}, \underline{x}\}, \underline{x}] = 2\underline{x}\,,$$

where now $\{X, Y\} = XY + YX$ is the anticommutator. This superalgebra contains as its even part the $\mathfrak{sl}(2)$ Lie algebra generated by $-\Delta$ and $|x|^2$.

The objective thus reads: Determine the Fourier-like operators intertwining the naturally coupled quantities Δ and $|x|^2$ on the one hand, and $\underline{D}$ and $\underline{x}$ on the other hand. Both for the Laplace and the Dirac case, the question regarding uniqueness was translated to one concerning symmetries, by means of the operator exponential formulation of $\mathcal{F}$. In general, a symmetry is understood in the sense of an operation leaving an object unchanged. For an operator T which itself acts on objects, by symmetries we mean operators commuting with T. In the setting of superalgebras, it is natural to consider also operators anticommuting with T. The composition of two symmetries of an operator T is of course again a symmetry, so we have an algebra: the symmetry algebra of T. This theme of symmetries will lead us to the next part of our journey.

After the project pertaining to Fourier transforms was finished, we set out with a new aim in mind. To this end, we were inspired by recent work on generalizations of the Dirac and Laplace operator involving reflection groups. The so-called Dirac-Dunkl and Laplace-Dunkl operator are obtained by replacing partial derivatives by deformations thereof in the form of Dunkl operators [12, 22]. The latter are differential-difference operators associated to a root system with a corresponding Weyl group of reflections. The reflection terms in this deformation are accompanied by parameters which can be set to zero to recover the regular partial derivatives. A precise definition of the Dunkl operators is given in Chapter 2.

In recent work [9], a specific case of a Dirac-Dunkl operator in three dimensions was considered, namely for the most straightforward root system, corresponding to the abelian reflection group $(\mathbb{Z}_2)^3$. Here, $\mathbb{Z}_2$ is the cyclic group of order 2. The symmetry algebra of this Dirac operator was shown to be equivalent with (a central extensions of) the so-called Bannai-Ito algebra [8, 23], which appeared as the algebraic structure encoding the bispectral properties of the Bannai-Ito polynomials. The latter were first proposed as a $q \to -1$ limit of the q-Racah polynomials and shown to be the most general orthogonal polynomial systems satisfying the Leonard duality property [6, 23]. In abstract form, the Bannai-Ito algebra is the associative algebra with three generators K_1, K_2, K_3 governed by the relations

$$\{K_1, K_2\} = K_3 + \omega_3\,, \ \{K_2, K_3\} = K_1 + \omega_1\,, \ \{K_3, K_1\} = K_2 + \omega_2\,,$$

and scalars $\omega_1, \omega_2, \omega_3$. In the realization as symmetry algebra of the Dirac-Dunkl operator, these scalars depend on the Dunkl deformation parameters.

When the Dunkl parameters are all set to zero, the $(\mathbb{Z}_2)^3$ Dirac-Dunkl operator reduces to the regular three-dimensional Dirac operator on Euclidean space. It is a classical result that the latter is invariant under a realization of the angular momentum algebra, the Lie algebra $\mathfrak{so}(3)$. Indeed, when $\omega_1 = \omega_2 = \omega_3 = 0$ the Bannai-Ito algebra is in fact equivalent to $\mathfrak{so}(3)$ (or $\mathfrak{su}(2)$) supplemented with an involution. The study of the symmetry algebra of generalizations of the Dirac operator thus provides a recipe to obtain extensions of the angular momentum algebra $\mathfrak{so}(3)$. For the Dunkl case, these extensions are deformations including parameters, which reduce to the classical $\mathfrak{so}(3)$ algebra when the parameters are set to zero. When moving up to arbitrary dimension n, the same holds true for the Lie algebra $\mathfrak{so}(n)$ as the symmetry algebra of the Dirac operator on $\mathbb{R}^n$. A higher rank version of the Bannai-Ito was postulated in this manner as the symmetry algebra of the $(\mathbb{Z}_2)^n$ Dirac-Dunkl operator [10].

We thus set out on our exploration for symmetries of a specific case of the Dirac-Dunkl operator, namely for the root system of type A_{n-1}. The associated Weyl group of this root system is the symmetric group on n elements, denoted by S_n. The three-dimensional setup then provides the toy model, being sufficiently far from trivial, yet inducing computations which remain somewhat manageable. The insights obtained here, led not only to a systematic approach for the n-dimensional case, but also to a casting of our problem into a bigger class of generalized Laplace and Dirac operators. To explain this relocation, we will take a slight detour to the quantum-mechanical world.

The Fourier transform, the Laplace and the Dirac operator all play a prominent role in the theory of quantum physics. One of the most important models in quantum mechanics is that of the quantum harmonic oscillator. It is one of the few quantum-mechanical systems where explicit expressions for the exact solution are known, and it forms a sound approximation for more involved systems when near a stable equilibrium. For the n-dimensional harmonic oscillator, the Hamiltonian corresponding to the total energy of the system is given by

$$\hat{H} = \frac{1}{2}\sum_{j=1}^{n} \hat{p}_j^2 + \frac{1}{2}\sum_{j=1}^{n} \hat{x}_j^2 ,$$

in units with mass, frequency and the reduced Planck constant $\hbar$ all equal to 1. When establishing a quantum mechanical analog of the classical harmonic oscillator, one has to accommodate, in a mathematical framework, for quantum phenomena at small scales and low energy levels. Experimental observations include objects having characteristics of both particles and waves, the discrete nature of certain physical quantities, and an inherent uncertainty as to the precision certain pairs of physical quantities can be known. Mathematically, observable physical quantities are translated to operator representations, with commuting operators corresponding to physical properties which can be defined with arbitrary precision. Operators do not necessarily commute, a property which is used to represent these uncertainty limits for certain pairs of observables.

In canonical quantization the position operators $\hat{x}_j$ and their associated momentum operators $\hat{p}_j$ are Fourier duals, satisfying the canonical commutation relations $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\delta_{i,j}$ (again with $\hbar = 1$). In the coordinate representation, the position operator $\hat{x}_j$ stands for multiplication by the position coordinate variable x_j and hence the associated momentum operator becomes $\hat{p}_j = -\mathrm{i}\partial_{x_j}$. The Hamiltonian $\hat{H}$ then corresponds, up to an additive constant, to the function H we encountered already in the exponent of the Fourier transform. The kinetic energy operator is thus being represented by the Laplace operator Δ. It is in this context that our next research questions will arise.

Eugene Wigner rightfully noticed that the canonical commutation relations are not the only way to accommodate quantization of a classical system [25]. Without a physical justification for imposing such relations, the equations of motion permit also other, more fundamental, quantization procedures. Another main argument to relinquish the canonical commutation relations, is that they are not compatible with finite-dimensional Hilbert spaces, and hence with discrete models having only a finite number of modes. We will get back to this second issue in Chapters 4–6. For the quantum harmonic oscillator, in the Wigner framework, the (self-adjoint) position operators $\hat{x}_1, \ldots, \hat{x}_n$ and momentum operators $\hat{p}_1, \ldots, \hat{p}_n$ are to satisfy the compatibility conditions of the Hamiltonian as generator of time evolution with the equations of motion, that is

$$[\hat{H}, \hat{x}_j] = -\mathrm{i}\hat{p}_j, \qquad [\hat{H}, \hat{p}_j] = \mathrm{i}\hat{x}_j \qquad (j = 1, \ldots, n).$$

These relations are also called the Hamilton-Lie equations. Here, canonical quantization is but one possible solution.

Now, the problem we were originally interested in was the study of symmetries of generalizations of the Dirac and Laplace operator involving reflection groups, namely the Dirac-Dunkl and Laplace-Dunkl operator. This problem may appear totally unrelated to our quantum-mechanical detour. However, it can be cast in the form of a Wigner system. The Dunkl operators are valid candidates for the momentum operators $\hat{p}_1, \ldots, \hat{p}_n$ under the previously stated compatibility conditions. In this Wigner system, the Laplace-Dunkl operator takes the role of the kinetic energy term, which was canonically played by the ordinary Laplace operator. Working in this more general framework not only helped us in finding symmetries for the Dirac-Dunkl and Laplace-Dunkl operators, but also for a bigger class of abstract operators. The formulation of symmetries of these Dirac-like and Laplace-like operators, and the algebraic relations governing the symmetry algebra, is the topic of Chapter 2.

While Chapter 2 deals with the symmetry algebra in an abstract fashion, the natural follow-up investigation is one of representation theoretic nature. The classification of all possible representations of an algebraic structure is valuable both from a purely mathematical point of view and because of potential applications in, for instance, physical models. This forms the impetus for Chapter 3. Armed with the results obtained in Chapter 2, we return to the three-dimensional case of the Dirac-Dunkl operator for the root system A_2 with non-abelian reflection group S_3. In the context of explicit realizations of these representations, it is natural to consider also the Dirac equation associated to this Dirac-like operator. The study of the S_3 Dirac-Dunkl operator symmetry algebra and its representations thus forms the starting point for Chapter 3.

Finally, we return to the other reason for resorting to Wigner quantization: quantum systems modeled by a finite-dimensional Hilbert space. For simplicity, we restrict ourselves to the one-dimensional case. Specifically, what we have in mind are models, or extensions, of the quantum harmonic oscillator, where the energy and the position variable can only take on a finite number of discrete values. Such models are of interest for applications in quantum optics and signal analysis [3–5]. This finiteness is achieved through working in a finite-dimensional representation of a suitable underlying algebraic structure. We will first elaborate on the context and give a brief overview of some different existing quantum oscillator models.

In the one-dimensional case, we have only one momentum operator $\hat{p}$ and one position operator which we will denote by $\hat{q}$. The Hamiltonian $\hat{H}$ is again the generator of time evolution. These operators are assumed to be essentially self-adjoint. The Hamilton-Lie equations, which state the compatibility of Hamilton's equations with the Heisenberg equations, now read

$$[\hat{H}, \hat{q}] = -\mathrm{i}\hat{p}\,, \qquad [\hat{H}, \hat{p}] = \mathrm{i}\hat{q}\,,$$

again in units with mass, frequency and the reduced Planck constant $\hbar$ all equal to 1. For the one-dimensional canonical quantum oscillator, these relations close into an algebraic structure by means of the canonical commutation relation $[\hat{q}, \hat{p}] = \mathrm{i}$. This structure is the oscillator Lie algebra of quantum mechanics, containing the Heisenberg algebra. The position wavefunctions of this model are given in terms of Hermite polynomials, a type of classical (continuous) orthogonal polynomials, contained also in the Askey-scheme. We will see that this is but one example of an intrinsic relation between oscillator models, their algebraic description and special functions.

An alternative model is the so-called parabose oscillator or Wigner quantum oscillator [20, 21, 25]. Wigner considered a system where the canonical commutation relation was dropped, but the Hamiltonian retained the classical form

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2).$$

The underlying algebraic structure then turns out to be the Lie superalgebra $\mathfrak{osp}(1|2)$. This leads to working in an infinite-dimensional representation of $\mathfrak{osp}(1|2)$, which inherently contains a parameter. This parameter induces a shift of the energy levels, which remain equidistant nonetheless. For a specific value of the representation parameter, the paraboson oscillator reduces to the canonical model. Likewise, the position wavefunctions are now given in terms of Laguerre polynomials, containing also this parameter, of which the Hermite polynomials are a special case for this specific parameter value.

In general, an algebraic recipe for a quantum oscillator model contains three (self-adjoint) operators $\hat{q}$, $\hat{p}$, $\hat{H}$ which satisfy the Hamilton-Lie equations. They are moreover assumed to belong to some algebraic structure, such as the enveloping algebra of a Lie algebra or Lie superalgebra. A physical property which is usually imposed is that the spectrum of $\hat{H}$ in a unitary representation of this algebra is equidistant. To obtain a finite oscillator model, we require an algebra which has finite-dimensional representations. The discrete position values then correspond to the spectrum of the position operator $\hat{q}$. These values are real due to the self-adjointness of $\hat{q}$, though a priori they are not necessarily equidistant. The usefulness as a practical model depends on the structure of these position values. We will see that the, now discrete, position wavefunctions in this case will be given in terms of discrete orthogonal polynomials.

The simplest example is the so-called $\mathfrak{su}(2)$ finite oscillator model [1, 2]. Here, the Hamiltonian, position and momentum operator are compatible with the Hamilton equations and the equations of motion, but instead of satisfying the canonical commutation relation, they close into the Lie algebra $\mathfrak{su}(2)$. This algebra is usually defined [15, 26] by its basis elements J_0, J_+, J_- with commutators $[J_0, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_0$. For a system with N equidistant possible position values, the model at hand is the standard finite (irreducible) $\mathfrak{su}(2)$ representation of dimension $N = 2j + 1$, where j is a non-negative integer or half-integer. The position (and momentum) operators then have as spectrum the set $\{-j, -j + 1, \ldots, +j\}$, corresponding to the N position values. The discrete wavefunctions can be written in terms of symmetric Krawtchouk polynomials, a specific case of a family of finite and discrete orthogonal polynomials classified also in the discrete side of the Askey-scheme of hypergeometric orthogonal polynomials. In the limit $N \longrightarrow \infty$, the Krawtchouk wavefunctions (after rescaling) tend to the common continuous wavefunctions of the harmonic oscillator in terms of Hermite functions.

It was observed that the $\mathfrak{su}(2)$ finite oscillator model could be extended by the addition of new elements to the algebra and through introducing parameters into the algebraic relations. This lead to the formulation of the one-parameter deformations $\mathfrak{u}(2)_\alpha$ and $\mathfrak{su}(2)_\alpha$, both of which are connected to a finite oscillator model [16, 17]. For a specific value of the parameter α, they coincide with the $\mathfrak{su}(2)$ model. The $\mathfrak{u}(2)_\alpha$ algebra, which was investigated first, was constructed in such a way that the representations of this algebra are suitable extensions of the $\mathfrak{su}(2)$ model. However, this was possible only for the even-dimensional representations. As it turned out, a different deformation was required to extend the odd-dimensional representations, which brought about the $\mathfrak{su}(2)_\alpha$ algebra. A remarkable property is that the discrete spectrum of the position operator in these representations no longer remains equidistant, and becomes dependent of the deformation parameter α.

For both models, the discrete position wavefunctions could be expressed in terms of another type of discrete orthogonal polynomials in the Askey-scheme, namely the so-called (dual) Hahn polynomials. These polynomials generally contain two independent parameters, though for the cases at hand, they appeared with both parameters specified in terms of the deformation parameter α. More specifically, the wave functions of even and odd degree were characterized by dual Hahn polynomials with different sets of parameters. The action of the position operator on these wavefunctions boiled down to a pair of recurrence relations for the sets of dual Hahn polynomials with different parameters. In turn, this is equivalent to a pair of contiguous relations for (generalized) hypergeometric functions similar to the ones explored already by Gauss. The question which thus pops up is: Are there other ways to combine dual Hahn polynomials with different parameters? Furthermore, can we use them to obtain interesting finite oscillator models?

The act of combining two sets of orthogonal polynomials to obtain a new family was investigated already by Chihara and others [7, 19, 24]. In essence, one obtains a kernel partner for a set of orthogonal polynomials by means of the Christoffel-Geronimus transform and a given parameter. A new family of orthogonal polynomials is then constructed by interweaving the original set of polynomials and this kernel partner. The question of combining dual Hahn polynomials with different parameters thus becomes: For which Christoffel parameter is the kernel partner of a dual Hahn polynomial again a dual Hahn polynomial with possibly different parameters? The follow-up question remains the same: Do these combinations lead to interesting finite oscillator models, in particular, having an equidistant position spectrum? The answers to these questions will be affirmative and form the basis for Chapter 4 and the subsequent chapters.

PLAN OF ACTION

Recall that for our first objective we want to investigate whether there are operators on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$, other than the Fourier transform, which intertwine $-\Delta$ and $|x|^2$. The latter two operators are invariant under $O(n)$, the group of orthogonal transformations on $\mathbb{R}^n$. This orthogonal invariance has two important consequences. First, there is a class of trivial solutions obtained through the composition of the classical Fourier transform with an orthogonal transformation, since this yields another operator with the desired property. In order to avoid such trivialities, we shall consider two solutions to our problem as being equivalent if they are linked by means of an orthogonal transformation. Second, the orthogonal invariance will provide a useful tool in the form of a classical result known as Howe-duality. The space $\mathcal{S}(\mathbb{R}^n)$ admits a natural decomposition in irreducible subspaces under the action of the dual pair $(O(n), \mathfrak{sl}(2))$. These subspaces have a basis in terms of Laguerre polynomials which are eigenfunctions of the Fourier transform with a natural accompanying action of $\mathfrak{sl}(2)$ as realized by $-\Delta$ and $|x|^2$. It will be helpful to describe operators by their action on this basis. Paired with this basis of eigenfunctions is the operator exponential formulation of the classical Fourier transform. Here in the exponent of the imaginary unit i appears a specific element of $\mathfrak{sl}(2)$ with a diagonal action on the aforementioned basis and integer eigenvalues. As an extension of this approach, we consider exponentials of other diagonal elements of $\mathfrak{sl}(2)$ and related algebraic structures.

To effectively determine solutions, the property of intertwining $-\Delta$ and $|x|^2$ is first translated to commuting or anticommuting with appropriate elements in the $\mathfrak{sl}(2)$ realization. In order to construct such symmetries we make use of the Casimir element Ω in the center of the universal enveloping algebra of $\mathfrak{sl}(2)$. In this way we obtain a class of solutions containing exponentials of polynomials or infinite series in Ω. These functions must satisfy the extra requirement of taking on only integer values when evaluated at the eigenvalues of Ω. Moreover, their values matter only up to a modulo 4 congruency. Depending on the parity of the dimension we work in, the eigenvalues of Ω are squares of either integers or half-integers. To describe all the solutions we thus define an appropriate polynomial basis for each situation. For every solution defined as an operator exponential, there exists a related formulation as an integral transform where its kernel is given by a plane wave decomposition in terms of Bessel functions and Gegenbauer polynomials. To further restrict the class of solutions to a finite set of interesting operators, we impose an extra condition in the form of a periodicity restriction on the eigenvalues, as is the case for the classical Fourier transform. We proceed with a complete classification and description of all solutions satisfying these extra conditions. For this subset of solutions, we are able to obtain explicit closed form expressions for the associated kernel when written as an integral transform. Finally, we demonstrate that these transforms satisfy an uncertainty principle.

Next, the same procedure is repeated for the Dirac operator in the more general setting of Clifford algebra-valued functions. Here, the Lie algebra $\mathfrak{sl}(2)$ is extended to the Lie superalgebra $\mathfrak{osp}(1|2)$. We now use the Casimir element C in the center of the universal enveloping algebra of $\mathfrak{osp}(1|2)$ to construct the desired operators. We again classify the subset of solutions which satisfy a periodicity restriction, for which we determine also explicit closed form expressions for the associated integral kernels.

In Chapter 2, we are dealing with a generalized Laplace operator

$$\Delta = \sum_{j=1}^{n} p_j^2 ,$$

in the context of a Wigner quantum system. For ease of notation and computation, we have substituted the physical momentum $\hat{p}_j$ by operators $p_j = \mathrm{i}\hat{p}_j$, and similarly we use x_j to denote the position operator $\hat{x}_j$. We work in the algebraic structure $\mathcal{A}$ generated by n commuting position operators $x_1, \ldots, x_n$ and n commuting momentum operators $p_1, \ldots, p_n$ subject to the relations

$$[\Delta, x_j] = 2p_j , \qquad [|x|^2, p_j] = -2x_j .$$

First, by means of these relations, it is shown that the generalized Laplace operator Δ and its partner the squared norm $|x|^2$ generate a realization of the Lie algebra $\mathfrak{sl}(2)$, just as was the case for the classical Laplacian.

Instead of looking for symmetries of just Δ, we will keep momentum and position operators on equal footing. The goal is to determine the elements in $\mathcal{A}$ which commute with both Δ and its conjugate partner $|x|^2$, and thus with the entire $\mathfrak{sl}(2)$ realization. Symmetries obtained in this way will contain an equal number of position and momentum operators, so we repress trivialities such as functions of $p_1, \ldots, p_n$ which obviously commute with Δ as $[p_i, p_j] = 0$. The most basic (non-trivial) symmetries in $\mathcal{A}$ will then be linear in the momentum operators, and thus also in the position operators. Next, we establish the algebraic relations satisfied by these symmetries and go over the main examples in the setting of Dunkl operators.

In the second part, the generalized Dirac operator $\underline{D}$ is realized as square root of Δ using a Clifford algebra $\mathcal{C}$. We show that the algebraic structure in the form of the tensor product $\mathcal{A} \otimes \mathcal{C}$ contains an $\mathfrak{osp}(1|2)$ realization, as was the case for the regular Dirac operator. In this setting of superalgebras, we consider also anticommuting symmetries and we start by classifying all operators that commute or anticommute with the $\mathfrak{osp}(1|2)$ realization. Next, we determine the quadratic relations satisfied by these symmetries, beginning with the simplest interactions of symmetries. A helpful guideline in doing so, are the results obtained already for the $\mathbb{Z}_2^N$ Dirac-Dunkl operator. For this relatively simple case, the symmetries and their algebraic relations have been determined and give rise to the (higher rank) Bannai-Ito algebra [9, 10]. Hence, our general algebraic structure should coincide with these results when $p_1, \ldots, p_n$ are set equal to the Dunkl operators associated to that specific root system.

In Chapter 3, we consider in detail a specific type of Dirac-Dunkl operator in three dimensions. The general relations for the symmetry algebra we obtained earlier then take on an explicit form. The case we have in mind is that of the root system A_2 with Weyl group S_3, the symmetric group on three elements. The symmetry algebra then becomes a one-parameter deformation of the classical angular momentum algebra, the Lie algebra $\mathfrak{so}(3)$, incorporating elements of S_3. To classify all finite-dimensional irreducible representations of this algebra in abstract fashion, we first construct a form of ladder operators. Starting from a highest weight vector, being an eigenvector of a set of mutually commuting elements of our algebra, we build our representation space using the ladder operators. Next, we determine whether the obtained representations are unitary. Finally, we return to the realization in the framework of Dunkl operators and we construct explicit expressions for the wavefunctions on which the symmetry algebra acts as a unitary irreducible representation.

The final chapters originated from the context of finite oscillator models. To construct new models, we want to classify all pairs of recurrence relations for two sets of dual Hahn polynomials with different parameters. Equivalently, such a pair of recurrence relations corresponds to a Christoffel-Geronimus transform where the kernel partner of a dual Hahn polynomial is again of this same family, with different parameters. On the level of hypergeometric series this translates to a pair of contiguous relations. For our classification, we begin with a general pair of recurrence relations linking two sets of polynomials and determine conditions for the coefficients. These two recurrence relations can be combined to obtain three-term recurrence relations. This, we can compare to the well-known three-term recurrence relation for the dual Hahn orthogonal polynomials, and solve to find all possible coefficients. The same classification process can then also be applied to other discrete orthogonal polynomials such as Hahn and Racah polynomials.

Next, we elaborate upon the link with finite oscillator models. In the previously obtained models, the dual Hahn polynomials appeared as discrete position wavefunctions. Being of discrete nature, the considered pairs of recurrence relations can be cast in matrix form. In this way, an eigenvalue equation arises where each eigenvector contains, in alternating order, the two families of dual Hahn polynomials evaluated in the discrete position values. Such an eigenvector corresponds to a polynomial in a new orthogonal system, of which the tridiagonal Jacobi matrix will play the role of the position operator in a finite oscillator model. The spectrum of this matrix then yields the possible position values, and its eigenvectors correspond to the position wavefunctions.

The matrices appearing in this way can be seen as representation matrices of deformations or extensions of $\mathfrak{su}(2)$ through which the algebraic relations can be determined. In general, the dual Hahn polynomials contain two independent parameters, which are also present in the algebraic structure and in the recurrence relations. The ensuing task is to check which parameters values lead to suitable algebraic relations and position values for constructing finite oscillator models. This is the subject of the next two chapters.

In Chapter 5, we investigate in particular a finite oscillator model which has the potential to have equidistant position values. The related algebraic structure is an extension of $\mathfrak{su}(2)$ by a parity operator P, which we refer to as the algebra $\mathfrak{su}(2)_P$. Before getting to the oscillator model, we classify all irreducible unitary finite-dimensional representations of this algebra. Next, we use the obtained odd-dimensional representations to construct a finite oscillator model with equidistant position values related to the algebra $\mathfrak{su}(2)_P$. The orthonormal eigenvectors of the position and momentum operator form the corresponding wavefunctions, which are given in terms of the previously determined pair of dual Hahn polynomials.

In Chapter 6, we develop a finite oscillator model pertaining to a pair of Racah polynomials. This pair was also obtained in the previous classification, where their Jacobi matrix was observed to have equidistant eigenvalues for specific values of the Racah polynomial parameters. Contrary to Chapter 5, we have models for an odd number as well as for an even number of position values. We investigate the properties of the discrete position wavefunctions and plot the lowest energy states for some specific parameter values.

The final chapter is a spin-off of the previously obtained results. Recall that the classified pairs of recurrence relations could be cast in the form of an eigenvalue equation. Here, we have a matrix with a well-defined tridiagonal structure and explicit expressions for both eigenvalues and eigenvectors. Moreover, as the original discrete polynomials contain parameters, these are present also in the eigenvalues and matrix entries. What we have are perfect candidates to test the accuracy of numerical eigenvalue computations. Our matrices are in fact extensions of a standard test matrix which goes by several names: the Sylvester-Kac matrix, the Kac matrix, the Clement matrix. This tridiagonal with zero diagonal matrix has simple integer entries and eigenvalues. We investigate the use of the new classes of test matrices by comparing the exact known eigenvalues with those computed using the inherent MATLAB function `eig()` for different parameter values.

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Part II

MAIN STORY

1

Generalized Fourier Transforms Arising from the Enveloping Algebras of $\mathfrak{sl}(2)$ and $\mathfrak{osp}(1|2)$

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abstract

The Howe dual pair ($\mathfrak{sl}(2)$, $O(m)$) allows the characterization of the classical Fourier transform (FT) on the space of rapidly decreasing functions as the exponential of a well-chosen element of $\mathfrak{sl}(2)$ such that the Helmholtz relations are satisfied. In this paper we first investigate what happens when instead we consider exponentials of elements of the universal enveloping algebra of $\mathfrak{sl}(2)$. This leads to a complete class of generalized Fourier transforms, that all satisfy properties similar to the classical FT. There is moreover a finite subset of transforms which very closely resemble the FT. We obtain operator exponential expressions for all these transforms by making extensive use of the theory of integer-valued polynomials. We also find a plane wave decomposition of their integral kernel and establish uncertainty principles. In important special cases we even obtain closed formulas for the integral kernels. In the second part of the paper, the same problem is considered for the dual pair ($\mathfrak{osp}(1|2)$, $Spin(m)$), in the context of the Dirac operator on $\mathbb{R}^m$. This connects our results with the Clifford-Fourier transform studied in previous work.

1 introduction

The classical Fourier transform (FT) over $\mathbb{R}^m$ is given by

$$(\mathcal{F}f)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \mathrm{e}^{i\langle x,y\rangle} f(x)\, \mathrm{d}x$$

and is of crucial importance in many aspects of harmonic analysis and signal processing.

Recently, various extensions of the FT have been investigated by rewriting the transform as an operator exponential. Indeed, introducing the operators

$$\Delta_x := \sum_{i=1}^{m} \partial_{x_i}^2, \qquad |x|^2 := \sum_{i=1}^{m} x_i^2, \qquad \mathbb{E}_x := \sum_{i=1}^{m} x_i \partial_{x_i}$$

with Δ_x the Laplace operator and $\mathbb{E}_x$ the Euler operator, one has

$$\mathcal{F} = e^{-i\frac{\pi}{4}m} e^{i\frac{\pi}{4}(-\Delta_x + |x|^2)} \tag{1.1}$$

which relates the transform with the representation theory of the Lie algebra $\mathfrak{sl}(2)$ as the operators $E = |x|^2/2$, $F = -\Delta_x/2$ and $H = \mathbb{E}_x + m/2$ satisfy

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H, \tag{1.2}$$

see for example [23, 28] for a detailed mathematical treatment.

By now well-established fields of research based on this observation include the fractional FT [39] and the family of linear canonical transforms [45]. Both have extensive practical applications in the design of optical systems and signal processing.

The observation in formula (1.1) has also led to many other theoretically oriented generalizations of the FT, by considering alternative realizations of $\mathfrak{sl}(2)$ in terms of differential or difference operators (or combinations thereof). Once such a realization is obtained, it is possible to define a generalized FT based on formula (1.1). It is then a challenging question to find a concrete integral transform expression for this operator. Research on this topic has a long history: it has been investigated in both the continuous and the discrete case, and also for more complicated algebras than $\mathfrak{sl}(2)$, such as the superalgebra $\mathfrak{osp}(1|2)$. Important examples in the continuous case include the Dunkl transform [2, 19] and the radially deformed Fourier transform [3, 14, 35, 36] as well as its Clifford deformation [15, 17]. In more complicated geometries, the super Fourier transform [12] and a q-deformed version [10] have been investigated. In the discrete case we mention the fractional Fourier–Kravchuk transform [1] and various further deformations [32, 33]. For a more detailed review of the followed strategy and the results in this line of investigation we refer the reader to [13, 37].

The present paper has a related yet different aim. We do not intend to change the realization of $\mathfrak{sl}(2)$ in formula (1.2), but instead wish to study precisely how unique the operator realization of the FT in (1.1) is and to what extent the interplay of $\mathfrak{sl}(2)$ and its Howe dual $O(m)$ fixes the FT (see [29]). We are specifically interested in determining whether any other operators portray similar behaviour.

A crucial property of the classical FT is its interaction with differential operators. In particular, we have for $j = 1, \dots, m$

$$\begin{aligned} \mathcal{F} \circ \partial_{x_j} &= -i\, y_j \circ \mathcal{F} \\ \mathcal{F} \circ x_j &= -i\, \partial_{y_j} \circ \mathcal{F}, \end{aligned} \tag{1.3}$$

which uniquely determines the FT and its integral kernel up to a normalization constant. We can relax (1.3) to a more general interaction with differential operators featuring now the elements of the $\mathfrak{sl}(2)$ realization that appear in the operator exponential formulation (1.1) of the FT (which also takes into account the $O(m)$ symmetry). Hereto, let

$$(Tf)(y)= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K(x,y)f(x)\,\mathrm{d}x$$

be an integral transform on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^m)\subset L^2(\mathbb{R}^m)$; we say that T satisfies the Helmholtz relations if

$$\begin{aligned} T\circ\Delta_x &= -|y|^2\circ T \\ T\circ|x|^2 &= -\Delta_y\circ T\,. \end{aligned} \tag{1.4}$$

These relations no longer have the classical FT as unique solution for the operator T. Only when imposing the extra condition that T must be the exponential of an element of $\mathfrak{sl}(2)=\operatorname{span}\{\Delta_x, |x|^2, [\Delta_x, |x|^2]\}$ do they yield as unique solution the FT (or its inverse) up to a normalization constant [28, 37].

The aim of our paper is to look for generalized Fourier transforms, satisfying properties similar to the classical FT. We do this by investigating what other solutions the Helmholtz relations (1.4) have, where we make explicit use of the $\mathfrak{sl}(2)$ relations in (1.2) and their $O(m)$ invariance. Indeed, we wish to analyse in detail what happens when considering T an operator exponential from the universal enveloping algebra of $\mathfrak{sl}(2)$. If we combine this with a periodicity restriction on the eigenvalues, as is the case for the FT, we are led to an interesting finite set of new generalized Fourier transforms that behave in a way very similar to the classical FT. In Theorem 2.13, we establish for all transforms in this set their operator exponential formulation. Subsequently, in Theorem 2.16, we are even able to determine explicit integral kernels for some of these transforms, thus realizing them as integral transforms. A crucial role for reaching these results is played by the Casimir Ω of $\mathfrak{sl}(2)$ and the (very technical) study of integer-valued polynomials in Ω.

In the more general context of Clifford-algebra valued functions, where the Dirac operator takes over the role of the Laplace operator (see [20, 27]) and the Lie superalgebra $\mathfrak{osp}(1|2)$ that of $\mathfrak{sl}(2)$, quite a bit of attention has already been paid to transforms consisting of a specific operator exponential from the universal enveloping algebra of $\mathfrak{osp}(1|2)$. Most notably, the so-called Clifford–Fourier transform [7, 8, 16, 18] is defined as such an operator exponential.

To provide a solid motivation for the study of this transform, we will also treat the case of Clifford analysis with the tools developed in the present paper. We again find an interesting class of generalized, now essentially non-scalar Fourier transforms that behave similarly to the classical FT, see Theorem 3.10 for their operator exponential formulation and Theorem 3.13 for explicit kernels realizing them as integral transforms.

The resulting generalized FTs that we obtain in both the harmonic and Clifford case exhibit properties and behaviour similar to the FT. To emphasize that, we will also show that they, for example, still satisfy a version of the uncertainty principle (see e.g., [24] for a review). This is achieved in Theorem 2.21 and Theorem 3.6.

The paper is organized as follows. We commence Section 2 by laying out the specific properties we will use as a starting point to determine generalized FTs and introducing the relevant $\mathfrak{sl}(2)$ and $O(m)$ representation theory. Next, we work out the requirements for our operators which brings us to the concept of integer-valued polynomials. These polynomials allow us to give an explicit expression for the desired operators. For a subset of solutions which satisfy a periodicity restriction we give a complete classification and we also obtain explicit integral kernels for these transforms. Finally, we establish a generalized uncertainty principle. In Section 3 we lift our objective to the setting of Clifford analysis and apply the strategy we developed in the previous section to obtain analogous results. In particular, we again give a complete classification of the subset of solutions which satisfy a periodicity restriction and we obtain explicit integral kernels. Finally, in Section 4 we present some conclusions regarding our results, while in the appendix we give an overview of main definitions and results used in the main text, as well as some technical proofs that have been omitted from the text.

2 FOURIER TRANSFORMS IN HARMONIC ANALYSIS

We want to look for generalized Fourier transforms, satisfying similar properties to the classical FT. As interaction with differential operators we require the Helmholtz relations (1.4) and we will further impose two additional important properties of the FT. In particular, we require the eigenfunction basis of the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^m) \subset L^2(\mathbb{R}^m)$ to be preserved, which will be explained in more detail shortly.

This gives our first goal, which is to determine all operators $T : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m)$ that satisfy the following properties:

(i) the Helmholtz relations

$$\begin{aligned} T \circ \Delta_x &= -|y|^2 \circ T \\ T \circ |x|^2 &= -\Delta_y \circ T \end{aligned}$$

(ii) $T\, \phi_{j,k,\ell} = \mu_{j,k}\, \phi_{j,k,\ell}$ with $\mu_{j,k} \in \mathbb{C}$

(iii) $T^4 = \mathrm{id}$

Here, the standard eigenfunction basis of $\mathcal{S}(\mathbb{R}^m) \subset L^2(\mathbb{R}^m)$ is given by the Hermite functions

$$\phi_{j,k,\ell} := 2^j j!\, L_j^{\frac{m}{2}+k-1}(|x|^2) H_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}. \tag{2.1}$$

Here $j, k \in \mathbb{Z}_{\geq 0}$, L_j^{α} is a generalized Laguerre polynomial and $\{ H_k^{(\ell)} \}_{\ell=1}^{\dim(\mathcal{H}_k)}$ is a basis for $\mathcal{H}_k$, the space of spherical harmonics of degree k, that is, homogeneous polynomial null-solutions of the Laplace operator of degree k. This basis $\{\phi_{j,k,\ell}\}$ realizes the complete decomposition of $\mathcal{S}(\mathbb{R}^m)$ in irreducible subspaces under the natural action of the dual pair $(\mathfrak{sl}(2), O(m))$ [3]. The action of the Fourier transform on the eigenfunction basis is given by

$$\mathcal{F}\phi_{j,k,\ell} = \mathrm{e}^{i\frac{\pi}{2}(2j+k)}\, \phi_{j,k,\ell} = i^{2j+k}\, \phi_{j,k,\ell}. \tag{2.2}$$

Another way to write the Hermite functions (see [11]) is

$$\phi_{j,k,\ell} = \left(-\frac{\Delta_x}{2} - \frac{|x|^2}{2} + \mathbb{E}_x + \frac{m}{2}\right)^j H_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}. \tag{2.3}$$

Now, consider the following linear combinations of the operators E, F, H satisfying (1.2)

$$h = -\frac{\Delta_x}{2} + \frac{|x|^2}{2}, \quad e = -\frac{\Delta_x}{4} - \frac{|x|^2}{4} + \frac{1}{2}\mathbb{E}_x + \frac{m}{4}, \quad f = \frac{\Delta_x}{4} + \frac{|x|^2}{4} + \frac{1}{2}\mathbb{E}_x + \frac{m}{4}. \tag{2.4}$$

This triple generates another operator realization of the Lie algebra $\mathfrak{sl}(2)$ (see also [2, 3] for a more general situation). Indeed, by means of (1.2) one easily verifies that they satisfy the commutation relations

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{2.5}$$

Using these operators, the Helmholtz property (i) translates to the (anti-)commutation relations

$$T \circ h = h \circ T, \qquad T \circ e = -e \circ T, \qquad T \circ f = -f \circ T. \tag{2.6}$$

Moreover, in terms of these operators we can write property (2.3) more compactly as $\phi_{j,k,\ell} = (2e)^j \phi_{0,k,\ell}$, and the operator exponential formulation of the classical Fourier transform as

$$\mathcal{F} = \mathrm{e}^{i\frac{\pi}{4}(-\Delta_x + |x|^2 - m)} = \mathrm{e}^{i\frac{\pi}{2}(h - \frac{m}{2})}. \tag{2.7}$$

The action of the operators (2.4) on the basis (2.1) is as follows

$$h\,\phi_{j,k,\ell} = \left(2j+k+\tfrac{m}{2}\right)\phi_{j,k,\ell}, \quad e\,\phi_{j,k,\ell} = \frac{1}{2}\,\phi_{j+1,k,\ell}, \quad f\,\phi_{j,k,\ell} = -j\,(2j-2+m+2k)\,\phi_{j-1,k,\ell}.$$

Note that h is a diagonal operator while e works as a raising operator on the j-index and f as a lowering operator.

REMARK 2.1: For every $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \{1, \ldots, \dim(\mathcal{H}_k)\}$, the set $\{\, \phi_{j,k,\ell} \mid j \in \mathbb{Z}_{\geq 0} \,\}$ forms a basis for the positive discrete series representation of $\mathfrak{sl}(2)$ with lowest weight $k + m/2$. This is an irreducible unitary representation of the real form $\mathfrak{su}_{1,1}$ of $\mathfrak{sl}(2)$ (determined by the $*$-conditions: $h^* = h$, $e^* = -f$, $f^* = -e$).

For every $j, k \in \mathbb{Z}_{\geq 0}$ the set $\{\, \phi_{j,k,\ell} \mid \ell = 1, \ldots, \dim(\mathcal{H}_k) \,\}$ forms a basis for an irreducible representation of $O(m)$ for $m > 2$.

Before moving on to the explicit computation of solutions for the operator T, we first glance at some important consequences that hold for every operator satisfying the properties (i)–(iii).

LEMMA 2.2: *For T an operator that satisfies the properties (i)–(iii), there are only four possible values for the eigenvalues $\mu_{j,k}$ of T, namely $\mu_{j,k} \in \{1, i, -1, -i\}$. Moreover, the spectrum of eigenvalues is completely determined by the eigenvalues $\mu_{0,k}$ for $k \in \mathbb{Z}_{\geq 0}$; the other eigenvalues for $j > 0$ follow from the relation*

$$\mu_{j,k} = (-1)^j \mu_{0,k}. \tag{2.8}$$

Proof. Property (iii) necessitates that the eigenvalues $\mu_{j,k}$ of T satisfy $(\mu_{j,k})^4 = 1$ and thus are integer powers of $i = e^{i\frac{\pi}{2}}$. Using property (ii) and the relations (2.3) and (2.6), we find

$$\mu_{j+1,k}\,\phi_{j+1,k,\ell} = T\phi_{j+1,k,\ell} = T\circ(2e)\phi_{j,k,\ell} = -2e\circ T\phi_{j,k,\ell} = -2e\,\mu_{j,k}\,\phi_{j,k,\ell} = -\mu_{j,k}\,\phi_{j+1,k,\ell}$$

for all valid j, k, ℓ. Relation (2.8) now follows from subsequent application of $\mu_{j+1,k} = -\mu_{j,k}$. □

PROPOSITION 2.3: *Let $T : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m)$ be an operator that satisfies the properties (i) and (ii). Then, on the basis $\{\phi_{j,k,\ell}\}$, the operator T coincides with the integral transform*

$$(Tf)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x, y)\, f(x)\, \mathrm{d}x$$

where

$$K_m(x, y) = 2^\lambda\, \Gamma(\lambda) \sum_{k=0}^{+\infty} (k + \lambda)\, \mu_{0,k}\, z^{-\lambda} J_{k+\lambda}(z)\, C_k^\lambda(w). \tag{2.9}$$

Here, the following notations are used: $\lambda = (m-2)/2$, $z = |x||y|$, $w = \langle x, y\rangle/z$, and $J_{k+\lambda}$ are Bessel functions while C_k^λ denote the so-called Gegenbauer or ultraspherical polynomials (see Appendix A.1).

Proof. For $\xi, \eta \in \mathbb{S}^{m-1}$ and $H_\ell \in \mathcal{H}_\ell$, the following reproducing kernel formula holds (see e.g. [21])

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{S}^{m-1}} 2^\lambda\, \Gamma(\lambda)\, (k + \lambda)\, C_k^\lambda(\langle \xi, \eta\rangle) H_\ell(\xi)\, \mathrm{d}\xi = \delta_{k\ell}\, H_\ell(\eta). \tag{2.10}$$

Using the explicit form (2.1) of the eigenfunctions, together with formula (2.10) and the identity

$$\int_0^\infty r^{2\lambda+1+k} L_j^{k+\lambda}(r^2)\, e^{-r^2/2}\, (rs)^{-\lambda}\, J_{k+\lambda}(rs)\, \mathrm{d}r = (-1)^j\, s^k\, L_j^{k+\lambda}(s^2)\, e^{-s^2/2} \tag{2.11}$$

(see e.g. [43, exercise 21, p. 380]), it follows that

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x, y)\, \phi_{j,k,\ell}(x)\, \mathrm{d}x = (-1)^j\, \mu_{0,k}\, \phi_{j,k,\ell}(y).$$

By relation (2.8) for the eigenvalues of T, this integral transform coincides with T on the Hermite basis. □

LEMMA 2.4: *The kernel K_m satisfies*

$$\begin{aligned} K_m(Ax, y) &= K_m(x, Ay), \qquad && A \in O(m) \\ K_m(cx, y) &= K_m(x, cy), && c \in \mathbb{R}. \end{aligned} \tag{2.12}$$

Proof. This follows from the explicit formula (2.9) for $K_m(x, y)$. □

PROPOSITION 2.5: *A continuous operator $T : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}(\mathbb{R}^m)$ satisfying (i)–(iii) has a unitary extension to $L^2(\mathbb{R}^m)$.*

Proof. This result follows from the fact that $\mathcal{S}(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m)$ and that all eigenvalues have unit norm. □

We now continue to determine new operators T that satisfy properties (i)–(iii). We already know one solution, namely the classical Fourier transform. In order to find more solutions, we first introduce a new operator as follows. Assume we have an operator T that satisfies (i)–(iii), we then put $\widetilde{T} := T \circ \mathcal{F}^{-1}$. As the classical Fourier transform is an automorphism on $\mathcal{S}(\mathbb{R}^m)$, the operator $\widetilde{T}$ uniquely defines T and we can retrieve the operator T by the relation

$$T = \widetilde{T} \circ \mathcal{F}. \tag{2.13}$$

Our objective to obtain operators T is thus equivalent with finding suitable operators $\widetilde{T}$. Hereto we determine what the requirements are for such an operator $\widetilde{T}$ in order to yield an operator T that satisfies properties (i)–(iii).

Clearly, the basis $\{\phi_{j,k,\ell}\}$ must also form an eigenbasis of $\widetilde{T}$. Next, as $\mathcal{F}$ already satisfies the set of equations (2.6), we have that an operator T decomposed as (2.13) will satisfy the set of equations (2.6) if the operator $\widetilde{T}$ commutes with the operators h, e, f. A consequence of imposing this commutative property on $\widetilde{T}$ combined with $\mathcal{F}$ being of the form (2.7), is that $\widetilde{T}$ also commutes with $\mathcal{F}$. Property (iii) together with $\mathcal{F}^4 = \mathrm{id}$ then gives us $\widetilde{T}^4 = \mathrm{id}$. These requisites for the operator $\widetilde{T}$ naturally lead to the following result.

PROPOSITION 2.6: *Every operator $\widetilde{T}$ of the form*

$$\widetilde{T} = \exp\left(i\tfrac{\pi}{2}F\right), \tag{2.14}$$

with F an operator that

- *commutes with (the generators of)* $\mathfrak{sl}(2) = \operatorname{span}\{\Delta_x, |x|^2, [\Delta_x, |x|^2]\}$
- *has integer eigenvalues on the functions $\{\phi_{j,k,\ell}\}$ (independent of ℓ)*

will yield an operator T by (2.13) *that satisfies the properties (i)–(iii).*

Furthermore, in the next section we will show that for each operator satisfying the properties (i)–(iii), we have an equivalent operator of the form (2.14).

Now, we want to establish an explicit expression for the operator F in (2.14). Hereto we start by looking at the first condition listed in Proposition 2.6, which requires commuting operators. We denote by $\mathcal{U}(\mathfrak{sl}(2))$ the universal enveloping algebra of $\mathfrak{sl}(2)$. The center of $\mathcal{U}(\mathfrak{sl}(2))$ is the subset consisting of the elements that commute with all elements of $\mathfrak{sl}(2)$, and hence also with all elements of $\mathcal{U}(\mathfrak{sl}(2))$. The center is finitely generated by the Casimir element (see [30]):

$$\Omega = 1 + h^2 + 2ef + 2fe, \tag{2.15}$$

or in the framework of our operator realization: $\Omega = \left(\mathbb{E}_x + \frac{m-2}{2}\right)^2 - |x|^2\Delta_x$. Every polynomial function of the operator Ω will yield an operator F that commutes with the generators of $\mathfrak{sl}(2)$. This notion can be further generalized to include infinite power series in Ω. Such operators live in the extension $\overline{\mathcal{U}}\big(\mathfrak{sl}(2)\big)$ of the universal enveloping algebra that also allows infinite power series in the elements of $\mathfrak{sl}(2)$ [31].

The second condition in Proposition 2.6 is also facilitated by operators of this form as the Casimir element is a diagonal operator on the representation space $\operatorname{span}\{\, \phi_{j,k,\ell} \mid j \in \mathbb{Z}_{\geq 0} \,\}$. The eigenvalues of Ω are given by

$$\Omega\, \phi_{j,k,\ell} = \left(k + \frac{m}{2} - 1\right)^2 \phi_{j,k,\ell} = (k+\lambda)^2\, \phi_{j,k,\ell}, \tag{2.16}$$

with $\lambda = m/2 - 1$. Note that for even dimensions m, the eigenvalues of Ω are squares of integers, while for odd dimension, we have squares of half-integers (elements of $\mathbb{Z}+\frac{1}{2}$). In order to find expressions for F that have integer eigenvalues on the functions $\{\phi_{j,k,\ell}\}$, we need functions that take on integer values when evaluated at the eigenvalues of Ω. To this end, we invoke the notion of integer-valued polynomials. This is the subject of the following subsection.

2.1 *Integer-valued polynomials*

An integer-valued polynomial on $\mathbb{Z}$ is a polynomial whose value at every integer $n \in \mathbb{Z}$ is again an integer. We denote the set of all such polynomials by $\operatorname{Int}(\mathbb{Z}) = \{\, f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z} \,\}$. It is an elementary result that the polynomials

$$\binom{x}{n} = \prod_{\ell=0}^{n-1} \frac{x-\ell}{n-\ell} = \frac{1}{n!}\prod_{\ell=0}^{n-1}(x-\ell) \tag{2.17}$$

are integer-valued; moreover, they form a basis of the $\mathbb{Z}$-module $\operatorname{Int}(\mathbb{Z})$ (see e.g. [9]). The polynomial $\binom{x}{n}$ has degree n and its roots are the integers $\{0, 1, \dots, n-1\}$. The first few polynomials are given by

$$\binom{x}{0} = 1, \qquad \binom{x}{1} = x, \qquad \binom{x}{2} = \frac{1}{2}x^2 - \frac{1}{2}x, \qquad \binom{x}{3} = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x.$$

Now, we are interested in functions that take on integer values when evaluated at the eigenvalues of the Casimir operator Ω. As there is a disparity pertaining to the form of the eigenvalues (2.16), depending on the dimension m, we first handle the case where the dimension is even.

2.1.1 *On squares of integers*

For even dimension m, the eigenvalues (2.16) of Ω are squares of integers, which are of course again integers, so every integer-valued polynomial with Ω substituted for x is a valid solution for F in (2.14). However, the condition to be integer-valued on squares of integers is less restrictive than the requirement of being integer-valued on all integers. In order to specify the exact class of solutions, we introduce a special type of integer-valued polynomials. For $n = 0$, put $E_0(x) \equiv 1$, and for $n \in \mathbb{Z}_{\geq 1}$, put

$$E_n(x) = \prod_{\ell=0}^{n-1} \frac{x^2 - \ell^2}{n^2 - \ell^2} = \frac{2}{(2n)!} \prod_{\ell=0}^{n-1} (x^2 - \ell^2). \tag{2.18}$$

The polynomial E_n has degree $2n$ and its roots are the integers $\{0, \pm 1, \pm 2, \ldots, \pm(n-1)\}$, while $E_n(n) = 1$. The first few of these polynomials are given by

$$E_0(x) = 1, \qquad E_1(x) = x^2, \qquad E_2(x) = \frac{1}{12}x^4 - \frac{1}{12}x^2, \qquad E_3(x) = \frac{1}{360}x^6 - \frac{1}{72}x^4 + \frac{1}{90}x^2.$$

PROPOSITION 2.7: *The polynomials $\{ E_n(x) | \, n \in \mathbb{Z}_{\geq 0} \}$ are integer-valued on $\mathbb{Z}$.*

Proof. The case $n = 0$ is obvious. For $n \geq 1$ one has

$$E_n(x) = \frac{2}{(2n)!} \prod_{\ell=0}^{n-1} (x^2 - \ell^2) = \frac{x}{n} \frac{1}{(2n-1)!} \prod_{\ell=0}^{2n-2} (x - \ell + n - 1) = \frac{x}{n} \binom{x+n-1}{2n-1}.$$

Using a property of binomial coefficients we get

$$\frac{x}{n} \binom{x+n-1}{2n-1} = \frac{x+n-n}{n} \binom{x+n-1}{2n-1} = 2\binom{x+n}{2n} - \binom{x+n-1}{2n-1},$$

which is clearly integer-valued. □

The prominent feature of the polynomials defined in (2.18) is that they contain only even powers of x. Because of this, substituting the operator Ω for x^2 in $E_n(x)$ yields an operator whose eigenvalues when acting on the eigenfunctions $\{\phi_{j,k,\ell}\}$ are all integers. We use the notation

$$E_n(\sqrt{\Omega}) = E_n(x)\Big|_{x^2 = \Omega}$$

to denote this substitution. Moreover, we have the following important property.

PROPOSITION 2.8: *Every integer-valued polynomial $P(x)$ that contains only even powers of x can be written as a $\mathbb{Z}$-linear combination of the polynomials $E_n(x)$, i.e.*

$$P(x) = \sum_{n=0}^{N} a_n E_n(x), \qquad a_n \in \mathbb{Z}. \tag{2.19}$$

Proof. Let $P(x)$ be an integer-valued polynomial that contains only even powers of x. The polynomial $P(x)$ necessarily has even degree, say $2N$. Now, we know that the polynomial E_n has degree $2n$ and its roots are the integers $\{0, \pm 1, \pm 2, \ldots, \pm(n-1)\}$, while $E_n(n) = 1$. Hence, the coefficients $a_0, a_1, \ldots, a_N \in \mathbb{Z}$ in the sum (2.19) can be recursively determined from the (integer) values $P(0), P(1), \ldots, P(N)$ such that when evaluated at $0 \leq x \leq N$ the sum (2.19) equals $P(x)$. Moreover, as $P(-x) = P(x)$, (2.19) coincides with $P(x)$ on at least $2N + 1$ points. A polynomial of degree $2N$ is uniquely determined by its values at $2N + 1$ points, which completes the proof. □

Note that every polynomial in x that is integer-valued on squares can be turned into a polynomial that is integer-valued on $\mathbb{Z}$ and that contains only even powers in x (by substituting x^2 for x). Hence, we have shown that the polynomials $E_n(x)$ suffice to construct every polynomial that takes on integer values when evaluated at the eigenvalues of the Casimir operator Ω.

One can further generalize the previous concept to construct any integer-valued function F satisfying $F(x) = F(-x)$, by specifying the coefficients $a_n \in \mathbb{Z}$ in the (possibly infinite) series

$$\sum_{n=0}^{\infty} a_n E_n(x).$$

Indeed, when $x = 0$ this series equals the coefficient a_0, while its value at $x = n$ is fixed by the coefficients $a_0, a_1, \ldots, a_n$ and thus can be specified by the choice of the value of a_n. Note that when evaluated at an integer $x \in \mathbb{Z}$, only a finite number of terms in this series is non-zero, as $E_n(x) = 0$ for $n > |x|$.

To conclude, we remark that there is an additional important aspect we have to consider for the goal we have in mind. In the operator formulation (2.14), the function F is used as an exponent of $e^{i\frac{\pi}{2}} = i$. For an integer n, the relation $i^n = i^{n \bmod 4}$ holds. Hence, given two functions such that on each square number their function values are integers in the same congruence class modulo 4, they will yield the same eigenvalues and thus equivalent operators $\widetilde{T}$. It thus suffices to consider the series

$$\sum_{n=0}^{\infty} a_n E_n(x), \qquad a_n \in \{0, 1, 2, 3\}, \tag{2.20}$$

which gives all possible functions modulo 4.

The values of the polynomials $E_n(x)$ modulo 4 can be computed by means of a computer algebra package. For the first few polynomials these are given in Table 1. Here, one can observe that $E_n(x)$ is periodic in x with period dependent on n. The exact relation is stated in Corollary A.2 (Appendix).

TABLE 1: The values of the polynomials $E_0, E_1, \ldots, E_5$ modulo 4 for $x = 0, \ldots, 15$.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
E_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
E_1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
E_2	0	0	1	2	0	2	1	0	0	0	1	2	0	2	1	0
E_3	0	0	0	1	0	3	0	2	0	2	0	3	0	1	0	0
E_4	0	0	0	0	1	2	2	2	0	2	2	2	1	0	0	0
E_5	0	0	0	0	0	1	0	1	0	3	0	3	0	2	0	2

2.1.2 *On squares of half-integers*

In the odd-dimensional case, the eigenvalues (2.16) of Ω are squares of half-integers. Therefore, we define the following polynomials. For $n \in \mathbb{Z}_{\geq 1}$, put

$$D_n(x) = \prod_{\ell=0}^{n-1} \frac{x^2 - (\ell+\frac{1}{2})^2}{(n+\frac{1}{2})^2 - (\ell+\frac{1}{2})^2} = \frac{1}{(2n)!} \prod_{\ell=0}^{n-1} \left(x^2 - (\ell+\tfrac{1}{2})^2\right). \tag{2.21}$$

and for $n = 0$ put $D_0(x) \equiv 1$. These polynomials are integer-valued on half-integers. Indeed, for $n \geq 1$ and $k \in \mathbb{Z}$ one has

$$D_n\left(k+\tfrac{1}{2}\right) = \frac{1}{(2n)!} \prod_{\ell=0}^{2n-1} \left(k + \tfrac{1}{2} + n - 1 - \ell + \tfrac{1}{2}\right) = \binom{k+n}{2n}.$$

The first few polynomials are given by $D_0(x) = 1$, $\quad D_1(x) = \dfrac{1}{2}x^2 - \dfrac{1}{8}$,

$$D_2(x) = \frac{1}{24}x^4 - \frac{5}{48}x^2 + \frac{3}{128}, \quad D_3(x) = \frac{1}{720}x^6 - \frac{7}{576}x^4 + \frac{259}{11520}x^2 - \frac{5}{1024}.$$

The polynomial D_n has degree $2n$ and its roots are the half-integers $\{\pm\frac{1}{2}, \pm(1+\frac{1}{2}), \pm(2+\frac{1}{2}), \ldots, \pm(n-\frac{1}{2})\}$, while $D_n(n+\frac{1}{2}) = 1$.

The values of the first few polynomials $D_n(x)$ modulo 4 are given in Table 2. Again, one clearly perceives the periodicity of $D_n(x)$, as stated in Corollary A.4 (Appendix).

TABLE 2: The values of the polynomials $D_0, D_1, \ldots, D_5$ modulo 4 for $x = \frac{1}{2}, \ldots, 15+\frac{1}{2}$

$x-1/2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
D_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_1	0	1	3	2	2	3	1	0	0	1	3	2	2	3	1	0
D_2	0	0	1	1	3	3	2	2	2	2	3	3	1	1	0	0
D_3	0	0	0	1	3	0	0	2	2	0	0	3	1	0	0	0
D_4	0	0	0	0	1	1	1	1	3	3	3	3	2	2	2	2
D_5	0	0	0	0	0	1	3	2	2	1	3	0	0	2	2	0

Similar to what we had for even dimension, the polynomials $D_n(x)$ contain only even powers of x, so we can again substitute the Casimir Ω for x^2 in $D_n(x)$. In this way, we arrive at the following general form

$$\sum_{n=0}^{\infty} a_n D_n(x), \qquad a_n \in \{0, 1, 2, 3\}. \tag{2.22}$$

Note that when evaluated at a half-integer $x \in \mathbb{Z}+\frac{1}{2}$, only a finite number of terms in this series is non-zero, as again $D_n(x) = 0$ for $n > |x|$.

2.1.3 *Conclusion*

In this way, we arrive at the following result for arbitrary dimension m.

THEOREM 2.9: *The properties*

(i) the Helmholtz relations

$$\begin{aligned} T \circ \Delta_x &= -|y|^2 \circ T \\ T \circ |x|^2 &= -\Delta_y \circ T \end{aligned}$$

(ii) $T\, \phi_{j,k,\ell} = \mu_{j,k}\, \phi_{j,k,\ell}$ *with* $\mu_{j,k} \in \mathbb{C}$

(iii) $T^4 = \mathrm{id}$

are satisfied by an operator T *of the form*

$$T = \mathrm{e}^{i\frac{\pi}{2}F(\sqrt{\Omega})}\mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})} \in \overline{\mathcal{U}}\big(\mathfrak{sl}(2)\big), \tag{2.23}$$

where $F(\sqrt{\Omega})$ *is an operator that consists of a function given by* (2.20) *(for even dimension) or* (2.22) *(for odd dimension) with the Casimir operator* Ω *substituted for* x^2*. Conversely, every operator that satisfies properties (i)–(iii) is equivalent with an operator of the form* (2.23).

Proof. Only the last part remains to be proved. For this, it suffices to note that for an operator satisfying properties (i)–(iii), by Lemma 2.2, its spectrum of eigenvalues is completely determined by the eigenvalues $\mu_{0,k}$ for $k \in \mathbb{Z}_{\geq 0}$. Now, the coefficients a_n in (2.20) or (2.22) can be chosen to yield every possible set of valid eigenvalues $\mu_{0,k}$ for $k \in \mathbb{Z}_{\geq 0}$. Indeed, recursively working upwards from $k = 0$, the coefficient $a_{\lfloor k+\lambda \rfloor}$ fixes the value of the function F evaluated at $k + \lambda$, with $\lambda = m/2 - 1$. This in turn fixes the eigenvalue $\mu_{0,k}$ of T as we have

$$T\, \phi_{0,k,\ell} = i^{F(k+\lambda)} i^k\, \phi_{0,k,\ell},$$

where we used that the eigenvalues of Ω are given by (2.16). □

Note that the preceding theorem naturally contains the classical Fourier transform. Indeed, for $F \equiv 0$ the operator (2.23) is precisely the operator exponential formulation (2.7) of the Fourier transform. We now investigate how we can further narrow down this class of integral transforms T, preferably to a finite set of interesting transforms. Throughout this process we seek inspiration in other useful properties of the Fourier transform.

2.2 *Periodicity restriction*

We now assume that T is an operator of the form (2.23) as described in Theorem 2.9. The behavior of the eigenvalues $\mu_{j,k}$ of T with regard to the index j is given in Lemma 2.2.

By successive application of relation (2.8) we find that the eigenvalues are two-periodic in j:

$$\mu_{j+2,k} = (-1)^2 \mu_{j,k} = \mu_{j,k}.$$

As far as the k index is concerned there are thus far no restrictions on the eigenvalues of the solutions (2.23) for T. From (2.2) one clearly sees that the eigenvalues of the Fourier transform are four-periodic in the index k. If we restrict the class of operators in Theorem 2.9 to those operators whose eigenvalues are four-periodic in the index k, we necessarily have but a finite number of valid operator solutions. Indeed, from Lemma 2.2 we know that every operator T whose eigenvalues are four-periodic in k, that is

$$\mu_{j,k+4} = \mu_{j,k}$$

will have its eigenvalue spectrum completely determined by the four values $\mu_{0,0}$, $\mu_{0,1}$, $\mu_{0,2}$, $\mu_{0,3}$ (or any other set of four eigenvalues with k indices that are mutually incongruent modulo 4) . Furthermore, each one of these four eigenvalues can take on only four possible values, namely $\{1, i, -1, -i\}$, as by Lemma 2.2 they must be integer powers of i. Hence, this leaves us with a finite number of valid operator solutions.

An additional advantage of imposing this periodicity restriction on T is that this will allow us to obtain a closed formula for the kernel when T is written as an integral transform. This will be discussed subsequently in Section 2.3.

First, to find these solutions, we again look at the decomposition (2.13) of T. As we now require the eigenvalues of the operator T to be four-periodic in k, and using that those of the Fourier transform already are four-periodic in k, this implies that the eigenvalues of the operator $\widetilde{T}$ must also be four-periodic in k. In order for this to hold, we need a function of the form (2.20) or (2.22) that is four-periodic modulo 4 when evaluated at (the square root of) the eigenvalues of Ω.

Depending on the parity of the dimension m one works in, the desired operators follow from the results in the following theorems. Here, we denote by $\mathbf{1}_A$ the indicator function of the set A, defined as

$$\mathbf{1}_A : A \longrightarrow \{0, 1\} : x \mapsto \mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

and also $4\mathbb{Z} + b = \{ 4a + b \mid a \in \mathbb{Z} \}$. For even dimension, the theorem is formulated as follows.

THEOREM 2.10: *For $x \in \mathbb{Z}_{\geq 0}$, one has*

$$E_1(x) \equiv \mathbf{1}_{4\mathbb{Z}+1}(x) + \mathbf{1}_{4\mathbb{Z}+3}(x) \pmod 4$$

$$E_2(x) + 2\,E_3(x) \equiv \mathbf{1}_{4\mathbb{Z}+2}(x) \pmod 4$$

$$\sum_{n=1}^{\infty} \Big(E_{2^n+1}(x) + \sum_{j=1}^{n-1} 2\,E_{2^n+1+2^j}(x) \Big) \equiv \mathbf{1}_{4\mathbb{Z}+3}(x) \pmod 4.$$

If one denotes

$$\mathcal{E}_{0101}(x) := E_1(x), \qquad \mathcal{E}_{0010}(x) := E_2(x) + 2\,E_3(x),$$

$$\mathcal{E}_{0001}(x) := \sum_{n=1}^{\infty} \Big(E_{2^n+1}(x) + \sum_{j=1}^{n-1} 2\,E_{2^n+1+2^j}(x)\Big),$$

then modulo 4 every integer-valued even function $F(x)$ on the integers that is four-periodic in x can be written as

$$F(x) = a + b\,\mathcal{E}_{0101}(x) + c\,\mathcal{E}_{0010}(x) + d\,\mathcal{E}_{0001}(x), \tag{2.24}$$

with $a, b, c, d \in \{0, 1, 2, 3\}$.

Proof. The proof of these results involves some long technical calculations and has therefore been omitted from the main text. It can be found in Appendix A.2. □

To illustrate this property, the first few values of these functions modulo 4 are listed in Table 3.

TABLE 3: The values of the functions $\mathcal{E}_{0101}$, $\mathcal{E}_{0010}$, $\mathcal{E}_{0001}$ modulo 4 for $x = 0, \ldots, 15$

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathcal{E}_{0101}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$\mathcal{E}_{0010}$	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
$\mathcal{E}_{0001}$	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1

In the same fashion, we have for odd dimension

THEOREM 2.11: *For $x \in \mathbb{Z}_{\geq 0}$, one has*

$$D_1\big(x+\tfrac{1}{2}\big) + 2\,D_2\big(x+\tfrac{1}{2}\big) \equiv \mathbf{1}_{4\mathbb{Z}+1}(x) + \mathbf{1}_{4\mathbb{Z}+2}(x) \pmod 4$$

$$\sum_{n=1}^{\infty} \Big(D_{2^n}\big(x+\tfrac{1}{2}\big) + \sum_{j=1}^{n-1} 2\,D_{2^n+2^j}\big(x+\tfrac{1}{2}\big)\Big) \equiv \mathbf{1}_{4\mathbb{Z}+2}(x) + \mathbf{1}_{4\mathbb{Z}+3}(x) \pmod 4$$

$$2D_3\big(x+\tfrac{1}{2}\big) + \sum_{n=0}^{\infty} \Big(D_{2^n+1}\big(x+\tfrac{1}{2}\big) + \sum_{j=1}^{n-1} 2\,D_{2^n+1+2^j}\big(x+\tfrac{1}{2}\big)\Big) \equiv \mathbf{1}_{4\mathbb{Z}+2}(x) \pmod 4.$$

If one denotes

$$\mathcal{D}_{0110}(x) := D_1\big(x+\tfrac{1}{2}\big) + 2\,D_2\big(x+\tfrac{1}{2}\big), \quad \mathcal{D}_{0011}(x) := \sum_{n=1}^{\infty} \Big(D_{2^n}\big(x+\tfrac{1}{2}\big) + \sum_{j=1}^{n-1} 2\,D_{2^n+2^j}\big(x+\tfrac{1}{2}\big)\Big),$$

$$\mathcal{D}_{0010}(x) := 2D_3\left(x+\tfrac{1}{2}\right) + \sum_{n=0}^{\infty}\left(D_{2^n+1}\left(x+\tfrac{1}{2}\right) + \sum_{j=1}^{n-1} 2\, D_{2^n+1+2^j}\left(x+\tfrac{1}{2}\right)\right),$$

then modulo 4 every integer-valued even function $F(x)$ on the half-integers that is four-periodic in x can be written as

$$F(x) = a + b\,\mathcal{D}_{0110}(x) + c\,\mathcal{D}_{0011}(x) + d\,\mathcal{D}_{0010}(x), \tag{2.25}$$

with $a, b, c, d \in \{0, 1, 2, 3\}$.

Proof. The proof of these results is similar to that of Theorem 2.10. □

The first few values of these functions modulo 4 are listed in Table 4.

TABLE 4: The values of the functions $\mathcal{D}_{0110}$, $\mathcal{D}_{0011}$, $\mathcal{D}_{0010}$ modulo 4 for $x = \frac{1}{2}, \ldots, 15+\frac{1}{2}$

$x-1/2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathcal{D}_{0110}$	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\mathcal{D}_{0011}$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$\mathcal{D}_{0010}$	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0

REMARK 2.12: The polynomials occurring in the preceding two theorems which are not an infinite series have as explicit form

$$\mathcal{E}_{0101}(x) = x^2, \quad \mathcal{E}_{0010}(x) = \frac{1}{180}x^6 + \frac{1}{18}x^4 - \frac{11}{180}x^2, \quad \mathcal{D}_{0110}(x) = \frac{1}{12}x^4 + \frac{7}{24}x^2 - \frac{5}{64}.$$

Putting everything together, we can summarize our results in the following theorem.

THEOREM 2.13: *Let $T : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}(\mathbb{R}^m)$ be an operator that satisfies the following properties:*

(i) the Helmholtz relations

$$\begin{aligned} T \circ \Delta_x &= -|y|^2 \circ T \\ T \circ |x|^2 &= -\Delta_y \circ T \end{aligned}$$

(ii) $T\,\phi_{j,k,\ell} = \mu_{j,k}\,\phi_{j,k,\ell}$ *with* $\mu_{j,k} \in \mathbb{C}$

(iii) $T^4 = \mathrm{id}$

(iv) the eigenvalues of T are 4-periodic in the index k: $\mu_{j,k+4} = \mu_{j,k}$.

Then T can be written as

$$T = e^{i\frac{\pi}{2}F(\sqrt{\Omega})} e^{i\frac{\pi}{2}(h-\frac{m}{2})}$$

where F consists of a function as specified in (2.24) *(Theorem 2.10) if m even and* (2.25) *(Theorem 2.11) if m odd.*

Conversely, every operator T of this form satisfies properties $(i)-(iv)$.

Proof. The four eigenvalues $\mu_{0,0}$, $\mu_{0,1}$, $\mu_{0,2}$, $\mu_{0,3}$ completely determine the eigenvalue spectrum of the operator T. By Theorem 2.10 or Theorem 2.11, depending on the parity of the dimension, we can construct a function F such that the eigenvalues of $e^{i\frac{\pi}{2}F(\sqrt{\Omega})} e^{i\frac{\pi}{2}(h-\frac{m}{2})}$ and T coincide. The rest is a direct consequence of Theorem 2.9. □

2.3 *Closed formulas for the kernel*

In Proposition 2.3 we already found a formulation as an integral transform for all of the operator exponentials we obtained. The kernel of these integral transforms, given in (2.9), consists of an infinite series of Bessel functions and Gegenbauer polynomials. Now, a natural question is whether it is possible to reduce these infinite series to a closed formula. For a select case of transforms that also satisfy Theorem 2.13 the answer to this is indeed positive. Our approach consists of first determining a formula for the kernel in the lowest dimension, followed by using a recursive relation to move up in dimension. Hereto, we first prove the following lemma.

LEMMA 2.14: *Let T be an operator of the form* (2.23) *as specified in Theorem 2.9. When T is written as an integral transform, its kernel satisfies the following recursive relation*

$$K_{m+2} = -i\, z^{-1} \partial_w K_m$$

for $m \geq 2$. Here K_{m+2} denotes the kernel in dimension $m+2$.

Proof. As T is of the form (2.23), its eigenvalues are given by

$$T\, \phi_{j,k,\ell} = e^{i\frac{\pi}{2}F(\sqrt{\Omega})} e^{i\frac{\pi}{2}(h-\frac{m}{2})}\, \phi_{j,k,\ell} = e^{i\frac{\pi}{2}F(k+\lambda)} e^{i\frac{\pi}{2}(2j+k)}\, \phi_{j,k,\ell},$$

where we used (2.16) for the eigenvalues of Ω, (2.2) for those of $\mathcal{F}$ and $\lambda = (m-2)/2$.

From Proposition 2.3 we know that T can be written as an integral transform

$$(Tf)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x,y)\, f(x)\, \mathrm{d}x$$

with kernel

$$K_m(x,y) = 2^{\lambda}\, \Gamma(\lambda) \sum_{k=0}^{+\infty} (k+\lambda)\, i^{F(k+\lambda)} i^k\, z^{-\lambda} J_{k+\lambda}(z)\, C_k^{\lambda}(w).$$

Using Appendix A.1, property (A.1) of the Gegenbauer polynomials, we find

$$-i\, z^{-1} \partial_w K_m(x,y) = 2^{\lambda}\, \Gamma(\lambda) \sum_{k=0}^{+\infty} (k+\lambda)\, i^{F(k+\lambda)} (-i)\, i^k\, z^{-\lambda-1} J_{k+\lambda}(z)\, \partial_w C_k^{\lambda}(w)$$

$$= 2^{\lambda}\,\Gamma(\lambda)\sum_{k=1}^{+\infty}(k+\lambda)\,i^{F(k+\lambda)}i^{k-1}\,z^{-(\lambda+1)}J_{k+\lambda}(z)\,2\lambda\,C_{k-1}^{\lambda+1}(w)$$

$$= 2^{\lambda+1}\,\Gamma(\lambda+1)\sum_{k=0}^{+\infty}(k+1+\lambda)\,i^{F(k+1+\lambda)}i^{k}\,z^{-(\lambda+1)}J_{k+1+\lambda}(z)\,C_{k}^{\lambda+1}(w)$$

As $\lambda+1=m/2$, this last expression is precisely $K_{m+2}(x,y)$. □

2.3.1 *Even dimension*

The previous lemma allows us to move up in dimension in steps of two. We now consider only even dimension, starting with the two-dimensional case. An important asset in the explicit computation of a formula for the kernels in dimension $m=2$ will be the property of 4-periodicity in the index k, as is the case for the solutions specified in Theorem 2.13. In the two-dimensional case, the Gegenbauer polynomials in the kernel (2.9) reduce to cosines. The 4-periodicity in k then allows us to make explicit use of the formulas

$$\cos(z\sin\theta)= J_0(z)+2\sum_{n=1}^{+\infty}J_{4n}(z)\cos(4n\theta)+2\sum_{n=0}^{+\infty}J_{4n+2}(z)\cos\big((4n+2)\theta\big), \tag{2.26}$$

$$\cos(z\cos\theta)= J_0(z)+2\sum_{n=1}^{+\infty}J_{4n}(z)\cos(4n\theta)-2\sum_{n=0}^{+\infty}J_{4n+2}(z)\cos\big((4n+2)\theta\big), \tag{2.27}$$

$$\sin(z\cos\theta)= 2\sum_{n=0}^{+\infty}J_{4n+1}(z)\cos\big((4n+1)\theta\big)-2\sum_{n=0}^{+\infty}J_{4n+3}(z)\cos\big((4n+3)\theta\big), \tag{2.28}$$

which can be found in [44, p. 22], formulas (1), (3) and (4). In this way we arrive at the following result.

THEOREM 2.15: *In dimension $m=2$, the operator exponential*

$$T_{abc}=e^{i\frac{\pi}{2}F_{abc}(\sqrt{\Omega})}e^{i\frac{\pi}{2}(h-1)}$$

with $F_{abc}(x)= a+b\,\mathcal{E}_{0101}(x)+c\,\mathcal{E}_{0010}(x)$ (as specified in Theorem 2.10) and $a,b,c\in\{0,1,2,3\}$, can be written as an integral transform whose kernel is given by

$$K_2(x,y)= i^{a}\left(\frac{1+i^{c}}{2}\cos(s)+i^{b+1}\sin(s)+\frac{1-i^{c}}{2}\cos(t)\right), \tag{2.29}$$

where $s=\langle x,y\rangle$ and $t=\sqrt{|x|^2|y|^2-s^2}$.

Proof. By Theorem 2.9, T_{abc} satisfies the properties (i)–(iii) and consequently, by Proposition 2.3, T_{abc} can be written as an integral transform with kernel (2.9). For $m=2$, we have $\lambda=(m-2)/2=0$ and using the identity, for $w=\cos\theta$ and integer $k\geq 1$,

$$\lim_{\lambda\to 0}\Gamma(\lambda)\,C_k^{\lambda}(\cos\theta)= \frac{2}{k}\cos(k\theta),$$

(see [22, Vol. I, section 3.15], formula (14)) this kernel reduces to

$$K_2(x,y) = \mu_{0,0}\, J_0(z) + 2\sum_{k=1}^{+\infty} \mu_{0,k}\, J_k(z)\cos(k\theta).$$

From Theorem 2.10 we see that the eigenvalues of T_{abc} are 4-periodic in k and for $m=2$ we have

$$\mu_{0,0} = i^a, \quad \mu_{0,1} = i^{a+b} i, \quad \mu_{0,2} = i^{a+c}(-1), \quad \mu_{0,3} = i^{a+b}(-i).$$

This allows us to rewrite the kernel as

$$\begin{aligned} K_2(x,y) = i^a \Big(& J_0(z) + 2\sum_{n=1}^{+\infty} J_{4n}(z)\cos(4n\theta) - i^c\, 2\sum_{n=0}^{+\infty} J_{4n+2}(z)\cos\big((4n+2)\theta\big) \\ & + i^{b+1}\Big(2\sum_{n=0}^{+\infty} J_{4n+1}(z)\cos\big((4n+1)\theta\big) - 2\sum_{n=0}^{+\infty} J_{4n+3}(z)\cos\big((4n+3)\theta\big)\Big)\Big) \end{aligned}$$

The formulas (2.26), (2.27), (2.28), $s = z\cos\theta$ and $t = z\sin\theta$ then yield

$$K_2(x,y) = i^a\Big(\frac{1}{2}\big(\cos(s)+\cos(t)\big) + i^c\frac{1}{2}\big(\cos(s)-\cos(t)\big) + i^{b+1}\sin(s)\Big). \qquad \square$$

Using Lemma 2.14 we now find the following theorem.

THEOREM 2.16: *In even dimension m, the operator exponential*

$$T_{abc} = e^{i\frac{\pi}{2}F_{abc}(\sqrt{\Omega})} e^{i\frac{\pi}{2}(h-\frac{m}{2})} \tag{2.30}$$

with $F_{abc}(x) = a + b\,\mathcal{E}_{0101}(x) + c\,\mathcal{E}_{0010}(x)$ (as specified in Theorem 2.10), can be written as an integral transform whose kernel is given by

$$K_m(x,y) = i^a(-i)^\lambda\Big(\frac{1+i^c}{2}\big(\partial_s\big)^\lambda\cos(s) + i^{b+1}\big(\partial_s\big)^\lambda\sin(s) + \frac{1-i^c}{2}\Big(\partial_s - \frac{s}{t}\,\partial_t\Big)^\lambda\cos(t)\Big), \tag{2.31}$$

with $\lambda = (m-2)/2$, $s = \langle x, y\rangle$ and $t = \sqrt{|x|^2|y|^2 - s^2}$. Moreover, one has

$$\Big(\partial_s - \frac{s}{t}\,\partial_t\Big)^\lambda\cos(t) = \sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{\lfloor\frac{\lambda}{2}\rfloor} s^{\lambda-2\ell}\,\frac{1}{2^\ell\,\ell!}\,\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-2\ell)}\,\frac{J_{\lambda-1/2-\ell}(t)}{t^{\lambda-1/2-\ell}}. \tag{2.32}$$

Proof. For $m=2$ we have $\lambda = 0$, and the expression (2.31) coincides with the kernel (2.29) which was obtained in Theorem 2.15. By successive application of Lemma 2.14 we have for even dimension $m \geq 2$

$$K_m = \big(-i\,z^{-1}\partial_w\big)^\lambda K_2.$$

From $s = zw$ and $t = z\sqrt{1-w^2}$, we easily find

$$z^{-1}\partial_w = z^{-1}\partial_w[s]\,\partial_s + z^{-1}\partial_w[t]\,\partial_t = \partial_s - \frac{w}{\sqrt{1-w^2}}\,\partial_t = \partial_s - \frac{s}{t}\,\partial_t,$$

which proves (2.31).

We now show (2.32) by induction. The statement holds for $m = 2$ as equation (2.32) then reduces to the identity (see (A.3))

$$\cos t = \sqrt{\frac{\pi}{2}}\,t^{1/2}J_{-1/2}(t).$$

Before we continue, we make a distinction between λ even and λ odd (or equivalently $m \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 4$), as the upper bound of the summation in (2.32) contains a floor function.

Consider the case $\lambda = 2j$ (or thus $m = 4j+2$) and assume (2.32) holds for this λ. We have, using property (A.4) of the Bessel function,

$$\begin{aligned}
\left(\partial_s - \frac{s}{t}\,\partial_t\right)^{\lambda+1}\cos(t) &= \left(\partial_s - \frac{s}{t}\,\partial_t\right)\left(\sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{j} s^{2j-2\ell}\frac{1}{2^\ell \ell!}\frac{\Gamma(2j+1)}{\Gamma(2j+1-2\ell)}\frac{J_{(m-2\ell-3)/2}(t)}{t^{(m-2\ell-3)/2}}\right)\\
&= \sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{j-1} s^{2j-2\ell-1}\frac{1}{2^\ell \ell!}\frac{\Gamma(2j+1)}{\Gamma(2j-2\ell)}\frac{J_{(m-2\ell-3)/2}(t)}{t^{(m-2\ell-3)/2}}\\
&\quad + \sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{j} s^{2j-2\ell+1}\frac{1}{2^\ell \ell!}\frac{\Gamma(2j+1)}{\Gamma(2j+1-2\ell)}\frac{J_{(m-2\ell-1)/2}(t)}{t^{(m-2\ell-1)/2}}\\
&= \sqrt{\frac{\pi}{2}}\sum_{\ell=1}^{j} s^{2j-2\ell+1}\frac{2\ell}{2^\ell \ell!}\frac{\Gamma(2j+1)}{\Gamma(2j+2-2\ell)}\frac{J_{(m-2\ell-1)/2}(t)}{t^{(m-2\ell-1)/2}}\\
&\quad + \sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{j} s^{2j-2\ell+1}\frac{(2j+1-2\ell)}{2^\ell \ell!}\frac{\Gamma(2j+1)}{\Gamma(2j+2-2\ell)}\frac{J_{(m-2\ell-1)/2}(t)}{t^{(m-2\ell-1)/2}}\\
&= \sqrt{\frac{\pi}{2}}\sum_{\ell=1}^{j} s^{2j-2\ell+1}\frac{(2j+1)}{2^\ell \ell!}\frac{\Gamma(2j+1)}{\Gamma(2j+2-2\ell)}\frac{J_{(m-2\ell-1)/2}(t)}{t^{(m-2\ell-1)/2}}\\
&\quad + \sqrt{\frac{\pi}{2}}\,s^{2j+1}\,(2j+1)\frac{J_{(m-1)/2}(t)}{t^{(m-1)/2}}\\
&= \sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{j} s^{2j-2\ell+1}\frac{1}{2^\ell \ell!}\frac{\Gamma(2j+2)}{\Gamma(2j+2-2\ell)}\frac{J_{(m-2\ell-1)/2}(t)}{t^{(m-2\ell-1)/2}},
\end{aligned}$$

as required. The inductive step in the case λ odd is treated similarly. □

REMARK 2.17: The explicit form for the kernel obtained in the preceding theorem, together with the Helmholtz relations, yields its polynomial boundedness similar to

Lemma 5.2 and Theorem 5.3 in [18], which can be proven in exactly the same fashion. This ensures the corresponding integral transforms to be well-defined and continuous on $\mathcal{S}(\mathbb{R}^m)$.

Next, we consider some specific cases of operators manifested in Theorem 2.16. The function F_{abc} in (2.30) contains three parameters, each having four possible values (up to modulo 4 congruence). The role of the parameter a is but a scalar multiplicative factor, so we take $a = 0$ in the following.

Putting $c = 0$, (2.30) reduces to the operator exponential

$$T_b = \mathrm{e}^{i\frac{\pi}{2}b\Omega}\mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})}, \tag{2.33}$$

where we used $\mathcal{E}_{0101}(x)= E_1(x)= x^2$. When written as an integral transform we find that its kernel is given by

$$K_m(x, y)= \left(-i\,\partial_s\right)^{\lambda}\left(\cos(s)+i^{b+1}\sin(s)\right) = i^{b\lambda^2}\cos(\langle x, y\rangle)+i^{b(\lambda+1)^2+1}\sin(\langle x, y\rangle). \tag{2.34}$$

Here, we distinguish four possible scenarios for the value of b:

- For $b \equiv 0 \pmod 4$, the operator exponential (2.33) is precisely the classical Fourier transform and (2.34) indeed gives

$$K_m(x, y)= \cos(\langle x, y\rangle)+i\sin(\langle x, y\rangle)= e^{i\langle x,y\rangle}.$$

- Taking $b \equiv 2 \pmod 4$ and multiplying (2.34) by $\mathrm{e}^{i\pi\lambda}$, we get the kernel

$$\cos(\langle x, y\rangle)-i\sin(\langle x, y\rangle)= e^{-i\langle x,y\rangle}$$

  of the inverse Fourier transform and hence

$$\mathcal{F}^{-1} = \mathrm{e}^{i\pi\lambda}\mathrm{e}^{i\pi\Omega}\mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})} = \mathrm{e}^{-i\frac{\pi}{2}(h-\frac{m}{2})}.$$

  As $\mathcal{F}^{-1} = \mathcal{F}^3$, we also have

$$\mathcal{F}^2 = \mathrm{e}^{i\pi(\Omega+\lambda)} = \mathrm{e}^{i\pi(h-\frac{m}{2})}.$$

- The other cases for the value of b, namely $b \equiv 1 \pmod 4$ and $b \equiv 3 \pmod 4$, give rise to another pair of interesting transforms. Taking $b = 2\lambda + 1$ and multiplying by an appropriate multiplicative factor, we find the kernel $\cos(\langle x, y\rangle)-\sin(\langle x, y\rangle)$, corresponding to the operator exponential

$$\mathrm{e}^{i\frac{\pi}{2}\lambda^2}\mathrm{e}^{i\frac{\pi}{2}(2\lambda+1)\Omega}\mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})}.$$

  Furthermore, the operator exponential

$$\mathrm{e}^{i\frac{\pi}{2}(\lambda^2+2\lambda)}\mathrm{e}^{i\frac{\pi}{2}(2\lambda-1)\Omega}\mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})}$$

  can be written as an integral transform whose kernel is given by $\cos(\langle x, y\rangle)+\sin(\langle x, y\rangle)$. This is the cosine-and-sine or Hartley kernel of the integral transform known as the Hartley transform. The Hartley transform is a real linear operator that is symmetric and Hermitian [4, 5]. Moreover, it is a unitary operator that is its own inverse.

We summarize this in the following table

$\widetilde{T} = T \circ \mathcal{F}^{-1}$	$K_m(x, y)$
$e^{i\frac{\pi}{2}b\Omega}$	$i^{b\lambda^2}\cos(\langle x, y\rangle)+i^{b(\lambda+1)^2+1}\sin(\langle x, y\rangle)$
1	$\cos(\langle x, y\rangle)+i\sin(\langle x, y\rangle)$
$e^{i\pi\lambda}e^{i\pi\Omega}$	$\cos(\langle x, y\rangle)-i\sin(\langle x, y\rangle)$
$e^{i\frac{\pi}{2}\lambda^2}e^{i\frac{\pi}{2}(2\lambda+1)\Omega}$	$\cos(\langle x, y\rangle)-\sin(\langle x, y\rangle)$
$e^{i\frac{\pi}{2}(\lambda^2+2\lambda)}e^{i\frac{\pi}{2}(2\lambda-1)\Omega}$	$\cos(\langle x, y\rangle)+\sin(\langle x, y\rangle)$

Note that the kernel of all of these integral transforms is of the form

$$K_m(x, y)= \cos(\langle x, y\rangle)+i^d\sin(\langle x, y\rangle)$$

for some integer d. As $\cos(\langle x, y\rangle)$ and $\sin(\langle x, y\rangle)$ are given by the real and imaginary parts of the Fourier transform, an integral transform with such a kernel coincides with

$$\frac{1}{2}\Big((\mathcal{F}f)(y)+(\mathcal{F}f)(-y)\Big) + i^{d-1}\frac{1}{2}\Big((\mathcal{F}f)(y)-(\mathcal{F}f)(-y)\Big).$$

Finally, we consider one more special instance. Putting $c = 2$ (and again $a = 0$), (2.31) reduces to the kernel

$$K_m(x, y)= i^{b+1}i^{-\lambda}\sin\Big(s + \lambda\frac{\pi}{2}\Big) + i^{-\lambda}\sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{\lfloor\frac{m-2}{4}\rfloor} s^{\frac{m}{2}-1-2\ell}\,\frac{1}{2^\ell\ell!}\,\frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}-2\ell)}\,\frac{J_{(m-2\ell-3)/2}(t)}{t^{(m-2\ell-3)/2}},$$

corresponding to the operator exponential

$$T_b = e^{i\frac{\pi}{2}(\frac{3}{2}\Omega^2-\frac{3}{2}\Omega+b\Omega)}e^{i\frac{\pi}{2}(h-\frac{m}{2})}.$$

For $m = 2$ this kernel becomes

$$K_2(x, y)= i^{b+1}\sin(\langle x, y\rangle)+\cos\Big(\sqrt{|x|^2|y|^2-\langle x, y\rangle^2}\Big).$$

2.3.2 *Odd dimension*

While Lemma 2.14 remains valid for odd dimension m, we have no specific formulas for dimension $m = 3$ comparable to those we used for dimension $m = 2$, that is, formulas (2.26)–(2.28). We do have another way to obtain a closed formula for a kernel of the form (2.9). This other approach holds for a more restricted class of operators (for even dimension this class is already included in Theorem 2.16) but it has the advantage that we can also use this in odd dimension.

LEMMA 2.18: *If the action of an operator T on the Hermite basis $\{\phi_{j,k,\ell}\}$ is given by*

$$T\,\phi_{j,k,\ell} = \mu_{j,k}\,\phi_{j,k,\ell},$$

with eigenvalues $\mu_{j,k} \in \mathbb{C}$ *that satisfy* $\mu_{j+1,k} = -\mu_{j,k}$ *and* $\mu_{j,k+2} = -\mu_{j,k}$ *for* $j, k \in \mathbb{Z}_{\geq 0}$. *Then, on* $\{\phi_{j,k,\ell}\}$, T *can be written as an integral transform*

$$(Tf)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x,y)\, f(x)\, \mathrm{d}x$$

with kernel

$$K_m(x,y) = \mu_{0,0} \cos(\langle x, y\rangle) + \mu_{0,1} \sin(\langle x, y\rangle). \tag{2.35}$$

Proof. Let T be as specified in the lemma. From the conditions on its eigenvalues we know that the spectrum of T is completely determined by its eigenvalues $\mu_{0,0}$ and $\mu_{0,1}$, while the others follows from

$$\mu_{j,2n} = (-1)^j(-1)^n \mu_{0,0}, \qquad \mu_{j,2n+1} = (-1)^j(-1)^n \mu_{0,1}.$$

From the proof of Proposition 2.3 we know that the integral transform

$$(Tf)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x,y)\, f(x)\, \mathrm{d}x$$

with kernel $K_m(x,y)$, given by (2.9), will have the same eigenvalue spectrum as that of T.

Distinguishing between even and odd values of k, the series in $K_m(x,y)$ becomes

$$\begin{aligned} K_m(x,y) &= \mu_{0,0}\, 2^\lambda\, \Gamma(\lambda) \sum_{n=0}^{+\infty} (2n+\lambda)\, (-1)^n\, z^{-\lambda} J_{2n+\lambda}(z)\, C_{2n}^\lambda(w) \\ &\quad + \mu_{0,1}\, 2^\lambda\, \Gamma(\lambda) \sum_{n=0}^{+\infty} (2n+1+\lambda)\, (-1)^n\, z^{-\lambda} J_{2n+1+\lambda}(z)\, C_{2n+1}^\lambda(w). \end{aligned}$$

The desired result now follows from formulas (44) and (45) of [22, Vol. II, section 7.15]. □

Note that the condition $\mu_{j,k+2} = -\mu_{j,k}$ for $j, k \in \mathbb{Z}_{\geq 0}$ immediately implies the eigenvalues being 4-periodic in the index k. In even dimension, one easily verifies that the operator exponential T_b given by (2.33) satisfies the conditions of Lemma 2.18. Indeed, we have already obtained its kernel to be (2.34), which is of the form (2.35).

Now, in the odd dimensional case, we find an operator with suitable eigenvalues by constructing a function that alternates between two values modulo 4, as is the case for the function in the exponent of T_b in even dimension. The function

$$F_{ab}(x) = a + b\left(\mathcal{D}_{0110}(x) + \mathcal{D}_{0011}(x) + 2\mathcal{D}_{0010}(x)\right).$$

meets this requirement as for $k \in \mathbb{Z}_{\geq 0}$, $F_{ab}\left(2k+\frac{1}{2}\right) \equiv a$ and $F_{ab}\left(2k+1+\frac{1}{2}\right) \equiv a+b$. This function allows us to state the following theorem, where we have plugged in values for a and b to yield the simplest form for the kernel function.

THEOREM 2.19: *In odd dimension, let T_d be the operator exponential*

$$T_d = \mathrm{e}^{i\frac{\pi}{2}dF(\sqrt{\Omega})}\mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})}$$

with $d \in \{0, 1, 2, 3\}$ and

$$F\big(\sqrt{\Omega}\big) = (m-2)\Big(\mathcal{D}_{0110}\big(\sqrt{\Omega}\big) + \mathcal{D}_{0011}\big(\sqrt{\Omega}\big) + 2\mathcal{D}_{0010}\big(\sqrt{\Omega}\big)\Big) + \Big(\frac{m+1}{2}\Big)^2.$$

Then T_d can be written as an integral transform

$$(Tf)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x, y)\, f(x)\, \mathrm{d}x$$

whose kernel is given by

$$K_m(x, y) = \cos\big(\langle x, y\rangle\big) + i^{d+1}\sin\big(\langle x, y\rangle\big).$$

Proof. The operator T_d is 4-periodic in the index k by Theorem 2.13 and its eigenvalues are given by

$$\mathrm{e}^{i\frac{\pi}{2}dF(\sqrt{\Omega})}\mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})}\,\phi_{j,k,\ell} = \mathrm{e}^{i\frac{\pi}{2}dF(k+\lambda)}\mathrm{e}^{i\frac{\pi}{2}(2j+k)}\,\phi_{j,k,\ell}\,.$$

The first four eigenvalues with $j = 0$ are

$$\mu_{0,0} = 1, \quad \mu_{0,1} = i^{d+1}, \quad \mu_{0,2} = -1, \quad \mu_{0,3} = -i^{d+1},$$

Using also Lemma 2.2, we see that T_d satisfies the conditions of Lemma 2.18 and thus find the desired kernel. □

2.4 *Uncertainty principle*

As an application of our previous results, we show how to obtain generalized uncertainty principles (following the same strategy as developed for the Dunkl transform in [40, 41] and later generalized and streamlined in [3]) for any continuous integral transform

$$(Tf)(y) = \int_{\mathbb{R}^m} K(x, y) f(x)\, \mathrm{d}x$$

on $\mathcal{S}(\mathbb{R}^m)$ that satisfies all the properties of Theorem 2.13. Hence, it applies in particular to the operators from e.g. Theorem 2.16 and Theorem 2.19 as they have polynomially bounded kernels (see Remark 2.17). By Proposition 2.5, such a T has a unitary extension to $L^2(\mathbb{R}^m)$ and we may now establish the following lemma.

LEMMA 2.20: *If the continuous integral transform T on $\mathcal{S}(\mathbb{R}^m)$ satisfies the properties (i)–(iii), then for its unitary extension to $L^2(\mathbb{R}^m)$ the following inequality holds:*

$$\||x|f\|^2 + \||x|T(f)\|^2 \geq m\|f\|^2, \qquad f \in \mathcal{S}(\mathbb{R}^m), \tag{2.36}$$

with $\|\cdot\|$ the L^2 norm. The inequality becomes an equality if and only if $f = \alpha e^{-|x|^2/2}$ with $\alpha \in \mathbb{R}$.

Proof. We can compute that for $f \in \mathcal{S}(\mathbb{R}^m)$

$$\begin{aligned}\||x|T(f)\|^2 &= \langle |x|T(f), |x|T(f)\rangle \\ &= \langle |x|^2 T(f), T(f)\rangle \\ &= -\langle T(\Delta_x f), T(f)\rangle \\ &= -\langle \Delta_x f, f\rangle\end{aligned}$$

where we used the Helmholtz relations and the unitarity of T. Using this result, the left-hand side of (2.36) equals

$$\begin{aligned}\||x|f\|^2 + \||x|T(f)\|^2 &= \langle |x|^2 f, f\rangle - \langle \Delta_x f, f\rangle \\ &= \langle (|x|^2 - \Delta_x) f, f\rangle \\ &= 2\langle hf, f\rangle \\ &\geq m\langle f, f\rangle\end{aligned}$$

as the smallest eigenvalue of h is $m/2$. The equality then follows as this minimal eigenvalue corresponds to the ground state $f = \alpha e^{-|x|^2/2}$. □

Subsequently we establish the following generalized uncertainty principle.

THEOREM 2.21: *Let*

$$(Tf)(y) = \int_{\mathbb{R}^m} K(x, y) f(x)\, \mathrm{d}x$$

be a continuous integral transform on $\mathcal{S}(\mathbb{R}^m)$ that satisfies the properties (i)–(iii), then one has the following uncertainty principle:

$$\||x|f\| \cdot \||x|T(f)\| \geq \frac{m}{2}\|f\|^2$$

for $f \in \mathcal{S}(\mathbb{R}^m)$.

Proof. Put, for $c > 0$, $f_c(x) = f(cx)$. An easy computation then shows that

$$\||x|f_c\|^2 = c^{-m-2}\||x|f\|^2$$

and similarly

$$\|f_c\|^2 = c^{-m}\|f\|^2.$$

We also have, using the homogeneity of the kernel as given in (2.12),

$$\begin{aligned}(Tf_c)(y) &= \int_{\mathbb{R}^m} K(cx, y/c) f(cx)\, \mathrm{d}x \\ &= c^{-m}\int_{\mathbb{R}^m} K(x, y/c) f(x)\, \mathrm{d}x \\ &= c^{-m}(Tf)(y/c).\end{aligned}$$

As a consequence we have

$$\begin{aligned}\||x|T(f_c)\|^2 &= \|c^{-m}|x|(Tf)(y/c)\|^2\\ &= c^{-2m}\int_{\mathbb{R}^m}|x|^2|(Tf)(x/c)|^2\,\mathrm{d}x\\ &= c^{-m+2}\||x|T(f)\|^2.\end{aligned}$$

Now substitute f_c for f in Lemma 2.20, and apply the previously established relations. This yields:

$$c^{-2}\||x|f\|^2+c^2\||x|T(f)\|^2\geq m\|f\|^2.$$

Finally put

$$c=\sqrt{\frac{\||x|f\|}{\||x|T(f)\|}}$$

and the theorem follows. □

REMARK 2.22: More specialized uncertainty principles can be developed following a strategy similar to the one used in, for example, [34].

3 FOURIER TRANSFORMS IN CLIFFORD ANALYSIS

In the previous section we obtained a class of operators satisfying the set of properties (i)–(iii). We did this by capitalizing on the relation between these properties, the classical Fourier transform and a realization of the Lie algebra $\mathfrak{sl}(2)$. Now, we wish to expand this train of thought to a broader setting. The aforementioned operator realization of $\mathfrak{sl}(2)$ possesses a natural generalization to the Lie superalgebra $\mathfrak{osp}(1|2)$. For this reason, we turn our attention to the context of Clifford analysis and aim to find solutions by applying the techniques developed in the previous section.

We start by giving a brief overview of the framework of Clifford analysis (see e.g. [6, 20]), a higher dimensional function theory where functions take on values in a Clifford algebra. The orthogonal Clifford algebra $\mathcal{C}l_m$ is generated by the canonical basis $\{\, e_i \mid i = 1, \ldots, m\,\}$ of $\mathbb{R}^m$ under the relations

$$\begin{aligned}&e_ie_j + e_je_i = 0 \qquad (i \neq j)\\ &e_i^2 = -1.\end{aligned}$$

This algebra has dimension 2^m as a vector space over $\mathbb{R}^m$, and we have

$$\mathcal{C}l_m = \bigoplus_{k=0}^{m}\operatorname{span}\{\, e_{i_1}e_{i_2}\cdots e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq m\,\}.$$

The empty product ($k = 0$) is defined as the multiplicative identity element.

Functions taking values in $\mathcal{C}l_m$ can be decomposed as

$$f = f_0 + \sum_{i=1}^{m} e_i f_i + \sum_{i<j} e_i e_j f_{ij} + \cdots + e_1 \cdots e_m f_{1\ldots m}$$

with $f_0, f_i, f_{ij}, \ldots, f_{1\ldots m}$ all real-valued functions on $\mathbb{R}^m$. We identify the point $x = (x_1, \ldots, x_m)$ in $\mathbb{R}^m$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j.$$

The Clifford product of two vectors splits into a scalar part and a bivector part, given by, respectively, minus the inner product of the two vectors and the outer product or wedge product:

$$\underline{x}\,\underline{y} = -\langle x, y\rangle + \underline{x} \wedge \underline{y},$$

with $-\langle x, y\rangle = -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y}+\underline{y}\,\underline{x})$ and $\underline{x}\wedge\underline{y} = \sum_{j<k} e_{jk}(x_j y_k - x_k y_j) = \frac{1}{2}(\underline{x}\,\underline{y}-\underline{y}\,\underline{x})$.

Furthermore, we introduce a first-order vector differential operator by

$$\partial_{\underline{x}} = \sum_{j=1}^{m} \partial_{x_j} e_j.$$

This operator is the so-called Dirac operator. Together with the vector variable they satisfy the relations

$$\partial_{\underline{x}}^2 = -\Delta_x, \qquad \underline{x}^2 = -|x|^2, \qquad \{\underline{x}, \partial_{\underline{x}}\} = -2\mathbb{E}_x - m,$$

where $\{a, b\} = ab + ba$, and hence they generate a realization of the Lie superalgebra $\mathfrak{osp}(1|2)$, which contains the Lie algebra $\mathfrak{sl}(2)=$ span$\{\Delta_x, |x|^2, [\Delta_x, |x|^2]\}$ as its even part [25].

As the Clifford-valued operators $\partial_{\underline{x}}$ and $\underline{x}$ factorize the operators $-\Delta_x$ and $-|x|^2$, they allow us to refine the Helmholtz relations to what we call the Clifford–Helmholtz relations

$$T \circ \partial_{\underline{x}} = -i\,\underline{y} \circ T$$
$$T \circ \underline{x} = -i\,\partial_{\underline{y}} \circ T$$

Every operator that satisfies this system will by definition also satisfy the Helmholtz relations. They form an intermediate step in the generalization of the properties of the Fourier transform:

$$\begin{cases} T \circ \partial_{x_j} = -i\,y_j \circ T \\ T \circ x_j = -i\,\partial_{y_j} \circ T \end{cases} \Longrightarrow \begin{cases} T \circ \partial_{\underline{x}} = -i\,\underline{y} \circ T \\ T \circ \underline{x} = -i\,\partial_{\underline{y}} \circ T \end{cases} \Longrightarrow \begin{cases} T \circ \Delta_x = -|y|^2 \circ T \\ T \circ |x|^2 = -\Delta_y \circ T \end{cases}$$

where in the first we assume it must hold for all $j \in \{1, \ldots, m\}$.

In the previous section we made extensive use of the eigenfunction basis of $\mathcal{S}(\mathbb{R}^m) \subset L^2(\mathbb{R}^m)$ given by the functions (2.1). In the framework of Clifford analysis the relevant function space is the space $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m$ which decomposes under the action of the dual pair $(\mathfrak{osp}(1|2), Spin(m))$. This action leads to the following important basis (see [42])

$$\begin{aligned} \psi_{2p,k,\ell}(x) &:= 2^p p!\, L_p^{\frac{m}{2}+k-1}(|x|^2)\, M_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}, \\ \psi_{2p+1,k,\ell}(x) &:= 2^p p! \sqrt{2}\, L_p^{\frac{m}{2}+k}(|x|^2)\, \underline{x}\, M_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}, \end{aligned} \tag{3.1}$$

where $p, k \in \mathbb{Z}_{\geq 0}$ and $\{ M_k^{(\ell)} \mid \ell = 1, \ldots, \dim(\mathcal{M}_k) \}$ is a basis for $\mathcal{M}_k$, the space of spherical monogenics of degree k, that is, homogeneous polynomial null-solutions of the Dirac operator of degree k. It is clear that every spherical monogenic is a spherical harmonic, indeed we have $\mathcal{H}_k \otimes \mathcal{C}l_m = \mathcal{M}_k \oplus \underline{x}\mathcal{M}_{k-1}$.

The action of the Fourier transform on the eigenfunctions $\{\psi_{j,k,\ell}\}$ is given by

$$\mathcal{F}\psi_{j,k,\ell} = \mathrm{e}^{i\frac{\pi}{2}(j+k)}\, \psi_{j,k,\ell} = i^{j+k}\, \psi_{j,k,\ell}.$$

Moreover, the functions $\{\psi_{j,k,\ell}\}$ have the following important property (see e.g. [42]), comparable to property (2.3),

$$\psi_{j,k,\ell}(x) = \left(\frac{\sqrt{2}}{2} \left(\underline{x} - \partial_{\underline{x}} \right) \right)^j M_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}. \tag{3.2}$$

Now, our aim is to find all operators $T \colon \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m \to \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m$ that satisfy

(I) the Clifford–Helmholtz relations

$$\begin{aligned} T \circ \partial_{\underline{x}} &= -i\, \underline{y} \circ T, \\ T \circ \underline{x} &= -i\, \partial_{\underline{y}} \circ T, \end{aligned}$$

(II) $T\, \psi_{j,k,\ell} = \mu_{j,k}\, \psi_{j,k,\ell}$ with $\mu_{j,k} \in \mathbb{C}$,

(III) $T^4 = \mathrm{id}$.

In line with Section 2, we first introduce some suitable linear combinations of the operators of interest, namely $\underline{x}$ and $\partial_{\underline{x}}$. Hereto put

$$b^+ = \frac{\sqrt{2}}{2}\left(\underline{x} - \partial_{\underline{x}}\right) \quad \text{and} \quad b^- = -\frac{\sqrt{2}}{2}\left(\underline{x} + \partial_{\underline{x}}\right).$$

They satisfy the relations

$$[\{b^-, b^+\}, b^{\pm}] = \pm 2 b^{\pm}, \tag{3.3}$$

and hence also generate a realization of the Lie superalgebra $\mathfrak{osp}(1|2)$, [38]. This Lie superalgebra contains an even subalgebra isomorphic with $\mathfrak{sl}(2)$ generated by the even (or “bosonic”) elements

$$h = \frac{1}{2}\{b^-, b^+\}, \qquad e = \frac{1}{4}\{b^+, b^+\}, \qquad f = -\frac{1}{4}\{b^-, b^-\}.$$

These are precisely the operators (2.4) we considered in the previous section.

Working in this realization of $\mathfrak{osp}(1|2)$ the Clifford–Helmholtz relations translate to the relations

$$\begin{aligned} T \circ b^{+} &= \ \ i\, b^{+} \circ T \\ T \circ b^{-} &= -i\, b^{-} \circ T, \end{aligned} \tag{3.4}$$

and property (3.2) can be written more compactly as $\psi_{j,k,\ell}(x) = \left(b^{+}\right)^{j} M_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}$.

Moreover, in this realization the operators act on the eigenfunctions (3.1) in a nice way. We have

$$b^{+}\, \psi_{j,k,\ell} = \psi_{j+1,k,\ell} \tag{3.5}$$

and for integer p,

$$b^{-}\, \psi_{2p,k,\ell} = 2p\, \psi_{2p-1,k,\ell}, \qquad b^{-}\, \psi_{2p+1,k,\ell} = (2p + m + 2k)\, \psi_{2p,k,\ell}.$$

The action of the other operators is then as follows

$$h\, \psi_{j,k,\ell} = (j + k + \tfrac{m}{2})\, \psi_{j,k,\ell}, \qquad e\, \psi_{j,k,\ell} = \frac{1}{2}\, \psi_{j+2,k,\ell}$$

and, again for integer p,

$$f\, \psi_{2p,k,\ell} = -p\,(2p - 2 + m + 2k)\, \psi_{2p-2,k,\ell}, \qquad f\, \psi_{2p+1,k,\ell} = -p\,(2p + m + 2k)\, \psi_{2p-1,k,\ell}.$$

Note that h again acts diagonally on $\psi_{j,k,\ell}$.

REMARK 3.1: For every $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \{1, \ldots, \dim(\mathcal{M}_k)\}$, the set $\{\, \psi_{j,k,\ell} \mid j \in \mathbb{Z}_{\geq 0} \,\}$ forms a basis for the irreducible representation of the Lie superalgebra $\mathfrak{osp}(1|2)$ with lowest weight $k + m/2$. This representation is a direct sum of two positive discrete series representations of $\mathfrak{su}_{1,1}$.

For every $j, k \in \mathbb{Z}_{\geq 0}$ the set $\{\, \psi_{j,k,\ell} \mid \ell = 1, \ldots, \dim(\mathcal{M}_k) \,\}$ forms a basis for an irreducible spinor representation of $Spin(m)$, when restricting the values $\mathcal{C}l_m$ to a spinor space.

Similar to the results obtained in the harmonic case, we immediately have some important consequences for an operator T that satisfies the properties (I)–(III).

LEMMA 3.2: *Let T be an operator satisfying the properties (I)–(III). There are only four possible values for the eigenvalues $\mu_{j,k}$ of the operator T, namely $\mu_{j,k} \in \{1, i, -1, -i\}$. Moreover, the spectrum of eigenvalues is completely determined by the eigenvalues $\mu_{0,k}$ for $k \in \mathbb{Z}_{\geq 0}$; the other eigenvalues for $j > 0$ follow from the relation*

$$\mu_{j,k} = i^{j}\, \mu_{0,k}. \tag{3.6}$$

Proof. Property (III) necessitates that the eigenvalues $\mu_{j,k}$ of T satisfy $(\mu_{j,k})^4 = 1$ and thus are integer powers of i. Using property (II) and the relations (3.2) and (3.5), we find

$$\mu_{j+1,k}\, \psi_{j+1,k,\ell} = T\, \psi_{j+1,k,\ell} = T \circ b^{+}\, \psi_{j,k,\ell} = i\, b^{+} \circ T\, \psi_{j,k,\ell} = i\, b^{+}\, \mu_{j,k}\, \psi_{j,k,\ell} = i\, \mu_{j,k}\, \psi_{j+1,k,\ell}.$$

The relation (3.6) now follows from subsequent application of $\mu_{j+1,k} = i\, \mu_{j,k}$. □

PROPOSITION 3.3: *Let $T : \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m \longrightarrow \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m$ be an operator that satisfies the properties (I) and (II). Then, on the basis $\{\psi_{j,k,\ell}\}$, the operator T coincides with the integral transform*

$$(Tf)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x, y)\, f(x)\, \mathrm{d}x$$

with

$$K_m(x, y) = A(w, z) + (\underline{x} \wedge \underline{y})\; B(w, z) \tag{3.7}$$

where

$$\begin{aligned} A(w, z) &= 2^\lambda\, \Gamma(\lambda) \sum_{k=0}^{+\infty} \frac{1}{2}\left(i\, k\, \mu_{0,k-1} + (k + 2\lambda)\mu_{0,k}\right)\, z^{-\lambda} J_{k+\lambda}(z) C_k^\lambda(w) \\ B(w, z) &= 2^{\lambda+1}\, \Gamma(\lambda + 1) \sum_{k=1}^{+\infty} \frac{1}{2}\left(i\, \mu_{0,k-1} - \mu_{0,k}\right)\, z^{-\lambda-1} J_{k+\lambda}(z) C_{k-1}^{\lambda+1}(w), \end{aligned} \tag{3.8}$$

Here, the notations $\lambda = (m-2)/2$, $z = |x||y|$ and $w = \langle x, y\rangle / z$ are used.

Proof. We refer to Theorem A.6 in Appendix A.3 for the full proof of this result. □

In the following result, $\mathrm{Spin}(m)$ denotes the spin group, which is a subgroup of the Clifford algebra $\mathcal{C}l_m$ that is a double cover of the special orthogonal group $\mathrm{SO}(m)$.

LEMMA 3.4: *The kernel satisfies*

$$\begin{aligned} K_m(x, cy) &= K_m(cx, y) \qquad \forall c \in \mathbb{R} \\ K_m(x, Ay) &= K_m(Ax, y) \qquad \forall A \in \mathrm{Spin}(m) \end{aligned}$$

Proof. This follows from the explicit formulas (3.7)–(3.8) for $K_m(x, y)$. □

PROPOSITION 3.5: *A continuous operator $T : \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m \longrightarrow \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m$ satisfying (I)–(III) has a unitary extension to $L^2(\mathbb{R}^m) \otimes \mathcal{C}l_m$.*

Proof. This result follows from the fact that $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m$ is dense in $L^2(\mathbb{R}^m) \otimes \mathcal{C}l_m$ and that all eigenvalues have unit norm. □

We also have an uncertainty principle, using the inner product (A.10) for $L^2(\mathbb{R}^m) \otimes \mathcal{C}l_m$:

THEOREM 3.6: *Let*

$$(Tf)(y) = \int_{\mathbb{R}^m} K(x, y) f(x)\, \mathrm{d}x$$

be a continuous integral transform on $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m$ that satisfies the properties (I)–(III). Then one has the following uncertainty principle:

$$\|\underline{x} f\|_2 \cdot \|\underline{x} T(f)\|_2 \geq \frac{m}{2}\left(\|f\|_2\right)^2$$

for $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_m$.

Proof. This follows by the same reasoning as used to prove Theorem 2.21, now using Lemma 3.4 and noting that the Clifford–Helmholtz relations imply the regular Helmholtz relations. □

REMARK 3.7: In the special case of the Clifford–Fourier transform of [7, 8, 16, 18] more specialized uncertainty principles can be developed following the strategy of [26].

Now, to determine operators that satisfy properties (I)–(III), we proceed in the same way as we did in the previous section for the harmonic case. We start by decomposing T as $T = \widetilde{T} \circ \mathcal{F}$, with again $\widetilde{T} := T \circ \mathcal{F}^{-1}$. From (3.4) and property (III), we have the following conditions for the operator $\widetilde{T}$: $\widetilde{T}$ has to commute with $b^{\pm}$ and $\widetilde{T}^4 = \mathrm{id}$. Therefore, we look at the universal enveloping superalgebra of the Lie superalgebra $\mathfrak{osp}(1|2)$, denoted by $\mathcal{U}(\mathfrak{osp}(1|2))$ (and its extension $\overline{\mathcal{U}}(\mathfrak{osp}(1|2))$ which also allows infinite power series). The center of $\mathcal{U}(\mathfrak{osp}(1|2))$ is finitely generated by the Casimir element [25]:

$$C = \frac{1}{4} + \frac{1}{2}\, b^- b^+ - \frac{1}{2}\, b^+ b^- + h^2 + 2ef + 2fe.$$

REMARK 3.8: The Casimir element C differs from the Casimir element Ω of $\mathfrak{sl}(2)$, given by (2.15), by an additional term. This extra term is related to another special element which we call the Scasimir element (see [25]). The Scasimir element

$$S = \frac{1}{2}\, b^- b^+ - \frac{1}{2}\, b^+ b^- - \frac{1}{2}$$

is a square root of the Casimir operator: $S^2 = C$. It commutes with the even ("bosonic") generators and anti-commutes with the odd ("fermionic") generators b^+ and b^-. The Scasimir S in our operator realization of $\mathfrak{osp}(1|2)$ is related to the angular Dirac operator or Gamma operator in Clifford analysis as follows:

$$S = \frac{m-1}{2} - \Gamma_x, \quad \text{with} \quad \Gamma_x = -\underline{x}\partial_{\underline{x}} - \mathbb{E}_x = -\sum_{j<k} e_{jk}(x_j \partial_{x_k} - x_k \partial_{x_j}).$$

The Casimir element C is a diagonal operator on the representation space spanned by $\{\, \psi_{j,k,\ell} \mid j \in \mathbb{Z}_{\geq 0} \,\}$. Its action is given by

$$C\, \psi_{j,k,\ell} = \left(k + \frac{m-1}{2}\right)^2 \psi_{j,k,\ell}. \tag{3.9}$$

Note that the eigenvalues of the Casimir operator C are again squares of integers or half-integers depending on the value of the dimension m. However, contrary to the eigenvalues of the Casimir element Ω in the harmonic case (2.16), the eigenvalues of C are now squares of half-integers for even dimension, while for odd dimension we have squares of integers.

The desired operators follow by using again the integer-valued polynomials defined in (2.18) and (2.21). We summarize this in the following theorem, which should be compared to Theorem 2.9.

THEOREM 3.9: *The properties*

(I) the Clifford–Helmholtz relations

$$T \circ \partial_{\underline{x}} = -i\,\underline{y} \circ T$$
$$T \circ \underline{x} = -i\,\partial_{\underline{y}} \circ T$$

(II) $T\,\psi_{j,k,\ell} = \mu_{j,k}\,\psi_{j,k,\ell}$ *with* $\mu_{j,k} \in \mathbb{C}$,

(III) $T^4 = \mathrm{id}$

are satisfied by operators T *of the form*

$$T = e^{i\frac{\pi}{2}F(\sqrt{C})} e^{i\frac{\pi}{2}(h-\frac{m}{2})} \in \overline{\mathcal{U}}\big(\mathfrak{osp}(1|2)\big) \tag{3.10}$$

where $F(\sqrt{C})$ *is an operator that consists of a function given by* (2.20) *(for odd dimension) or* (2.22) *(for even dimension) with the Casimir operator* C *substituted for* x^2*. Moreover, every operator that satisfies properties (I)–(III) is equivalent with an operator of the form* (2.23).

Proof. The last part follows by exactly the same reasoning as used in the proof of Theorem 2.9. □

In line with our approach in the previous section we now proceed by imposing a periodicity restriction to further narrow down this set of operators T. This will again aid us in finding closed formulas for their kernels when written as integral transforms.

3.1 *Periodicity restriction*

The behavior of the eigenvalues $\mu_{j,k}$ of T with regard to the index j is given in Lemma 3.2. By successive application of (3.6) we find that these eigenvalues are four-periodic in j:

$$\mu_{j+4,k} = (i)^4\,\mu_{j,k} = \mu_{j,k}.$$

For the same reasons as in the harmonic case we can again impose that the eigenvalues of T should be 4-periodic in k. Depending on the parity of the dimension m one works in, the desired operators follow from the results in either Theorem 2.10 or Theorem 2.11, which of course remain valid. We summarize this in the following theorem.

THEOREM 3.10: *Let* T *be an operator that satisfies the following properties:*

(I) the Helmholtz relations

$$T \circ \Delta_x = -|y|^2 \circ T,$$
$$T \circ |x|^2 = -\Delta_y \circ T,$$

(II) $T\,\phi_{j,k,\ell} = \mu_{j,k}\,\phi_{j,k,\ell}$ *with* $\mu_{j,k} \in \mathbb{C}$,

(III) $T^4 = \mathrm{id}$,

(IV) the eigenvalues of T *are 4-periodic in the index* k: $\mu_{j,k+4} = \mu_{j,k}$.

Then T *can be written as*

$$T = e^{i\frac{\pi}{2}F(\sqrt{C})}e^{i\frac{\pi}{2}(h-\frac{m}{2})}$$

where F *consists of a function of type* (2.24) *as specified in Theorem 2.10 (for odd dimension) or of type* (2.25) *as in Theorem 2.11 (for even dimension).*

Conversely, every operator T *of this form satisfies properties* $(I)-(IV)$.

Proof. Completely analogous to the proof of Theorem 2.13. □

3.2 *Closed formulas for the kernel*

We are again interested in finding closed formulas for the kernel of the integral transforms corresponding to the determined operator solutions. It turns out that in even dimensions we have a result which can be compared with Theorem 2.16 in the harmonic case. Hereto we first rewrite the kernel (3.7) of

$$(Tf)(y)= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m(x, y)\, f(x)\, \mathrm{d}x$$

as obtained in Proposition 3.3. We put

$$K_m(x, y)= A_m(w, z)-\lambda\, B_m(w, z)+(\underline{x} \wedge \underline{y})\, z^{-1}\partial_w B_m(w, z)$$

where

$$\begin{aligned}
A_m(w, z) &= 2^{\lambda}\,\Gamma(\lambda)\sum_{k=0}^{+\infty}(k+\lambda)\frac{1}{2}(i\,\mu_{0,k-1} + \mu_{0,k})\, z^{-\lambda} J_{k+\lambda}(z) C_k^{\lambda}(w)\\
B_m(w, z) &= 2^{\lambda}\,\Gamma(\lambda)\sum_{k=0}^{+\infty}\frac{1}{2}\left(i\,\mu_{0,k-1} - \mu_{0,k}\right) z^{-\lambda} J_{k+\lambda}(z) C_k^{\lambda}(w) \qquad (3.11)\\
z^{-1}\partial_w B_m(w, z) &= 2^{\lambda+1}\,\Gamma(\lambda+1)\sum_{k=1}^{+\infty}\frac{1}{2}\left(i\,\mu_{0,k-1} - \mu_{0,k}\right) z^{-\lambda-1} J_{k+\lambda}(z) C_{k-1}^{\lambda+1}(w),
\end{aligned}$$

using the notations $\lambda = (m-2)/2$, $z = |x||y|$ and $w = \langle x, y\rangle/z$. We then find the recursive relations listed in the following lemma.

LEMMA 3.11: *Let* T *be an operator of the form* (3.10) *as specified in Theorem 3.9. When written as an integral transform, the components of its kernel as specified in* (3.11) *satisfy the following recursive relations*

$$\begin{aligned}
A_{m+2} &= -i\, z^{-1}\partial_w A_m\\
B_{m+2} &= -i\, z^{-1}\partial_w B_m
\end{aligned}$$

for $m \geq 2$.

Proof. As T is of the form (3.10), its eigenvalues are given by

$$T\,\psi_{j,k,\ell} = e^{i\frac{\pi}{2}F(\sqrt{C})}e^{i\frac{\pi}{2}(h-\frac{m}{2})}\,\psi_{j,k,\ell} = e^{i\frac{\pi}{2}F(k+\lambda+\frac{1}{2})}e^{i\frac{\pi}{2}(j+k)}\,\psi_{j,k,\ell} = \mu_{j,k}\,\psi_{j,k,\ell},$$

where we used (3.9) for the eigenvalues of C.

Using property (A.1) of the Gegenbauer polynomials, we find for $-i\,z^{-1}\partial_w A_m(w,z)$ the expression

$$\begin{aligned}
&2^{\lambda}\,\Gamma(\lambda)\sum_{k=0}^{+\infty}(k+\lambda)\frac{1}{2}(i\,i^{F(k-1+\lambda+\frac{1}{2})}i^{k-1} + i^{F(k+\lambda+\frac{1}{2})}i^{k})\; z^{-\lambda-1}J_{k+\lambda}(z)\,\partial_w C_k^{\lambda}(w)\\
&= 2^{\lambda}\,\Gamma(\lambda)\sum_{k=1}^{+\infty}(k+\lambda)\,\frac{1}{2}(i^{F(k-1+\lambda+\frac{1}{2})}i^{k} + i^{F(k+\lambda+\frac{1}{2})}i^{k})\; z^{-(\lambda+1)}J_{k+\lambda}(z)\,2\lambda\,C_{k-1}^{\lambda+1}(w)\\
&= 2^{\lambda+1}\Gamma(\lambda+1)\sum_{k=0}^{+\infty}(k+1+\lambda)\frac{1}{2}(i^{F(k+\lambda+\frac{1}{2})}i^{k} + i^{F(k+1+\lambda+\frac{1}{2})}i^{k})z^{-(\lambda+1)}J_{k+1+\lambda}(z)C_k^{\lambda+1}(w).
\end{aligned}$$

This is precisely $A_{m+2}(w,z)$. The second relation follows in exactly the same way. □

To obtain closed formulas for the kernels in even dimension, we again start by looking at dimension $m = 2$ where we want to use the formulas (2.26), (2.27), (2.28). An additional complication is that when written as an integral transform the kernel contains a non-scalar part. In dimension $m = 2$ the Gegenbauer polynomials occurring in this bivector part reduce to sine functions. With this in mind we aim to use the formula

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2j+1}(z)\,\sin\big((2n+1)\theta\big), \tag{3.12}$$

which can be found in [44, p. 22, formula (2)]. In $m = 2$, the component $z^{-1}\partial_w B_m$ of (3.11) will take on the form (3.12) only if the eigenvalues of the corresponding operator T satisfy, for $k \geq 1$,

$$\begin{cases} i\,\mu_{0,k-1} - \mu_{0,k} = 0 & \text{if } k \text{ is even}\\ i\,\mu_{0,k-1} - \mu_{0,k} = c_0 & \text{if } k \text{ is odd,}\end{cases}$$

with c_0 a constant independent of k. The operators of the form (3.10) whose eigenvalues satisfy these requirements are the subject of the following theorem. In what follows we use the notations $s = \langle x, y\rangle$ and $t = \sqrt{|x|^2|y|^2 - s^2}$.

THEOREM 3.12: *In dimension $m = 2$, the operator exponential*

$$T_{ab} = e^{i\frac{\pi}{2}F_{ab}(\sqrt{C})}e^{i\frac{\pi}{2}(h-1)}$$

with $F_{ab}(x) = a + b\,\mathcal{D}_{0110}(x)$ (as specified in Theorem 2.11) and $a, b \in \mathbb{Z}_4$, can be written as an integral transform whose kernel is given by

$$K_2(x,y) = i^a\Big(\frac{1+i^b}{2}\big(\cos(s) + i\sin(s)\big) + \frac{1-i^b}{2}\cos(t) + (\underline{x}\wedge\underline{y})\,\frac{1-i^b}{2}\,\frac{i\sin(t)}{t}\Big). \tag{3.13}$$

Proof. The proof goes along the same lines as the proof of Theorem 2.15.

By Theorem 3.9, T_{ab} satisfies the properties (I)–(III). For $m = 2$, the kernel $K_2(x, y)$ obtained in Proposition 3.3 (for the formulation of T as an integral transform), reduces to

$$\mu_{0,0}\, J_0(z)+\sum_{k=1}^{+\infty}(i\,\mu_{0,k-1}+\mu_{0,k})\, J_k(z)\cos(k\theta)+(\underline{x}\wedge\underline{y})\sum_{k=1}^{+\infty}(i\,\mu_{0,k-1}-\mu_{0,k})\, J_k(z)\,\frac{\sin(k\theta)}{z\sin\theta}.$$

This follows from $\lambda = 0$ and the identities (see [22, Vol. I, section 3.15], formulas (14) and (15)), for $w = \cos\theta$ and integer $k \geq 1$,

$$\lim_{\lambda\to 0}\Gamma(\lambda)\, C_k^{\lambda}(\cos\theta)=\frac{2}{k}\cos(k\theta), \qquad \text{and} \qquad C_{k-1}^{1}(\cos\theta)=\frac{\sin(k\theta)}{\sin\theta}.$$

Now, using the fact that the eigenvalues of T_{ab} are 4-periodic in k, with (for $m = 2$)

$$\mu_{0,0} = i^a, \quad \mu_{0,1} = i^{a+b}i, \quad \mu_{0,2} = i^{a+b}(-1), \quad \mu_{0,3} = i^a(-i),$$

we can rewrite the kernel as

$$\begin{aligned} K_2(x, y)= i^a\Big(J_0(z)&+2\sum_{n=1}^{+\infty} J_{4n}(z)\cos(4n\theta)-i^b\, 2\sum_{n=0}^{+\infty} J_{4n+2}(z)\cos\big((4n+2)\theta\big)\\ &+ (1+i^b)\, i\,\Big(\sum_{n=0}^{+\infty} J_{4n+1}(z)\cos\big((4n+1)\theta\big) - \sum_{n=0}^{+\infty} J_{4n+3}(z)\cos\big((4n+3)\theta\big)\Big)\\ &+ (\underline{x}\wedge\underline{y})\,(1-i^b)\, i\sum_{n=0}^{+\infty} J_{2n+1}(z)\,\frac{\sin\big((2n+1)\theta\big)}{z\sin\theta}\Big). \end{aligned}$$

The formulas (2.26), (2.27), (2.28), (3.12) and $s = zw = z\cos\theta$, $t = z\sqrt{1-w^2} = z\sin\theta$ then yield for $K_2(x, y)$ the expression

$$i^a\Big(\frac{1}{2}\big(\cos(s)+\cos(t)\big) + \frac{i^b}{2}\big(\cos(s)-\cos(t)\big) + \frac{(1+i^b)i}{2}\sin(s)\Big) + (\underline{x}\wedge\underline{y})\,\frac{1-i^b}{2}\,\frac{i\sin(t)}{t}\Big).$$

□

Using Lemma 3.11 we now find the following theorem.

THEOREM 3.13: *In even dimension m, the operator exponential*

$$T_{ab} = e^{i\frac{\pi}{2}F_{ab}(\sqrt{C})}e^{i\frac{\pi}{2}(h-\frac{m}{2})}$$

with $F_{ab}(x)= a + b\,\mathcal{D}_{0110}(x)$ *(as specified in Theorem 2.11), can be written as an integral transform whose kernel is given by*

$$\begin{aligned} K_m(x, y)= i^{a-\lambda}\Big(&\frac{1-i^b}{2}\Big(\partial_s - \frac{s}{t}\,\partial_t\Big)^{\lambda}\cos(t)-\lambda\frac{1-i^b}{2}\Big(\partial_s - \frac{s}{t}\,\partial_t\Big)^{\lambda-1}\frac{i\sin(t)}{t}\\ &+\frac{1+i^b}{2}\big(\cos(s)+i\,\sin(s)\big) + (\underline{x}\wedge\underline{y})\,\frac{1-i^b}{2}\Big(\partial_s - \frac{s}{t}\,\partial_t\Big)^{\lambda}\frac{i\sin(t)}{t}\Big), \end{aligned} \tag{3.14}$$

with $\lambda = (m-2)/2$. Moreover, one has

$$\left(\partial_s - \frac{s}{t}\,\partial_t\right)^{\lambda} \cos(t) = \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\lfloor \frac{m-2}{4} \rfloor} s^{\frac{m}{2}-1-2\ell} \frac{1}{2^{\ell}\ell!} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}-2\ell)} \frac{J_{(m-2\ell-3)/2}(t)}{t^{(m-2\ell-3)/2}}.$$

$$\left(\partial_s - \frac{s}{t}\,\partial_t\right)^{\lambda-1} \frac{\sin(t)}{t} = \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\lfloor \frac{m-4}{4} \rfloor} s^{\frac{m}{2}-2-2\ell} \frac{1}{2^{\ell}\ell!} \frac{\Gamma(\frac{m}{2}-1)}{\Gamma(\frac{m}{2}-1-2\ell)} \frac{J_{(m-2\ell-3)/2}(t)}{t^{(m-2\ell-3)/2}}.$$

$$\left(\partial_s - \frac{s}{t}\,\partial_t\right)^{\lambda} \frac{\sin(t)}{t} = \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\lfloor \frac{m-2}{4} \rfloor} s^{\frac{m}{2}-1-2\ell} \frac{1}{2^{\ell}\ell!} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}-2\ell)} \frac{J_{(m-2\ell-1)/2}(t)}{t^{(m-2\ell-1)/2}}.$$

Proof. This is analogous to the proof of Theorem 2.16 in the harmonic case. For $m = 2$ we have $\lambda = 0$, and the expression (3.14) coincides with (3.13) which was obtained in Theorem 3.12. Starting from K_2, we find the kernel in every even dimension $m > 2$ by means of the recursive relations obtained in Lemma 3.11. Note that when moving from dimension $m = 2$ to $m = 4$, although we have no explicit expression for B_2, we can obtain B_4 directly from the bivector part in K_2:

$$K_2(x, y) = A_2(w, z) + (\underline{x} \wedge \underline{y})\, z^{-1} \partial_w B_2(w, z) = A_2(w, z) + (\underline{x} \wedge \underline{y})\, i\, B_4(w, z). \qquad \square$$

3.2.1 *Specific cases*

The kernel (3.14) in Theorem 3.13 consists of a linear interpolation of two terms, with coefficients $(1+i^b)/2$ and $(1-i^b)/2$. The first term is precisely the kernel of the classical Fourier transform, while the second term contains a non-scalar part and can be written as a finite sum of Bessel functions. This second term closely resembles an expression found for the kernel of the Clifford-Fourier transform in [18] and also for similar transforms devised in [16].

We can isolate this second term by putting $b = 2$ in (3.14). Multiplying by $i^{\lambda-a}$ to eliminate a scalar multiplicative factor, the kernel reduces to

$$\begin{aligned} K_m(x, y) = {} & \left(\left(\partial_s - \frac{s}{t}\,\partial_t\right)^{\lambda} \cos(t) - \lambda\, i \left(\partial_s - \frac{s}{t}\,\partial_t\right)^{\lambda-1} \frac{\sin(t)}{t} \right) \\ & + (\underline{x} \wedge \underline{y})\, i \left(\partial_s - \frac{s}{t}\,\partial_t\right)^{\lambda} \frac{\sin(t)}{t}, \end{aligned}$$

which corresponds to the operator exponential

$$T = e^{i\frac{\pi}{2}\lambda} e^{i\frac{\pi}{2}(C-\frac{1}{4})} e^{i\frac{\pi}{2}(h-\frac{m}{2})} = e^{i\frac{\pi}{2}(C-\frac{5}{4}+h)}.$$

For dimension $m = 2$ this kernel reduces to

$$K_2(x, y) = \cos(t) + (\underline{x} \wedge \underline{y})\, i\, \frac{\sin(t)}{t}.$$

4 CONCLUSIONS AND REMARKS

We started our investigation from a list of properties (i)–(iii) which specifically highlights two symmetric structures underlying the higher dimensional Fourier transform, namely the orthogonal symmetry and an algebraic symmetry with respect to an operator realization of the Lie algebra $\mathfrak{sl}(2)$. We then obtained a complete set of solutions satisfying (i)–(iii) in the form of operator exponentials, which includes in particular the classical Fourier transform. The transforms we have constructed also demonstrate other interesting features. For instance, for each of them, we found a corresponding formulation as an integral transform. Moreover, for a select set of operators when written as an integral transform the kernel could even be reduced to a closed formula being polynomially bounded.

In the process of describing these solutions, we gave a brief overview on the subject of integer-valued polynomials and a generalization thereof, i.e. polynomials that are integer-valued on the set of square numbers, the set of half-integers or the set of squares of half-integers.

These results were subsequently lifted to the setting of Clifford analysis where the Lie algebra $\mathfrak{sl}(2)$ in the algebraic symmetry was generalized to the Lie superalgebra $\mathfrak{osp}(1|2)$, resulting in a new list of properties (I)–(III). Also in this case the obtained operators form a complete set of solutions and they all have an equivalent formulation as an integral transform. For a select set of transforms we again find a polynomially bounded formula for the kernel. The findings obtained here coalesce with those for similar generalized Fourier transforms in the context of Clifford analysis as described in [16, 18].

We note that in both the regular and the Clifford setting more closed formulas for kernels are found for even dimension than in the odd dimensional case. The reason for this is that, although the recursive relations obtained in Lemma 2.14 and Lemma 3.11 also hold for odd dimension, we have no formula for dimension $m = 3$ to use as starting point and as a consequence no such form is found.

Finally, we note that when moving to Clifford analysis we replaced the Helmholtz relations by their more restrictive variant called the Clifford–Helmholtz relations. Nevertheless, one may also consider solutions to the regular Helmholtz relations in the context of Clifford analysis. Following the same strategy we applied before, this leads to the subset of $\mathcal{U}\big(\mathfrak{osp}(1|2)\big)$ consisting of the elements that commute with all elements of $\mathfrak{sl}(2)$, instead of $\mathfrak{osp}(1|2)$. This subset is generated by S, the Scasimir element of $\mathfrak{osp}(1|2)$, whose eigenvalues on the basis $\{\psi_{j,k,\ell}\}$ square to those of the Casimir C, given by (3.9). Accordingly, this brings us to polynomials in S that again take integer values when acting on the basis $\{\psi_{j,k,\ell}\}$. This gives rise to a bigger class of more general operator exponentials which includes in particular the operators we obtained by imposing the Clifford–Helmholtz relations.

A APPENDIX

Here we give an overview of some definitions and results used in the main text, as well as some proofs that have been omitted from the text.

A.1 *Special functions*

For $\alpha > -1$, p a positive integer and $\Gamma(\cdot)$ the Gamma function, the generalized Laguerre polynomials are given by

$$L_j^{\alpha}(x)= \sum_{n=0}^{j} \frac{\Gamma(j+\alpha+1)}{n!\,(j-n)!\,\Gamma(n+\alpha+1)}(-x)^n.$$

For $k \in \mathbb{Z}_{\geq 0}$ and $\lambda > -1/2$, the Gegenbauer polynomials are defined as

$$C_k^{\lambda}(w)= \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^j \frac{\Gamma(k-j+\lambda)}{\Gamma(\lambda)\,j!\,(k-2j)!}(2w)^{n-2j}.$$

They are a special case of the Jacobi polynomials and satisfy the differentiation property (see e.g. [21, 43]):

$$\frac{\mathrm{d}}{\mathrm{d}w}C_k^{\lambda}(w)= 2\lambda\, C_{k-1}^{\lambda+1}(w). \tag{A.1}$$

The Bessel function can be defined by the power series

$$J_{\nu}(t)= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{t}{2}\right)^{2n+\nu}, \tag{A.2}$$

where $\nu \in \mathbb{C}$ is the order of the Bessel function (see e.g. [44]). In particular, for $\nu = 1/2$ and $\nu = -1/2$, the power series (A.2) reduces to

$$J_{1/2}(t)= \sqrt{\frac{2}{\pi t}}\sin t, \qquad J_{-1/2}(t)= \sqrt{\frac{2}{\pi t}}\cos t, \tag{A.3}$$

as found in [44, p. 54]. The Bessel functions satisfy (see [44, p. 45])

$$-\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}[t^{-\nu}J_{\nu}(t)] = t^{-(\nu+1)}J_{\nu+1}(t). \tag{A.4}$$

A.2 *Integer-valued polynomials*

We will now elaborate on the identities showcased in Theorem 2.10 and Theorem 2.11 of which the proofs were omitted from the main text. Before delving into the actual proofs, we first show some auxiliary results. In subsection 2.1 we already mentioned the polynomials $E_n(x)$ and $D_n(x)$ exhibiting a property of periodicity when evaluated modulo 4. We now prove a similar result for the more general integer-valued polynomials $\binom{x}{n}$ from which the other periodicity results follow.

PROPOSITION A.1: *For $n, N \in \mathbb{Z}_{\geq 0}$ such that $n < 2^N$, and $x \in \mathbb{Z}$, one has*

$$\binom{x}{n} \equiv \binom{x + 2^{N+1}}{n} \pmod 4$$

Proof. Take $N \in \mathbb{Z}_{\geq 0}$ and $n = 2^N - 1$, the proof for every $n < 2^N - 1$ is similar.

From (2.17), we have by definition

$$\binom{x + 2^{N+1}}{2^N - 1} = \frac{1}{(2^N - 1)!} \prod_{k=0}^{2^N-2} (2^{N+1} + x - k). \tag{A.5}$$

When expanding this product, each term consists of a number of factors of the form $x - k$ multiplied with a power of 2^{N+1}. We will show that every term containing a factor 2^{N+1} is divisible by four and hence congruent 0 modulo 4.

First, note that the denominator $(2^N - 1)!$ in (A.5) contains $2^N - N - 1$ times the factor 2. This follows from adding the number of integers smaller than $2^N - 1$ that are divisible by the powers 2^j for j ranging from 1 to $N - 1$.

Next, we determine when a term containing a factor 2^{N+1} in the expansion of the numerator of (A.5) has a minimal number of factors 2 in its prime factorization. This is the case when its other factors in the product (A.5) are the integers of the set $A_x := \{ x - k \mid k \in \mathbb{Z}_{\geq 0} \text{ and } 0 \leq k \leq 2^N - 2 \}$ that contain less than $N + 1$ factors 2 and its remaining factors being 2^{N+1}. For a given $x \in \mathbb{Z}$, the set A_x consists of $2^N - 1$ consecutive integers. The absolute minimal case thus ensues when the set A_x contains $2^N - 2$ integers that are not divisible by 2^{N-1}. The product of these integers gives a minimum of $2^N - 2N$ factors 2. Hence, a factor 2^{N+1} multiplied by such a product of integers contains at least $2^N - N + 1$ times the factor 2.

As the denominator in (A.5) contains $2^N - N - 1$ times the factor 2, every term in the expansion of (A.5) with a factor 2^{N+1} must contain at least two factors 2 or thus be divisible by four. □

COROLLARY A.2: *For natural numbers $n, N \in \mathbb{Z}_{\geq 0}$ such that $n \leq 2^N$, and $x \in \mathbb{Z}_{\geq 0}$, one has*

$$E_n(x) \equiv E_n\left(x + 2^{N+2}\right) \pmod 4,$$

and

$$2\,E_n(x) \equiv 2\,E_n\left(x + 2^{N+1}\right) \pmod 4,$$

Proof. Writing (2.18) as

$$E_n(x) = \frac{2}{(2n)!} \prod_{k=0}^{n-1} (x^2 - k^2) = \frac{2}{(2n)!}\, x \prod_{k=0}^{2n-2} (x - n + 1 + k),$$

the periodicity follows by the same reasoning as used in the proof of Proposition A.1. □

We need one more lemma which will prove to be useful in the proof of Theorem 2.10.

LEMMA A.3: *For n an odd integer and k an even integer, we have*

$$E_n(k) \equiv 0 \pmod 4.$$

Proof. Put $n = 2N + 1$ and $k = 2K$ for some integers N, K. Then

$$E_{2N+1}(2K) = \frac{2K}{2N+1}\binom{2K+2N}{4N+1} = \frac{2K(2K+2N)}{(2N+1)(4N+1)}\binom{2K+2N-1}{4N} \equiv 0 \pmod 4$$

as the numerator contains a factor 4 while the denominator contains only odd factors. □

Proof of Theorem 2.10. Note first that for $a, b, c, d \in \{0, 1, 2, 3\}$ the function

$$F(x) = a + b\,(\mathbf{1}_{4\mathbb{Z}+1}(x) + \mathbf{1}_{4\mathbb{Z}+3}(x)) + c\,\mathbf{1}_{4\mathbb{Z}+2}(x) + d\,\mathbf{1}_{4\mathbb{Z}+3}(x)$$

is 4-periodic with its first four values given by

$$F(0) = a, \quad F(1) = a + b, \quad F(2) = a + c, \quad F(3) = a + b + d.$$

By specifying $a, b, c, d \in \{0, 1, 2, 3\}$ one can clearly obtain every possible combination of first four values modulo 4.

The first identity of Theorem 2.10 follows easily from the fact that $E_1(x) = x^2$, as every $x \in \mathbb{Z}_{\geq 0}$ is equal to either $2k$ or $2k + 1$ for some $k \in \mathbb{Z}_{\geq 0}$, so

$$(2k)^2 = 4k^2 \equiv 0 \pmod 4, \quad \text{while} \quad (2k+1)^2 = 4k^2 + 4k + 1 \equiv 1 \pmod 4.$$

For the second identity, by (2.18) we have

$$E_2(x) + 2\,E_3(x) = \frac{1}{12}x^2(x^2-1) + \frac{1}{180}x^2(x^2-1)(x^2-4) = \frac{x^2(x^2-1)(x^2+11)}{4\cdot 45}.$$

Now for $k \in \mathbb{Z}_{\geq 0}$, we find

$$E_2(2k+1) + 2\,E_3(2k+1) = \frac{(4k^2+4k+1)(4k^2+4k)(4k^2+4k+12)}{4\cdot 45} \equiv 0 \pmod 4.$$

Moreover, again for $k \in \mathbb{Z}_{\geq 0}$, it holds that

$$E_2(2k) + 2\,E_3(2k) = \frac{4k^2(4k^2-1)(4k^2+11)}{4\cdot 45} \equiv k^2 \pmod 4.$$

We already know that k^2 is congruent 1 modulo 4 when k is odd, which corresponds to $2k \equiv 2 \pmod 4$, and congruent 0 modulo 4 when k is even, corresponding to $2k \equiv 0 \pmod 4$.

For the last identity, we will show that

$$\mathcal{E}_{0001}(x) \equiv \mathbf{1}_{4\mathbb{Z}+3}(x) \pmod 4.$$

Hereto we write

$$\mathcal{E}_{0001}(x) = \sum_{n=1}^{\infty} T_n(x) \qquad \text{with} \qquad T_n(x) := E_{2^n+1}(x) + \sum_{j=1}^{n-1} 2\, E_{2^n+1+2^j}(x).$$

By Lemma A.3 we immediately have that for x an even integer each term in $T_n(x)$ is congruent 0 modulo 4. What remains to be shown is that for k an integer one has $\mathcal{E}_{0001}(4k+1) \equiv 0 \pmod 4$ and $\mathcal{E}_{0001}(4k+3) \equiv 1 \pmod 4$. Hereto we look at the values of the functions E_{2^n+1} and $2\, E_{2^n+1+2^j}$ that constitute T_n and in this way generate the values of T_n and ultimately those of $\mathcal{E}_{0001}$.

Recall that

$$E_n(x) = \frac{\prod_{\ell=0}^{n-1}(x^2 - \ell^2)}{\prod_{\ell=0}^{n-1}(n^2 - \ell^2)}.$$

We are dealing with functions that consist of a fraction with integer numerator and denominator which evaluates to an integer. To determine the value of this integer up to modulo 4 congruence, we will look at the numerator and the denominator separately and factorize them into a power of 2 and a remaining odd part. For instance, if a/b is such a fraction and we have the following factorization $a = 2^k a'$ and $b = 2^\ell b'$ with a' and b' odd integers, then

$$\frac{a}{b} \mod 4 \equiv \begin{cases} 0 & \text{if } k - \ell \geq 2 \\ 2 & \text{if } k - \ell = 1 \\ a' \cdot (b')^{-1} & \text{if } k = \ell. \end{cases}$$

Note that $k \geq \ell$ must hold as 2^{-1} is not defined modulo 4 and we know that a/b must evaluate to an integer. Moreover, modulo 4 we have for odd integers $(1)^{-1} \equiv 1 \pmod 4$ and $(3)^{-1} \equiv 3 \pmod 4$.

We first look at the denominator of the function $E_{2^n+1}(x)$ for $n \geq 1$ which is given by

$$\prod_{\ell=0}^{2^n}\left((2^n+1)^2 - \ell^2\right) \equiv \prod_{j=0}^{2^{n-1}-1}\left((2^n+1)^2 - (2j+1)^2\right) \pmod 4. \tag{A.6}$$

Here, we carried out a first simplification where we removed a number of factors congruent 1 modulo 4 as they have no impact on the final value modulo 4 of the product. Indeed, for ℓ even, say $\ell = 2j$, we have $(2^n+1)^2 - (2j)^2 = 4n^2 + 4n + 1 - 4j^2 \equiv 1 \pmod 4$. If $n = 1$, (A.6) reduces to just one factor, yielding

$$\prod_{j=0}^{0}\left((2+1)^2 - (2j+1)^2\right) = \left(3^2 - 1^2\right) = 2^3$$

for the denominator of $E_3(x)$. For $n \geq 2$ we work out (A.6) as follows

$$\prod_{j=0}^{2^{n-1}-1}\left((2^n+1)^2 - (2j+1)^2\right) = \prod_{j=0}^{2^{n-1}-1}\left(2^n + 1 - 2j - 1\right)\left(2^n + 1 + 2j + 1\right)$$

$$
\begin{aligned}
&= \prod_{j=0}^{2^{n-1}-1} \left(2^n - 2j\right)\left(2^n + 2j + 2\right) \\
&= (2^{n+1})!! \\
&= 2^{(2^n)}(2^n)!
\end{aligned}
$$

Using repeatedly the recursive relation (for $n \geq 1$)

$$
(2^n)! = 2^{(2^{n-1})}(2^n - 1)!!\,(2^{n-1})!
$$

we find

$$
\begin{aligned}
2^{(2^n)}(2^n)! &= 2^{(2^n)} \prod_{j=1}^{n} 2^{(2^{j-1})}(2^j - 1)!! \\
&= 2^{(2^{n+1}-1)} \prod_{j=1}^{n} (2^j - 1)!!
\end{aligned}
$$

Now, the double factorial $(2^j - 1)!!$ consists of a product of 2^{j-1} odd consecutive integers. For $j = 1$ this product consists of just one factor, namely 1. The product of two odd consecutive integers is always congruent 3 modulo 4, hence $j = 2$ contributes a factor 3, while for $\ell \geq 3$ this double factorial is thus necessarily congruent 1 modulo 4. In this way, for the denominator of $E_{2^n+1}(x)$ we arrive at

$$
2^{2^{n+1}-1} \prod_{\ell=0}^{n-1} \prod_{j=0}^{2^\ell - 1} (2j + 1) \equiv 2^{(2^{n+1}-1)} \cdot 3 \pmod 4. \tag{A.7}
$$

Using this information we will show that for integers k and x

$$
E_{2^n+1}(2^n k + 2x + 1) \equiv \frac{k(k+1)}{2} \pmod 4 \qquad (0 \leq x < 2^{n-1}).
$$

For $n = 1$ we immediately have

$$
E_3(2k + 1) \equiv \frac{1}{2^3} \prod_{j=0}^{2} \left((2k+1)^2 - j^2\right) \equiv \frac{4k^2 + 4k + 1 - 1}{2^3} \equiv \frac{k(k+1)}{2} \pmod 4,
$$

while for $n \geq 2$ we find for the numerator of $E_{2^n+1}(2^n k + 2x + 1)$:

$$
\begin{aligned}
\prod_{j=0}^{2^n} \left((2^n k + 2x + 1)^2 - j^2\right) &\equiv \prod_{j=0}^{2^{n-1}-1} \left((2^n k + 2x + 1)^2 - (2j + 1)^2\right) \pmod 4 \\
&= \prod_{j=0}^{2^{n-1}-1} \left(2^n k + 2x - 2j\right)\left(2^n k + 2x + 2j + 2\right)
\end{aligned}
$$

$$= \prod_{j=1}^{2^n} \left(2^n(k-1)+2x+2j\right).$$

Here we have a product of 2^n even consecutive integers starting at $2^n(k-1)+2x+2$. For $x=0$ this product starts at $2^n(k-1)+2$ and goes up to $2^n(k-1)+2^{n+1}$. Compared to the case $x=0$, a shift occurs when $x>0$ where for j from 1 up to x the factor $2^n(k-1)+2j$ in this product gets replaced by $2^n(k-1)+2j+2^{n+1}$. Now, our goal is to factor out all powers of 2 and look at the remaining odd part modulo 4. As long as $2j < 2^n$, or thus $x < 2^{n-1}$, we see that $2^n(k-1)+2j$ contains at most a power of 2 equal to 2^{n-1}. Hence, when factoring out all powers of 2 the added term 2^{n+1} in the replacement factor $2^n(k-1)+2j+2^{n+1}$ can at most reduce to 2^2. This means that the odd part remaining after factoring out all powers of 2 of the replacement factor and the original factor give the same value modulo 4. Note that this no longer holds for $x=2^{n-1}$ as one can then factor out 2^n.

For $0 \le x < 2^{n-1}$ we have

$$\prod_{j=1}^{2^n} \left(2^n(k-1)+2x+2j\right) \equiv \prod_{j=1}^{2^n} \left(2^n(k-1)+2j\right) \pmod 4$$

Using repeatedly the recursive relation (for $n \ge 1$)

$$\prod_{j=1}^{2^n} \left(2^n(k-1)+2j\right) = 2^{(2^n)} \prod_{j=1}^{2^{n-1}} \left(2^{n-1}(k-1)+2j-1\right) \prod_{j=1}^{2^{n-1}} \left(2^{n-1}(k-1)+2j\right)$$

we find

$$\begin{aligned}\prod_{j=1}^{2^n} \left(2^n(k-1)+2j\right) &= (k+1)\prod_{\ell=1}^{n} 2^{(2^\ell)} \prod_{j=1}^{2^{\ell-1}} \left(2^{\ell-1}(k-1)+2j-1\right)\\ &= (k+1)2^{(2^{n+1}-2)} \prod_{\ell=1}^{n}\prod_{j=1}^{2^{\ell-1}} \left(2^{\ell-1}(k-1)+2j-1\right).\end{aligned}$$

Now, for $\ell=1$ the inner most product reduces to one factor $2^0(k-1)+1=k$, while for $\ell=2$ we have $\left(2(k-1)+1\right)\left(2(k-1)+3\right) \equiv 3 \pmod 4$. For all other values of ℓ, we have a product of $2^{\ell-1}$ odd consecutive integers which is always congruent 1 modulo 4. In this way, we arrive at

$$2^{(2^{n+1}-2)} \cdot 3 \cdot k(k+1)$$

for the numerator of $E_{2^n+1}(2^nk+2x+1)$. Together with what we obtained for the denominator of $E_{2^n+1}(x)$ in (A.7) we ultimately find

$$E_{2^n+1}(2^nk+2x+1) \equiv \frac{k(k+1)}{2} \pmod 4 \qquad \left(0 \le x < 2^{n-1}\right).$$

What this means is that when looking at the values of E_{2^n+1} modulo 4 evaluated at the odd integers, starting at 1 we have 2^{n-1} times the value 0, followed by 2^{n-1} times the value 1, then 2^{n-1} times the value 3 and 2^{n-1} times the value 2 after which the sequence mirrors and repeats due to the periodicity result in Corollary A.2 and the fact that $E_n(-x) = E_n(x)$. This is illustrated in Table A1.

Using the same techniques as above one shows that

$$2E_{2^n+1+2^j}\left(2^n + 2^{n+1}k + 2x + 2^j y + 1\right) \equiv (k+1)^2 y(y+1) \pmod 4,$$

for $0 \le x < 2^{j-1}$ and $0 \le y < 2^{n-j+1}$. Note that because of the added factor 2 in front, it suffices to count the powers of 2 and we do not need to take into account the congruence class modulo 4 of the remaining odd part when decomposing the numerator and the denominator. This result translates to the evaluation at odd integers as follows: starting at 1, for $2E_{2^n+1+2^j}$ we have 2^{n-1} times the value 0, followed by 2^{n-j+1} times a sequence consisting of 2^{j-1} values 0, 2^j values 2 and again 2^{j-1} values 0. This is followed by 2^{n-1} times the value 0 and the other values follow from Corollary A.2.

Together with what we obtained for E_{2^n+1}, this gives the following sequence of values for T_n with $n \ge 2$ evaluated at the integers $4k+1$ starting at 1 ($k = 0$): we have 2^{n-2} times the value 0, followed by 2^{n-1} times the value 1, 2^{n-1} times the value 2, 2^{n-1} times the value 3 and finally 2^{n-2} times the value 0, after which the sequence repeats due to the periodicity result in Corollary A.2. This is illustrated in Table A2. A similar result holds for the values of T_n evaluated at the integers $4k+3$ which is illustrated in table A3. More formally, we can also write this as (for $n \ge 2$)

$$\begin{aligned} T_n(2^n k + 4x + 1) &\equiv \frac{k(k+1)}{2} + \frac{k^2(k^2-1)}{2} \pmod 4 \qquad (0 \le x < 2^{n-2}) \\ T_n(2^n k + 4x + 3) &\equiv \frac{k(k-1)}{2} + \frac{k^2(k^2-1)}{2} \pmod 4 \qquad (0 \le x < 2^{n-2}). \end{aligned} \tag{A.8}$$

Putting

$$\mathcal{E}_{0001}^N(x) := \sum_{n=1}^{N} T_n(x),$$

the function $\mathcal{E}_{0001}^N$ has periodicity 2^{N+3} due to Corollary A.2. When evaluated at integers congruent 1 modulo 4 it thus suffices to know the first 2^{N+1} values. We now show that the values $\mathcal{E}_{0001}^N(4k+1)$ for k from 0 to 2^{N+1} are 2^{N-1} times 0, followed by 2^{N-1} times 3, followed by 2^{N-1} times 2 and finally 2^{N-1} times 1.

This obviously holds for the case $N = 1$, as then we have $T_1 = E_3$ which evaluated at $4k+1$ gives the sequence 0, 3, 2, 1 repeated indefinitely. Next, we assume this holds for $\mathcal{E}_{0001}^N(4k+1)$ and we show that it also holds for $\mathcal{E}_{0001}^{N+1}(4k+1)$. We have by definition

$$\mathcal{E}_{0001}^{N+1}(x) = T_{N+1}(x) + \sum_{n=1}^{N} T_n(x).$$

Owing to Corollary A.2, it suffices to look at T_{N+1} evaluated at the values $4k+1$ for k from 0 to 2^{N+2}, which we found in (A.8). The addition of these values with the assumed values of $\mathcal{E}_{0001}^{N}$ now give the desired values of $\mathcal{E}_{0001}^{N+1}$ modulo 4. This is illustrated in Table A4.

Hence, for an integer congruent 1 modulo 4, say $4k+1$, we thus find that $\mathcal{E}_{0001}^{N}(4k+1) \equiv 0 \pmod 4$ for N such that $4k+1 \leq 2^{N-1}$. As $\mathcal{E}_{0001} = \lim_{N\to\infty} \mathcal{E}_{0001}^{N}$, we have $\mathcal{E}_{0001}(4k+1) \equiv 0 \pmod 4$ for every integer k. In exactly the same manner one finds that $\mathcal{E}_{0001}(4k+3) \equiv 1 \pmod 4$. □

The proof of Theorem 2.11 is analogous to the one of Theorem 2.10, now using the following periodicity property of the polynomials D_n.

COROLLARY A.4: *For natural numbers n, $N \in \mathbb{Z}_{\geq 0}$ such that $n \leq 2^N$, and $x \in \mathbb{Z}_{\geq 0}$, one has*

$$D_n\left(x+\tfrac{1}{2}\right) \equiv D_n\left(x+\tfrac{1}{2}+2^{N+2}\right) \pmod 4,$$

and

$$2\,D_n\left(x+\tfrac{1}{2}\right) \equiv 2\,D_n\left(x+\tfrac{1}{2}+2^{N+1}\right) \pmod 4,$$

Proof. Writing (2.21) as

$$D_n\left(x+\tfrac{1}{2}\right) = \frac{1}{(2n)!}\prod_{k=0}^{n-1}\left((x+\tfrac{1}{2})^2-(k+\tfrac{1}{2})^2\right) = \frac{1}{(2n)!}\prod_{k=0}^{2n-1}(x-n+1+k),$$

the periodicity follows by the same reasoning as used in the proof of Proposition A.1. □

A.3 *Clifford analysis*

To show Proposition 3.3, we first give some auxiliary results. For these results, let T be an integral transform

$$T[f(x)](y) = \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} K(x,y)\, f(x)\, \mathrm{d}x$$

with kernel given by

$$K(x,y) = A(w,z) + (\underline{x} \wedge \underline{y})\, B(w,z)$$

where, for α_k, $\beta_k \in \mathbb{C}$,

$$A(w,z) = 2^{\lambda}\,\Gamma(\lambda)\sum_{k=0}^{+\infty}(k+\lambda)\,\alpha_k\, z^{-\lambda} J_{k+\lambda}(z) C_k^{\lambda}(w)$$

$$B(w,z) = 2^{\lambda+1}\,\Gamma(\lambda+1)\sum_{k=1}^{+\infty}(k+\lambda)\,\beta_k\, z^{-\lambda-1} J_{k+\lambda}(z) C_{k-1}^{\lambda+1}(w).$$

The eigenvalues of T can be determined using the following proposition, which is a generalization of Bochner's formulas for the classical Fourier transform.

TABLE A1: The values (modulo 4) of the polynomials $E_3, E_5, \ldots, E_{31}$ evaluated on the odd integers 1 to 33

$2k+1$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33
E_3	0	1	3	2	2	3	1	0	0	1	3	2	2	3	1	0	0
E_5	0	0	1	1	3	3	2	2	2	2	3	3	1	1	0	0	0
E_9	0	0	0	0	1	1	1	1	3	3	3	3	2	2	2	2	2
E_{17}	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	3
E_{31}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE A2: The values (modulo 4) of the polynomials $T_1, T_2, \ldots, T_6$ evaluated on the integers congruent 1 modulo 4, ranging from 1 to 65

$4k+1$	1	5	9	13	17	21	25	29	33	37	41	45	49	53	57	61	65
T_1	0	3	2	1	0	3	2	1	0	3	2	1	0	3	2	1	0
T_2	0	1	1	2	2	3	3	0	0	1	1	2	2	3	3	0	0
T_3	0	0	1	1	1	1	2	2	2	2	3	3	3	3	0	0	0
T_4	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2
T_5	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
T_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE A3: The values (modulo 4) of the polynomials $T_1, T_2, \ldots, T_6$ evaluated on the integers congruent 3 modulo 4, ranging from 3 to 67

$4k+3$	3	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67
T_1	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	0
T_2	0	3	3	2	2	1	1	0	0	3	3	2	2	1	1	0	0
T_3	0	0	3	3	3	3	2	2	2	2	1	1	1	1	0	0	0
T_4	0	0	0	0	3	3	3	3	3	3	3	3	2	2	2	2	2
T_5	0	0	0	0	0	0	0	0	3	3	3	3	3	3	3	3	3
T_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3

TABLE A4: The values (modulo 4) of $\mathcal{E}_{0001}^{N+1}$ evaluated on the first 2^{N+2} integers congruent 1 modulo 4

	$\overbrace{}^{2^{N-1}}$	$\overbrace{}^{2^{N-1}}$	$\overbrace{}^{2^{N-1}}$	$\overbrace{}^{2^{N-1}}$	$\overbrace{}^{2^{N-1}}$	$\overbrace{}^{2^{N-1}}$	$\overbrace{}^{2^{N-1}}$	$\overbrace{}^{2^{N-1}}$
$\mathcal{E}_{0001}^{N}(4k+1)$	$0\ldots0$	$3\ldots3$	$2\ldots2$	$1\ldots1$	$0\ldots0$	$3\ldots3$	$2\ldots2$	$1\ldots1$
$T_{N+1}(4k+1)$	$0\ldots0$	$1\ldots1$	$1\ldots1$	$2\ldots2$	$2\ldots2$	$3\ldots3$	$3\ldots3$	$0\ldots0$
$\mathcal{E}_{0001}^{N+1}(4k+1)$	$\underbrace{0\ldots0 \quad 0\ldots0}_{2^N}$		$\underbrace{3\ldots3 \quad 3\ldots3}_{2^N}$		$\underbrace{2\ldots2 \quad 2\ldots2}_{2^N}$		$\underbrace{1\ldots1 \quad 1\ldots1}_{2^N}$	

PROPOSITION A.5: *Let $M_k \in \mathcal{M}_k$ be a spherical monogenic of degree k. Let $f(x) = f_0(|x|)$ be a real-valued radial function in $\mathcal{S}(\mathbb{R}^m)$. Further, put $\underline{\xi} = \underline{x}/|x|$, $\underline{\eta} = \underline{y}/|y|$ and $r = |x|$. Then one has*

$$T[f(x)M_k(x)](y) = \left(\alpha_k - k\,\beta_k\right) M_k(\eta) \int_0^{+\infty} r^{m+k-1} f_0(r) z^{-\lambda} J_{k+\lambda}(z)\, \mathrm{d}r$$

and

$$T[f(x)\underline{x}M_k(x)](y) = (\alpha_{k+1} + (k+m-1)\beta_{k+1})\underline{\eta} M_k(\eta) \int_0^{+\infty} r^{m+k} f_0(r) z^{-\lambda} J_{k+1+\lambda}(z)\, \mathrm{d}r$$

with $z = r|y|$ and $\lambda = (m-2)/2$.

Proof. The proof relies on (2.10) and goes along similar lines as the proof of Theorem 6.4 in [18]. □

We then have the following theorem.

THEOREM A.6: *The functions $\{\psi_{j,k,\ell}\}$ are eigenfunctions of T. One has, putting $\beta_0 = 0$,*

$$\begin{aligned} (T\psi_{2p,k,\ell})(y) &= (-1)^p \left(\alpha_k - k\,\beta_k\right) \psi_{2p,k,\ell}(y) \\ (T\psi_{2p+1,k,\ell})(y) &= (-1)^p \left(\alpha_{k+1} + (k+m-1)\beta_{k+1}\right) \psi_{2p+1,k,\ell}(y). \end{aligned} \tag{A.9}$$

Proof. The functions $\{\psi_{j,k,\ell}\}$ are of the form $f(x)M_k(x)$ or $f(x)\underline{x}M_k(x)$ with $f(x) = f_0(|x|)$ a real-valued radial function in $\mathcal{S}(\mathbb{R}^m)$. Hence we can apply Proposition A.5 and we find, using $\lambda = (m-2)/2$,

$$(T\psi_{2p,k,\ell})(y) = \left(\alpha_k - k\,\beta_k\right) M_k^{(\ell)}(\eta) \int_0^{\infty} r^{2\lambda+1+k} L_p^{k+\lambda}(r^2)\, e^{-r^2/2} z^{-\lambda} J_{k+\lambda}(z)\, \mathrm{d}r.$$

Substituting $z = r|y|$, the integral becomes

$$\int_0^{\infty} r^{2\lambda+1+k} L_p^{k+\lambda}(r^2)\, e^{-r^2/2} (r|y|)^{-\lambda} J_{k+\lambda}(r|y|)\, \mathrm{d}r.$$

Now we can apply the identity (2.11) to give the final result of

$$\begin{aligned} (T\psi_{2p,k,\ell})(y) &= (-1)^p \left(\alpha_k - k\,\beta_k\right) M_k^{(\ell)}(y) L_p^{k+\lambda}(|y|^2) e^{-|y|^2/2} \\ &= (-1)^p \left(\alpha_k - k\,\beta_k\right) \psi_{2p,k,\ell}(y). \end{aligned}$$

The expression for $(T\psi_{2p+1,k,\ell})(y)$ follows similarly. □

Using these results, we finally arrive at:

Proof of Proposition 3.3. Denote the eigenvalues of T by $\mu_{j,k}$ for $j, k \in \mathbb{Z}_{\geq 0}$. From Lemma 3.2, we see that these eigenvalues satisfy

$$\mu_{j+2,k} = -\mu_{j,k}.$$

Moreover, as $\mu_{j+1,k} = i\,\mu_{j,k}$ we find that putting

$$\alpha_k = \frac{1}{2(k+\lambda)}\big(i\,k\,\mu_{0,k-1} + (k+2\lambda)\mu_{0,k}\big), \qquad \beta_k = \frac{1}{2(k+\lambda)}\big(i\,\mu_{0,k-1} - \mu_{0,k}\big)$$

in (A.9), gives an integral transform that coincides with T on the eigenfunction basis. □

We conclude with a note relevant to the space $L^2(\mathbb{R}^m)\otimes\mathcal{C}l_m$ in Theorem 3.6. This space is equipped with the inner product

$$\langle f, g\rangle = \left[\int_{\mathbb{R}^m} \overline{f^c}\, g \,\mathrm{d}x\right]_0 . \tag{A.10}$$

Here, $u \mapsto \bar{u}$ is the anti-involution on the Clifford algebra $\mathcal{C}l_m$ defined by

$$\overline{uv} = \overline{v}\,\overline{u} \qquad \text{and} \qquad \overline{e_j} = -e_j \quad (j = 1, \ldots, m).$$

Furthermore, f^c denotes the complex conjugate of the function f and $u \mapsto [u]_0$ is the projection on the space of 0-vectors (scalars). The functions $\{\psi_{j,k,\ell}\}$ defined in formula (2.1) are after suitable normalization an orthonormal basis for $L^2(\mathbb{R}^m)\otimes\mathcal{C}l_m$ (see e.g. [42]), satisfying

$$\langle \psi_{j_1,k_1,\ell_1}, \psi_{j_2,k_2,\ell_2}\rangle = \delta_{j_1 j_2}\delta_{k_1 k_2}\delta_{\ell_1 \ell_2}.$$

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On the algebra of symmetries of Laplace and Dirac operators

2

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Abstract

We consider a generalization of the classical Laplace operator, which includes the Laplace-Dunkl operator defined in terms of the differential-difference operators associated with finite reflection groups called Dunkl operators. For this Laplace-like operator, we determine a set of symmetries commuting with it, which are generalized angular momentum operators, and we present the algebraic relations for the symmetry algebra. In this context, the generalized Dirac operator is then defined as a square root of our Laplace-like operator. We explicitly determine a family of graded operators which commute or anti-commute with our Dirac-like operator depending on their degree. The algebra generated by these symmetry operators is shown to be a generalization of the standard angular momentum algebra and the recently defined higher rank Bannai-Ito algebra.

1 Introduction

The study of solutions of the Laplace equation or of the Dirac equation, in any context or setting, is a major topic of investigation. For that purpose, a crucial role is played by the symmetries of the Laplace operator Δ or of the Dirac operator $\underline{D}$, i.e. operators commuting with Δ or (anti)commuting with $\underline{D}$. The symmetries involved and the algebras they generate lead to topics such as separation of variables and special functions. Without claiming completeness we refer the reader to [2, 3, 6, 7, 10, 18].

For this paper, the context we have in mind is that of Dunkl derivatives [9, 23], i.e. where the ordinary derivative $\frac{\partial}{\partial x_i}$ is replaced by a Dunkl derivative $\mathcal{D}_i$ in the expression of the Laplace or Dirac operator. One often refers to these operators as the Laplace-Dunkl and the Dirac-Dunkl operator. The chief purpose of this paper is to determine the symmetries of the Laplace-Dunkl operator and of the Dirac-Dunkl operator, and moreover study the algebra generated by these symmetries.

In the process of this investigation, it occurred to us that it is advantageous to treat this problem in a more general context, which we shall describe here in the introduction. For this purpose, let us first turn to a standard topic in quantum mechanics: the description of the N-dimensional (isotropic) harmonic oscillator. The Hamiltonian $\hat{H}$ of this oscillator (with the common convention $m = \omega = \hbar = 1$ for mass, frequency and the reduced Planck constant) is given by

$$\hat{H} = \frac{1}{2}\sum_{j=1}^{N} \hat{p}_j^2 + \frac{1}{2}\sum_{j=1}^{N} \hat{x}_j^2. \tag{1.1}$$

In canonical quantum mechanics, the coordinate operators $\hat{x}_j$ and momentum operators $\hat{p}_j$ are required to be (essentially self-adjoint) operators satisfying the canonical commutation relations

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = \mathrm{i}\delta_{ij}. \tag{1.2}$$

So in the "coordinate representation", where $\hat{x}_j$ is represented by multiplication with the variable x_j, the operator $\hat{p}_j$ is represented by $\hat{p}_j = -\mathrm{i}\frac{\partial}{\partial x_j}$.

Because the canonical commutation relations are sometimes considered as "unphysical" or "imposed without a physical motivation", more fundamental ways of quantization have been the topic of various research fields (such as geometrical quantization). One of the pioneers of a more fundamental quantization procedure was Wigner, who introduced a method that later became known as "Wigner quantization" [21, 20, 24, 26, 27]. Briefly said, in Wigner quantization one preserves all axioms of quantum mechanics, except that the canonical commutation relations are replaced by a more fundamental principle: the compatibility of the (classical) Hamilton equations with the Heisenberg equations of motion. Concretely, these compatibility conditions read

$$[\hat{H}, \hat{x}_j] = -\mathrm{i}\hat{p}_j, \qquad [\hat{H}, \hat{p}_j] = \mathrm{i}\hat{x}_j \qquad (j = 1, \ldots, N). \tag{1.3}$$

Thus for the quantum oscillator, one keeps the Hamiltonian (1.1), but replaces the relations (1.2) by (1.3). When the canonical commutation relations (1.2) hold, the compatibility relations (1.3) are automatically valid (this is a version of the Ehrenfest theorem), but not vice versa. Hence Wigner quantization is a generalization of canonical quantization, and canonical quantization is just one possible solution of Wigner quantization. Note that in Wigner quantization the coordinate operators $\hat{x}_j$ (and the momentum operators) in general do not commute, so this is of particular significance in the field of non-commutative quantum mechanics.

In a mathematical context, as in this paper, one usually replaces the physical momentum operator components $\hat{p}_j$ by operators $p_j = \mathrm{i}\hat{p}_j$, and one also denotes the coordinate operators $\hat{x}_j$ by x_j. Then the operator H takes the form

$$H = -\frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{j=1}^{N} x_j^2. \tag{1.4}$$

So in the canonical case, where x_j stands for multiplication by the variable x_j, p_j is just the derivative $\frac{\partial}{\partial x_j}$, and the first term of H is (up to a factor $-1/2$) equal to the Laplace operator $\sum_{j=1}^N p_j^2 = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} = \Delta$. In the more general case, the compatibility conditions (1.3) read

$$[H, x_j] = -p_j, \qquad [H, p_j] = -x_j \qquad (j = 1, \ldots, N). \tag{1.5}$$

We are now in a position to describe our problem in a general framework:

Given N commuting operators $x_1, \ldots, x_N$ and N commuting operators $p_1, \ldots, p_N$, consider the operator $H = -\frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{j=1}^{N} x_j^2$, and suppose that the compatibility conditions (1.5) *hold. Classify the symmetries of the generalized Laplace operator, i.e. classify the operators that commute with $\sum_{j=1}^N p_j^2$.*

In other words, we are given N operators $x_1, \ldots, x_N$ and N operators $p_1, \ldots, p_N$ that satisfy

$$[x_i, x_j] = 0, \qquad [p_i, p_j] = 0, \tag{1.6}$$

$$\left[\frac{1}{2}\sum_{i=1}^{N} p_i^2, x_j\right] = p_j, \qquad \left[\frac{1}{2}\sum_{i=1}^{N} x_i^2, p_j\right] = -x_j. \tag{1.7}$$

Under these conditions, the *first problem* is: determine the operators that commute with the generalized Laplace operator

$$\Delta = \sum_{i=1}^{N} p_i^2. \tag{1.8}$$

Our two major examples of systems satisfying (1.6) and (1.7) are the "canonical case" and the "Dunkl case".

For the first example, x_i is just the multiplication by the variable x_i, and p_i is the derivative with respect to x_i: $p_i = \frac{\partial}{\partial x_i}$. Clearly, these operators satisfy (1.6) and (1.7), and the operator Δ in (1.8) coincides with the classical Laplace operator.

For the second example, x_i is again the multiplication by the variable x_i, but p_i is the Dunkl derivative $p_i = \mathcal{D}_i$, which is a certain differential-difference operator with an underlying reflection group determined by a root system (a precise definition follows later in this paper). Conditions (1.6) still hold: the commutativity of the operators x_i is trivial, but the commutativity of the operators p_i is far from trivial [9, 11]. Following [9], also the conditions (1.7) are valid in the Dunkl case. The operator Δ in (1.8) now takes the form $\sum_{i=1}^N \mathcal{D}_i^2$ and is known as the Dunkl Laplacian or the Laplace-Dunkl operator. By the way, it is no surprise that the operators x_i and $\mathcal{D}_j$ do not satisfy the canonical commutation relations. It is, however, very surprising that they satisfy the more general Wigner quantization relations (for a Hamiltonian of oscillator type).

So the solution of the first problem in the general context will in particular lead to the determination of symmetries of the Laplace-Dunkl operator.

Since we are dealing with these operators in an algebraic context, it is worthwhile to move to a closely related operator, the Dirac operator. For this purpose, consider a set of N generators of a Clifford algebra, i.e. N elements e_i satisfying

$$\{e_i, e_j\} = e_i e_j + e_j e_i = \epsilon\, 2\delta_{ij}$$

where ϵ is $+1$ or -1. The generators e_i are supposed to commute with the general operators x_j and p_j. Under the general conditions (1.6) and (1.7), the *second problem* is now: determine the operators that commute (or anti-commute) with the generalized Dirac operator

$$\underline{D} = \sum_{i=1}^{N} e_i p_i. \tag{1.9}$$

Obviously, this is a refinement of the first problem, since $\underline{D}^2 = \epsilon\, \Delta$.

For our two major examples, in the canonical case the operator (1.9) is just the classical Dirac operator; in the Dunkl case, the operator (1.9) is known as the Dirac-Dunkl operator.

In the present paper we solve both problems in the general framework (1.6)–(1.7), and even go beyond it by determining the algebraic relations satisfied by the symmetries. In section 2 we consider the generalized Laplace operator Δ and determine all symmetries, i.e. all operators commuting with Δ (Theorem 2.3). Next, in Theorem 2.5 the algebraic relations satisfied by these symmetries are established. For the generalized Dirac operator $\underline{D}$, the determination of the symmetries is computationally far more involved. In section 3, Theorem 3.7 classifies essentially all operators that commute or anti-commute with $\underline{D}$. In the following subsections, we derive the quadratic relations satisfied by the symmetries of the Dirac operator. The computations of these relations are very intricate, and involve subtle techniques. Fortunately, there is a case to compare with. For the Dunkl case, in which the underlying reflection group is the simplest possible example (namely $\mathbb{Z}_2^N$), the symmetries and their algebraic relations have been determined in [6, 7] and give rise to the so-called (higher rank) Bannai-Ito algebra. Our results can be considered as an extension of these relations to an arbitrary underlying reflection group, in fact in an even more general context.

2 SYMMETRIES OF LAPLACE OPERATORS

We start by formally describing the operator algebra that will contain the desired symmetries of a generalized Laplace operator (1.8), as brought up in the introduction.

DEFINITION 2.1: *We define the algebra $\mathcal{A}$ to be the unital (over the field $\mathbb{R}$ or $\mathbb{C}$) associative algebra generated by the $2N$ elements $x_1, \ldots, x_N$ and $p_1, \ldots, p_N$ subject to the following relations:*

$$[x_i, x_j] = 0, \qquad [p_i, p_j] = 0,$$

$$\left[\frac{1}{2}\sum_{i=1}^{N} p_i^2, x_j\right] = p_j, \qquad \left[\frac{1}{2}\sum_{i=1}^{N} x_i^2, p_j\right] = -x_j.$$

Note that an immediate consequence of the relations of $\mathcal{A}$ is

$$[x_i, p_j] = [x_i, -[H, x_j]] = -[[x_i, H], x_j] = -[p_i, x_j] = [x_j, p_i],$$

where H is given by (1.4). This reciprocity

$$[x_i, p_j] = [x_j, p_i] \tag{2.1}$$

will be useful for many ensuing calculations, starting with the following theorem.

THEOREM 2.2: *The algebra $\mathcal{A}$ contains a copy of the Lie algebra $\mathfrak{sl}(2)$ generated by the elements*

$$\frac{|x|^2}{2} = \frac{1}{2}\sum_{i=1}^{N} x_i^2, \qquad -\frac{\Delta}{2} = -\frac{1}{2}\sum_{i=1}^{N} p_i^2, \qquad \mathbb{E} = \frac{1}{2}\sum_{i=1}^{N}\{p_i, x_i\}, \tag{2.2}$$

satisfying the relations

$$\left[\mathbb{E}, \frac{|x|^2}{2}\right] = |x|^2, \qquad \left[\mathbb{E}, -\frac{\Delta}{2}\right] = -\Delta, \qquad \left[\frac{|x|^2}{2}, -\frac{\Delta}{2}\right] = \mathbb{E}.$$

Proof. By direct computation we have

$$\frac{1}{4}[\Delta, |x|^2] = \frac{1}{4}\sum_{i=1}^{N}[\Delta, x_i^2] = \frac{1}{4}\sum_{i=1}^{N}\{[\Delta, x_i], x_i\} = \frac{1}{2}\sum_{i=1}^{N}\{p_i, x_i\}.$$

Using the commutativity of $p_1, \ldots, p_N$ and relation (2.1), we have

$$\begin{aligned}
[\mathbb{E}, \Delta] &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}[\{p_i, x_i\}, p_j^2] = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{[\{p_i, x_i\}, p_j], p_j\} \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{\big(p_i(x_ip_j - p_jx_i)+(x_ip_j - p_jx_i)p_i\big), p_j\} \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{\big(p_i(x_jp_i - p_ix_j)+(x_jp_i - p_ix_j)p_i\big), p_j\} \\
&= -\frac{1}{2}\sum_{j=1}^{N}\left\{\left[\sum_{i=1}^{N} p_i^2, x_j\right], p_j\right\} = -\sum_{j=1}^{N}\{p_j, p_j\} = -2\Delta.
\end{aligned}$$

In the same manner, using now the commutativity of $x_1, \ldots, x_N$, we find $[\mathbb{E}, |x|^2] = 2|x|^2$. □

In the spirit of Howe duality [16, 17], our objective is to determine the subalgebra of $\mathcal{A}$ which commutes with the Lie algebra $\mathfrak{sl}(2)$ realized by Δ and $|x|^2$ as appearing in Theorem 2.2. As mentioned in the introduction, the element Δ corresponds to a generalized version of the Laplace operator, which reduces to the classical Laplace operator for a specific choice of the elements $p_1, \ldots, p_N$. In the (Euclidean) coordinate representation, $|x|^2$ of course represents the norm squared.

2.1 *Symmetries*

As $p_1, \ldots, p_N$ are commuting operators, by definition they also commute with Δ. However, in general they are not symmetries of $|x|^2$. An immediate first example of an operator which does commute with both Δ and $|x|^2$ is given by the Casimir operator (in the universal enveloping algebra) of their $\mathfrak{sl}(2)$ realization

$$\Omega = \mathbb{E}^2 - 2\mathbb{E} - |x|^2\Delta \in \mathcal{U}(\mathfrak{sl}(2)) \subset \mathcal{A}. \tag{2.3}$$

Note that this operator is of the same order in both $x_1, \ldots, x_N$ and $p_1, \ldots, p_N$ as it has to commute with both Δ and $|x|^2$. More precisely it is of fourth order in the generators of $\mathcal{A}$, being quadratic in $x_1, \ldots, x_N$ and quadratic in $p_1, \ldots, p_N$. We now set out to consider the most elementary symmetries, those which are of second order in the generators of $\mathcal{A}$.

THEOREM 2.3: *In the algebra $\mathcal{A}$, the elements quadratic in the generators $x_1, \ldots, x_N$ and $p_1, \ldots, p_N$, which commute with Δ and $|x|^2$ are spanned by*

$$L_{ij} = x_i p_j - x_j p_i, \qquad C_{ij} = [p_i, x_j] = p_i x_j - x_j p_i \qquad (i, j \in \{1, \ldots, N\}). \tag{2.4}$$

Note that when $i = j$ we have $L_{ii} = 0$, while $C_{ii} = [p_i, x_i]$ does not necessarily vanish. Moreover, as $L_{ij} = -L_{ji}$, every symmetry L_{ij} is up to a sign equal to one of the $N(N-1)/2$ symmetries $\{L_{ij} \mid 1 \le i < j \le N\}$. By relation (2.1), $C_{ij} = C_{ji}$ and thus we have $N(N+1)/2$ symmetries $\{C_{ij} \mid 1 \le i \le j \le N\}$. In total, this gives N^2 generically distinct symmetries.

Proof. It is trivial that there are no nonzero quadratic elements in $x_1, \ldots, x_N$ that commute with Δ, and no nonzero quadratic elements in $p_1, \ldots, p_N$ that commute with $|x|^2$. Now, as Δ commutes with $p_1, \ldots, p_N$ and using condition (1.7), we have for $i, j \in \{1, \ldots, N\}$

$$[\Delta, x_i p_j - x_j p_i] = x_i[\Delta, p_j] + [\Delta, x_i]p_j - x_j[\Delta, p_i] - [\Delta, x_j]p_i = 2p_i p_j - 2p_j p_i = 0.$$

In the same manner we have $[\Delta, p_i x_j - x_j p_i] = 0$. The relations for $|x|^2$ follow similarly. □

2.2 *Symmetry algebra*

For the following results, we make explicit use of the symmetry $C_{ij} = [p_i, x_j]$ being symmetric in its two indices, by relation (2.1). This is the case for p_i corresponding to classical partial derivatives, but also for their generalization in the form of Dunkl operators. We will return in detail to these examples in section 2.3. Another consequence of relation (2.1) pertains to the form of the other symmetries of Theorem 2.3. By means of $x_i p_j - p_j x_i = x_j p_i - p_i x_j$ we readily observe that

$$L_{ij} = x_i p_j - x_j p_i = p_j x_i - p_i x_j. \tag{2.5}$$

Given these symmetry properties, the symmetries of Theorem 2.3 generate an algebraic structure within $\mathcal{A}$ whose relations we present after the following lemma.

LEMMA 2.4: *In the algebra $\mathcal{A}$, the symmetries* (2.4) *satisfy the following relations for all $i, j, k \in \{1, \dots, N\}$*

$$[C_{ij}, p_k] = [C_{kj}, p_i], \qquad \text{and} \qquad [C_{ij}, x_k] = [C_{kj}, x_i].$$

Moreover, we also have

$$L_{ij}p_k + L_{ki}p_j + L_{jk}p_i = 0 = p_kL_{ij} + p_jL_{ki} + p_iL_{jk},$$

and

$$x_kL_{ij} + x_jL_{ki} + x_iL_{jk} = 0 = L_{ij}x_k + L_{ki}x_j + L_{jk}x_i.$$

Proof. For the first relation, writing out the commutators in $[[p_k, x_j], p_i] - [[p_i, x_j], p_k]$ we find

$$p_kx_jp_i - x_jp_kp_i - p_ip_kx_j + p_ix_jp_k - p_ix_jp_k + x_jp_ip_k + p_kp_ix_j - p_kx_jp_i.$$

We see that all terms cancel due to the mutual commutativity of the operators $p_1, \dots, p_N$. The other relation of the first line follows in the same way.

For the other two relations, the identities follow immediately by choosing the appropriate expression for L_{ij} of (2.5) and making use of the commutativity of either $x_1, \dots, x_N$ or $p_1, \dots, p_N$. □

THEOREM 2.5: *In the algebra $\mathcal{A}$, the symmetries* (2.4) *satisfy the following relations for all $i, j \in \{1, \dots, N\}$,*

$$\begin{aligned}[L_{ij}, L_{kl}] &= L_{il}C_{jk} + L_{jk}C_{il} + L_{ki}C_{lj} + L_{lj}C_{ki} \\ &= C_{jk}L_{il} + C_{il}L_{jk} + C_{lj}L_{ki} + C_{ki}L_{lj},\end{aligned} \tag{2.6}$$

and

$$\{L_{ij}, L_{kl}\} + \{L_{ki}, L_{jl}\} + \{L_{jk}, L_{il}\} = 0, \tag{2.7}$$

and

$$[L_{ij}, C_{kl}] + [L_{ki}, C_{jl}] + [L_{jk}, C_{il}] = 0. \tag{2.8}$$

Proof. We will prove the first line of the first relation, i.e. (2.6), the second line follows in a similar manner. We have

$$\begin{aligned}[x_ip_j - x_jp_i, x_kp_l - x_lp_k] &= [x_ip_j, x_kp_l] - [x_ip_j, x_lp_k] - [x_jp_i, x_kp_l] + [x_jp_i, x_lp_k] \\ &= x_i[p_j, x_k]p_l + x_k[x_i, p_l]p_j - x_i[p_j, x_l]p_k - x_l[x_i, p_k]p_j \\ &\quad - x_j[p_i, x_k]p_l - x_k[x_j, p_l]p_i + x_j[p_i, x_l]p_k + x_l[x_j, p_k]p_i \\ &= x_iC_{jk}p_l - x_kC_{li}p_j - x_iC_{jl}p_k + x_lC_{ki}p_j \\ &\quad - x_jC_{ik}p_l + x_kC_{lj}p_i + x_jC_{il}p_k - x_lC_{kj}p_i.\end{aligned}$$

Swapping all operators p_l with C_{jk}, we find

$$\begin{aligned}
[L_{ij}, L_{kl}] &= x_i p_l C_{jk} + x_i[C_{jk}, p_l] - x_k p_j C_{li} - x_k[C_{li}, p_j] - x_i p_k C_{jl} - x_i[C_{jl}, p_k] \\
&\quad + x_l p_j C_{ki} + x_l[C_{ki}, p_j] - x_j p_l C_{ik} - x_j[C_{ik}, p_l] + x_k p_i C_{lj} \\
&\quad + x_k[C_{lj}, p_i] + x_j p_k C_{il} + x_j[C_{il}, p_k] - x_l p_i C_{kj} - x_l[C_{kj}, p_i] \\
&= x_i p_l C_{jk} - x_k p_j C_{li} - x_i p_k C_{jl} + x_l p_j C_{ki} - x_j p_l C_{ik} + x_k p_i C_{lj} + x_j p_k C_{il} - x_l p_i C_{kj} \\
&\quad + x_i\big([C_{jk}, p_l] - [C_{jl}, p_k]\big) + x_k\big(- [C_{il}, p_j] + [C_{lj}, p_i]\big) \\
&\quad - x_l\big(- [C_{ki}, p_j] + [C_{kj}, p_i]\big) - x_j\big([C_{ik}, p_l] - [C_{il}, p_k]\big) \\
&= L_{il}C_{jk} + L_{jk}C_{il} + L_{ki}C_{lj} + L_{lj}C_{ki},
\end{aligned}$$

where we used $C_{jk} = C_{kj}$ and Lemma 2.4.

The identities (2.7) and (2.8) follow by making explicit use of both expressions of (2.5) for L_{ij}. For the left-hand side of (2.7) we have

$$\begin{aligned}
& L_{ij}L_{kl} + L_{kl}L_{ij} + L_{ki}L_{jl} + L_{jl}L_{ki} + L_{jk}L_{il} + L_{il}L_{jk} \\
&= (x_ip_j - x_jp_i)(p_lx_k - p_kx_l) + (x_kp_l - x_lp_k)(p_jx_i - p_ix_j) + (x_kp_i - x_ip_k)(p_lx_j - p_jx_l) \\
&\quad + (x_jp_l - x_lp_j)(p_ix_k - p_kx_i) + (x_jp_k - x_kp_j)(p_lx_i - p_ix_l) + (x_ip_l - x_lp_i)(p_kx_j - p_jx_k),
\end{aligned}$$

where again one observes that all terms vanish due to the commutativity of $p_1, \dots, p_N$.

Working out the commutators, the left-hand side of (2.8) becomes

$$\begin{aligned}
& L_{ij}[p_l, x_k] - [p_l, x_k]L_{ij} + L_{ki}[p_l, x_j] - [p_l, x_j]L_{ki} + L_{jk}[p_l, x_i] - [p_l, x_i]L_{jk} \\
&= L_{ij}p_lx_k - p_lx_kL_{ij} + L_{ki}p_lx_j - p_lx_jL_{ki} + L_{jk}p_lx_i - p_lx_iL_{jk} \\
&\quad - L_{ij}x_kp_l + x_kp_lL_{ij} - L_{ki}x_jp_l + x_jp_lL_{ki} - L_{jk}x_ip_l + x_ip_lL_{jk}.
\end{aligned}$$

Hence, plugging in suitable choices for the symmetries L_{ij}, this becomes

$$\begin{aligned}
& (x_ip_j - x_jp_i)p_lx_k - p_lx_k(x_ip_j - x_jp_i) + (x_kp_i - x_ip_k)p_lx_j - p_lx_j(x_kp_i - x_ip_k) \\
&+ (x_jp_k - x_kp_j)p_lx_i - p_lx_i(x_jp_k - x_kp_j) - (p_jx_i - p_ix_j)x_kp_l + x_kp_l(p_jx_i - p_ix_j) \\
&- (p_ix_k - p_kx_i)x_jp_l + x_jp_l(p_ix_k - p_kx_i) - (p_kx_j - p_jx_k)x_ip_l + x_ip_l(p_kx_j - p_jx_k).
\end{aligned}$$

One observes that all terms vanish due to the commutativity of $x_1, \dots, x_N$ and $p_1, \dots, p_N$ respectively. □

2.3 *Examples*

EXAMPLE 2.1: As a first example, we consider N mutually commuting variables $x_1, \dots, x_N$, doubling as operators acting on functions by left multiplication with the respective variable and p_j being just the derivative $\partial/\partial x_j$ for $j \in \{1, \dots, N\}$. In this case obviously $p_1, \dots, p_N$ mutually commute and the operators of interest are

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}, \qquad |x|^2 = \sum_{i=1}^{N} x_i^2, \qquad H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2,$$

which satisfy

$$\frac{1}{2}[\Delta, x_i] = \frac{\partial}{\partial x_i} = p_i, \qquad \frac{1}{2}[|x|^2, p_i] = -x_i.$$

By Theorem 2.3, we have the following symmetries:

$$L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \qquad C_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

While C_{ij} is a scalar for every i, j, the L_{ij} symmetries are the standard angular momentum operators whose symmetry algebra is the Lie algebra $\mathfrak{so}(N)$:

$$[L_{ij}, L_{kl}] = L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ik}.$$

This is in accordance with Theorem 2.5 as in this case evidently $C_{ij} = C_{ji}$.

Note that Δ and $|x|^2$ are also invariant under $\mathrm{O}(N)$, the group of orthogonal transformations on $\mathbb{R}^N$, but these transformations are not contained in the algebra $\mathcal{A}$.

EXAMPLE 2.2: A more intriguing example is given by a generalization of partial derivatives to differential-difference operators associated to a Coxeter or Weyl group W. Let R be a (reduced) root system and k a multiplicity function which is invariant under the natural action of the Weyl group W consisting of all reflections associated to R,

$$\sigma_\alpha(x) = x - 2\langle x, \alpha\rangle\alpha/\|\alpha\|^2, \qquad \alpha \in R, x \in \mathbb{R}^N.$$

For $\xi \in \mathbb{R}^N$, the Dunkl operator [9, 23] is defined as

$$\mathcal{D}_\xi f(x) := \frac{\partial}{\partial \xi} f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle} \langle \alpha, \xi\rangle,$$

where the summation is taken over all roots in R_+, a fixed positive subsystem of R. For a fixed root system and function k, the Dunkl operators associated to any two vectors commute, see [9]. Hence, they form potential candidates for the operators $p_1, \ldots, p_N$ satisfying condition (1.6). The operator of interest is the Laplace-Dunkl operator Δ_k, which can be written as

$$\Delta_k = \sum_{i=1}^{N} (\mathcal{D}_{\xi_i})^2$$

for any orthonormal basis $\{\xi_1, \ldots, \xi_N\}$ of $\mathbb{R}^N$. For the orthonormal basis associated to the coordinates $x_1, \ldots, x_N$ we use the notation

$$\mathcal{D}_i f(x) := \frac{\partial}{\partial x_i} f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle} \alpha_i \qquad i \in \{1, \ldots, N\}. \tag{2.9}$$

For our purpose, let x_j again stand for multiplication by the variable x_j and take now $p_j = \mathcal{D}_j$ for $j \in \{1, \ldots, N\}$. Besides condition (1.6), condition (1.7) is also satisfied (see for instance [9, 23]). We note that the $\mathfrak{sl}(2)$ relations in this context were already obtained by [14].

By Theorem 2.3 we have as symmetries, on the one hand, the Dunkl version of the angular momentum operators

$$L_{ij} = x_i\mathcal{D}_j - x_j\mathcal{D}_i.$$

On the other hand, the symmetries

$$C_{ij} = [\mathcal{D}_i, x_j] = \delta_{ij} + \sum_{\alpha\in R_+} 2k(\alpha)\alpha_i\alpha_j\sigma_\alpha$$

consist of linear combinations of the reflections in the Weyl group, with coefficients determined by the multiplicity function k and the roots of the root system. This is of course in agreement with Δ_k being W-invariant [23]. The Weyl group is a subgroup of $\mathrm{O}(N)$, and in this case the algebra $\mathcal{A}$ does contain these reflections in W.

Note that indeed $C_{ij} = C_{ji}$, in accordance with relation (2.1). Theorem 2.5 now yields the Dunkl version of the angular momentum algebra for an arbitrary Weyl group or root system:

$$\begin{aligned}[L_{ij}, L_{kl}] &= L_{il}C_{jk} + L_{jk}C_{il} + L_{ki}C_{lj} + L_{lj}C_{ki}\\ &= L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ki}\\ &\quad + \sum_{\alpha\in R_+} 2k(\alpha)\big(L_{il}\alpha_j\alpha_k + L_{jk}\alpha_i\alpha_l + L_{ki}\alpha_l\alpha_j + L_{lj}\alpha_k\alpha_i\big)\sigma_\alpha.\end{aligned}$$

This relation states the interaction of the L_{ij} symmetries amongst one another. The interaction between the symmetries C_{kl} is governed by the group multiplication of the Weyl group, while the relations for the symmetries L_{ij} and C_{kl} follow from the action of a reflection σ_α for a root α:

$$\sigma_\alpha\mathcal{D}_\xi = \mathcal{D}_{\sigma_\alpha(\xi)}\sigma_\alpha, \qquad \text{hence} \qquad \sigma_\alpha L_{ij} = L_{\sigma_\alpha(\xi_i)\sigma_\alpha(\xi_j)}\sigma_\alpha,$$

where $\{\xi_1, \dots, \xi_N\}$ is the orthonormal basis associated to the coordinates $x_1, \dots, x_N$. This allows us to interchange any two symmetries of the form L_{ij} and C_{kl}.

Specific cases of this result have been considered before, namely for $W = (\mathbb{Z}_2)^3$ in [13] and $W = S_N$ in [12].

3 SYMMETRIES OF DIRAC OPERATORS

We now turn to a closely related operator of the generalized Laplace operator considered in the preceding section, namely the Dirac operator. For an operator of the form (1.8), one can construct a "square root" by introducing a set of elements $e_1, \dots, e_N$ which commute with x_i and p_i for all $i \in \{1, \dots, N\}$ and which satisfy the following relations

$$\{e_i, e_j\} = e_ie_j + e_je_i = \epsilon\, 2\delta_{ij}, \tag{3.1}$$

where $\epsilon = \pm 1$, or thus for $i \neq j$

$$(e_i)^2 = \epsilon = \pm 1, \qquad e_i e_j + e_j e_i = 0.$$

We use these elements to define the following two operators

$$\underline{D} = \sum_{i=1}^{N} e_i p_i, \qquad \underline{x} = \sum_{i=1}^{N} e_i x_i,$$

whose squares equal

$$\underline{D}^2 = \epsilon \sum_{i=1}^{N} (p_i)^2 = \epsilon \Delta, \qquad \underline{x}^2 = \epsilon \sum_{i=1}^{N} (x_i)^2 = \epsilon |x|^2,$$

by means of the anti-commutation relations (3.1) of $e_1, \ldots, e_N$ and condition (1.6). For the classical case where p_i is the ith partial derivative, the operator $\underline{D}$ is the standard Dirac operator.

The elements $e_1, \ldots, e_N$ in fact generate what is known as a Clifford algebra [22], which we will denote as $\mathcal{C} = \mathcal{C}\ell(\mathbb{R}^N)$. A general element in this algebra is a linear combination of products of $e_1, \ldots, e_N$. The standard convention is to denote for instance $e_1 e_2 e_3$ simply as e_{123}. Hereto, we introduce the concept of a 'list' for use as index of Clifford numbers.

DEFINITION 3.1: *We define a list to indicate a finite sequence of distinct elements of a given set, in our case the set* $\{1, \ldots, N\}$. *For a list* $A = a_1 \cdots a_n$ *of* $\{1, \ldots, N\}$ *with* $0 \leq n \leq N$, *we will use the notation*

$$e_A = e_{a_1} e_{a_2} \cdots e_{a_n}. \tag{3.2}$$

REMARK 3.2: Note that in a list the order matters as the Clifford generators $e_1, \ldots, e_N$ anti-commute. Moreover, duplicate elements would cancel out as they square to $\epsilon = \pm 1$, so we consider only lists containing distinct elements. For a set $A = \{a_1, \ldots, a_n\} \subset \{1, \ldots, N\}$, the notation e_A stands for $e_{a_1} e_{a_2} \cdots e_{a_n}$ with $a_1 < a_2 < \cdots < a_n$.

The collection $\{e_A \mid A \subset \{1, \ldots, N\}\}$ forms a basis of the Clifford algebra $\mathcal{C}$, where for the empty set we put $e_\emptyset = 1$.

REMARK 3.3: In general, the square of each individual element e_i ($i \in \{1, \ldots, N\}$) can independently be chosen equal to either $+1$ or -1. This corresponds to an underlying space with arbitrary signature defined by the specified signs. The original Dirac operator was constructed as a square root of the wave operator by means of the gamma or Dirac matrices which form a matrix realization of the Clifford algebra for $N = 4$ with Minkowski signature. To simplify notations in the following, we have chosen the square of all e_i ($i \in \{1, \ldots, N\}$) to be equal to ϵ which can be either $+1$ or -1. One can generalize all results to arbitrary signature by making the appropriate substitutions.

In order to consider symmetries of the generalized Dirac operator (1.9) we will work in the tensor product $\mathcal{A} \otimes \mathcal{C}$ with the algebra $\mathcal{A}$ as defined in Definition 2.1. To avoid overloading on notations, we will omit the tensor symbol $\otimes$ when writing down elements of $\mathcal{A} \otimes \mathcal{C}$ and use regular product notation. In this notation $e_1, \dots, e_N$ indeed commutes with x_i and p_i for all $i \in \{1, \dots, N\}$.

Akin to the realization of the Lie algebra $\mathfrak{sl}(2)$ in the algebra $\mathcal{A}$ given by Theorem 2.2, we have something comparable in this case.

THEOREM 3.4: *The algebra $\mathcal{A} \otimes \mathcal{C}$ contains a copy of the Lie superalgebra $\mathfrak{osp}(1|2)$ generated by the (odd) elements $\underline{D}$ and $\underline{x}$ satisfying the relations*

$$\begin{aligned} \{\underline{x}, \underline{x}\} &= \epsilon\, 2|x|^2 & \{\underline{D}, \underline{D}\} &= \epsilon\, 2\Delta & \{\underline{x}, \underline{D}\} &= \epsilon\, 2\mathbb{E} \\ [|x|^2, \underline{x}] &= 0 & [|x|^2, \underline{D}] &= -2\underline{D} & [\mathbb{E}, \underline{x}] &= \underline{x} \\ [\Delta, \underline{x}] &= 2\underline{x} & [\Delta, \underline{D}] &= 0 & [\mathbb{E}, \underline{D}] &= -\underline{D} \end{aligned}$$

and containing as an even subalgebra the Lie algebra $\mathfrak{sl}(2)$ in the algebra $\mathcal{A}$ given by Theorem 2.2 with relations

$$\left[\mathbb{E}, \frac{|x|^2}{2}\right] = |x|^2 \qquad \left[\mathbb{E}, -\frac{\Delta}{2}\right] = -\Delta \qquad \left[\frac{|x|^2}{2}, -\frac{\Delta}{2}\right] = \mathbb{E}.$$

Proof. The relations follow by straightforward computations. By means of the anti-commutation relations (3.1) one finds that

$$\begin{aligned} \{\underline{x}, \underline{D}\} &= \sum_{i=1}^{N} x_i e_i \sum_{j=1}^{N} p_j e_j + \sum_{j=1}^{N} p_j e_j \sum_{i=1}^{N} x_i e_i \\ &= \sum_{i=1}^{N} \epsilon (x_i p_i + p_i x_i) + \sum_{1 \le i < j \le N} (x_i p_j - p_j x_i - x_j p_i + p_i x_j) e_i e_j. \end{aligned}$$

Looking back at (2.2), the first summation is precisely $\epsilon\, 2\mathbb{E}$, while the second summation vanishes by relation (2.1). Moreover, by relation (2.1) we have

$$\begin{aligned} [\mathbb{E}, \underline{D}] &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} [\{p_i, x_i\}, p_j] e_j = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{p_i, [x_i, p_j]\} e_j \\ &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{p_i, [x_j, p_i]\} e_j = -\frac{1}{2} \sum_{j=1}^{N} \left[\sum_{i=1}^{N} p_i^2, x_j \right] e_j = -\underline{D}, \end{aligned}$$

and in the same manner $[\mathbb{E}, \underline{x}] = \underline{x}$. □

3.1 *Symmetries*

We wish to determine symmetries in the algebra $\mathcal{A} \otimes \mathcal{C}$ for the Dirac operator $\underline{D}$ which are linear in both $x_1, \dots, x_N$ and $p_1, \dots, x_N$. Given the Lie superalgebra framework, it is natural to consider operators which either commute or anti-commute with $\underline{D}$. Indeed, the Lie superalgebra $\mathfrak{osp}(1|2)$ has both a Scasimir and a Casimir element in its universal enveloping algebra [1]. The Scasimir operator

$$\mathcal{S} = \frac{1}{2}\left([\underline{D}, \underline{x}] - \epsilon\right) \in \mathcal{U}(\mathfrak{osp}(1|2)) \subset \mathcal{A} \otimes \mathcal{C}, \tag{3.3}$$

anti-commutes with odd elements and commutes with even elements. In the classical case, the Scasimir operator is up to a constant term equal to the angular Dirac operator Γ, i.e. $\underline{D}$ restricted to the sphere. The Scasimir $\mathcal{S}$ is a symmetry which is linear in both $x_1, \ldots, x_N$ and $p_1, \ldots, x_N$ and we will get back to it before the end of this subsection. Finally, the square of the Scasimir element yields the Casimir element $\mathcal{C} = \mathcal{S}^2$, which commutes with all elements of $\mathfrak{osp}(1|2)$.

Note that another symmetry is obtained by means of the anti-commutation relations (3.1) of the Clifford algebra. The so-called pseudo-scalar $e_1 \cdots e_N$ is easily seen to commute with $\underline{D}$ for N odd and anti-commute with $\underline{D}$ for N even.

The Dirac operator is defined such that it squares to the Laplace operator, $\underline{D}^2 = \epsilon\Delta$. This allows us to readily make use of the properties of Δ by means of the following straightforward relations. For an operator Z, we have that

$$[\underline{D}, \{\underline{D}, Z\}] = \underline{D}(\underline{D}Z + Z\underline{D}) - (\underline{D}Z + Z\underline{D})\underline{D} = \left[\underline{D}^2, Z\right] \tag{3.4}$$

and

$$\{\underline{D}, [\underline{D}, Z]\} = \underline{D}(\underline{D}Z - Z\underline{D}) + (\underline{D}Z - Z\underline{D})\underline{D} = \left[\underline{D}^2, Z\right]. \tag{3.5}$$

A direct consequence of these relations is that every symmetry of the Laplace operator Δ yields symmetries of the Dirac operator $\underline{D}$.

PROPOSITION 3.5: *If Z commutes with Δ and $\underline{D}^2 = \epsilon\Delta$, then the operator $\{\underline{D}, Z\}$ commutes with $\underline{D}$, while the operator $[\underline{D}, Z]$ anti-commutes with $\underline{D}$.*

Letting Z be one of the symmetries of Theorem 2.3, we indeed obtain symmetries of $\underline{D}$, but they are not of the same order in $x_1, \ldots, x_N$ as in $p_1, \ldots, p_N$. These symmetries are in fact combinations of the obvious symmetries $p_1, \ldots, p_N$ and symmetries which are linear in $x_1, \ldots, x_N$ and in $p_1, \ldots, p_N$. We set forth to determine the latter explicitly. Hereto, a first observation is that the elements of the Clifford algebra also commute with the Laplace operator, by definition as it is a scalar (non-Clifford) operator. For A a list of distinct elements of $\{1, \ldots, N\}$, we have

$$\underline{D}(\underline{D}e_A \pm e_A\underline{D}) \mp (\underline{D}e_A \pm e_A\underline{D})\underline{D} = \left[\underline{D}^2, e_A\right] = 0.$$

The explicit expressions of these operators follow from the anti-commutation relations (3.1) as

$$e_A\underline{D} = e_A\sum_{l=1}^{N} p_l e_l = (-1)^{|A|-1}\sum_{a\in A} p_a e_a e_A + (-1)^{|A|}\sum_{a\notin A} p_a e_a e_A, \tag{3.6}$$

with $|A|$ denoting the number of elements of the list A, so

$$\underline{D}e_A - (-1)^{|A|}e_A\underline{D} = \sum_{a\in A} 2p_a e_a e_A \qquad \text{and} \qquad \underline{D}e_A + (-1)^{|A|}e_A\underline{D} = \sum_{a\notin A} 2p_a e_a e_A, \quad (3.7)$$

where (here and throughout the paper) the summation index $a \notin A$ is meant to run over all elements of $\{1, \ldots, N\} \setminus A$. Note that for a list of one element $A = i$, we have $\underline{D}e_i + e_i\underline{D} = \epsilon 2p_i$. With this information, the relations (3.4) and (3.5) now also lend themselves to the construction of more intricate symmetries of both $\underline{D}$ and $\underline{x}$.

THEOREM 3.6: *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for $i \in \{1, \ldots, N\}$, the operator*

$$O_i = \frac{\epsilon}{2}\left([\underline{D}, x_i] - e_i\right) = \frac{\epsilon}{2}\left([p_i, \underline{x}] - e_i\right) = \frac{\epsilon}{2}\left(\sum_{l=1}^{N} e_l C_{li} - e_i\right) \qquad (3.8)$$

anti-commutes with $\underline{D}$ and $\underline{x}$.

Proof. The equalities in (3.8) follow immediately from $C_{ij} = [p_i, x_j] = [p_j, x_i] = C_{ji}$. By direct computation, using (3.5) and the anti-commutation relations (3.1), we have

$$\{\underline{D}, O_i\} = \frac{\epsilon}{2}\{\underline{D}, [\underline{D}, x_i]\} - \epsilon\frac{1}{2}\{\underline{D}, e_i\} = \frac{1}{2}[\Delta, x_i] - p_i = 0.$$

In the same manner, one finds that $O_i = \frac{\epsilon}{2}\left([p_i, \underline{x}] - e_i\right)$ anti-commutes with $\underline{x}$. □

The symmetries O_i with one index $i \in \{1, \ldots, N\}$ defined in (3.8) can be generalized to symmetries with multiple indices. Hereto, we define the operators

$$\underline{D}_A = \sum_{a\in A} p_a e_a \qquad \text{and} \qquad \underline{x}_A = \sum_{a\in A} x_a e_a, \qquad (3.9)$$

for A a subset of $\{1, \ldots, N\}$, and by extension for A a list of $\{1, \ldots, N\}$ as the order does not matter in the summation. By means of the operators (3.9) we state the following result.

THEOREM 3.7: *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for A a list of distinct elements of $\{1, \ldots, N\}$, the operator*

$$O_A = \frac{1}{2}\left(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A\right) \qquad (3.10)$$

$$= \frac{1}{2}\left(e_A \underline{D}_A\,\underline{x} - \underline{x}\,\underline{D}_A e_A - \epsilon e_A\right) \qquad (3.11)$$

$$= \frac{1}{2}\Big(-\epsilon + \sum_{j\in A}\sum_{i\notin A\setminus\{j\}} C_{ij}e_ie_j - \sum_{\{i,j\}\subset A} 2L_{ij}e_ie_j\Big)e_A \qquad (3.12)$$

satisfies

$$\underline{D}\,O_A = (-1)^{|A|}O_A\underline{D} \qquad \textit{and} \qquad \underline{x}\,O_A = (-1)^{|A|}O_A\underline{x}.$$

Proof. We first show the equivalence of the three expressions (3.10) and (3.12). Starting from (3.10), up to a factor 1/2, and using $C_{ij} = C_{ji}$, we have

$$
\begin{aligned}
&\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A = \sum_{l=1}^{N} p_l e_l \sum_{a\in A} x_a e_a e_A - e_A \sum_{a\in A} x_a e_a \sum_{l=1}^{N} p_l e_l - \epsilon e_A \\
&= \sum_{a\in A}\Big(\epsilon(p_a x_a - x_a p_a)e_A + \sum_{l\in A\setminus\{a\}} (p_l x_a + x_a p_l) e_l e_a e_A + \sum_{l\notin A} (p_l x_a - x_a p_l) e_l e_a e_A\Big) - \epsilon e_A \\
&= \sum_{a\in A}\Big(\epsilon(p_a x_a - x_a p_a)e_A + \sum_{l\in A\setminus\{a\}} (p_a x_l + x_l p_a) e_a e_l e_A + \sum_{l\notin A} (p_a x_l - x_l p_a) e_l e_a e_A\Big) - \epsilon e_A \\
&= e_A \sum_{a\in A} p_a e_a \sum_{l=1}^{N} x_l e_l - \sum_{l=1}^{N} x_l e_l \sum_{a\in A} p_a e_a e_A - \epsilon e_A = e_A \underline{D}_A \underline{x} - \underline{x}\,\underline{D}_A e_A - \epsilon e_A.
\end{aligned}
$$

Again starting from (3.10), up to a factor 1/2, we have

$$
\begin{aligned}
&\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A = -\epsilon e_A + \sum_{l=1}^{N} p_l e_l \sum_{a\in A} x_a e_a e_A - e_A \sum_{a\in A} x_a e_a \sum_{l=1}^{N} p_l e_l \\
&= \Big(-\epsilon + \sum_{l=1}^{N} p_l e_l \sum_{a\in A} x_a e_a - \sum_{a\in A} x_a e_a \sum_{l\in A} p_l e_l + \sum_{a\in A} x_a e_a \sum_{l\notin A} p_l e_l\Big) e_A \\
&= \Big(-\epsilon + \epsilon \sum_{a\in A}(p_a x_a - x_a p_a) + \sum_{a\in A}\sum_{l\in A\setminus\{a\}} (p_l x_a + x_a p_l) e_l e_a + \sum_{a\in A}\sum_{l\notin A} (p_l x_a - x_a p_l) e_l e_a\Big) e_A \\
&= \Big(-\epsilon + \sum_{a\in A}\sum_{l\notin A\setminus\{a\}} C_{la} e_l e_a + \sum_{\{a,l\}\subset A} \big((p_l x_a + x_a p_l) e_l e_a + (p_a x_l + x_l p_a) e_a e_l\big)\Big) e_A
\end{aligned}
$$

which equals (3.12), up to a factor 1/2, when using $L_{ij} = x_i p_j - x_j p_i = p_j x_i - p_i x_j$ and $C_{ij} = C_{ji}$.

Now for the proof itself, the case where A is the empty set is trivial, as $O_\emptyset = -\epsilon/2$ obviously commutes with $\underline{D}$ and $\underline{x}$. For A a singleton the result is given by Theorem 3.6 so let now $|A| \geq 2$. Using $\underline{x}_A e_A = (-1)^{|A|-1} e_A \underline{x}_A$ and (3.6), we have

$$
\begin{aligned}
&\underline{D}\, O_A - (-1)^{|A|} O_A \underline{D} \\
&= \frac{1}{2}\underline{D}\big(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A\big) - (-1)^{|A|}\frac{1}{2}\big(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A\big)\underline{D} \\
&= \frac{1}{2}\big(\underline{D}^2\, \underline{x}_A e_A - \underline{x}_A e_A \underline{D}^2\big) - \frac{\epsilon}{2}\big(\underline{D} e_A - (-1)^{|A|} e_A \underline{D}\big) \\
&= \frac{\epsilon}{2}\sum_{a\in A}[\Delta, x_a] e_a e_A - \epsilon \sum_{a\in A} p_a e_a e_A,
\end{aligned}
$$

which vanishes because of condition (1.7). In the same manner, using now the form (3.11) for O_A, the expression $\underline{x}\, O_A - (-1)^{|A|} O_A \underline{x}$ vanishes. □

REMARK 3.8: Note that if the order of the list A is altered, O_A changes but only in sign. Say π is a permutation on the list A, we have $O_A = \operatorname{sign}(\pi) O_{\pi(A)}$, where $\operatorname{sign}(\pi)$ is positive for an even permutation π and negative for an odd one. Hence up to a sign all the symmetries of this form are given by $\{O_A \mid A \subset \{1, \ldots, N\}\}$ where the elements of A are in ascending order in accordance with the standard order for natural numbers.

For the special case where $A = \{1, \ldots, N\}$, the operator (3.10) is seen to correspond precisely to the Scasimir element (3.3) of $\mathfrak{osp}(1|2)$ multiplied by the pseudo-scalar

$$O_{1\cdots N} = \frac{1}{2}\left([\underline{D}, \underline{x}] - \epsilon\right) e_1 \cdots e_N.$$

For a list A of $\{1, \ldots, N\}$, the operator O_A either commutes or anti-commutes with $\underline{D}$ and $\underline{x}$. The subsequent corollary is useful if one is interested solely in commuting symmetries.

COROLLARY 3.9: *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for A a list of distinct elements of $\{1, \ldots, N\}$, we have*

$$\left[\underline{D}, O_A \prod_{i \in A} O_i\right] = 0.$$

Note that the order of the product matters in general, but not for the result.

Proof. Follows immediately from Theorem 3.6 and Theorem 3.7. □

Expression (3.12) shows that the symmetries O_A are constructed using the symmetries C_{ij} and L_{ij} from the previous section, together with the Clifford algebra generators $e_1, \ldots, e_N$. The factor $1/2$ is chosen such that for a list of $\{1, \ldots, N\}$ consisting of just two distinct elements i and j, the symmetry O_{ij} corresponds to the generalized angular momentum symmetry L_{ij}, up to additive terms. Indeed, we have that

$$O_{ij} = L_{ij} - \epsilon \frac{1}{2} e_i e_j + \epsilon \frac{1}{2} \sum_{l \neq j} C_{li} e_l e_j - \epsilon \frac{1}{2} \sum_{l \neq i} C_{lj} e_l e_i.$$

This can be written more compactly by means of the explicit expression (3.8) for O_i as

$$O_{ij} = L_{ij} + \epsilon \frac{1}{2} e_i e_j + O_i e_j - O_j e_i \tag{3.13}$$

Together with the Clifford algebra generators $e_1, \ldots, e_N$, the symmetries O_i with one index and O_{ij} with two indices in fact suffice to build up all other symmetries O_A. Indeed, plugging in the expression (3.8) for O_i, one easily verifies that

$$O_A = \left(\epsilon \frac{|A| - 1}{2} + \epsilon \sum_{i \in A} O_i e_i - \sum_{\{i,j\} \subset A} L_{ij} e_i e_j\right) e_A \tag{3.14}$$

reduces to (3.12). Using now the form (3.13) to substitute L_{ij}, the operator O_A can also be written as

$$O_A = \left(- \epsilon \frac{(|A| - 1)(|A| - 2)}{4} - \epsilon(|A| - 2) \sum_{i \in A} O_i e_i - \sum_{\{i,j\} \subset A} O_{ij} e_i e_j\right) e_A. \tag{3.15}$$

Finally, the symmetries can also be constructed recursively. If we denote by $A \setminus \{a\}$ the list A with the element a omitted, and we define $\mathrm{sign}(A, a)$ as either $+1$ or -1 such that $\mathrm{sign}(A, a)e_{A\setminus\{a\}}e_a = e_A$, then it follows that

$$\begin{aligned}\sum_{a\in A}\mathrm{sign}(A, a)O_{A\setminus\{a\}}e_a &= \sum_{a\in A}\Big(\epsilon\frac{|A|-2}{2} + \epsilon\sum_{i\in A\setminus\{a\}}O_ie_i - \sum_{\{i,j\}\subset A\setminus\{a\}}L_{ij}e_ie_j\Big)e_A\\ &= \Big(\epsilon|A|\frac{|A|-2}{2} + \epsilon(|A|-1)\sum_{i\in A}O_ie_i - (|A|-2)\sum_{\{i,j\}\subset A}L_{ij}e_ie_j\Big)e_A\\ &= \epsilon\frac{|A|-2}{2}e_A + \epsilon\sum_{i\in A}O_ie_ie_A + (|A|-2)O_A.\end{aligned}$$

Using this relation, Theorem 3.7 can also be proved by induction on the cardinality of A, starting from $|A| = 3$.

3.2 *Symmetry algebra*

Before establishing the algebraic structure generated by the symmetries O_A, we first introduce some helpful relations with Clifford numbers. From the definition (3.8) of O_j we have

$$\{e_i, O_j\} = e_iO_j + O_je_i = [p_i, x_j] - \delta_{ij}.$$

The property $[p_i, x_j] = [p_j, x_i]$ then implies that $\{e_i, O_j\} = \{e_j, O_i\}$, or by a reordering of terms $O_ie_j - O_je_i = e_iO_j - e_jO_i$. This is in fact a special case of the following useful result.

LEMMA 3.10: *In the algebra $\mathcal{A}\otimes\mathcal{C}$, for A a list of distinct elements of $\{1, \ldots, N\}$, we have*

$$\sum_{a\in A}O_ae_ae_A = e_A\sum_{a\in A}e_aO_a.$$

Proof. The identity follows by direct calculation using the definition (3.8) of O_a and the commutation relations of $e_1, \ldots, e_N$:

$$\begin{aligned}\sum_{a\in A}O_ae_ae_A &= \sum_{a\in A}\epsilon\frac{1}{2}\sum_{l=1}^{N}[p_l, x_a]e_le_ae_A - \sum_{a\in A}\epsilon\frac{1}{2}e_ae_ae_A\\ &= \sum_{a\in A}\epsilon\frac{1}{2}\sum_{l\in A}[p_l, x_a]e_Ae_le_a - \sum_{a\in A}\epsilon\frac{1}{2}\sum_{l\notin A}[p_l, x_a]e_Ae_le_a - \sum_{a\in A}\frac{1}{2}e_A\\ &= \sum_{a\in A}\epsilon\frac{1}{2}\sum_{l\in A}[p_a, x_l]e_Ae_ae_l + \sum_{a\in A}\epsilon\frac{1}{2}\sum_{l\notin A}[p_l, x_a]e_Ae_ae_l - \sum_{a\in A}\epsilon\frac{1}{2}e_Ae_ae_a\\ &= e_A\sum_{a\in A}e_aO_a.\end{aligned}$$

□

Note that by means of this lemma, the symmetry O_A, in the form (3.14), can equivalently be written with e_A in front, that is

$$O_A = e_A\Big(\epsilon\frac{|A|-1}{2} + \epsilon\sum_{i\in A} e_iO_i - \sum_{\{i,j\}\subset A} L_{ij}e_ie_j\Big).$$

3.2.1 *Relations for symmetries with one or two indices*

Next, we present some relations which hold for symmetries with one or two indices.

THEOREM 3.11: *In the algebra $\mathcal{A}\otimes\mathcal{C}$, for $i, j, k \in \{1,\dots,N\}$ we have*

$$[O_{ij}, O_k] + [O_{jk}, O_i] + [O_{ki}, O_j] = 0.$$

Proof. If any of the indices are equal, the identity becomes trivial as $O_{ij} = -O_{ji}$. For distinct i, j, k, we have by (3.8), (3.13) and using $O_ie_j - O_je_i = e_iO_j - e_jO_i$

$$\begin{aligned}
&[O_{ij}, O_k] + [O_{jk}, O_i] + [O_{ki}, O_j]\\
&= \epsilon\frac{1}{2}\sum_{l=1}^{N} e_l([L_{ij}, C_{lk}] + [L_{jk}, C_{li}] + [L_{ki}, C_{lj}])\\
&\quad + \epsilon\frac{1}{2}([e_ie_j, O_k] + [e_je_k, O_i] + [e_ke_i, O_j])\\
&\quad + (O_ie_j - O_je_i)O_k - O_k(e_iO_j - e_jO_i) + (O_je_k - O_ke_j)O_i\\
&\quad - O_i(e_jO_k - e_kO_j) + (O_ke_i - O_ie_k)O_j - O_j(e_kO_i - e_iO_k).
\end{aligned}$$

The first line of the right-hand side vanishes by Theorem 2.5, while the second line does so by direct calculation plugging in the definition of O_i and using $C_{ij} = C_{ji}$. Finally, the remaining terms of the last two lines cancel out pairwise. □

For the next result, we first write out the form (3.14) of O_A for A a list of three elements, say i, j, k which are all distinct:

$$O_{ijk} = \epsilon e_ie_je_k + O_ie_je_k - O_je_ie_k + O_ke_ie_j + L_{ij}e_k - L_{ik}e_j + L_{jk}e_i. \tag{3.16}$$

The commutation relations for symmetries with two indices are as follows.

THEOREM 3.12: *In the algebra $\mathcal{A}\otimes\mathcal{C}$, for $i, j, k, l \in \{1,\dots,N\}$ the symmetries satisfy*

$$\begin{aligned}
[O_{ij}, O_{kl}] &= (O_{il} + \epsilon[O_i, O_l])\delta_{jk} + (O_{jk} + \epsilon[O_j, O_k])\delta_{il} + (O_{ki} + \epsilon[O_k, O_i])\delta_{lj} + (O_{lj}\\
&\quad + \epsilon[O_l, O_j])\delta_{ki} + \frac{1}{2}(\{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} - \{O_{ijl}, O_k\} + \{O_{ijk}, O_l\}).
\end{aligned}$$

Proof. For the cases where $i = j$ or $k = l$ or $\{i, j\} = \{k, l\}$, both sides of the equation reduce to zero, so from now on we assume that $i \neq j$ and $k \neq l$ and $\{i, j\} \neq \{k, l\}$. Plugging in (3.13) we have

$$[O_{ij}, O_{kl}] = \Big[L_{ij} + \epsilon\frac{1}{2}e_ie_j + O_ie_j - O_je_i,\ L_{kl} + \epsilon\frac{1}{2}e_ke_l + O_ke_l - O_le_k\Big]$$

$$
\begin{aligned}
&= [L_{ij}, L_{kl}] + [L_{ij}, O_k e_l - O_l e_k] + [O_i e_j - O_j e_i, L_{kl}] + [O_i e_j - O_j e_i, O_k e_l - O_l e_k] \\
&\quad + \frac{1}{4}[e_i e_j, e_k e_l] + \epsilon\frac{1}{2}\left([e_i e_j, O_k e_l] - [e_i e_j, O_l e_k] + [O_i e_j, e_k e_l] - [O_j e_i, e_k e_l]\right)
\end{aligned}
$$

By Theorem 2.5, and using $\{e_i, O_j\} = [p_i, x_j] - \delta_{ij}$, we have

$$
\begin{aligned}
[L_{ij}, L_{kl}] &= L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ki} \\
&\quad + L_{il}\{e_j, O_k\} + L_{jk}\{e_i, O_l\} + L_{ki}\{e_l, O_j\} + L_{lj}\{e_k, O_i\}.
\end{aligned}
$$

Using $\{e_i, O_j\} = \{e_j, O_i\} = 1/2\{e_i, O_j\} + 1/2\{e_j, O_i\}$, the terms in the last line can be rewritten as

$$
\begin{aligned}
&\frac{1}{2}((L_{il}e_j + L_{lj}e_i)O_k + (L_{jk}e_i + L_{ki}e_j)O_l + (L_{lj}e_k + L_{jk}e_l)O_i + (L_{ki}e_l + L_{il}e_k)O_j) \\
&+ \frac{1}{2}((L_{jk}O_l + L_{lj}O_k)e_i + (L_{il}O_k + L_{ki}O_l)e_j + (L_{lj}O_i + L_{il}O_j)e_k + (L_{ki}O_j + L_{jk}O_i)e_l).
\end{aligned}
$$

Together with

$$
\begin{aligned}
[L_{ij}, O_k e_l - O_l e_k] &= [L_{ij}, O_k]e_l - [L_{ij}, O_l]e_k = \{L_{ij}e_k, O_l\} - \{L_{ij}e_l, O_k\} \\
[O_i e_j - O_j e_i, L_{kl}] &= [O_i, L_{kl}]e_j - [O_j, L_{kl}]e_i = \{L_{kl}e_j, O_i\} - \{L_{kl}e_i, O_j\},
\end{aligned}
$$

we find that $[L_{ij}, L_{kl}] + [L_{ij}, O_k e_l - O_l e_k] + [O_i e_j - O_j e_i, L_{kl}]$ equals

$$
\begin{aligned}
&L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ki} \\
&+ \frac{1}{2}((L_{il}e_j + L_{lj}e_i)O_k + \{-L_{ij}e_l, O_k\} + (L_{jk}e_i + L_{ki}e_j)O_l + \{L_{ij}e_k, O_l\} \\
&+ (L_{lj}e_k + L_{jk}e_l)O_i + \{O_i, L_{kl}e_j\} + (L_{ki}e_l + L_{il}e_k)O_j + \{O_j, -L_{kl}e_i\}) \\
&+ \frac{1}{2}((L_{jk}O_l + L_{lj}O_k + [L_{kl}, O_j])e_i + (L_{il}O_k + L_{ki}O_l - [L_{kl}, O_i])e_j \\
&+ (L_{lj}O_i + L_{il}O_j - [L_{ij}, O_l])e_k + (L_{ki}O_j + L_{jk}O_i + [L_{ij}, O_k])e_l).
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
&L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ki} + \frac{1}{2}(\{L_{il}e_j + L_{lj}e_i + L_{ji}e_l, O_k\} \\
&+ \{L_{jk}e_i + L_{ki}e_j + L_{ij}e_k, O_l\} + \{O_i, L_{lj}e_k + L_{jk}e_l + L_{kl}e_j\} + \{O_j, L_{ki}e_l + L_{il}e_k + L_{lk}e_i\}),
\end{aligned}
$$

by means of

$$
\begin{aligned}
&([L_{jk}, O_l] + [L_{lj}, O_k] + [L_{kl}, O_j])e_i + ([L_{il}, O_k] + [L_{ki}, O_l] - [L_{kl}, O_i])e_j \\
&+ ([L_{lj}, O_i] + [L_{il}, O_j] - [L_{ij}, O_l])e_k + ([L_{ki}, O_j] + [L_{jk}, O_i] + [L_{ij}, O_k])e_l = 0,
\end{aligned}
$$

which is a direct consequence of Theorem 2.5 after plugging in the definition (3.8) of O_i.

Now, the other terms appearing in $[O_{ij}, O_{kl}]$ can be expanded as follows. First,

$$[e_ie_j, e_ke_l] = 2\epsilon(\delta_{jk}e_ie_l - \delta_{il}e_ke_j - \delta_{lj}e_ie_k + \delta_{ik}e_le_j).$$

Moreover, we have

$$\begin{aligned}
[e_ie_j, O_ke_l] &= e_ie_jO_ke_l - O_ke_le_ie_j \\
&= e_ie_j\{e_l, O_k\} - e_ie_je_lO_k + O_ke_ie_le_j - O_k(\epsilon 2\delta_{il})e_j \\
&= e_ie_j\{e_l, O_k\} - e_ie_je_lO_k - O_ke_ie_je_l + O_ke_i(\epsilon 2\delta_{jl}) - O_k(\epsilon 2\delta_{il})e_j.
\end{aligned}$$

As $\{e_k, O_l\} = \{e_l, O_k\}$, after interchanging k and l in this result, subtraction yields the following

$$[e_ie_j, O_ke_l - O_le_k] = \epsilon 2(\delta_{jl}O_ke_i - \delta_{il}O_ke_j - \delta_{jk}O_le_i + \delta_{ik}O_le_j) + \{e_ie_je_k, O_l\} - \{e_ie_je_l, O_k\}.$$

For the last term, $[O_ie_j - O_je_i, O_ke_l - O_le_k]$, we use Lemma 3.10 to find

$$\begin{aligned}
2[O_ie_j - O_je_i, O_ke_l - O_le_k] &= 2(O_ie_j - O_je_i)(e_kO_l - e_lO_k) - 2(O_ke_l - O_le_k)(e_iO_j - e_jO_i) \\
&= (O_ke_le_j - O_le_ke_j)O_i + O_i(e_je_kO_l - e_je_lO_k) \\
&\quad + (-O_ke_le_i + O_le_ke_i)O_j + O_j(-e_ie_kO_l + e_ie_lO_k) \\
&\quad + (-O_ie_je_l + O_je_ie_l)O_k + O_k(-e_le_iO_j + e_le_jO_i) \\
&\quad + (O_ie_je_k - O_je_ie_k)O_l + O_l(e_ke_iO_j - e_ke_jO_i).
\end{aligned}$$

We first consider the case where one of i, j is equal to either k or l, for instance say $i = l$:

$$\begin{aligned}
&(O_ke_ie_j - O_ie_ke_j)O_i + O_i(e_je_kO_i - e_je_iO_k) \\
&\quad + (-O_k\epsilon + O_ie_ke_i)O_j + O_j(-e_ie_kO_i + \epsilon O_k) \\
&\quad + (-O_ie_je_i + O_j\epsilon)O_k + O_k(-\epsilon O_j + e_ie_jO_i) \\
&\quad + (O_ie_je_k - O_je_ie_k)O_i + O_i(e_ke_iO_j - e_ke_jO_i) \\
&= (O_ke_ie_j - O_ie_ke_j - O_je_ie_k + O_ie_je_k - O_je_ie_k + O_ke_ie_j)O_i \\
&\quad + O_i(e_je_kO_i - e_je_iO_k + e_ke_iO_j + e_ke_iO_j - e_ke_jO_i - e_je_iO_k) + \epsilon 2[O_j, O_k] \\
&= (O_ke_ie_j - O_ie_ke_j - O_je_ie_k + O_ie_je_k - O_je_ie_k + O_ke_ie_j)O_i \\
&\quad + O_i(O_ie_je_k - O_ke_je_i + O_je_ke_i + O_je_ke_i - O_ie_ke_j - O_ke_je_i) + \epsilon 2[O_j, O_k] \\
&= 2\{O_ke_ie_j - O_je_ie_k + O_ie_je_k, O_i\} + \epsilon 2[O_j, O_k].
\end{aligned}$$

Finally, when i, j, k, l are all distinct, $2[O_ie_j - O_je_i, O_ke_l - O_le_k]$ equals

$$\begin{aligned}
&(O_ke_le_j - O_le_ke_j)O_i + O_i(-e_ke_jO_l + e_le_jO_k + e_ke_lO_j - e_ke_lO_j) \\
&\quad + (-O_ke_le_i + O_le_ke_i)O_j + O_j(e_ke_iO_l - e_le_iO_k + e_le_kO_i - e_le_kO_i) \\
&\quad + (-O_ie_je_l + O_je_ie_l)O_k + O_k(e_ie_lO_j - e_je_lO_i + e_je_iO_l - e_je_iO_l) \\
&\quad + (O_ie_je_k - O_je_ie_k)O_l + O_l(-e_ie_kO_j + e_je_kO_i + e_ie_jO_k - e_ie_jO_k) \\
&= (O_ke_le_j - O_le_ke_j - O_je_le_k)O_i + O_i(-O_le_ke_j + O_ke_le_j + O_je_ke_l)
\end{aligned}$$

$$
\begin{aligned}
&+ (-O_k e_l e_i + O_l e_k e_i - O_i e_k e_l)O_j + O_j(O_l e_k e_i - O_k e_l e_i + O_i e_l e_k)\\
&+ (-O_i e_j e_l + O_j e_i e_l - O_l e_i e_j)O_k + O_k(O_j e_i e_l - O_i e_j e_l + O_l e_j e_i)\\
&+ (O_i e_j e_k - O_j e_i e_k - O_k e_j e_i)O_l + O_l(-O_j e_i e_k + O_i e_j e_k + O_k e_i e_j)\\
= &\{O_k e_l e_j - O_l e_k e_j + O_j e_k e_l, O_i\} + \{O_l e_k e_i - O_k e_l e_i + O_i e_l e_k, O_j\}\\
&+ \{O_j e_i e_l - O_i e_j e_l + O_l e_j e_i, O_k\} + \{O_i e_j e_k - O_j e_i e_k + O_k e_i e_j, O_l\}.
\end{aligned}
$$

This boils down to $[O_i e_j - O_j e_i, O_k e_l - O_l e_k]$ being equal to

$$
\begin{aligned}
&\epsilon\delta_{jk}[O_i, O_l] + \epsilon\delta_{il}[O_j, O_k] + \epsilon\delta_{lj}[O_k, O_i] + \epsilon\delta_{ki}[O_l, O_j]\\
&+ \frac{1}{2}(\{O_k e_l e_j - O_l e_k e_j + O_j e_k e_l, O_i\} + \{O_l e_k e_i - O_k e_l e_i + O_i e_l e_k, O_j\}\\
&+ \{O_j e_i e_l - O_i e_j e_l + O_l e_j e_i, O_k\} + \{O_i e_j e_k - O_j e_i e_k + O_k e_i e_j, O_l\}).
\end{aligned}
$$

Hence, combining all of the above, we find for $[O_{ij}, O_{kl}]$

$$
\begin{aligned}
&L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ki} + \frac{1}{2}(\{L_{il}e_j + L_{lj}e_i + L_{ji}e_l, O_k\}\\
&+ \{L_{jk}e_i + L_{ki}e_j + L_{ij}e_k, O_l\} + \{L_{ki}e_l + L_{il}e_k + L_{lk}e_i, O_j\}\\
&+ \{L_{lj}e_k + L_{jk}e_l + L_{kl}e_j, O_i\}) + \epsilon\frac{1}{2}(\delta_{jk}e_i e_l - \delta_{il}e_k e_j - \delta_{lj}e_i e_k + \delta_{ik}e_l e_j)\\
&+ \epsilon\delta_{jk}[O_i, O_l] + \epsilon\delta_{il}[O_j, O_k] + \epsilon\delta_{lj}[O_k, O_i] + \epsilon\delta_{ki}[O_l, O_j]\\
&+ \frac{1}{2}(\{O_k e_l e_j - O_l e_k e_j + O_j e_k e_l, O_i\} + \{O_l e_k e_i - O_k e_l e_i + O_i e_l e_k, O_j\}\\
&+ \{O_j e_i e_l - O_i e_j e_l + O_l e_j e_i, O_k\} + \{O_i e_j e_k - O_j e_i e_k + O_k e_i e_j, O_l\})\\
&+ \epsilon\frac{1}{2}(\delta_{jl}\epsilon 2O_k e_i - \delta_{il}\epsilon 2O_k e_j - \{e_i e_j e_l, O_k\} - \delta_{jk}\epsilon 2O_l e_i + \delta_{ik}\epsilon 2O_l e_j + \{e_i e_j e_k, O_l\})\\
&- \epsilon\frac{1}{2}(\delta_{lj}\epsilon 2O_i e_k - \delta_{kj}\epsilon 2O_i e_l - \{e_k e_l e_j, O_i\} - \delta_{li}\epsilon 2O_j e_k + \delta_{ki}\epsilon 2O_j e_l + \{e_k e_l e_i, O_j\}).
\end{aligned}
$$

Collecting the appropriate terms, we recognize all ingredients to make symmetries with three indices (3.16) and we arrive at the desired result

$$
\begin{aligned}
[O_{ij}, O_{kl}] = &(O_{il} + \epsilon[O_i, O_l])\delta_{jk} + (O_{jk} + \epsilon[O_j, O_k])\delta_{il} + (O_{ki} + \epsilon[O_k, O_i])\delta_{lj}\\
&+ (O_{lj} + \epsilon[O_l, O_j])\delta_{ki} + \frac{1}{2}(\{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} - \{O_{ijl}, O_k\} + \{O_{ijk}, O_l\}).
\end{aligned}
$$

□

In summary, for i, j, k, l all distinct elements of $\{1, \ldots, N\}$ we have

$$
\begin{aligned}
[O_{ij}, O_{ki}] &= O_{jk} + \epsilon[O_j, O_k] + \{O_{ijk}, O_i\}\\
[O_{ij}, O_{kl}] &= \frac{1}{2}(\{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} - \{O_{ijl}, O_k\} + \{O_{ijk}, O_l\}).
\end{aligned}
$$

3.2.2 *Relations for symmetries with general index*

Now, we are interested in relations for general symmetries with an arbitrary index list A, i.e.

$$O_A = \frac{1}{2}\left(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A\right).$$

Before doing so, we make a slight detour clearing up some conventions and notations. An important fact to take into account is the interaction of the appearing Clifford numbers with different lists as index (recall Definition 3.1). For instance, if A and B denote two lists of $\{1, \ldots, N\}$, the following properties are readily shown to hold by direct computation:

$$e_B e_A = (-1)^{|A||B|-|A\cap B|} e_A e_B, \tag{3.17}$$

and

$$(e_A)^2 = \epsilon^{|A|}(-1)^{\frac{|A|^2-|A|}{2}}. \tag{3.18}$$

The product $e_A e_B$ in fact reduces (up to a sign) to contain only e_i with i an index in the set $(A \cup B)\backslash(A \cap B)$, since all indices in $A \cap B$ appear twice and cancel out (for these set operations we disregard the order of the lists A and B and view them just as sets). The remaining indices are what is called the symmetric difference of the sets A and B. We will denote this associative operation by $A \triangle B = (A \cup B)\backslash(A \cap B) = (A \setminus B) \cup (B \setminus A)$. When applied to two lists A and B, we view them as sets and the resultant $A\triangle B$ is a set. Note that $|A\triangle B| = |A| + |B| - 2|A \cap B|$.

When dealing with interactions between symmetries O_A and O_B for two lists A and B, products of the kind $e_A e_B$ are exactly what we encounter. Not wanting to overburden notations, but still taking into account all resulting signs due to the anti-commutation relations (3.1) if one were to work out the reduction $e_A e_B$, we introduce the following notation

$$O_{A,B} = \frac{1}{2}\left(\underline{D}\,\underline{x}_{A\triangle B} e_A e_B - e_A e_B \underline{x}_{A\triangle B} \underline{D} - \epsilon e_A e_B\right). \tag{3.19}$$

Note that the order of A and B matters, as by (3.17) we have $O_{A,B} = (-1)^{|A||B|-|A\cap B|} O_{B,A}$. Moreover, up to a sign $O_{A,B}$ is equal to $O_{A\triangle B}$, where the elements of $A\triangle B$ are in ascending order when used as a list, see Remark 3.2. Since the symmetric difference operation on sets is associative, one easily extends this definition to an arbitrary number of lists, e.g.

$$O_{A,B,C} = \frac{1}{2}\left(\underline{D}\,\underline{x}_{A\triangle B\triangle C} e_A e_B e_C - e_A e_B e_C \underline{x}_{A\triangle B\triangle C} \underline{D} - \epsilon e_A e_B e_C\right).$$

As an example, if we consider the lists $A = 234$ and $B = 31$, then the set $A\triangle B$ contains the elements 2, 4, 1 but not 3 as 3 appears in both A and B, so we have

$$\begin{aligned} O_{234,31} &= \frac{1}{2}\left(\underline{D}\,\underline{x}_{\{2,4,1\}} e_{234} e_{31} - e_{234} e_{31} \underline{x}_{\{2,4,1\}} \underline{D} - \epsilon e_{234} e_{31}\right) \\ &= -\epsilon \frac{1}{2}\left(\underline{D}\,(x_2 e_2 + x_4 e_4 + x_1 e_1) e_{241} - e_{241}(x_2 e_2 + x_4 e_4 + x_1 e_1)\underline{D} - \epsilon e_{241}\right) \\ &= -\epsilon O_{241} = -\epsilon O_{124}. \end{aligned}$$

We elaborate on one final convention. If A and B denote two lists of $\{1, \ldots, N\}$, then when viewed as ordinary sets, the intersection $A \cap B$ contains all elements of A that also belong to B (or equivalently, all elements of B that also belong to A). As a list appearing as index of a Clifford number, we distinguish between $A \cap B$ and $B \cap A$ in the sense that we understand the elements of $A \cap B$ to be in the sequential order of A, while those of $B \cap A$ are in the sequential order of B. If $A = 124$ and $B = 231$, then $e_{A\cap B} = e_{12}$, while $e_{B\cap A} = e_{21}$.

The framework where operators of the form (3.19) make their appearance is one where both commutators and anti-commutators are considered. Inspired by property (3.17), we define the "supercommutators"

$$[\![O_A, O_B]\!]_- = O_A O_B - (-1)^{|A||B|-|A\cap B|} O_B O_A, \tag{3.20}$$

$$[\![O_A, O_B]\!]_+ = O_A O_B + (-1)^{|A||B|-|A\cap B|} O_B O_A. \tag{3.21}$$

The algebraic relations we obtained in Theorem 3.12 can now be generalized to higher index versions. We start with a generalization of Theorem 3.11.

THEOREM 3.13: *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for A a list of distinct elements of $\{1, \ldots, N\}$, we have*

$$\sum_{a\in A} [\![O_a, O_{a,A}]\!]_- = 0.$$

Note that $O_{a,A}$ is up to a sign equal to $O_{A\setminus\{a\}}$ since $a \in A$, where $A \setminus \{a\}$ is the list A with the element a removed.

Proof. By direct calculation and using (3.8), (3.14) and Lemma 3.10 to arrive at the second line we have

$$\begin{aligned}
&\sum_{a\in A}(O_a O_{a,A} - (-1)^{|A|-1} O_{a,A} O_a) = \epsilon \frac{|A|-2}{2}\Big(\sum_{a\in A} O_a e_a e_A - (-1)^{|A|-1}\sum_{a\in A} e_a e_A O_a\Big)\\
&+ \epsilon \sum_{a\in A}\sum_{b\in A\setminus\{a\}} O_a e_a e_A e_b O_b - (-1)^{|A|-1}\epsilon \sum_{a\in A}\sum_{b\in A\setminus\{a\}} O_b e_b e_a e_A O_a\\
&+ \epsilon \frac{1}{2}\sum_{a\in A}\sum_{\{i,j\}\subset A\setminus\{a\}}\sum_{l\notin A\setminus\{a,i,j\}} [L_{ij}, C_{la}] e_l e_i e_j e_a e_A\\
&+ \epsilon \frac{1}{2}\sum_{a\in A}\sum_{\{i,j\}\subset A\setminus\{a\}}\sum_{l\in A\setminus\{a,i,j\}} \{L_{ij}, C_{la}\} e_l e_i e_j e_a e_A\\
&- \epsilon \frac{1}{2}\sum_{a\in A}\sum_{\{i,j\}\subset A\setminus\{a\}} L_{ij}(e_a e_i e_j e_a e_A - (-1)^{|A|-1} e_i e_j e_a e_A e_a).
\end{aligned}$$

As $(-1)^{|A|-1} e_a e_A = e_A e_a$ for $a \in A$, the last line vanishes identically. Similarly, the first line of the right-hand side vanishes by Lemma 3.10. Moreover, the second line then vanishes by interchanging the summations. In the third line, we write the first two summations as one summation running over all three-element subsets of A as follows

$$\epsilon\frac{1}{2}\sum_{\{a,i,j\}\subset A}\sum_{l\notin A\setminus\{a,i,j\}}([L_{ij},C_{la}]+[L_{ai},C_{lj}]+[L_{ja},C_{li}])e_le_ie_je_ae_A,$$

which vanishes by Theorem 2.5. Finally, rewriting the four summations in the fourth line as one summation over all four-element subsets of A, one sees that all terms in this summation cancel out using $C_{ij}=C_{ji}$ and $L_{ij}=-L_{ji}$.

□

Using Theorem 3.13 for the list $A=ijkl$ yields

$$\{O_i,O_{jkl}\}-\{O_j,O_{ikl}\}+\{O_k,O_{ijl}\}-\{O_l,O_{ijk}\}=0.$$

By means of this identity, the relation of Theorem 3.12 for four distinct indices can be cast also in two other formats:

$$[\![O_{ij},O_{kl}]\!]_-=[O_{ij},O_{kl}]=\{O_i,O_{jkl}\}-\{O_j,O_{ikl}\}=-\{O_{ijl},O_k\}+\{O_{ijk},O_l\}. \quad (3.22)$$

The results of Theorem 3.12 are actually special cases of two different more general relations. By means of the following three theorems and the supercommutators (3.20)–(3.21), one is able to swap two symmetry operators O_A and O_B for A, B arbitrary lists of $\{1,\dots,N\}$. Moreover, the supercommutators reduce to explicit expressions in terms of the symmetries and supercommutators containing (at least) one symmetry with just one index. First, we generalize (3.22) to arbitrary disjoint lists.

THEOREM 3.14: *In the algebra $\mathcal{A}\otimes\mathcal{C}$, for two lists of $\{1,\dots,N\}$, denoted by A and B, such that $A\cap B=\emptyset$ as sets, we have*

$$[\![O_A,O_B]\!]_-=\epsilon\sum_{a\in A}[\![O_a,O_{a,A,B}]\!]_-\,.$$

Note that in this case $a\triangle A\triangle B=(A\setminus\{a\})\cup B$. Moreover, following a similar strategy (or using Theorem 3.13) one also obtains

$$[\![O_A,O_B]\!]_-=\epsilon\sum_{b\in B}[\![O_{A,B,b},O_b]\!]_-\,.$$

Proof. A practical property for this proof and the following ones is (3.7). By definition, plugging in (3.10), we find

$$\begin{aligned}[\![O_A,O_B]\!]_-&=O_AO_B-(-1)^{|A||B|-|A\cap B|}O_BO_A\\&=\frac{1}{4}\left(\underline{D}\,\underline{x}_Ae_A-e_A\underline{x}_A\underline{D}-\epsilon e_A\right)\left(\underline{D}\,\underline{x}_Be_B-e_B\underline{x}_B\underline{D}-\epsilon e_B\right)\\&\quad-(-1)^{|A||B|}\frac{1}{4}\left(\underline{D}\,\underline{x}_Be_B-e_B\underline{x}_B\underline{D}-\epsilon e_B\right)\left(\underline{D}\,\underline{x}_Ae_A-e_A\underline{x}_A\underline{D}-\epsilon e_A\right).\end{aligned}$$

First, note that for the terms of $[\![O_A, O_B]\!]_-$ which do not contain $\underline{D}$, we have

$$e_A e_B - (-1)^{|A||B|} e_B e_A = 0.$$

For the terms with a single occurrence of $\underline{D}$, we have (up to a factor $-\epsilon/4$)

$$\begin{aligned}
&(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D}) e_B - (-1)^{|A||B|} e_B (\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D}) \\
&- (-1)^{|A||B|} (\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D}) e_A + e_A (\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D}) \\
= &\sum_{a \in A} \left(\underline{D}\, x_a e_a e_A e_B - x_a e_A e_a \underline{D} e_B - (-1)^{|A||B|} e_B \underline{D}\, x_a e_a e_A + (-1)^{|A||B|} e_B e_A e_a x_a \underline{D}\right) \\
&- (-1)^{|A||B|} \underline{D}\,\underline{x}_B e_B e_A + (-1)^{|A||B|} e_B \underline{x}_B \underline{D} e_A + e_A \underline{D}\,\underline{x}_B e_B - e_A e_B \underline{x}_B \underline{D} \\
= &\sum_{a \in A} \left(\underline{D}\, x_a e_a e_A e_B - (-1)^{|A|-1} x_a \underline{D} e_A e_a e_B - (-1)^{|A||B|+|A|-1} e_B e_a e_A \underline{D}\, x_a + e_A e_B e_a x_a \underline{D}\right) \\
&+ (-1)^{|A|-1} \sum_{a \in A} x_a \sum_{l \in A \setminus \{a\}} 2 p_l e_l e_A e_a e_B - (-1)^{|A||B|} \sum_{a \in A} e_B \sum_{l \in A \setminus \{a\}} 2 p_l e_l e_a e_A x_a \\
&- \underline{D}\,\underline{x}_B e_A e_B + (-1)^{|A||B|+|A|} e_B \underline{x}_B e_A \underline{D} + (-1)^{|A|} \underline{D}\, e_A \underline{x}_B e_B - e_A e_B \underline{x}_B \underline{D} \\
&+ (-1)^{|A||B|} e_B \underline{x}_B \sum_{a \in A} 2 p_a e_a e_A - (-1)^{|A|} \sum_{a \in A} 2 p_a e_a e_A \underline{x}_B e_B \\
= &\sum_{a \in A} \left(\underline{D}\, x_a e_a e_A e_B - x_a \underline{D} e_a e_A e_B - (-1)^{|B|+|A|-1} e_a e_A e_B \underline{D}\, x_a + (-1)^{|A|+|B|-1} e_a e_A e_B x_a \underline{D}\right) \\
&+ (-1)^{|A|+|B|-1} \sum_{a \in A} e_a e_A e_B \sum_{l \in A \setminus \{a\}} x_l e_l \epsilon \{\underline{D}, e_a\} + \sum_{a \in A} \epsilon \{\underline{D}, e_a\} \sum_{l \in A \setminus \{a\}} e_l x_l e_a e_A e_B \\
&+ \sum_{a \in A} (-1)^{|B|+|A|-1} e_a e_A e_B \underline{x}_B \epsilon \{\underline{D}, e_a\} + \sum_{a \in A} \epsilon \{\underline{D}, e_a\} \underline{x}_B e_a e_A e_B \\
= &\sum_{a \in A} \left(\underline{D}\, x_a e_a e_A e_B - x_a \underline{D} e_a e_A e_B - (-1)^{|B|+|A|-1} e_a e_A e_B \underline{D}\, x_a + (-1)^{|A|+|B|-1} e_a e_A e_B x_a \underline{D}\right. \\
&+ \epsilon (-1)^{|A|+|B|-1} e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D}\, e_a - \epsilon e_a e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D} \\
&- \epsilon (-1)^{|A|+|B|-1} \underline{D}\,\underline{x}_{a \triangle A \triangle B} e_a e_A e_B e_a + \epsilon e_a \underline{D}\,\underline{x}_{a \triangle A \triangle B} e_a e_A e_B,
\end{aligned}$$

where we made use of (3.7), $\{\underline{D}, e_a\} = \epsilon 2 p_a$ and $\underline{x}_{a \triangle A \triangle B} = \underline{x}_{A \setminus \{a\}} + \underline{x}_B$ for $a \in A$ and $A \cap B = \emptyset$.

Next, we work out the terms of $[\![O_A, O_B]\!]_-$ containing two occurrences of $\underline{D}$, that is

$$\left(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D}\right)\left(\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D}\right) - (-1)^{|A||B|} \left(\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D}\right)\left(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D}\right).$$

For the terms having $\underline{DD}$ in the middle we readily find

$$\begin{aligned}
&- e_A \underline{x}_A \underline{DD}\,\underline{x}_B e_B + (-1)^{|A||B|} e_B \underline{x}_B \underline{DD}\,\underline{x}_A e_A \\
= &- \sum_{a \in A} \left(e_A x_a e_a \underline{DD}\,\underline{x}_B e_B - (-1)^{|A||B|} e_B \underline{x}_B \underline{DD}\, x_a e_a e_A\right)
\end{aligned}$$

$$= -\sum_{a\in A}\left(x_a\underline{D}\underline{D}\,\underline{x}_B e_a e_A e_B - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_B \underline{D}\underline{D}\,x_a\right)$$
$$= -\sum_{a\in A}\left(x_a\underline{D}\underline{D}\,\underline{x}_{a\triangle A\triangle B} e_a e_A e_B - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_{a\triangle A\triangle B} \underline{D}\underline{D}\,x_a\right),$$

since

$$\sum_{a\in A}\left(x_a\underline{D}\underline{D}\,\underline{x}_{a\triangle A} e_a e_A e_B - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_{a\triangle A} \underline{D}\underline{D}\,x_a\right)$$
$$= \sum_{a\in A}\sum_{l\in A\setminus\{a\}} x_a\underline{D}\underline{D}\,x_l e_l e_a e_A e_B - \sum_{a\in A}\sum_{l\in A\setminus\{a\}} x_l\underline{D}\underline{D}\,x_a e_a e_l e_A e_B$$
$$= 0.$$

Similarly, we have

$$-\underline{D}\,\underline{x}_A e_A e_B \underline{x}_B \underline{D} + (-1)^{|A||B|}\underline{D}\,\underline{x}_B e_B e_A \underline{x}_A \underline{D}$$
$$= -\sum_{a\in A}\left(\underline{D}\,x_a e_a e_A e_B \underline{x}_B \underline{D} - (-1)^{|A||B|}\underline{D}\,\underline{x}_B e_B e_A x_a e_a \underline{D}\right)$$
$$= -\sum_{a\in A}\left(\underline{D}\,x_a e_a e_A e_B \underline{x}_{a\triangle A\triangle B} \underline{D} - (-1)^{|A|-1+|B|}\underline{D}\,\underline{x}_{a\triangle A\triangle B} e_a e_A e_B x_a \underline{D}\right).$$

Finally, we manipulate the remaining terms as follows

$$\underline{D}\,\underline{x}_A e_A \underline{D}\,\underline{x}_B e_B + e_A \underline{x}_A \underline{D} e_B \underline{x}_B \underline{D} - (-1)^{|A||B|}\underline{D}\,\underline{x}_B e_B \underline{D}\,\underline{x}_A e_A - (-1)^{|A||B|} e_B \underline{x}_B \underline{D} e_A \underline{x}_A \underline{D}$$
$$= \sum_{a\in A}\Big(\underline{D}\,x_a e_a e_A \underline{D}\,\underline{x}_B e_B + e_A x_a e_a \underline{D} e_B \underline{x}_B \underline{D}$$
$$- (-1)^{|A||B|}(\underline{D}\,\underline{x}_B e_B \underline{D}\,x_a e_a e_A + e_B \underline{x}_B \underline{D} e_A x_a e_a \underline{D})\Big)$$
$$= \sum_{a\in A}\Big((-1)^{|A|-1}\underline{D}\,x_a \underline{D}\,e_a e_A \underline{x}_B e_B + (-1)^{|A|-1} x_a \underline{D} e_A e_a e_B \underline{x}_B \underline{D}$$
$$- (-1)^{|A||B|+|A|-1}\underline{D}\,\underline{x}_B e_B e_a e_A \underline{D}\,x_a - (-1)^{|A||B|+|A|-1} e_B \underline{x}_B e_A e_a \underline{D} x_a \underline{D}\Big)$$
$$+ \sum_{a\in A}\sum_{l\in A\setminus\{a\}}\Big(-(-1)^{|A|-1}\underline{D}\,x_a 2p_l e_l e_a e_A \underline{x}_B e_B - (-1)^{|A|-1} x_a 2p_l e_l e_A e_a e_B \underline{x}_B \underline{D}$$
$$- (-1)^{|A||B|}\underline{D}\,\underline{x}_B e_B 2p_l e_l e_a e_A x_a - (-1)^{|A||B|} e_B \underline{x}_B 2p_l e_l e_A e_a x_a \underline{D}\Big)$$
$$= \sum_{a\in A}\Big(\underline{D}\,x_a \underline{D}\,\underline{x}_{a\triangle A\triangle B} e_a e_A e_B + x_a \underline{D} e_a e_A e_B \underline{x}_{a\triangle A\triangle B} \underline{D}$$
$$- (-1)^{|B|+|A|-1}\underline{D}\,\underline{x}_{a\triangle A\triangle B} e_a e_A e_B \underline{D}\,x_a - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_{a\triangle A\triangle B} \underline{D} x_a \underline{D}\Big).$$

In the last step we used the following (and a similar result for the other two terms)

$$\sum_{a\in A}\underline{D}\,x_a\underline{D}\sum_{l\in A\setminus\{a\}} x_l e_l e_a e_A e_B - (-1)^{|B|+|A|-1}\sum_{a\in A}\underline{D}\sum_{l\in A\setminus\{a\}} x_l e_l e_a e_A e_B \underline{D}\,x_a$$
$$= \sum_{a\in A}\sum_{l\in A\setminus\{a\}}\underline{D}\,x_a\underline{D}\,x_l e_l e_a e_A e_B + \sum_{a\in A}\sum_{l\in A\setminus\{a\}}\underline{D}\,x_l\underline{D}\,e_l e_a e_A e_B x_a$$

$$
\begin{aligned}
&- \sum_{a\in A} \sum_{l\in A\setminus\{a\}} \underline{D}\, x_l \sum_{b\in A\setminus\{a,l\}\cup B} 2p_b e_b e_l e_a e_A e_B x_a \\
&= -\sum_{a\in A} \sum_{l\in A\setminus\{a\}} \underline{D}\, x_l \sum_{b\in A\setminus\{a,l\}} 2p_b e_b e_l e_a e_A e_B x_a - \sum_{a\in A} \sum_{l\in A\setminus\{a\}} \underline{D}\, x_l \sum_{b\in B} 2p_b e_b e_l e_a e_A e_B x_a \\
&= -\sum_{\{a,l,b\}\subset A} 2\underline{D}\,(x_l p_b x_a - x_a p_b x_l + x_b p_a x_l - x_b p_l x_a + x_a p_l x_b - x_l p_a x_b) e_b e_l e_a e_A e_B \\
&\quad - \sum_{\{a,l\}\subset A} \sum_{b\in B} 2\underline{D}\,(x_l p_b x_a - x_a p_b x_l) e_b e_l e_a e_A e_B \\
&= -\sum_{\{a,l\}\subset A} \sum_{b\in B} 2\underline{D}\,(x_l(p_b x_a - x_a p_b) - x_a(p_b x_l - x_l p_b)) e_b e_l e_a e_A e_B \\
&= -\sum_{\{a,l\}\subset A} \sum_{b\in B} 2\underline{D}\,(x_l(p_a x_b - x_b p_a) - x_a(p_l x_b - x_b p_l)) e_b e_l e_a e_A e_B \\
&= -\sum_{\{a,l\}\subset A} \sum_{b\in B} 2\underline{D}\,(x_l p_a - x_a p_l) x_b e_b e_l e_a e_A e_B - \sum_{\{a,l\}\subset A} \sum_{b\in B} x_b (x_a p_l - x_l p_a) e_b e_l e_a e_A e_B \\
&= \sum_{a\in A} \sum_{l\in A\setminus\{a\}} \underline{D}\, x_a 2p_l \underline{x}_B e_l e_a e_A e_B - \sum_{a\in A} \sum_{l\in A\setminus\{a\}} \underline{D}\, \underline{x}_B 2p_l x_a e_l e_a e_A e_B .
\end{aligned}
$$

To arrive at this result we made use of property (3.7), Lemma 2.4, the commutativity of $x_1, \ldots, x_N$ and $C_{ij} = C_{ji}$.

Putting everything together and comparing with

$$
\begin{aligned}
&\sum_{a\in A} [\![O_a, O_{a,A,B}]\!]_- \\
&= \sum_{a\in A} \frac{\epsilon}{4}\Big((\underline{D}\, x_a - x_a\underline{D} - e_a)(\underline{D}\, \underline{x}_{a\triangle A\triangle B} e_a e_A e_B - e_a e_A e_B \underline{x}_{a\triangle A\triangle B}\underline{D} - \epsilon e_a e_A e_B) \\
&\quad - (-1)^{|A|+|B|-1}(\underline{D}\, \underline{x}_{a\triangle A\triangle B} e_a e_A e_B - e_a e_A e_B \underline{x}_{a\triangle A\triangle B}\underline{D} - \epsilon e_a e_A e_B)(\underline{D}\, x_a - x_a\underline{D} - e_a)\Big),
\end{aligned}
$$

the proof is completed. □

THEOREM 3.15: *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for two lists of $\{1, \ldots, N\}$, denoted by A and B, such that $A \subset B$ as sets, we have*

$$
[\![O_A, O_B]\!]_- = \epsilon \sum_{a\in A} [\![O_a, O_{a,A,B}]\!]_- .
$$

Note that in this case $a\triangle A\triangle B = \{a\} \cup (B \setminus A)$.

Proof. To prove this result, one is not limited to just the form (3.10) for O_A and O_B as we did in the proof of Theorem 3.14. One may also use for instance the form (3.14), and employ a strategy similar to the one used in the proof of Theorem 3.12. A proof in this style can be found in the Appendix A. □

THEOREM 3.16: *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for two lists of $\{1, \ldots, N\}$, denoted by A and B, we have*

$$[\![O_A, O_B]\!]_+ = \epsilon O_{A,B} + (e_{A\cap B})^2 [\![O_{A,(A\cap B)}, O_{(A\cap B),B}]\!]_+ + (e_{A\cap B})^2 [\![O_{A\cap B}, O_{(A\cap B),A,B}]\!]_+ .$$

Note that $A\triangle B = (A \setminus B)\cup(B \setminus A)$, while $A\triangle(A \cap B) = A \setminus B$ and $(A \cap B)\triangle B = B \setminus A$, and finally $(A \cap B)\triangle A\triangle B = A \cup B$.

Proof. Because of its length and as it employs a similar strategy as used already in the proof of Theorem 3.14, we have moved the proof of this result to the Appendix A. □

COROLLARY 3.17: *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for two lists of $\{1, \ldots, N\}$, denoted by A and B, we have*

$$\begin{aligned}[\![O_A, O_B]\!]_+ = \epsilon O_{A,B} &+ (e_{A\cap B})^2 2O_{A,(A\cap B)}O_{(A\cap B),B} + (e_{A\cap B})^2 2O_{A\cap B}O_{(A\cap B),A,B} \\ &- \epsilon (e_{A\cap B})^2 \sum_{a\in A\setminus B} [\![O_a, O_{(A\cap B),B}]\!]_- - \epsilon (e_{A\cap B})^2 \sum_{a\in A\cap B} [\![O_a, O_{(A\cap B),A,B}]\!]_- .\end{aligned}$$

Proof. Note first that $[\![O_A, O_B]\!]_- + [\![O_A, O_B]\!]_+ = 2O_AO_B$. Now, combine Theorem 3.16 with Theorem 3.14 and Theorem 3.15, using $(A \setminus B)\cap(B \setminus A)= \emptyset$ and $(A \cap B)\subset (A \cup B)$. □

By means of Theorem 3.14, Theorem 3.15 and Theorem 3.16 (or thus Corollary 3.17) we can swap any two operators O_A and O_B where A or B is not a list of just one element. We briefly explain the need for three such theorems. Theorem 3.16 yields an empty identity in two cases, when $A \cap B = \emptyset$ or when either A or B is contained in the other as sets. For example, say $A \cap B = \emptyset$, then we have

$$\begin{aligned}[\![O_A, O_B]\!]_+ &= \epsilon O_{A,B} + (e_{A\cap B})^2 [\![O_{A,(A\cap B)}, O_{(A\cap B),B}]\!]_+ + (e_{A\cap B})^2 [\![O_{A\cap B}, O_{(A\cap B),A,B}]\!]_+ \\ &= \epsilon O_{A,B} + [\![O_A, O_B]\!]_+ + O_\emptyset O_{A,B} + O_{A,B}O_\emptyset \\ &= [\![O_A, O_B]\!]_+ ,\end{aligned}$$

as $e_\emptyset = 1$ and $O_\emptyset = -\epsilon/2$. For these cases we can resort to Theorem 3.14 or Theorem 3.15. However, if A is a list of a single element a, Theorem 3.14 and Theorem 3.15 are empty identities:

$$[\![O_A, O_B]\!]_- = \epsilon \sum_{a\in A} [\![O_a, O_{a,A,B}]\!]_- = \epsilon [\![O_a, O_{a,a,B}]\!]_- = [\![O_a, O_B]\!]_-,$$

but so is Theorem 3.16 as either $a \in B$ or $a\cap B = \emptyset$. Now, for the case $a \notin B$ Theorem 3.13 yields

$$O_aO_B = (-1)^{|B|}O_BO_a - \epsilon \sum_{b\in B} [\![O_b, O_{b,a,B}]\!]_- ,$$

while by definition we also have

$$O_aO_B = \pm(-1)^{|B|-|a\cap B|}O_BO_a + [\![O_a, O_B]\!]_\mp .$$

We see that all expressions involving supercommutators can be reduced to sums of supercommutators containing (at least) one symmetry with just one index. In the following section we review again examples of specific realizations of the elements $x_1, \dots, x_N$ and $p_1, \dots, p_N$ of the algebra $\mathcal{A}$. For these examples, the one index symmetries in particular take on an explicit form whose interaction with other symmetries can be computed explicitly. This form depends on the makeup of the symmetries $C_{ij} = [p_i, x_j]$ as O_i is given by (3.8).

4 EXAMPLES

We recall the two examples from the previous section.
EXAMPLE 4.1: Let again Δ be the classical Laplace operator, then

$$\underline{D} = \sum_{i=1}^{N} e_i p_i = \sum_{i=1}^{N} e_i \frac{\partial}{\partial x_i}$$

is the classical Dirac operator. The $\mathfrak{osp}(1|2)$ structure of Theorem 3.4 here was obtained already in [15]. The commutator C_{li} in the definition (3.8) of O_i reduces to $[p_i, x_j] = \delta_{ij}$, so

$$O_i = \epsilon \frac{1}{2} \left(\sum_{l=1}^{N} e_l \delta_{li} - e_i \right) = \epsilon \frac{1}{2} (e_i - e_i) = 0.$$

Moreover, (3.13) then becomes

$$O_{ij} = L_{ij} + \epsilon \frac{1}{2} e_i e_j,$$

in accordance with results obtained in [8], while for a general subset $A \subset \{1, \dots, N\}$ one has

$$O_A = \left(\epsilon \frac{|A| - 1}{2} - \sum_{\{i,j\} \subset A} L_{ij} e_i e_j \right) e_A.$$

Since $[p_i, x_j] = 0$ for $i \neq j$, given a subset $A \subset \{1, \dots, N\}$ the operators $\underline{x}_A$ and $\underline{D}_A$ as defined by (3.9) in fact also generate a copy of the Lie superalgebra $\mathfrak{osp}(1|2)$ whose Scasimir element we will denote by

$$\mathcal{S}_A = \frac{1}{2} \left([\underline{D}_A, \underline{x}_A] - \epsilon \right).$$

From (3.10), we see that in this case O_A equals $\mathcal{S}_A e_A$.

As in this case the one-index symmetries are identically zero, the algebraic relations simplify accordingly. The symmetries with two indices generate a realization of the Lie algebra $\mathfrak{so}(N)$, as seen from the relations of Theorem 3.12. For general symmetries Theorem 3.16 now yields

$$[\![O_A, O_B]\!]_+ = \epsilon O_{A,B} + (e_{A\cap B})^2 2 O_{A,(A\cap B)} O_{(A\cap B),B} + (e_{A\cap B})^2 2 O_{A\cap B} O_{(A\cap B),A,B},$$

as by Theorem 3.14 and Theorem 3.15 $[\![O_A, O_B]\!]_- = 0$ for $A \cap B = \emptyset$, or $A \subset B$, or $B \subset A$. This corresponds to (a special case of) the higher rank Bannai-Ito algebra of [7], which strictly speaking is not included in the results obtained there.

To conclude, we mention another group of symmetries of $\underline{D}$ and $\underline{x}$ which are not inside the algebra $\mathcal{A} \otimes \mathcal{C}$ in this case, but which will also be useful for the next example. For a normed vector α, its embedding in the Clifford algebra

$$\underline{\alpha} = \sum_{l=1}^{N} e_l \alpha_l$$

is an element of the so-called Pin group, which forms a double cover of the orthogonal group $\mathrm{O}(N)$. These elements have the property that

$$\underline{\alpha}\, \underline{v} = -\underline{w}\, \underline{\alpha},$$

where $w = \sigma_\alpha(v) = v - 2\langle v, \alpha\rangle \alpha / \|\alpha\|^2$. Hence, if we define the operators S_α as follows

$$S_\alpha = \sum_{l=1}^{N} e_l \alpha_l \sigma_\alpha = \underline{\alpha}\, \sigma_\alpha, \tag{4.1}$$

then it is immediately clear that they satisfy the following properties

$$S_\alpha \underline{v} = -\underline{w} S_\alpha, \qquad S_\alpha f(x) = f(\sigma_\alpha(x)) S_\alpha, \qquad (S_\alpha)^2 = \epsilon.$$

where again $w = \sigma_\alpha(v)$ and f is a function or operator which does not interact with the Clifford generators. By direct computation, one can show that the operator S_α anti-commutes with the Dirac operator. Moreover, the interaction of S_α and a symmetry operator O_A is simply given by the action of the reflection associated to α on the coordinate vectors corresponding to the elements of A.

EXAMPLE 4.2: We consider again the case where $p_1, \dots, p_N$ are given by the Dunkl operators (2.9) associated to a given root system R and with multiplicity function k. Here, the $\mathfrak{osp}(1|2)$ structure of Theorem 3.4 was obtained already in [5, 19].

The commutator in the definition (3.8) of O_i is then given by

$$C_{ij} = [\mathcal{D}_i, x_j] = \delta_{ij} + \sum_{\alpha \in R_+} 2k(\alpha) \alpha_i \alpha_j \sigma_\alpha.$$

The symmetries of the Dunkl Dirac operator $\sum_{i=1}^{N} \mathcal{D}_i e_i$ with one index thus become

$$O_i = \epsilon \frac{1}{2} \left(\sum_{l=1}^{N} e_l \delta_{li} + \sum_{l=1}^{N} e_l \sum_{\alpha \in R_+} 2k(\alpha) \alpha_l \alpha_i \sigma_\alpha - e_i \right) = \epsilon \sum_{\alpha \in R_+} k(\alpha) \alpha_i \sum_{l=1}^{N} e_l \alpha_l \sigma_\alpha.$$

On the right-hand side we see the operators (4.1) appear for the roots $\alpha \in R_+$. By direct computation, one can show that for a root α, the operator S_α anti-commutes with the

Dirac-Dunkl operator. The one-index symmetry O_i thus consist of linear combinations of the operators (4.1) determined by the root system and by the multiplicity function k

$$O_i = \epsilon \sum_{\alpha \in R_+} k(\alpha)\alpha_i S_\alpha .$$

Higher-index symmetries O_A contain the Dunkl angular momentum operators, appended with the anti-commuting symmetries S_α for $\alpha \in R_+$. For instance, if $A = \{i, j\}$ we have

$$O_{ij} = x_i \mathcal{D}_j - x_j \mathcal{D}_i + \epsilon \frac{1}{2} e_i e_j + \sum_{\alpha \in R_+} k(\alpha)(e_i \alpha_j - e_j \alpha_i) S_\alpha .$$

The algebraic relations of Theorem 3.14 and Theorem 3.15 can now be worked out explicitly as the action of the one-index symmetries is given by the reflections associated to the roots of the root system.

EXAMPLE 4.2.1: For the root system with Weyl group $W = (\mathbb{Z}_2)^N$, our results are in accordance with what was already obtained in [6, 7]. Here, the Dunkl operators (2.9) are given by

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \frac{\mu_i}{x_i}(1 - r_i) \qquad i \in \{1, \ldots, N\}$$

where r_i is the reflection operator in the $x_i = 0$ hyperplane and μ_i is the value of the multiplicity function on the conjugacy class of r_i. The one-index symmetry (3.8) in this case reduces to

$$O_i = \epsilon \sum_{\alpha \in R_+} k(\alpha)\alpha_i S_\alpha = \epsilon \mu_i r_i e_i .$$

Here, we also have $[p_i, x_j] = 0$ for $i \neq j$, so for a given subset $A \subset \{1, \ldots, N\}$ the operators $\underline{x}_A$ and $\underline{D}_A$ as defined by (3.9) generate a copy of the Lie superalgebra $\mathfrak{osp}(1|2)$ with the Scasimir element

$$\mathcal{S}_A = \frac{1}{2}\left([\underline{D}_A, \underline{x}_A] - \epsilon\right) .$$

From (3.10), we see that in this case O_A equals $\mathcal{S}_A e_A$. The relation with the symmetries denoted by Γ_A in [7] is

$$\Gamma_A = \mathcal{S}_A \prod_{i \in A} r_i = \mathcal{S}_A \prod_{i \in A} \frac{1}{\mu_i} O_i e_i = \mathcal{S}_A e_A \prod_{i \in A} \frac{1}{\mu_i} O_i = O_A \prod_{i \in A} \frac{1}{\mu_i} O_i ,$$

where the product over $i \in A$ is taken according to the order of A. The operator Γ_A commutes with the Dunkl Dirac operator by Corollary 3.9. The algebraic structure generated by the operators Γ_A and corresponding to the relations of Theorem 3.16 is the higher rank Bannai-Ito algebra of [7]. For the case $N = 3$, see [6, 13], this is the regular Bannai-Ito algebra [25].

EXAMPLE 4.2.2: For the root system of type A_{N-1}, with positive subsystem given, for instance, by

$$R_+ = \left\{ \frac{1}{\sqrt{2}}(\xi_i - \xi_j) \,\middle|\, 1 \leq i < j \leq N \right\}$$

where $\{\xi_1, \dots, \xi_N\}$ is an orthonormal basis of $\mathbb{R}^N$, the associated Weyl group is the symmetric group S_N of permutations on N elements. All permutations in S_N are conjugate so the multiplicity function $k(\alpha)$ has the same value for all roots, which we will denote by κ. The Dunkl operators (2.9) are then given by

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \kappa \sum_{j \neq i} \frac{1 - g_{ij}}{x_i - x_j} \qquad i \in \{1, \dots, N\}$$

where g_{ij} denotes the reflection corresponding to the root $1/\sqrt{2}(\xi_i - \xi_j)$. The related operator of the form (4.1) will be denoted as

$$G_{ij} = \frac{1}{\sqrt{2}}(e_i - e_j) g_{ij} = -G_{ji}.$$

In this case, the commutator $[p_i, x_j]$ does not reduce to zero for $i \neq j$. It is given by

$$[\mathcal{D}_i, x_j] = \begin{cases} 1 + \kappa \sum\limits_{l \neq i} g_{il} & \text{if } i = j \\ -\kappa g_{ij} & \text{if } i \neq j \end{cases}$$

and the one-index symmetry (3.8) becomes

$$O_i = \epsilon \sum_{\alpha \in R_+} k(\alpha) \alpha_i S_\alpha = \epsilon\kappa \sum_{1 \leq l < i} \frac{-1}{\sqrt{2}} G_{li} + \epsilon\kappa \sum_{i < l \leq N} \frac{1}{\sqrt{2}} G_{il} = \frac{\epsilon\kappa}{\sqrt{2}} \sum_{l=1}^{N} G_{il},$$

where $G_{ii} = 0$, while for the symmetry (3.13) we have

$$O_{ij} = x_i \mathcal{D}_j - x_j \mathcal{D}_i + \epsilon \frac{1}{2} e_i e_j + \frac{\epsilon\kappa}{\sqrt{2}} \sum_{l=1}^{N} (e_i G_{jl} - e_j G_{il}).$$

The relations for two-index symmetries of Theorem 3.12 are

$$[O_{ij}, O_{kl}] = \frac{\kappa}{\sqrt{2}} \big((O_{lij} - O_{lik}) G_{jk} + (O_{kji} - O_{kjl}) G_{il} + (O_{ikl} - O_{ikj}) G_{lj} + (O_{jlk} - O_{jli}) G_{ki} \big)$$

for four distinct indices, and when $l = i$ we have

$$[O_{ij}, O_{ki}] = O_{jk} + \frac{\kappa}{\sqrt{2}} 2 O_{ijk} (G_{ij} - G_{ki}) + \frac{\kappa}{\sqrt{2}} \sum_{a \neq i, j, k} (O_{ijk} - O_{ajk}) G_{ia} + \frac{\kappa^2}{2} \sum_{a=1}^{N} \sum_{b=1}^{N} [G_{ja}, G_{kb}].$$

5 SUMMARY AND OUTLOOK

The replacement of ordinary derivatives by Dunkl derivatives $\mathcal{D}_i$ in the expressions of the Laplace and the Dirac operator in N dimensions gives rise to the Laplace-Dunkl

Δ and Dirac-Dunkl operator $\underline{D}$. This paper was devoted to the study of the symmetry algebras of these two operators, i.e. to the algebraic relations satisfied by the operators commuting or anti-commuting with Δ or $\underline{D}$.

In the case of Dunkl derivatives, the underlying object is the reflection group G acting in N-dimensional space, characterized by a reduced root system. The Dunkl derivatives themselves then consist of an ordinary derivative plus a number of difference operators depending on this reflection group. So it can be expected that the reflection group G plays an essential role in the structure of the symmetry algebra.

One of the leading examples was for $N = 3$ and $G = \mathbb{Z}_2^3$. Even for this quite simple reflection group, the study of the symmetry algebras was already non-trivial, and led to the celebrated Bannai-Ito algebra [6, 13]. Following this, the second case where the symmetry algebra could be determined was for general N and $G = \mathbb{Z}_2^N$ [7], leading to a "higher rank Bannai-Ito algebra", of which the structure is already highly non-trivial.

The question that naturally arises is whether the symmetry algebras of the Laplace-Dunkl and Dirac-Dunkl operators for other reflection groups G can still be determined, and what their structure is. We considered it as a challenge to study this problem. Originally, we were hoping to solve the problem for the case $G = S_N$, which would already be a significant breakthrough. Herein, S_N is the symmetric group acting on the coordinates x_i by permuting the indices; as a reflection group it is associated with the root system of type A_{N-1}.

Our initial attempts and computations for the case $G = S_N$ were not promising, and the situation looked extremely complicated, particularly because the explicit actions of the Dunkl derivatives $\mathcal{D}_i$ are already very complex. Fortunately, at that moment we followed the advice "if you cannot solve the problem, generalize it." So we went back to the general case, with arbitrary reflection group G, and no longer focused on the explicit actions of the Dunkl derivatives $\mathcal{D}_i$, but on the algebraic relations among the coordinate operators x_i and the $\mathcal{D}_i$. Then we realized that we could still jump one level higher in the generalization, and just work with coordinate operators x_i and "momentum operators" p_i in the framework of a Wigner quantum system, by identifying the p_i with $\mathcal{D}_i$. As a consequence, we could forget about the actual meaning of the Dunkl derivatives, and just work and perform our computations in the associative algebra $\mathcal{A}$ (Definition 2.1). This general or "more abstract" setting enabled us to determine the elements (anti)commuting with Δ or $\underline{D}$, and to construct the algebraic relations satisfied by these elements. The resulting symmetry algebra, obtained in the paper, is still quite complicated. But we managed to determine it (for general G), going far beyond our initial goal. For the general Laplace-Dunkl operator, the symmetries and the symmetry algebra are described in Theorem 2.3 and Theorem 2.5. For the general Dirac-Dunkl operator, the symmetries are determined in Theorem 3.6 and Theorem 3.7. The relations for these symmetries (i.e. the symmetry algebra) are established in Section 3.2, and follow from Theorems 3.14, 3.15 and 3.16.

The results of this paper open the way to several new investigations. In particular, one could now go back to the interesting case $G = S_N$, and investigate how the symmetry algebra specializes. One of the purposes is to study representations of the symmetry algebra in that case, since this leads to null solutions of the Dirac operator. As in the case of $G = \mathbb{Z}_2^N$ [6, 13], one can expect that interesting families of orthogonal polynomials should arise. Furthermore, note that for the case of $G = \mathbb{Z}_2^3$ a superintegrable model on the two-sphere was obtained [4]. It is definitely worthwhile to investigate possible superintegrable systems for other groups G.

In a different direction, one can examine whether the context of Wigner quantum systems, as used here for rational Dunkl operators, is still of use for other types of operators. Possible examples are trigonometric Dunkl operators, or the Dunkl operators appearing in discrete function theory.

Finally, it is known that solutions of Wigner quantum systems with a Hamiltonian of type (1.1) can be described in terms of unitary representations of the Lie superalgebra $\mathfrak{osp}(1|2N)$ [24, 26]. The action and restriction of the coordinate operators x_i and the Dunkl operators $\mathcal{D}_i$ in these representations should be studied further.

A APPENDIX

This appendix contains the proofs of Theorem 3.15 and Theorem 3.16, which were omitted from the main text due to their length.

Proof of Theorem 3.15. We systematically go over every term appearing in $[\![O_A, O_B]\!]_-$, using now the form (3.14) for O_A and O_B, that is

$$\left[\!\!\left[\left(\epsilon\frac{|A|-1}{2} + \epsilon\sum_{a\in A} O_a e_a - \sum_{\{i,j\}\subset A} L_{ij}e_ie_j\right)e_A, \left(\epsilon\frac{|B|-1}{2} + \epsilon\sum_{b\in B} O_b e_b - \sum_{\{k,l\}\subset B} L_{kl}e_ke_l\right)e_B\right]\!\!\right]_- .$$

Starting with the terms which contain no O_i or L_{ij}, we have using property (3.17) and $A \subset B$

$$\frac{|A|-1}{2}\frac{|B|-1}{2}(e_Ae_B - (-1)^{|A||B|-|A\cap B|}e_Be_A) = \frac{|A|-1}{2}\frac{|B|-1}{2}(e_Ae_B - e_Ae_B) = 0.$$

Next, using property (3.17) and Lemma 3.10 we have

$$\begin{aligned}
\left[\!\!\left[e_A, \sum_{b\in B} O_b e_b e_B\right]\!\!\right]_- &= \sum_{b\in B} e_Ae_Be_bO_b - (-1)^{|A||B|-|A\cap B|}\sum_{b\in B} O_be_be_Be_A \\
&= \sum_{b\in B} e_Ae_Be_bO_b - \sum_{b\in B} O_be_be_Ae_B \\
&= \sum_{b\in A} e_Ae_Be_bO_b + \sum_{b\in B\setminus A} e_Ae_Be_bO_b - \sum_{b\in B\setminus A} e_Ae_Be_bO_b - \sum_{b\in A} O_be_be_Ae_B \\
&= \sum_{b\in A} e_Ae_Be_bO_b - \sum_{b\in A} O_be_be_Ae_B,
\end{aligned}$$

while

$$\left[\!\!\left[\sum_{a\in A} O_a e_a e_A, e_B\right]\!\!\right]_- = \sum_{a\in A} O_ae_ae_Ae_B - (-1)^{|A||B|-|A\cap B|}\sum_{a\in A} e_Be_Ae_aO_a$$

$$= \sum_{a\in A} O_a e_a e_A e_B - \sum_{a\in A} e_A e_B e_a O_a.$$

Hence, we have

$$\begin{aligned}
&\frac{|A|-1}{2}\left[\!\left[e_A, \sum_{b\in B} O_b e_b e_B\right]\!\right]_- + \frac{|B|-1}{2}\left[\!\left[\sum_{a\in A} O_a e_a e_A, e_B\right]\!\right]_- \\
&= \frac{|B|-1-(|A|-1)}{2}\left(\sum_{a\in A} O_a e_a e_A e_B - \sum_{a\in A} e_A e_B e_a O_a\right) \\
&= \frac{|B|-|A|+1-1}{2}\sum_{a\in A}\left(O_a e_a e_A e_B - (-1)^{|B|-|A|} e_a e_A e_B O_a\right) \\
&= \sum_{a\in A}\left[\!\left[O_a, \frac{|B|-|A|+1-1}{2} e_a e_A e_B\right]\!\right]_-.
\end{aligned}$$

For the next part, using the notation $A' = A \setminus \{a\}$, we find

$$\begin{aligned}
\left[\!\left[\sum_{a\in A} O_a e_a e_A, \sum_{b\in B} O_b e_b e_B\right]\!\right]_- &= \sum_{a\in A}\sum_{b\in B}(O_a e_a e_A e_B e_b O_b - (-1)^{|A||B|-|A\cap B|} O_b e_b e_B e_A e_a O_a) \\
&= \sum_{a\in A}\sum_{b\in B}(O_a e_a e_A e_B e_b O_b - O_b e_b e_A e_B e_a O_a) \\
&= \sum_{a\in A}\left[\!\left[O_a, \sum_{b\in B\setminus A'} O_b e_b e_a e_A e_B\right]\!\right]_- \\
&\quad + \sum_{a\in A}\sum_{b\in A'} O_a e_a e_A e_B e_b O_b - \sum_{a\in A}\sum_{b\in A'} O_b e_b e_A e_B e_a O_a.
\end{aligned}$$

One easily sees that the summations in the last line cancel out.

Now, for the parts containing "L_{ij}-terms", we have

$$\left[\!\left[\sum_{\{i,j\}\subset A} L_{ij} e_i e_j e_A, e_B\right]\!\right]_- = \sum_{\{i,j\}\subset A} L_{ij}\left(e_i e_j e_A e_B - (-1)^{|A||B|-|A\cap B|} e_B e_i e_j e_A\right) = 0,$$

as, using property (3.17) and $A \subset B$,

$$\begin{aligned}
(-1)^{|A||B|-|A\cap B|} e_B e_i e_j e_A &= (-1)^{|A||B|-|A\cap B|+|A\setminus\{i,j\}||B|-|(A\setminus\{i,j\})\cap B|} e_i e_j e_A e_B \\
&= e_i e_j e_A e_B.
\end{aligned}$$

Moreover, using Lemma 3.10 and property (3.17) while denoting $A' = A \setminus \{a\}$, we have

$$\left[\!\left[\epsilon \sum_{a\in A} O_a e_a e_A, -\sum_{\{k,l\}\subset B} L_{kl} e_k e_l e_B\right]\!\right]_-$$

$$
\begin{aligned}
&= -\epsilon \sum_{a\in A} O_a \sum_{\{k,l\}\subset B\setminus A'} L_{kl}e_ke_le_ae_Ae_B + \epsilon \sum_{a\in A} \sum_{\{k,l\}\subset B\setminus A'} L_{kl}e_ke_le_Ae_Be_aO_a \\
&\quad -\epsilon \sum_{a\in A} O_a \sum_{\{k,l\}\subset A'} L_{kl}e_ke_le_ae_Ae_B + \epsilon \sum_{a\in A} \sum_{\{k,l\}\subset A'} L_{kl}e_ke_le_Ae_Be_aO_a \\
&\quad +\epsilon \sum_{a\in A} O_a \sum_{k\in A'} \sum_{l\in B\setminus A'} L_{kl}e_ke_le_ae_Ae_B + \epsilon \sum_{a\in A} \sum_{k\in A'} \sum_{l\in B\setminus A'} L_{kl}e_ke_le_Ae_Be_aO_a \\
&= -\epsilon \sum_{a\in A} \left[\!\left[O_a, \sum_{\{k,l\}\subset B\setminus A'} L_{kl}e_ke_le_ae_Ae_B \right]\!\right]_- \\
&\quad -\epsilon \sum_{a\in A} O_a \sum_{\{k,l\}\subset A'} L_{kl}e_ke_le_ae_Ae_B + (-1)^{|B|-|A|}\epsilon \sum_{a\in A} \sum_{\{k,l\}\subset A'} L_{kl}e_ke_le_ae_Ae_BO_a \qquad \text{(A.1)} \\
&\quad +\epsilon \sum_{a\in A} O_a \sum_{k\in A'} \sum_{l\in B\setminus A'} L_{kl}e_ke_le_ae_Ae_B + (-1)^{|B|-|A|}\epsilon \sum_{a\in A} \sum_{k\in A'} \sum_{l\in B\setminus A'} L_{kl}e_ke_le_ae_Ae_BO_a. \qquad \text{(A.2)}
\end{aligned}
$$

We see that the final part to make $\epsilon \sum_{a\in A} [\![O_a, O_{a\triangle A\triangle B}]\!]_-$ appears here. To complete the proof we show that the last two lines, (A.1) and (A.2), cancel out with the remaining terms of $[\![O_A, O_B]\!]_-$.

Hereto, we use Lemma 3.10, to find (denoting again $A' = A \setminus \{a\}$)

$$
\begin{aligned}
&\left[\!\left[-\sum_{\{i,j\}\subset A} L_{ij}e_ie_je_A, \epsilon \sum_{b\in B} O_be_be_B \right]\!\right]_- \\
&= -\epsilon \sum_{\{i,j\}\subset A} L_{ij}e_ie_je_Ae_B \sum_{a\in A\setminus\{i,j\}} e_aO_a - \epsilon \sum_{\{i,j\}\subset A} L_{ij}e_ie_je_Ae_B \sum_{b\in B\setminus(A\setminus\{i,j\})} e_bO_b \\
&\quad + (-1)^{|A||B|-|A\cap B|}\epsilon \sum_{\{i,j\}\subset A} \sum_{a\in A\setminus\{i,j\}} O_aL_{ij}e_ae_ie_je_Be_A \\
&\quad + (-1)^{|A||B|-|A\cap B|}\epsilon \sum_{\{i,j\}\subset A} \sum_{b\in B\setminus(A\setminus\{i,j\})} O_bL_{ij}e_be_ie_je_Be_A \\
&= -(-1)^{|B|-|A|}\epsilon \sum_{a\in A} \sum_{\{i,j\}\subset A'} L_{ij}e_ie_je_ae_Ae_BO_a + \epsilon \sum_{a\in A} \sum_{\{i,j\}\subset A'} O_aL_{ij}e_ie_je_ae_Ae_B \\
&\quad -\epsilon \sum_{\{i,j\}\subset A} L_{ij} \sum_{b\in B\setminus(A\setminus\{i,j\})} O_be_be_ie_je_Ae_B + \epsilon \sum_{\{i,j\}\subset A} \sum_{b\in B\setminus(A\setminus\{i,j\})} O_bL_{ij}e_be_ie_je_Ae_B \\
&= \epsilon \sum_{a\in A} O_a \sum_{\{i,j\}\subset A'} L_{ij}e_ie_je_ae_Ae_B - (-1)^{|B|-|A|}\epsilon \sum_{a\in A} \sum_{\{i,j\}\subset A'} L_{ij}e_ie_je_ae_Ae_BO_a \\
&\quad -\epsilon \sum_{\{i,j\}\subset A} \sum_{b\in B\setminus(A\setminus\{i,j\})} [L_{ij}, O_b]e_be_ie_je_Ae_B. \qquad \text{(A.3)}
\end{aligned}
$$

This already causes (A.1) to vanish, so (A.2) and (A.3) remain.

Next, we look at

$$\sum_{\{i,j\}\subset A}\sum_{\{k,l\}\subset B} L_{ij}L_{kl}e_ie_je_Ae_ke_le_B - (-1)^{|A||B|-|A\cap B|}L_{kl}L_{ij}e_ke_le_Be_ie_je_A. \tag{A.4}$$

According to the sign of

$$\begin{aligned} &(-1)^{|A||B|-|A\cap B|}e_ke_le_Be_ie_je_A \\ &= (-1)^{|A||B|-|A\cap B|+|A\setminus\{i,j\}||B\setminus\{k,l\}|-|(A\setminus\{i,j\})\cap(B\setminus\{k,l\})|}e_ie_je_Ae_ke_le_B \\ &= (-1)^{|A\cap B|-|(A\setminus\{i,j\})\cap(B\setminus\{k,l\})|}e_ie_je_Ae_ke_le_B, \end{aligned}$$

the summation (A.4) reduces to a combination of commutators and anti-commutators involving L_{ij} and L_{kl}. We first treat the anti-commutators

$$\begin{aligned} &-\sum_{k\in A}\sum_{\{i,j\}\subset A\setminus\{k\}}\sum_{l\in B\setminus A}\{L_{ij}, L_{kl}\}e_ie_je_ke_le_Ae_B \\ &-\sum_{i\in A}\sum_{\{j,k\}\subset A\setminus\{i\}}\epsilon(\{L_{ik}, L_{ji}\}e_ke_je_Ae_B + \{L_{ij}, L_{ki}\}e_je_ke_Ae_B). \end{aligned}$$

The last line reduces to

$$\sum_{i\in A}\sum_{\{j,k\}\subset A\setminus\{i\}}\epsilon(\{L_{ik}, L_{ji}\} - \{L_{ij}, L_{ki}\})e_ke_je_Ae_B = 0,$$

while the first line vanishes by Theorem 2.5 as it can be rewritten as

$$\begin{aligned} &\sum_{\{k,i,j\}\subset A}\sum_{l\in B\setminus A}(\{L_{ij}, L_{kl}\}e_ie_je_ke_le_Ae_B + \{L_{kj}, L_{il}\}e_ke_je_ie_le_Ae_B + \{L_{ik}, L_{jl}\}e_ie_ke_je_le_Ae_B) \\ = &\sum_{\{k,i,j\}\subset A}\sum_{l\in B\setminus A}(\{L_{ij}, L_{kl}\} + \{L_{jk}, L_{il}\} + \{L_{ki}, L_{jl}\})e_ie_je_ke_le_Ae_B. \end{aligned}$$

Next, we treat the remaining terms of the summation (A.4) which reduce to four different summations of commutators

$$\begin{aligned} &\sum_{\{i,j\}\subset A}\sum_{\{k,l\}\subset B\setminus A}[L_{ij}, L_{kl}]e_ie_je_ke_le_Ae_B + \sum_{\{i,j\}\subset A}\sum_{\{k,l\}\subset(B\cap A)\setminus\{i,j\}}[L_{ij}, L_{kl}]e_ie_je_ke_le_Ae_B \\ &-\sum_{i\in A}\sum_{j\in A\setminus\{i\}}\sum_{l\in B\setminus A}[L_{ij}, L_{jl}]e_ie_je_je_le_Ae_B + \sum_{\{i,j\}\subset A}[L_{ij}, L_{ij}]e_ie_je_ie_je_Ae_B. \end{aligned} \tag{A.5}$$

Note that while the last summation obviously vanishes, the second one does also as

$$\begin{aligned} &\sum_{\{i,j\}\subset A\cap B}\sum_{\{k,l\}\subset(B\cap A)\setminus\{i,j\}}[L_{ij}, L_{kl}]e_ie_je_ke_le_Ae_B \\ = &\sum_{\{i,j,k,l\}\subset A\cap B}([L_{ij}, L_{kl}] + [L_{ik}, L_{lj}] + [L_{il}, L_{jk}] + [L_{jk}, L_{il}] + [L_{jl}, L_{ki}] + [L_{kl}, L_{ij}])e_ie_je_ke_le_Ae_B. \end{aligned}$$

By Theorem 2.5 and using $\{e_i, O_j\} = [p_i, x_j] - \delta_{ij}$, the summand of the first summation of (A.5) can be written as

$$L_{jk}\{e_i, O_l\}e_je_ke_ie_le_Ae_B + L_{il}\{e_j, O_k\}e_ie_le_je_ke_Ae_B$$
$$L_{lj}\{e_k, O_i\}e_le_je_ie_ke_Ae_B + L_{ki}\{e_l, O_j\}e_ke_ie_je_le_Ae_B.$$

Using $\{e_j, O_k\} = \{e_k, O_j\}$ and $\{e_i, O_l\} = \{e_l, O_i\}$ each term is altered to one containing O_b where $b \in B \setminus A$. In this way, we find

$$\begin{aligned}&\epsilon(L_{jk}O_le_je_ke_le_Ae_B + L_{lj}O_ke_le_je_ke_Ae_B + L_{il}O_ke_ie_le_ke_Ae_B + L_{ki}O_le_ke_ie_le_Ae_B)\\ &+ L_{jk}e_iO_le_je_ke_ie_le_Ae_B + L_{lj}e_iO_ke_le_je_ie_ke_Ae_B + L_{il}e_jO_ke_ie_le_je_ke_Ae_B + L_{ki}e_jO_le_ke_ie_je_le_Ae_B.\end{aligned}$$

The summation, as in (A.5), of the first line reduces to

$$\begin{aligned}&\epsilon\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{l\in(B\setminus A)\setminus\{k\}}\sum_{i\in A\setminus\{j\}} L_{jk}O_le_je_ke_le_Ae_B\\ &= \epsilon(|A|-1)\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{l\in(B\setminus A)\setminus\{k\}} L_{jk}O_le_je_ke_le_Ae_B,\end{aligned}$$

while, using Lemma 3.10 for the last line, we find

$$\begin{aligned}&-\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}}\sum_{l\in(B\setminus A)\setminus\{k\}} L_{jk}e_iO_le_le_ie_je_ke_Ae_B\\
&= \sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}} L_{jk}e_iO_ie_ie_ie_je_ke_Ae_B + \sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}} L_{jk}e_iO_je_je_ie_je_ke_Ae_B\\
&\quad -\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}}\sum_{l\in\{i,j\}\cup(B\setminus A)\setminus\{k\}} L_{jk}e_ie_ie_je_ke_Ae_Be_lO_l\\
&= \epsilon\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}} L_{jk}e_i(O_ie_j - O_je_i)e_ke_Ae_B - \epsilon\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}} L_{jk}e_je_ke_Ae_Be_iO_i\\
&\quad -\epsilon(|A|-1)\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{l\in\{j\}\cup(B\setminus A)\setminus\{k\}} L_{jk}O_le_le_je_ke_Ae_B\\
&= -\epsilon\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}} L_{jk}e_ie_jO_ie_ke_Ae_B - \epsilon(-1)^{|B|-|A|}\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}} L_{jk}e_je_ke_ie_Ae_BO_i\\
&\quad -\epsilon(|A|-1)\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{l\in(B\setminus A)\setminus\{k\}} L_{jk}O_le_le_je_ke_Ae_B.\end{aligned}$$

In total, the first summation of (A.5) thus yields

$$-\epsilon\sum_{j\in A}\sum_{k\in B\setminus A}\sum_{i\in A\setminus\{j\}} L_{jk}e_ie_jO_ie_ke_Ae_B - (-1)^{|B|-|A|}\epsilon\sum_{j\in A}\sum_{k\in B}\sum_{i\in A\setminus\{j\}} L_{jk}e_je_ke_ie_Ae_BO_i. \quad (A.6)$$

The final result is obtained following essentially the same strategy as used in the proof of Theorem 3.12 and Theorem 3.14, now applied to the third summation of (A.5):

$$-\epsilon\sum_{i\in A}\sum_{j\in A\setminus\{i\}}\sum_{l\in B\setminus A}[L_{ij}, L_{jl}]e_ie_le_Ae_B.$$

By Theorem 2.5, and using $\{e_i, O_j\} = [p_i, x_j] - \delta_{ij}$, the summand of this can be written as

$$-\epsilon L_{il}e_ie_le_Ae_B - \epsilon L_{il}\{e_j, O_j\}e_ie_le_Ae_B + \epsilon L_{ij}\{e_j, O_l\}e_ie_le_Ae_B - \epsilon L_{lj}\{e_i, O_j\}e_ie_le_Ae_B. \tag{A.7}$$

Here, the summation of the first term cancels out with

$$\left[\!\!\left[\epsilon\frac{|A|-1}{2}e_A, -\sum_{\{k,l\}\subset B} L_{kl}e_ke_le_B\right]\!\!\right]_- = \epsilon(|A|-1)\sum_{k\in A}\sum_{l\in B\setminus A} L_{kl}e_ke_le_Ae_B,$$

which, using property (3.17) and $A \subset B$, follows from

$$\begin{aligned}(-1)^{|A||B|-|A\cap B|}e_ke_le_Be_A &= (-1)^{|A||B|-|A|+|A||B\setminus\{k,l\}|-|A\cap(B\setminus\{k,l\})|}e_Ae_ke_le_B\\ &= (-1)^{|A|-|A\cap(B\setminus\{k,l\})|}e_Ae_ke_le_B.\end{aligned}$$

For the summation of the second and the fourth term of (A.7), using $O_ie_j - O_je_i = e_iO_j - e_jO_i$, we have

$$\begin{aligned}&-\sum_{i\in A}\sum_{j\in A\setminus\{i\}}\sum_{l\in B\setminus A}(\epsilon L_{il}O_je_je_i + L_{lj}O_j)e_le_Ae_B - \epsilon\sum_{i\in A}\sum_{j\in A\setminus\{i\}}\sum_{l\in B\setminus A}(L_{il}e_j + L_{lj}e_i)O_je_ie_le_Ae_B\\
&= -\sum_{\{i,j\}\in A}\sum_{l\in B\setminus A}(\epsilon L_{il}O_je_je_i + L_{lj}O_j + \epsilon L_{jl}O_ie_ie_j + L_{li}O_i)e_le_Ae_B\\
&\quad -\epsilon\sum_{\{i,j\}\in A}\sum_{l\in B\setminus A}(L_{il}e_j + L_{lj}e_i)(O_je_i - O_ie_j)e_le_Ae_B.\\
&= -\epsilon\sum_{\{i,j\}\in A}\sum_{l\in B\setminus A}(L_{il}O_je_je_i - L_{lj}O_ie_ie_j - L_{il}e_je_iO_j + L_{lj}e_ie_jO_i)e_le_Ae_B\\
&= \epsilon\sum_{i\in A}\sum_{j\in A\setminus\{i\}}\sum_{l\in B\setminus A} L_{lj}(O_ie_ie_j - e_ie_jO_i)e_le_Ae_B.\end{aligned} \tag{A.8}$$

When summed over l in $B \setminus A$, the third term of (A.7) yields, using Lemma 3.10,

$$\begin{aligned}&-\epsilon\sum_{l\in B\setminus A} L_{ij}e_jO_le_le_ie_Ae_B + \epsilon\sum_{l\in B\setminus A} L_{ij}O_le_le_je_ie_Ae_B\\
&= \epsilon L_{ij}e_jO_ie_ie_ie_Ae_B - \epsilon\sum_{l\in B\setminus A\cup\{i\}} L_{ij}e_je_ie_Ae_Be_lO_l\\
&\quad -\epsilon L_{ij}O_ie_ie_je_ie_Ae_B - \epsilon L_{ij}O_je_je_je_ie_Ae_B + \epsilon\sum_{l\in B\setminus A\cup\{i,j\}} L_{ij}e_je_ie_Ae_Be_lO_l\\
&= L_{ij}e_iO_je_Ae_B + (-1)^{|B|-|A|+1}L_{ij}e_ie_Ae_BO_j.\end{aligned}$$

Summing this over i in A and j in $A \setminus \{i\}$ and combined with (A.2), (A.3), (A.6), and (A.8), one observes that all terms cancel out.

□

Proof of Theorem 3.16. By definition, plugging in (3.10), the left-hand side expands to

$$\begin{aligned}[\![O_A, O_B]\!]_+ &= O_A O_B + (-1)^{|A||B|-|A\cap B|} O_B O_A \\ &= \frac{1}{4}\big(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A\big)\big(\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D} - \epsilon e_B\big) \\ &\quad + (-1)^{|A||B|-|A\cap B|}\frac{1}{4}\big(\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D} - \epsilon e_B\big)\big(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A\big).\end{aligned}$$

The idea of the proof is now as follows. We split up $\underline{x}_A$ and $\underline{x}_B$ into $\underline{x}_A = \underline{x}_{A\setminus B} + \underline{x}_{A\cap B}$ and $\underline{x}_B = \underline{x}_{B\setminus A} + \underline{x}_{A\cap B}$. We then combine the appropriate terms to make $\underline{x}_{A\triangle B} = \underline{x}_{A\setminus B} + \underline{x}_{B\setminus A}$ and $\underline{x}_{A\cup B} = \underline{x}_{A\setminus B} + \underline{x}_{B\setminus A} + \underline{x}_{A\cap B}$, and in turn all terms that make up the right-hand side. In doing so, we continually make use of the following facts: by property (3.18) we have $(e_{A\cap B})^4 = 1$, hence $(e_{A\cap B})^2$ is just a sign;

$$\underline{D} e_A - (-1)^{|A|} e_A \underline{D} = \sum_{a\in A} 2 p_a e_a e_A\,;$$

and for integer n one has $(-1)^n = (-1)^{n^2}$ and $(-1)^{n(n+1)} = 1$.

For the terms of $[\![O_A, O_B]\!]_+$ which do not contain $\underline{D}$, we have

$$\frac{1}{4} e_A e_B + (-1)^{|A||B|-|A\cap B|}\frac{1}{4} e_B e_A = \frac{1}{2} e_A e_B = -\frac{1}{2} e_A e_B + \frac{1}{2}(e_{A\cap B})^4 e_A e_B + \frac{1}{2}(e_{A\cap B})^4 e_A e_B.$$

Next, for the terms of $[\![O_A, O_B]\!]_+$ containing two occurrences of $\underline{D}$, we have

$$\begin{aligned}&\big(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D}\big)\big(\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D}\big) \\ &\quad + (-1)^{|A||B|-|A\cap B|}\big(\underline{D}\,\underline{x}_B e_B - e_B \underline{x}_B \underline{D}\big)\big(\underline{D}\,\underline{x}_A e_A - e_A \underline{x}_A \underline{D}\big) \\ &= \underline{D}\,\underline{x}_A e_A \underline{D}\,\underline{x}_B e_B - e_A \underline{x}_A \underline{D}\underline{D}\,\underline{x}_B e_B - \underline{D}\,\underline{x}_A e_A e_B \underline{x}_B \underline{D} + e_A \underline{x}_A \underline{D} e_B \underline{x}_B \underline{D} \\ &\quad + (-1)^{|A||B|-|A\cap B|}\big(\underline{D}\,\underline{x}_B e_B \underline{D}\,\underline{x}_A e_A - e_B \underline{x}_B \underline{D}\underline{D}\,\underline{x}_A e_A - \underline{D}\,\underline{x}_B e_B e_A \underline{x}_A \underline{D} + e_B \underline{x}_B \underline{D} e_A \underline{x}_A \underline{D}\big).\end{aligned}$$

We first look at the terms having $\underline{DD}$ in the middle:

$$\begin{aligned}& - e_A \underline{x}_A \underline{DD}\,\underline{x}_B e_B - (-1)^{|A||B|-|A\cap B|} e_B \underline{x}_B \underline{DD}\,\underline{x}_A e_A \\ =\; & - e_A(\underline{x}_{A\setminus B} + \underline{x}_{A\cap B})\underline{DD}\,(\underline{x}_{B\setminus A} + \underline{x}_{A\cap B}) e_B \\ & - (-1)^{|A||B|-|A\cap B|} e_B (\underline{x}_{B\setminus A} + \underline{x}_{A\cap B})\underline{DD}\,(\underline{x}_{A\setminus B} + \underline{x}_{A\cap B}) e_A \\ =\; & -(e_{A\cap B})^2\big(e_A e_{A\cap B} \underline{x}_{A\setminus B} \underline{DD}\,\underline{x}_{B\setminus A} e_{A\cap B} e_B \\ & - (-1)^{|A||B|-|A\cap B|+|A\cap B|(|A|+|B|)} e_{A\cap B} e_B \underline{x}_{B\setminus A} \underline{DD}\,\underline{x}_{A\setminus B} e_A e_{A\cap B} \\ & - (-1)^{1+|A\cap B|(|A|+|B|-2|A\cap B|+1)+|A\cap B|-1} e_{A\cap B} e_A e_B \underline{x}_{A\setminus B} \underline{DD}\,\underline{x}_{A\cap B} e_{A\cap B} \\ & - e_{A\cap B} \underline{x}_{A\cap B} \underline{DD}\,\underline{x}_{B\setminus A} e_{A\cap B} e_A e_B - e_{A\cap B} \underline{x}_{A\cap B} \underline{DD}\,\underline{x}_{A\cap B} e_{A\cap B} e_A e_B \\ & - (-1)^{1+|A\cap B|(|A|+|B|-2|A\cap B|+1)+|A\cap B|-1} e_{A\cap B} e_A e_B \underline{x}_{B\setminus A} \underline{DD}\,\underline{x}_{A\cap B} e_{A\cap B} \\ & - e_{A\cap B} \underline{x}_{A\cap B} \underline{DD}\,\underline{x}_{A\setminus B} e_{A\cap B} e_A e_B\end{aligned}$$

$$
\begin{aligned}
&\quad - (-1)^{|A\cap B|(|A|+|B|-2|A\cap B|)} e_{A\cap B} e_A e_B \underline{x}_{A\cap B} \underline{D} \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B}\big) \\
= &\ - (e_{A\cap B})^2 \big(e_A e_{A\cap B} \underline{x}_{A\setminus B} \underline{D} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B \\
&\quad - (-1)^{(|A|-|A\cap B|)(|B|-|A\cap B|)} e_{A\cap B} e_B \underline{x}_{B\setminus A} \underline{D} \underline{D}\, \underline{x}_{A\setminus B} e_A e_{A\cap B} \\
&\quad - e_{A\cap B} \underline{x}_{A\cap B} \underline{D} \underline{D}\, \underline{x}_{A\cup B} e_{A\cap B} e_A e_B \\
&\quad - (-1)^{|A\cap B|(|A|+|B|-|A\cap B|)-|A\cap B|} e_{A\cap B} e_A e_B \underline{x}_{A\cup B} \underline{D} \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B}\big).
\end{aligned}
$$

In exactly the same manner, we have

$$
\begin{aligned}
&\underline{D}\, \underline{x}_A e_A e_B \underline{x}_B \underline{D} + (-1)^{|A||B|-|A\cap B|} \underline{D}\, \underline{x}_B e_B e_A \underline{x}_A \underline{D} \\
&= \underline{D}\, (\underline{x}_{A\setminus B} + \underline{x}_{A\cap B}) e_A e_B (\underline{x}_{B\setminus A} + \underline{x}_{A\cap B}) \underline{D} \\
&\quad + (-1)^{|A||B|-|A\cap B|} \underline{D}\, (\underline{x}_{B\setminus A} + \underline{x}_{A\cap B}) e_B e_A (\underline{x}_{A\setminus B} + \underline{x}_{A\cap B}) \underline{D} \\
&= (e_{A\cap B})^2 \big(\underline{D}\, \underline{x}_{A\setminus B} e_A (e_{A\cap B})^2 e_B \underline{x}_{B\setminus A} \underline{D} - \underline{D}\, \underline{x}_{A\cap B} (e_{A\cap B})^2 e_A e_B \underline{x}_{A\cup B} \underline{D} \\
&\quad - (-1)^{(|A|-|A\cap B|)(|B|-|A\cap B|)} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B e_A e_{A\cap B} \underline{x}_{A\setminus B} \underline{D} \\
&\quad - (-1)^{|A\cap B|(|A|+|B|-|A\cap B|)-|A\cap B|} \underline{D}\, \underline{x}_{A\cup B} e_{A\cap B} e_A e_B e_{A\cap B} \underline{x}_{A\cap B} \underline{D}\big).
\end{aligned}
$$

Next, for the remaining terms with two occurrences of $\underline{D}$, we first have

$$
\begin{aligned}
&\underline{D}\, \underline{x}_A e_A \underline{D}\, \underline{x}_B e_B + (-1)^{|A||B|-|A\cap B|} \underline{D}\, \underline{x}_B e_B \underline{D}\, \underline{x}_A e_A \\
&= \underline{D}\, (\underline{x}_{A\setminus B} + \underline{x}_{A\cap B}) e_A \underline{D}\, (\underline{x}_{B\setminus A} + \underline{x}_{A\cap B}) e_B \\
&\quad + (-1)^{|A||B|-|A\cap B|} \underline{D}\, (\underline{x}_{B\setminus A} + \underline{x}_{A\cap B}) e_B \underline{D}\, (\underline{x}_{A\setminus B} + \underline{x}_{A\cap B}) e_A \\
&= (e_{A\cap B})^2 \big(\underline{D}\, \underline{x}_{A\setminus B} e_A e_{A\cap B} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B \\
&\quad + (-1)^{|A\cap B|(|A|+|B|-|A\cap B|-1)} \underline{D}\, \underline{x}_{A\setminus B} e_{A\cap B} e_A e_B \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} \\
&\quad + \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_A e_B + \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} e_A e_B \\
&\quad + (-1)^{|A||B|-|A\cap B|+|A\cap B|(|B|+|A|)} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B \underline{D}\, \underline{x}_{A\setminus B} e_A e_{A\cap B} \\
&\quad + (-1)^{|A\cap B|(|A|+|B|)} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_A e_B \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} + \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} \underline{D}\, \underline{x}_{A\setminus B} e_{A\cap B} e_A e_B \\
&\quad + (-1)^{|A\cap B|(|A|+|B|)} \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} e_A e_B \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B}\big) \\
&\quad - \underline{D}\, \underline{x}_{A\setminus B} \sum_{l\in A\cap B} 2p_l e_l\, \underline{x}_{B\setminus A} e_A e_B - \underline{D}\, \underline{x}_{A\setminus B} \sum_{l\in B\setminus A} 2p_l e_l\, \underline{x}_{A\cap B} e_A e_B \\
&\quad - \underline{D}\, \underline{x}_{A\cap B} \sum_{l\in A\setminus B} 2p_l e_l\, \underline{x}_{B\setminus A} e_A e_B + \underline{D}\, \underline{x}_{A\cap B} \sum_{l\in A\setminus B} 2p_l e_l\, \underline{x}_{A\cap B} e_A e_B \\
&\quad - \underline{D}\, \underline{x}_{B\setminus A} \sum_{l\in A\cap B} 2p_l e_l\, \underline{x}_{A\setminus B} e_A e_B - \underline{D}\, \underline{x}_{B\setminus A} \sum_{l\in A\setminus B} 2p_l e_l\, \underline{x}_{A\cap B} e_A e_B \\
&\quad - \underline{D}\, \underline{x}_{A\cap B} \sum_{l\in B\setminus A} 2p_l e_l\, \underline{x}_{A\setminus B} e_A e_B - \underline{D}\, \underline{x}_{A\cap B} \sum_{l\in A\setminus B} 2p_l e_l\, \underline{x}_{A\cap B} e_A e_B \\
&= (e_{A\cap B})^2 \big(\underline{D}\, \underline{x}_{A\setminus B} e_A e_{A\cap B} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B + \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} \underline{D}\, \underline{x}_{A\cup B} e_{A\cap B} e_A e_B
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^{(|A|-|A\cap B|)(|B|-|A\cap B|)} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B \underline{D}\, \underline{x}_{A\setminus B} e_A e_{A\cap B} \\
&+ (-1)^{|A\cap B|(|A|+|B|-|A\cap B|)-|A\cap B|} \underline{D}\, \underline{x}_{A\cup B} e_{A\cap B} e_A e_B \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B}\big) \\
&- 2\underline{D} \sum_{a\in A\setminus B} \sum_{c\in A\cap B} \sum_{b\in B\setminus A} \big(x_b(p_c x_a - p_a x_c) + x_c(p_a x_b - p_b x_a) - x_a(p_c x_b - p_b x_c)\big) e_a e_b e_c e_A e_B
\end{aligned}
$$

where the last line vanishes using $L_{ij} = x_i p_j - x_j p_i = p_j x_i - p_i x_j$.
Similarly one finds

$$
\begin{aligned}
& e_A \underline{x}_A \underline{D} e_B \underline{x}_B \underline{D} + (-1)^{|A||B|-|A\cap B|} e_B \underline{x}_B \underline{D} e_A \underline{x}_A \underline{D} \\
&= e_A(\underline{x}_{A\setminus B} + \underline{x}_{A\cap B})\underline{D}\, e_B(\underline{x}_{B\setminus A} + \underline{x}_{A\cap B})\underline{D} \\
&\quad + (-1)^{|A||B|-|A\cap B|} e_B(\underline{x}_{B\setminus A} + \underline{x}_{A\cap B})\underline{D}\, e_A(\underline{x}_{A\setminus B} + \underline{x}_{A\cap B})\underline{D} \\
&= (e_{A\cap B})^2 \big(e_A e_{A\cap B} \underline{x}_{A\setminus B} \underline{D}\, e_{A\cap B} e_B \underline{x}_{B\setminus A} \underline{D} + e_{A\cap B} \underline{x}_{A\cap B} \underline{D}\, e_{A\cap B} e_A e_B \underline{x}_{A\cup B} \underline{D} \\
&\quad + (-1)^{(|A|-|A\cap B|)(|B|-|A\cap B|)} e_{A\cap B} e_B \underline{x}_{B\setminus A} \underline{D}\, e_A e_{A\cap B} \underline{x}_{A\setminus B} \underline{D} \\
&\quad + (-1)^{|A\cap B|(|A|+|B|-|A\cap B|)-|A\cap B|} e_{A\cap B} e_A e_B \underline{x}_{A\cup B} \underline{D}\, e_{A\cap B} \underline{x}_{A\cap B} \underline{D}\big).
\end{aligned}
$$

Finally, for the terms with a single occurrence of $\underline{D}$, we have (up to a factor $-\epsilon$)

$$
\begin{aligned}
& \big(\underline{D}\, \underline{x}_A e_A - e_A \underline{x}_A \underline{D}\big) e_B + (-1)^{|A||B|-|A\cap B|} e_B \big(\underline{D}\, \underline{x}_A e_A - e_A \underline{x}_A \underline{D}\big) \\
& + e_A\big(\underline{D}\, \underline{x}_B e_B - e_B \underline{x}_B \underline{D}\big) + (-1)^{|A||B|-|A\cap B|}\big(\underline{D}\, \underline{x}_B e_B - e_B \underline{x}_B \underline{D}\big) e_A \\
= & - \underline{D}\, \underline{x}_{A\setminus B} e_A e_B - \underline{D}\, \underline{x}_{B\setminus A} e_A e_B + e_A e_B \underline{x}_{A\setminus B} \underline{D} + e_A e_B \underline{x}_{B\setminus A} \underline{D} + (e_{A\cap B})^2 \big(\underline{D}\, \underline{x}_{A\setminus B} e_A (e_{A\cap B})^2 e_B \\
& + (-1)^{|A\cap B|(|A|+|B|)} \underline{D}\, \underline{x}_{A\setminus B} e_{A\cap B} e_A e_B e_{A\cap B} + \underline{D}\, \underline{x}_{A\cap B} (e_{A\cap B})^2 e_A e_B - e_A e_{A\cap B} \underline{x}_{A\setminus B} \underline{D} e_{A\cap B} e_B \\
& - e_{A\cap B} \underline{x}_{A\cap B} \underline{D} e_{A\cap B} e_A e_B + (-1)^{|A||B|+|A\cap B|(|B|+|A|-1)} e_{A\cap B} e_B \underline{D}\, \underline{x}_{A\setminus B} e_A e_{A\cap B} \\
& + (-1)^{|A\cap B|(|A|+|B|)} e_{A\cap B} e_A e_B \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} - (-1)^{|A\cap B|(|A|+|B|)} e_{A\cap B} e_A e_B \underline{x}_{A\setminus B} \underline{D}\, e_{A\cap B} \\
& - (-1)^{|A||B|+|A\cap B|(|B|+|A|-1)} e_{A\cap B} e_B e_A e_{A\cap B} \underline{x}_{A\setminus B} \underline{D} - (e_{A\cap B})^2 e_A e_B \underline{x}_{A\cap B} \underline{D} \\
& + e_A e_{A\cap B} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B + e_{A\cap B} \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} e_A e_B - e_A (e_{A\cap B})^2 e_B \underline{x}_{B\setminus A} \underline{D} \\
& - (-1)^{|A\cap B|(|A|+|B|)} e_{A\cap B} e_A e_B e_{A\cap B} \underline{x}_{A\cap B} \underline{D} - (-1)^{|A\cap B|(|A|+|B|)} e_{A\cap B} e_A e_B \underline{x}_{A\cap B} \underline{D} e_{A\cap B} \\
& + (-1)^{|A\cap B|(|A|+|B|)} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_A e_B e_{A\cap B} + (-1)^{|A\cap B|(|A|+|B|)} \underline{D}\, \underline{x}_{A\cap B} e_{A\cap B} e_A e_B e_{A\cap B} \\
& - (e_{A\cap B})^2 e_A e_B \underline{x}_{B\setminus A} \underline{D} + (-1)^{|A||B|+|A\cap B|(|A|+|B|-1)} \underline{D}\, \underline{x}_{B\setminus A} e_{A\cap B} e_B e_A e_{A\cap B} \\
& - (-1)^{|A||B|+|A\cap B|(|B|+|A|-1)} e_{A\cap B} e_B \underline{x}_{B\setminus A} \underline{D} e_A e_{A\cap B}\big)
\end{aligned}
$$

where, for instance, we made use of the following computation

$$
\begin{aligned}
& - e_A \underline{x}_{A\cap B} \underline{D} e_B - (-1)^{|A||B|-|A\cap B|} e_B \underline{x}_{A\cap B} \underline{D} e_A \\
= & - (e_{A\cap B})^2 \big((e_{A\cap B})^2 e_A \underline{x}_{A\cap B} \underline{D} e_B + (-1)^{|A||B|-|A\cap B|} e_B \underline{x}_{A\cap B} \underline{D} (e_{A\cap B})^2 e_A\big) \\
= & - (e_{A\cap B})^2 \big((-1)^{|A|-|A\cap B|} e_{A\cap B} \underline{x}_{A\cap B} e_{A\cap B} e_A \underline{D} e_B \\
& + (-1)^{|A||B|+|A\cap B||A|} e_B \underline{x}_{A\cap B} \underline{D} e_{A\cap B} e_A e_{A\cap B}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= -(e_{A\cap B})^2\big(e_{A\cap B}\underline{x}_{A\cap B}\underline{D}e_{A\cap B}e_Ae_B + (-1)^{|A||B|+|A\cap B||A|+|A|-|A\cap B|}e_B\underline{x}_{A\cap B}e_{A\cap B}e_A\underline{D}e_{A\cap B}\\
&\quad - e_{A\cap B}\underline{x}_{A\cap B}\sum_{l\in A\setminus B}2p_le_le_{A\cap B}e_Ae_B + (-1)^{|A||B|+|A\cap B||A|}e_B\underline{x}_{A\cap B}\sum_{l\in A\setminus B}2p_le_le_{A\cap B}e_Ae_{A\cap B}\big)\\
&= (e_{A\cap B})^2\big(-e_{A\cap B}\underline{x}_{A\cap B}\underline{D}e_{A\cap B}e_Ae_B - (-1)^{|A\cap B|(|A|+|B|)}e_{A\cap B}e_Ae_B\underline{x}_{A\cap B}\underline{D}e_{A\cap B}\big).
\end{aligned}
$$

Putting everything together and comparing with

$$
O_{A,B} = \frac{1}{2}\big(\underline{D}\,\underline{x}_{A\triangle B}e_Ae_B - e_Ae_B\underline{x}_{A\triangle B}\underline{D} - \epsilon e_Ae_B\big),
$$

and

$$
\begin{aligned}
[\![O_{A,(A\cap B)}, O_{(A\cap B),B}]\!]_+ &= \frac{1}{4}\big(\underline{D}\,\underline{x}_{A\setminus B}e_Ae_{A\cap B} - e_Ae_{A\cap B}\underline{x}_{A\setminus B}\underline{D} - \epsilon e_Ae_{A\cap B}\big)\\
&\qquad\times\big(\underline{D}\,\underline{x}_{B\setminus A}e_{A\cap B}e_B - e_{A\cap B}e_B\underline{x}_{B\setminus A}\underline{D} - \epsilon e_{A\cap B}e_B\big)\\
&+ (-1)^{(|A|-|A\cap B|)(|B|-|A\cap B|)}\frac{1}{4}\big(\underline{D}\,\underline{x}_{B\setminus A}e_{A\cap B}e_B - e_{A\cap B}e_B\underline{x}_{B\setminus A}\underline{D} - \epsilon e_{A\cap B}e_B\big)\\
&\qquad\times\big(\underline{D}\,\underline{x}_{A\setminus B}e_Ae_{A\cap B} - e_Ae_{A\cap B}\underline{x}_{A\setminus B}\underline{D} - \epsilon e_Ae_{A\cap B}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
[\![O_{A\cap B}, O_{(A\cap B),A,B}]\!]_+ &= \frac{1}{4}\big(\underline{D}\,\underline{x}_{A\cap B}e_{A\cap B} - e_{A\cap B}\underline{x}_{A\cap B}\underline{D} - \epsilon e_{A\cap B}\big)\\
&\qquad\times\big(\underline{D}\,\underline{x}_{A\cup B}e_{A\cap B}e_Ae_B - e_{A\cap B}e_Ae_B\underline{x}_{A\cup B}\underline{D} - \epsilon e_{A\cap B}e_Ae_B\big)\\
&+ (-1)^{|A\cap B|(|A|+|B|-|A\cap B|)-|A\cap B|}\frac{1}{4}\big(\underline{D}\,\underline{x}_{A\cup B}e_{A\cap B}e_Ae_B - e_{A\cap B}e_Ae_B\underline{x}_{A\cup B}\underline{D} - \epsilon e_{A\cap B}e_Ae_B\big)\\
&\qquad\times\big(\underline{D}\,\underline{x}_{A\cap B}e_{A\cap B} - e_{A\cap B}\underline{x}_{A\cap B}\underline{D} - \epsilon e_{A\cap B}\big),
\end{aligned}
$$

the proof is completed. □

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3

A Dirac equation on the two-sphere: the S_3 Dirac-Dunkl operator symmetry algebra

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abstract

In recent work, we obtained the symmetry algebra for a class of Dirac operators, containing in particular the Dirac-Dunkl operator for arbitrary root system. We now consider the three-dimensional case of the Dirac-Dunkl operator associated to the root system A_2, and the associated Dirac equation. The corresponding Weyl group is S_3, the symmetric group on three elements. The explicit form of the symmetry algebra in this case is a one-parameter deformation of the classical angular momentum algebra $\mathfrak{so}(3)$ incorporating elements of S_3. For this algebra, we classify all finite-dimensional, irreducible representations and determine the conditions for the representations to be unitarizable. The eigenfunctions of the Dirac-Dunkl operator on the two-sphere form a realization of the unitary irreducible representation of the symmetry algebra, as realized in the framework of Dunkl operators. Using a Cauchy-Kowalevsky extension theorem we obtain explicit expressions for these eigenfunctions in terms of Jacobi polynomials.

1 introduction

It is a classical result that the three-dimensional Dirac operator on Euclidean space is invariant under a realization of the angular momentum algebra, the Lie algebra $\mathfrak{so}(3)$. The setting changes when one considers generalizations of the Dirac operator containing, instead of regular partial derivatives, more advanced expressions. A specific example of this is the Dirac-Dunkl operator, defined in terms of Dunkl operators. These operators retain a desirable commutative property but allow for non-local effects through reflection terms. In N-dimensional Euclidean space, the system of Dunkl operators (and thus the reflection terms) depend on the choice of the (reduced) root system, or explicitly on the generators of the underlying reflection group G.

In recent work [3], the symmetry algebra of the Dirac-Dunkl operator for $N = 3$ and $G = (\mathbb{Z}_2)^3$ (and of the associated Dirac equation on the two-sphere) was identified as the so-called Bannai-Ito algebra [22]. This led to the construction of representations of the Bannai-Ito algebra using the actions of the Dunkl operators. Moreover, by moving up in dimension a higher rank version of the Bannai-Ito algebra was postulated as the symmetry algebra of the $(\mathbb{Z}_2)^N$ Dirac-Dunkl operator [4]. For a recent overview of the Bannai-Ito algebra and its applications, we refer the reader to [13].

This interaction inspired our investigation into the Dunkl version of the Dirac operator for another reflection group, the symmetric group on three elements S_3, associated to the root system A_2. Doing so, this provided a stepping stone towards the determination of the symmetry algebra for a bigger class of generalized Laplace and Dirac operators in general dimension N, in the framework of Wigner systems [5]. These results contain in particular the Dunkl versions for arbitrary root system (that is, for arbitrary N and general G). Armed with these new tools we now return to the three-dimensional case with the objective of finding representations, and explicit realizations, of these abstract symmetry algebras.

In three dimensions, the symmetry algebra of the generalized Dirac operator $\underline{D}$ forms an extension of the classical angular momentum algebra, the Lie algebra $\mathfrak{so}(3)$. The algebraic relations were obtained in abstract form in [5] and are given by

$$\begin{aligned} [O_{23}, O_{12}] &= O_{31} + \{O_{123}, O_2\} + [O_3, O_1] \\ [O_{31}, O_{23}] &= O_{12} + \{O_{123}, O_3\} + [O_1, O_2] \\ [O_{12}, O_{31}] &= O_{23} + \{O_{123}, O_1\} + [O_2, O_3] \end{aligned} \tag{1.1}$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$ are respectively the commutator and the anti-commutator of A and B. This algebra, which we will denote by $\mathcal{O}_3$, is generated by seven generally non-trivial elements: $O_1, O_2, O_3, O_{12}, O_{23}, O_{31}, O_{123}$. The general expressions of these symmetries are given in [5, formulas (3.8) and (3.10)]. For the classical Dirac operator in terms of the standard partial derivatives, the one-index symmetries O_1, O_2, O_3 were seen to be identically zero and thus in this case the commutation relations (1.1) indeed reduce to those of the Lie algebra $\mathfrak{so}(3)$. For other types of Dirac operators, the relations (1.1) form an extension of $\mathfrak{so}(3)$ whose nature depends on the explicit form of the one-index symmetries O_1, O_2, O_3 in particular.

The aforementioned generalized Dirac operator $\underline{D}$ contains as a specific case the Dunkl version of the Dirac operator, which has appeared also in another context, e.g. [2, 17]. Here, the Dunkl operators [7, 19] are a generalization of partial derivatives in the form of differential-difference operators associated to a root system and invariant under its Weyl group G. These operators have seen numerous applications since their introduction in for instance physical models involving reflections [11–15, 18–20]. When dealing with Dunkl operators, the choice of root system and associated reflection group G is what determines the structure and the explicit form of the one-index symmetries O_1, O_2, O_3 and as a consequence also of the symmetry algebra, as seen from the right-hand side of the algebraic relations (1.1).

For the current paper, this is the root system A_2 with Weyl group $G = S_3$, the symmetric group on three elements. We will use the notation $\mathcal{O}S_3$ to denote the specific form the abstract algebra $\mathcal{O}_3$ takes on in the case of the S_3 Dirac-Dunkl operator. The explicit relations of the algebra generators are given in (3.5). From these expressions it is clear that one can speak of a one-parameter deformation of the angular momentum algebra $\mathfrak{so}(3)$ incorporating the symmetric group S_3. When the deformation parameter κ is set to zero, one recovers the ordinary $\mathfrak{so}(3)$ algebra, as the Dunkl operators then reduce to regular partial derivatives. For non-zero κ, the algebra relations (3.5) provide an interesting and exciting new structure: a deformation of $\mathfrak{so}(3)$ by means of elements of the S_3 group algebra. This algebra $\mathcal{O}S_3$ is the main object of study in this paper. In particular, we shall classify (finite-dimensional irreducible) representations of this algebra, and provide an explicit realization of a class of representations in terms of orthogonal polynomials.

We briefly elaborate upon other cases which have been considered. For the case of $(\mathbb{Z}_2)^3$ Dirac-Dunkl operator, the commutators in the left-hand side of the algebraic relations (1.1) become anti-commutators through the use of commuting involutions present in the reflection group. This yields the Bannai-Ito algebra as the symmetry algebra [3]. In more recent work [13], the root system of type B_3 was also considered to define extensions of the Bannai-Ito algebra. A crucial ingredient in this paper is again the existence of commuting involutions in the reflection group. The lack of such involutions characterizes the case at hand of the symmetric group, and its importance as a reference work for future investigations in higher dimensions.

In the subsequent section, we go over the definitions and notions required to introduce the Dirac-Dunkl operator related to S_3. In section 3, we elaborate on the explicit expressions of the symmetries of this operator and give the algebraic relations (1.1) for this specific case. In section 4, we construct a form of ladder operators and use them to classify all finite-dimensional irreducible representations of the symmetry algebra in abstract form. In the last section we determine explicit expressions for wavefunctions which form a unitary irreducible representation of the symmetry algebra, as realized in the framework of Dunkl operators.

2 THE S_3 DIRAC-DUNKL OPERATOR

We consider three-dimensional space $\mathbb{R}^3$ with coordinates x_1, x_2, x_3. The symmetric group S_3 is generated by the transpositions g_{12}, g_{23}, g_{31} which act on functions on $\mathbb{R}^3$ in the following way:

$$\begin{aligned} g_{12}f(x_1,x_2,x_3) &= f(x_2,x_1,x_3)\,, \\ g_{23}f(x_1,x_2,x_3) &= f(x_1,x_3,x_2)\,, \\ g_{31}f(x_1,x_2,x_3) &= f(x_3,x_2,x_1)\,. \end{aligned}$$

Denoting the two even elements by $g_{123} = g_{12}g_{23} = g_{31}g_{12} = g_{23}g_{31}$ and $g_{321} = g_{23}g_{12} = g_{12}g_{31} = g_{31}g_{23}$, the six elements of S_3 are $\{1, g_{12}, g_{23}, g_{31}, g_{123}, g_{321}\}$. For convenience we give the multiplication table of S_3 in Table 1.

TABLE 1: Multiplication table of S_3.

$\nearrow$	1	g_{12}	g_{23}	g_{31}	g_{123}	g_{321}
1	1	g_{12}	g_{23}	g_{31}	g_{123}	g_{321}
g_{12}	g_{12}	1	g_{123}	g_{321}	g_{23}	g_{31}
g_{23}	g_{23}	g_{321}	1	g_{123}	g_{31}	g_{12}
g_{31}	g_{31}	g_{123}	g_{321}	1	g_{12}	g_{23}
g_{123}	g_{123}	g_{31}	g_{12}	g_{23}	g_{321}	1
g_{321}	g_{321}	g_{23}	g_{31}	g_{12}	1	g_{123}

The symmetric group S_3 arises as the Weyl group of the root system A_2. The associated Dunkl operators are explicitly given by [7, 19]

$$\begin{aligned}\mathcal{D}_1 &= \partial_{x_1} + \kappa\left(\frac{1-g_{12}}{x_1-x_2} + \frac{1-g_{13}}{x_1-x_3}\right), \qquad \mathcal{D}_2 = \partial_{x_2} + \kappa\left(\frac{1-g_{12}}{x_2-x_1} + \frac{1-g_{23}}{x_2-x_3}\right),\\ \mathcal{D}_3 &= \partial_{x_3} + \kappa\left(\frac{1-g_{31}}{x_3-x_1} + \frac{1-g_{23}}{x_3-x_2}\right).\end{aligned} \tag{2.1}$$

Here the parameter κ denotes the value of the multiplicity function on the single conjugacy class all transpositions of the symmetric group share. This multiplicity function is usually taken to be real and non-negative, in order to have favorable properties such as intertwining operators [8]. Now, the property that makes these generalizations of partial derivatives so special is that they commute with one another, $[\mathcal{D}_i, \mathcal{D}_j] = 0$ for $i, j \in \{1, 2, 3\}$. Moreover, for i, j, k a cyclic permutation of 1, 2, 3, the action of S_3 on the Dunkl operators is simply given by

$$g_{ij}\mathcal{D}_i = \mathcal{D}_j g_{ij}, \quad g_{ij}\mathcal{D}_j = \mathcal{D}_i g_{ij}, \quad g_{ij}\mathcal{D}_k = \mathcal{D}_k g_{ij}\,.$$

The commutation relations with the coordinate variables are easily shown to be

$$[\mathcal{D}_i, x_j] = \mathcal{D}_i x_j - x_j \mathcal{D}_i = \begin{cases} 1 + \kappa \sum_{k\neq i} g_{ik} & i = j \\ -\kappa g_{ij} & i \neq j \end{cases} \tag{2.2}$$

for $i, j, k \in \{1, 2, 3\}$. Note that when $\kappa = 0$ these reduce to the standard relations as the Dunkl operators then reduce to ordinary partial derivatives.

The Laplace-Dunkl operator is given by

$$\Delta = (\mathcal{D}_1)^2 + (\mathcal{D}_2)^2 + (\mathcal{D}_3)^2, \tag{2.3}$$

which is obviously invariant under the action of S_3. In this setting, the Dirac-Dunkl operator $\underline{D}$ is defined as a square root of the Dunkl Laplacian as follows:

$$\underline{D} = e_1\mathcal{D}_1 + e_2\mathcal{D}_2 + e_3\mathcal{D}_3,$$

where e_1, e_2, e_3 generate the three-dimensional Euclidean Clifford algebra and hence satisfy the anti-commutation relations $\{e_i, e_j\} = 2\delta_{ij}$ for $i, j \in \{1, 2, 3\}$. (This corresponds to the positive choice of sign, $\epsilon = +1$ in the notation of [5]. The other choice of sign is readily obtained by making the appropriate substitutions.) The three-dimensional Euclidean Clifford algebra can be realized by means of the well-known Pauli matrices. For the first part of this paper, we can work with abstract Clifford elements e_1, e_2, e_3. We will use the Pauli matrices for the explicit construction of representation spaces in Section 5.

Together with the vector variable $\underline{x} = e_1x_1 + e_2x_2 + e_3x_3$, the operator $\underline{D}$ generates a realization of the $\mathfrak{osp}(1|2)$ Lie superalgebra [2, 10], since

$$[\{\underline{D}, \underline{x}\}, \underline{D}] = -2\underline{D}, \qquad [\{\underline{D}, \underline{x}\}, \underline{x}] = 2\underline{x}. \tag{2.4}$$

Here, the anti-commutator $\{\underline{D}, \underline{x}\}$ can be written in terms of the Euler operator $\mathbb{E}$ as follows

$$\{\underline{D}, \underline{x}\} = 2\mathbb{E} + 3 + 6\kappa\,, \qquad \mathbb{E} = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}\,. \tag{2.5}$$

The angular Dirac-Dunkl operator, which we denote by Γ, appears when the Dirac-Dunkl operator $\underline{D}$ is restricted to the two-sphere S^2. The operator Γ is up to an additive constant equal to the so-called Scasimir operator of the $\mathfrak{osp}(1|2)$ realization above [10],

$$\Gamma + 1 = \frac{1}{2}\left([\underline{D}, \underline{x}] - 1\right)\,, \tag{2.6}$$

satisfying $\{\Gamma + 1, \underline{D}\} = 0$ and $\{\Gamma + 1, \underline{x}\} = 0$, while commuting with the even elements of $\mathfrak{osp}(1|2)$. Working out the commutator in the right-hand side of (2.6) using (2.2) we obtain

$$\Gamma + 1 = 1 + \kappa(g_{12} + g_{23} + g_{31}) - e_1e_2L_{12} - e_2e_3L_{23} - e_3e_1L_{31}\,. \tag{2.7}$$

Here the Dunkl versions of the angular momentum operators are defined as

$$L_{12} = x_1\mathcal{D}_2 - x_2\mathcal{D}_1, \quad L_{23} = x_2\mathcal{D}_3 - x_3\mathcal{D}_2, \quad L_{31} = x_3\mathcal{D}_1 - x_1\mathcal{D}_3,$$

and are easily shown to commute with the Dunkl Laplacian Δ.

The space of homogeneous polynomials in the kernel of the Dirac-Dunkl operator are the solutions to the massless Dirac equation $\underline{D}\psi = 0$. Furthermore, this space forms an eigenspace of the angular Dirac-Dunkl operator Γ. This is easily shown as follows. Let $\mathcal{P}_n(\mathbb{R}^N)$ denote the space of homogeneous polynomials of degree n in N variables. The Dunkl monogenics of degree n are homogeneous spinor-valued polynomials of degree n in the kernel of the Dirac-Dunkl operator, which we will denote by $\mathcal{M}_n(\mathbb{R}^N, \mathbb{S}) = \ker \underline{D} \cap (\mathcal{P}_n(\mathbb{R}^N) \otimes \mathbb{S})$. Here $\mathbb{S}$ is a spinor representation of the Clifford algebra. For the three-dimensional Clifford algebra realized by the Pauli matrices a two-dimensional Dirac spinor representation is simply $\mathbb{S} \cong \mathbb{C}^2$, with basis spinors $\chi_+ = (1, 0)^T$ and $\chi_- = (0, 1)^T$. Now, for $\psi_n \in \mathcal{M}_n(\mathbb{R}^3, \mathbb{S})$ we have using $\underline{D}\psi_n = 0$ and (2.5)

$$\begin{aligned}(\Gamma + 1)\psi_n &= \frac{1}{2}([\underline{D}, \underline{x}] - 1)\psi_n = \frac{1}{2}(\underline{D}\,\underline{x} - 1)\psi_n \\ &= \frac{1}{2}(\{\underline{D}, \underline{x}\} - 1)\psi_n = \frac{1}{2}(2\mathbb{E} + 3 + 6\kappa - 1)\psi_n\,.\end{aligned}$$

As $\mathbb{E}\psi_n = n\psi_n$, this gives the following Dirac equation on the two-sphere

$$(\Gamma + 1)\psi_n = (n + 1 + 3\kappa)\psi_n \,. \tag{2.8}$$

We will construct explicit expressions for ψ_n in section 5.1.

3 SYMMETRY ALGEBRA OF THE S_3 DIRAC-DUNKL OPERATOR

In general, the symmetry algebra $\mathcal{O}_3$ is governed by the relations (1.1). It consists of three one-index symmetries O_1, O_2, O_3, three two-index symmetries O_{12}, O_{23}, O_{31} and a three-index symmetry O_{123}. We will elaborate upon the explicit form these symmetries and the relations (1.1) take on for the S_3 case of the Dirac-Dunkl operator, with symmetry algebra denoted by $\mathcal{O}S_3$.

Following [5, Theorem 3.6] the one-index symmetries anti-commute with the Dirac-Dunkl operator. For the S_3 case they are explicitly given by

$$O_1 = \frac{\kappa}{\sqrt{2}}(G_{12} - G_{31})\,,\ O_2 = \frac{\kappa}{\sqrt{2}}(G_{23} - G_{12})\,,\ O_3 = \frac{\kappa}{\sqrt{2}}(G_{31} - G_{23})\,, \tag{3.1}$$

where

$$G_{12} = \frac{1}{\sqrt{2}}g_{12}(e_1 - e_2)\,,\ G_{23} = \frac{1}{\sqrt{2}}g_{23}(e_2 - e_3)\,,\ G_{31} = \frac{1}{\sqrt{2}}g_{31}(e_3 - e_1)\,.$$

Note that the three one-index symmetries are not independent as $O_3 = -O_1 - O_2$, and moreover $O_1O_2O_1 = (3\kappa^2/2)O_3$.

The operators G_{12}, G_{23}, G_{31} appearing here consist of a transposition of S_3 appended with the Clifford element corresponding to the normed vector in the root system associated to the reflection in question (which is an element of the Pin group of the Clifford algebra). It was observed already in [5] that they also anti-commute with $\underline{D}$ (one easily verifies this by direct computation)

$$\{\underline{D}, G_{12}\} = 0, \qquad \{\underline{D}, G_{23}\} = 0, \qquad \{\underline{D}, G_{31}\} = 0.$$

The symmetries G_{12}, G_{23}, G_{31} in fact generate a new copy of the symmetric group S_3, which extends its action to affect also Clifford algebra elements but with an extra minus sign. Indeed, we have

$$(G_{ij})^2 = 1, \qquad G_{ij}e_iG_{ij} = -e_j, \qquad G_{ij}e_jG_{ij} = -e_i, \qquad G_{ij}e_kG_{ij} = -e_k$$

where i, j, k is a cyclic permutation of 1, 2, 3. Moreover, $G_{12}G_{23}G_{12} = G_{31}$ with analogous relations for conjugation with G_{23} and G_{31}. The symmetries corresponding to the two even elements of S_3 are

$$G_{123} = G_{12}G_{23} = \frac{1}{2}g_{123}(e_1e_2 + e_2e_3 + e_3e_1 - 1) = G_{23}G_{31} = G_{31}G_{12},$$

$$G_{321} = G_{23}G_{12} = \frac{1}{2}g_{321}(e_2e_1 + e_3e_2 + e_1e_3 - 1) = G_{31}G_{23} = G_{12}G_{31},$$

which both commute with the S_3 Dirac-Dunkl operator.

The individual symmetries G_{12}, G_{23}, G_{31} are not contained in the algebra $\mathcal{O}S_3$. However, the one-index symmetries O_1, O_2, O_3 are built up from G_{12}, G_{23}, G_{31}, so it is useful to extend the symmetry algebra to contain also this realization of S_3. We will denote this extension by $\mathcal{O}St_3$.

The two-index symmetries are analogues of the Dunkl angular momentum operators that commute with the S_3 Dirac-Dunkl operator. They are explicitly given by [5, Example 4.2.2]

$$\begin{aligned} O_{ij} &= L_{ij} + \frac{1}{2}e_ie_j + O_ie_j - O_je_i \\ &= L_{ij} + \frac{1}{2}e_ie_j + \frac{\kappa}{\sqrt{2}}(G_{ij}e_i - G_{jk}e_i + G_{ij}e_j - G_{ki}e_j) \\ &= L_{ij} + \frac{1}{2}e_ie_j + \kappa(g_{12} + g_{23} + g_{31})e_ie_j - O_ke_1e_2e_3 \end{aligned} \tag{3.2}$$

where i, j, k is a cyclic permutation of 1, 2, 3 and the last line follows by means of the identity

$$O_1e_1 + O_2e_2 + O_3e_3 = \kappa(g_{12} + g_{23} + g_{31}) .$$

The one-index and two-index symmetries together satisfy the relation [5, Theorem 3.11]

$$[O_{ij}, O_k] + [O_{jk}, O_i] + [O_{ki}, O_j] = 0 \qquad (i, j, k \in \{1, 2, 3\}).$$

The final symmetry is the three-index symmetry

$$O_{123} = e_1e_2e_3 + O_1e_2e_3 + O_2e_3e_1 + O_3e_1e_2 + L_{12}e_3 + L_{23}e_1 + L_{31}e_2$$

which anti-commutes with the S_3 Dirac-Dunkl operator. It was obtained in [5] that this symmetry is equal to the Scasimir of $\mathfrak{osp}(1|2)$ given by (2.6), multiplied by the pseudo-scalar $e_1e_2e_3$ (a symmetry of $\underline{D}$ inherent to the Clifford algebra):

$$O_{123} = \frac{1}{2}\left([\underline{D}, \underline{x}] - 1\right)\, e_1e_2e_3 = (\Gamma + 1)\, e_1e_2e_3$$

Using the anticommutation relations of e_1, e_2, e_3, one immediately has $[\underline{D}, e_1e_2e_3] = 0$ and moreover $(e_1e_2e_3)^2 = -1$. In fact, in the realization by means of the Pauli matrices, $e_1e_2e_3$ is just i times the identity matrix. As Γ is the restriction of the Dirac operator to the sphere, one readily shows that all the elements of $\mathcal{O}St_3$ commute with the symmetry O_{123}. Moreover, it can be written in terms of the other symmetries as follows

$$O_{123} = -\frac{1}{2}e_1e_2e_3 - O_1e_2e_3 - O_2e_3e_1 - O_3e_1e_2 + O_{12}e_3 + O_{31}e_2 + O_{23}e_1,$$

and by direct computation one finds

$$(O_{123})^2 = -\frac{1}{4} + O_1^2 + O_2^2 + O_3^2 + O_{12}^2 + O_{23}^2 + O_{31}^2 \tag{3.3}$$

$$= O_{12}^2 + O_{23}^2 + O_{31}^2 - \frac{3}{2}\kappa^2(G_{123} + G_{321}) + 3\kappa^2 - \frac{1}{4}\,.$$

In this framework, the algebraic relations (1.1) of the general algebra $\mathcal{O}_3$ translate to the following result.

THEOREM 3.1: *The algebra $\mathcal{O}St_3$ generated by the symmetries G_{12}, G_{23}, G_{31} and O_{12}, O_{23}, O_{31}, O_{123} is governed by the following relations:*

- *O_{123} commutes with the other symmetries,*
- *G_{12}, G_{23}, G_{31} generate a copy of S_3 and act on the indices of O_{12}, O_{23}, O_{31} by an S_3 action with minus sign, i.e.*

$$G_{12}O_{12} = -O_{12}G_{12}, \quad G_{12}O_{23} = -O_{31}G_{12}, \quad G_{12}O_{31} = -O_{23}G_{12}, \tag{3.4}$$

  and analogous actions of G_{23} and G_{31},
- *the commutation relations*

$$\begin{aligned}
[O_{12}, O_{31}] &= O_{23} + \sqrt{2}\kappa O_{123}(G_{12} - G_{31}) + \frac{3}{2}\kappa^2(G_{123} - G_{321}) \\
[O_{23}, O_{12}] &= O_{31} + \sqrt{2}\kappa O_{123}(G_{23} - G_{12}) + \frac{3}{2}\kappa^2(G_{123} - G_{321}) \\
[O_{31}, O_{23}] &= O_{12} + \sqrt{2}\kappa O_{123}(G_{31} - G_{23}) + \frac{3}{2}\kappa^2(G_{123} - G_{321}).
\end{aligned} \tag{3.5}$$

where κ is a scalar factor.

Proof. This follows immediately from (1.1), the explicit expressions (3.1) and

$$[O_1, O_2] = \frac{3}{2}\kappa^2(G_{123} - G_{321}) = [O_2, O_3] = [O_3, O_1]\,.$$

□

Note that for $\kappa = 0$ the commutation relations (3.5) reduce to the well-known relations of the Lie algebra $\mathfrak{so}(3)$, which is isomorphic to $\mathfrak{su}(2)$. For the sequel we will consider κ to be non-zero.

4 REPRESENTATIONS

Both from a purely mathematical point of view and because of their use in constructing eigenfunctions of the Dirac operator (2.8), we are interested in determining all finite dimensional irreducible representations of the algebra $\mathcal{O}St_3$ in abstract form, in particular the unitary representations. From (3.4), we see that the linear combination $O_{12} + O_{23} + O_{31}$ anti-commutes with G_{12}, G_{23}, G_{31} and thus commutes with the even elements G_{123} and G_{321} of S_3. We will build up irreducible representations starting from a mutual eigenvector of these commuting symmetries. In order to do this we construct a form of ladder operators. We start by defining some auxiliary operators.

DEFINITION 4.1: *Say $\omega = e^{2\pi i/3}$, so*

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\,, \qquad \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \overline{\omega} = \omega^{-1}\,, \qquad \omega^3 = 1\,.$$

We define the following linear combinations in the algebra $\mathcal{O}St_3$, with inverse relations on the right,

$$\begin{aligned} O_0 &= \frac{-i}{\sqrt{3}}(O_{12} + O_{23} + O_{31}), & O_{12} &= \frac{i}{\sqrt{6}}(\sqrt{2}O_0 + O_+ + O_-), \\ O_+ &= -i\sqrt{\frac{2}{3}}(O_{12} + \omega O_{23} + \omega^2 O_{31}), & O_{23} &= \frac{i}{\sqrt{6}}(\sqrt{2}O_0 + \omega^2 O_+ + \omega\, O_-), \\ O_- &= -i\sqrt{\frac{2}{3}}(O_{12} + \omega^2 O_{23} + \omega O_{31}), & O_{31} &= \frac{i}{\sqrt{6}}(\sqrt{2}O_0 + \omega\, O_+ + \omega^2 O_-)\,. \end{aligned} \tag{4.1}$$

We also define a set of linear combinations of G_{12}, G_{23}, G_{31}

$$N_+ = G_{12} + \omega G_{23} + \omega^2 G_{31}\,, \qquad N_- = G_{12} + \omega^2 G_{23} + \omega G_{31}\,. \tag{4.2}$$

Note that N_+ and N_- generate the same subset of the group algebra $\mathbb{C}S_3$ as O_1, O_2, O_3 do. The addition of G_{12} yields the full S_3 realization.

PROPOSITION 4.2: *The elements of the algebra $\mathcal{O}St_3$ defined in Definition 4.1 satisfy the relations*

$$[O_0, O_\pm] = \pm O_\pm + 2\kappa O_{123} N_\pm \tag{4.3}$$

$$[O_+, O_-] = 2\,O_0 + \kappa^2 [N_+, N_-] \tag{4.4}$$

where $[N_+, N_-] = -i\,3\sqrt{3}(G_{123} - G_{321})$.

Moreover, the elements $N_\pm$ are nilpotent, that is $N_\pm^2 = 0$, and satisfy

$$(N_\pm N_\mp)^2 = 9 N_\pm N_\mp\,. \tag{4.5}$$

The interaction with O_0, O_+, O_- is as follows

$$N_\pm O_0 = -O_0 N_\pm\,, \qquad N_\pm O_\pm = -O_\mp N_\mp\,, \qquad N_\pm O_\mp = -O_\pm N_\mp\,. \tag{4.6}$$

Finally, the square (3.3) *can be rewritten in the following forms*

$$\begin{aligned} (O_{123})^2 &= -O_0^2 - \frac{1}{2}\{O_+, O_-\} + \kappa^2 \frac{1}{2}\{N_+, N_-\} - \frac{1}{4} \\ &= -O_0^2 - O_+ O_- + O_0 + \kappa^2 N_+ N_- - \frac{1}{4} \\ &= -O_0^2 - O_- O_+ - O_0 + \kappa^2 N_- N_+ - \frac{1}{4}\,. \end{aligned} \tag{4.7}$$

Proof. The relations (4.3) and (4.4) are proved by straightforward computations using the commutation relations (3.5). For the commutator of N_+ and N_- we have

$$N_+N_- = (G_{12} + \omega G_{23} + \omega^2 G_{31})(G_{12} + \omega^2 G_{23} + \omega G_{31}) = 3 + 3\omega^2 G_{123} + 3\omega G_{321}\,, \tag{4.8}$$

while similarly

$$N_-N_+ = (G_{12} + \omega^2 G_{23} + \omega G_{31})(G_{12} + \omega G_{23} + \omega^2 G_{31}) = 3 + 3\omega G_{123} + 3\omega^2 G_{321}\,, \tag{4.9}$$

which leads to $[N_+, N_-] = -i\,3\sqrt{3}(G_{123} - G_{321})$, and also $\{N_+, N_-\} = 6 - 3(G_{123} + G_{321})$.

We illustrate the nilpotency of N_+, the result for N_- is similar,

$$\begin{aligned} N_+^2 &= \left(G_{12} + \omega G_{23} + \omega^2 G_{31}\right)^2 \\ &= 1 + \omega G_{12}G_{23} + \omega^2 G_{12}G_{31} + \omega G_{23}G_{12} + \omega^2 + G_{23}G_{31} + \omega^2 G_{31}G_{12} + G_{31}G_{23} + \omega \\ &= 1 + \omega + \omega^2 + (1 + \omega + \omega^2)G_{123} + (1 + \omega + \omega^2)G_{321} = 0. \end{aligned}$$

In the same way, starting now from the expressions (4.8) and (4.9), we obtain (4.5).

The interactions in (4.6) follow immediately from

$$\begin{aligned} G_{12}O_0 &= -O_0G_{12} & G_{23}O_0 &= -O_0G_{23} & G_{31}O_0 &= -O_0G_{31} \\ G_{12}O_+ &= -O_-G_{12} & G_{23}O_+ &= -\omega^2 O_-G_{23} & G_{31}O_+ &= -\omega O_-G_{31} \\ G_{12}O_- &= -O_+G_{12} & G_{23}O_- &= -\omega O_+G_{23} & G_{31}O_- &= -\omega^2 O_+G_{31}\,, \end{aligned} \tag{4.10}$$

which are direct consequences of (3.4) and the definitions (4.1), (4.2).

Finally, the square (3.3) is rewritten using the inverse relations (4.1). We find

$$\begin{aligned} &-6(O_{12}^2 + O_{23}^2 + O_{31}^2) \\ &= (\sqrt{2}O_0 + O_+ + O_-)^2 + (\sqrt{2}O_0 + \omega^2 O_+ + \omega O_-)^2 + (\sqrt{2}O_0 + \omega O_+ + \omega^2 O_-)^2 \\ &= (2 + 2 + 2)O_0^2 + (1 + \omega + \omega^2)O_+^2 + (1 + \omega^2 + \omega)O_-^2 \\ &\quad + (1 + \omega^2 + \omega)\{\sqrt{2}O_0, O_+\} + (1 + \omega + \omega^2)\{\sqrt{2}O_0, O_-\} + (1 + 1 + 1)\{O_+, O_-\}. \end{aligned}$$

The results now follow using the expression for $\{N_+, N_-\}$ and (4.3). □

An essential ingredient for the construction and classification of representation spaces is the existence of a couple of ladder operators.

PROPOSITION 4.3: *The elements in the algebra $\mathcal{O}\mathrm{St}_3$*

$$K_+ = \frac{1}{2}\{O_0, O_+\} \qquad\qquad K_- = \frac{1}{2}\{O_0, O_-\} \tag{4.11}$$

satisfy the relation

$$[O_0, K_\pm] = \pm K_\pm\,. \tag{4.12}$$

Moreover, we have the factorization

$$K_+K_- = -\left(O_{123}^2 + (O_0 - 1/2)^2\right)\left((O_0 - 1/2)^2 - \kappa^2 N_+N_-\right) \tag{4.13}$$

$$K_-K_+ = -\left(O_{123}^2 + (O_0 + 1/2)^2\right)\left((O_0 + 1/2)^2 - \kappa^2 N_-N_+\right)\,. \tag{4.14}$$

Proof. We immediately find that

$$[O_0, K_\pm] = \frac{1}{2}[O_0, \{O_0, O_\pm\}] = \frac{1}{2}\{O_0, [O_0, O_\pm]\} = \pm\frac{1}{2}\{O_0, O_\pm\} + \kappa\{O_0, O_{123}N_\pm\} = \pm K_\pm$$

as O_{123} commutes with O_0 and $N_\pm$, and $N_\pm$ anti-commutes with O_0, see (4.6).

The factorization of K_+K_- and K_-K_+ follows by long and tedious, but otherwise straightforward computations starting from the definitions (4.11), and using the relations (4.3), (4.6), and the expression (4.7). □

From (4.10) we find the interaction of the S_3 realization with $K_\pm$ to be as follows

$$\begin{aligned}
&G_{12}K_+ = K_-G_{12}\,, && G_{123}K_+ = \omega^2 K_+G_{123}\,, && G_{321}K_+ = \omega K_+G_{321}\,,\\
&G_{23}K_+ = \omega^2 K_-G_{23}\,, && G_{123}K_- = \omega K_-G_{123}\,, && G_{321}K_- = \omega^2 K_-G_{321}\,,\\
&G_{31}K_+ = \omega K_-G_{31}\,.
\end{aligned} \tag{4.15}$$

Our aim is now to determine all finite-dimensional irreducible representations of $\mathcal{O}\mathrm{St}_3$. Hereto, let (V, ρ_V) be a representation of $\mathcal{O}\mathrm{St}_3$. From here on, we consider V as an $\mathcal{O}\mathrm{St}_3$ module by setting $G \cdot v = \rho_V(G)v$ for $G \in \mathcal{O}\mathrm{St}_3$ and $v \in V$.

The element O_{123} commutes with all of the algebra $\mathcal{O}\mathrm{St}_3$ so its action on an invariant subspace V_0 of the representation V will be multiplication by a constant Λ. The constant Λ will later be determined in terms of other parameters characterizing the representation.

Following the results obtained in Proposition 4.2, our starting point will be the element O_0, given by (4.1), which commutes with the even S_3 elements G_{123} and G_{321}. Hence, without loss of generality, we can consider a mutual eigenvector for all these elements. Take $v_0 \in V$ to be such an eigenvector with eigenvalue λ for O_0. The eigenvalue for G_{123} is restricted to the set $\{1, \omega, \omega^2\}$ as $G_{123}^3 = G_{123}G_{321} = 1$ and if $G_{123}v_0 = \alpha v_0$ then $G_{321}v_0 = \alpha^{-1}v_0$.

We will construct the $\mathcal{O}\mathrm{St}_3$ invariant subspace containing v_0. If V is irreducible this space must be either V or trivial. The trivial case results from v_0 being the zero vector, so from now on we assume that v_0 is not the zero vector.

If $O_0v_0 = \lambda v_0$, then for a positive integer k, the vector $(K_\pm)^k v_0$ is also an eigenvector of O_0. Indeed, using $[O_0, (K_\pm)^k] = \pm k(K_\pm)^k$, which follows directly from (4.12), we have

$$O_0(K_\pm)^k v_0 = \big((K_\pm)^k O_0 + [O_0, (K_\pm)^k]\big)v_0 = K_\pm O_0 v_0 \pm k(K_\pm)^k K_\pm v_0 = (\lambda \pm k)(K_\pm)^k v_0\,. \tag{4.16}$$

The set of vectors $\{(K_+)^k v_0 \mid k \in \mathbb{N}\}$ must be linearly independent because they have distinct eigenvalues as eigenvectors of O_0. If we impose V to be finite-dimensional, then $(K_+)^k v_0 = 0$ for some $k \in \mathbb{N}$. Without loss of generality we may assume that $K_+v_0 = 0$. Following the same reasoning, the sequence $\{(K_-)^k v_0 \mid k \in \mathbb{N}\}$ must also be linearly independent and thus must terminate. Hence $K_-(K_-^n v_0) = 0$ for some $n \in \mathbb{N}$ and we may assume without loss of generality that n is minimal in this aspect, i.e. $K_-^n v_0 \neq 0$.

So far, we have obtained the following vectors of the representation V:

$$\{v_k := (K_-)^k v_0 \mid k = 0, \ldots, n\}\,. \tag{4.17}$$

The space spanned by these vectors is invariant under the action of O_0, G_{123}, G_{321}, O_{123} and K_-, with $O_0 v_k = (\lambda - k) v_k$. Recall that $G_{123} v_0 = \alpha v_0$ for $\alpha \in \{1, \omega, \omega^2\}$, or thus $\alpha = \omega^\ell$ for some integer ℓ. By (4.15), we then have

$$G_{123} v_k = G_{123}(K_-)^k v_0 = \omega^k (K_-)^k G_{123} v_0 = \omega^{\ell+k} v_k, \qquad G_{321} v_k = \omega^{-\ell-k} v_k\,.$$

The transpositions G_{12}, G_{23}, G_{31} all square to the identity and anti-commute with O_0. Let $v_k^- = G_{12} v_k$, then $G_{12} v_k^- = v_k$ and $O_0 v_k^- = O_0 G_{12} v_k = -G_{12} O_0 v_k = -(\lambda - k) v_k^-$. Moreover, $G_{23} v_k$ and $G_{31} v_k$ must both be proportional to v_k^- since the compositions G_{123} and G_{321} act diagonally on v_k, and in turn also on v_k^-. Indeed, we have

$$G_{123} v_k^- = G_{123} G_{12} v_k = G_{12} G_{321} v_k = \omega^{-\ell-k} v_k^-, \qquad G_{321} v_k^- = \omega^{\ell+k} v_k^-\,.$$

It follows from $G_{12} K_- = K_+ G_{12}$ that $v_k^- = G_{12}(K_-)^k v_0 = (K_+)^k G_{12} v_0 = (K_+)^k v_0^-$ or thus $K_+ v_k^- = v_{k+1}^-$. In this way, we arrive at the following set of vectors of V:

$$\mathcal{B} = \{v_k^+ := v_k = (K_-)^k v_0 \mid k = 0, \ldots, n\} \cup \{v_k^- := G_{12} v_k^+ = (K_+)^k G_{12} v_0 \mid k = 0, \ldots, n\}\,. \tag{4.18}$$

All these vectors are eigenvectors of the mutually commuting elements O_0 and G_{123}:

$$O_0 v_k^\pm = \pm(\lambda - k) v_k^\pm\,, \tag{4.19}$$

for $k \in \{0, \ldots, n\}$, while

$$G_{123} v_k^\pm = \omega^{\pm(\ell+k)} v_k^\pm, \qquad G_{321} v_k^\pm = \omega^{\mp(\ell+k)} v_k^\pm\,. \tag{4.20}$$

Note that the representation V is characterized or labeled by (λ, n, ℓ) where n is a non-negative integer and $\ell \in \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ with $3\mathbb{Z} = \{3z \mid z \in \mathbb{Z}\}$ the set of multiples of 3.

We will show that the set $\mathcal{B}$ spans the $\mathcal{O}St_3$ invariant subspace containing v_0, which if V is irreducible must be all of V. Moreover, in case the O_0 eigenvalues are all distinct then $\mathcal{B}$ forms a basis for the irreducible representation V. Hereto, we determine the action of all elements on $\mathcal{B}$.

The explicit action of G_{23} and G_{31} follows from (4.20) as

$$G_{23} v_k^\pm = G_{12} G_{123} v_k^\pm = \omega^{\pm(\ell+k)} G_{12} v_k^\pm = \omega^{\pm(\ell+k)} v_k^\mp, \quad G_{31} v_k^\pm = G_{12} G_{321} v_k^\pm = \omega^{\mp(\ell+k)} v_k^\mp\,, \tag{4.21}$$

and in turn the action of $N_\pm$ as defined by (4.2),

$$N_+ v_k^\pm = (G_{12} + \omega G_{23} + \omega^2 G_{31}) v_k^\pm = (1 + \omega^{1\pm\ell\pm k} + \omega^{-1\mp\ell\mp k}) v_k^\mp = 3\, \mathbf{1}_{3\mathbb{Z}}(\ell + k \pm 1) v_k^\mp\,. \tag{4.22}$$

where we employ the notation,

$$\mathbf{1}_{3\mathbb{Z}}(k) = \frac{1 + \omega^k + \omega^{-k}}{3} = \begin{cases} 1 & \text{if} \quad k \equiv 0 \pmod 3 \iff k \in 3\mathbb{Z} \\ 0 & \text{if} \quad k \equiv 1, 2 \pmod 3 \iff k \notin 3\mathbb{Z}\,. \end{cases}$$

Similarly, we have

$$N_- v_k^{\pm} = 3\,\mathbf{1}_{3\mathbb{Z}}(\ell + k \mp 1) v_k^{\mp}\,. \tag{4.23}$$

By (4.5), we find that the linear combinations of G_{123} and G_{321} denoted by N_+N_- and N_-N_+, see (4.8) and (4.9), satisfy the polynomial equation $X^2 - 9X = 0$. Consequently their eigenvalues are 0 and 9. Following (4.22) and (4.23), we obtain the diagonal actions

$$N_+N_- v_k^{\pm} = 9\,\mathbf{1}_{3\mathbb{Z}}(\ell + k \mp 1) v_k^{\pm}\,, \qquad \text{and} \qquad N_-N_+ v_k^{\pm} = 9\,\mathbf{1}_{3\mathbb{Z}}(\ell + k \pm 1) v_k^{\pm}\,. \tag{4.24}$$

We already know that $K_- v_k^+ = v_{k+1}^+$ and $K_+ v_k^- = v_{k+1}^-$ with $v_l^{\pm} = 0$ for $l > n$. Using (4.13) we find the action of K_+ and K_- on the rest of the basis $\mathcal{B}$:

$$\begin{aligned} K_+ v_k^+ &= K_+K_- v_{k-1}^+ = -\big(O_{123}^2 + (O_0 - 1/2)^2\big)\big((O_0 - 1/2)^2 - \kappa^2 N_+N_-\big) v_{k-1}^+ \\ &= -\big(\Lambda^2 + (\lambda - k + 1/2)^2\big)\big((\lambda - k + 1/2)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + k - 2)\big) v_{k-1}^+\,, \end{aligned} \tag{4.25}$$

and similarly

$$\begin{aligned} K_- v_k^- &= K_-K_+ v_{k-1}^- = -\big(O_{123}^2 + (O_0 + 1/2)^2\big)\big((O_0 + 1/2)^2 - \kappa^2 N_-N_+\big) v_{k-1}^- \\ &= -\big(\Lambda^2 + (\lambda - k + 1/2)^2\big)\big((\lambda - k + 1/2)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + k - 2)\big) v_{k-1}^-\,. \end{aligned} \tag{4.26}$$

For ease of notation, we define the expression $A(k)$ to denote these actions, that is

$$A(k) = -\big(\Lambda^2 + (\lambda - k + 1/2)^2\big)\big((\lambda - k + 1/2)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + k + 1)\big)\,, \tag{4.27}$$

such that

$$\begin{aligned} K_+ v_k^+ &= A(k) v_{k-1}^+ = K_+K_- v_{k-1}^+\,, \qquad & K_- v_k^- &= A(k) v_{k-1}^- = K_-K_+ v_{k-1}^- \\ K_+K_- v_k^- &= A(k) K_+ v_{k-1}^- = A(k) v_k^-\,, \qquad & K_-K_+ v_k^+ &= A(k) K_- v_{k-1}^+ = A(k) v_k^+\,. \end{aligned} \tag{4.28}$$

For the action of O_+ and O_- on $\mathcal{B}$, we set out as follows. Using (4.3) we have

$$K_{\pm} = \frac{1}{2}\{O_0, O_{\pm}\} = O_{\pm}O_0 + \frac{1}{2}[O_0, O_{\pm}] = O_{\pm}O_0 \pm \frac{1}{2}O_{\pm} + \kappa O_{123} N_{\pm}\,. \tag{4.29}$$

As $K_- v_k^+ = v_{k+1}^+$ for $k \leq n - 1$, we find

$$\begin{aligned} v_{k+1}^+ = K_- v_k^+ &= O_-\Big(O_0 - \frac{1}{2}\Big) v_k^+ + \kappa O_{123} N_- v_k^+ \\ &= (\lambda - k - 1/2) O_- v_k^+ + 3\kappa\Lambda \mathbf{1}_{3\mathbb{Z}}(\ell + k - 1) v_k^-\,. \end{aligned}$$

Hence, for $\lambda - k - 1/2 \neq 0$ (we will handle the zero case after determining the possible values for λ)

$$O_- v_k^+ = \frac{1}{\lambda - k - 1/2} v_{k+1}^+ - \frac{3\kappa\Lambda}{\lambda - k - 1/2} \mathbf{1}_{3\mathbb{Z}}(\ell + k - 1) v_k^-\,. \tag{4.30}$$

The action of O_- on v_n^+ is consistent with (4.30) by letting $v_{n+1}^+ = 0$ as by

$$0 = K_- v_n^+ = O_-\Big(O_0 - \frac{1}{2}\Big)v_n^+ + \kappa O_{123}N_- v_n^+ = (\lambda - n - 1/2)O_- v_n^+ + 3\kappa\Lambda \mathbf{1}_{3\mathbb{Z}}(\ell + n - 1)v_n^-$$

we have, for $\lambda - n - 1/2 \neq 0$

$$O_- v_n^+ = -\frac{3\kappa\Lambda}{\lambda - n - 1/2}\mathbf{1}_{3\mathbb{Z}}(\ell + n - 1)v_n^- \, .$$

The action (4.26) together with (4.29) yields the action of O_- on v_k^-. On the one hand

$$K_- v_k^- = O_-\Big(O_0 - \frac{1}{2}\Big)v_k^- + \kappa O_{123}N_- v_k^- = -(\lambda - k + 1/2)O_- v_k^- + 3\kappa\Lambda \mathbf{1}_{3\mathbb{Z}}(\ell + k + 1)v_k^+ \, ,$$

while on the other hand for $k \geq 1$ we have $K_- v_k^- = K_- K_+ v_{k-1}^- = A(k)v_{k-1}^-$, so for $\lambda - k + 1/2 \neq 0$

$$O_- v_k^- = \frac{-A(k)}{\lambda - k + 1/2}v_{k-1}^- + \frac{3\kappa\Lambda}{\lambda - k + 1/2}\mathbf{1}_{3\mathbb{Z}}(\ell + k + 1)v_k^+ \, . \tag{4.31}$$

For $\lambda \neq 1/2$, this is consistent with the action of O_- on v_0^- by letting $v_{-1}^- = 0$ as

$$0 = K_- v_0^- = O_-\Big(O_0 - \frac{1}{2}\Big)v_0^- + \kappa O_{123}N_- v_0^- = \Big(-\lambda - \frac{1}{2}\Big)O_- v_0^- + 3\kappa\Lambda \mathbf{1}_{3\mathbb{Z}}(\ell + 1)v_0^+ \, .$$

In a similar way we obtain the action of O_+ to be given by

$$O_+ v_k^- = \frac{-1}{\lambda - k - 1/2}v_{k+1}^- + \frac{3\kappa\Lambda}{\lambda - k - 1/2}\mathbf{1}_{3\mathbb{Z}}(\ell + k - 1)v_k^+ \tag{4.32}$$

and since $K_+ v_k^+ = K_+ K_- v_{k-1}^+ = A(k)v_{k-1}^+$

$$O_+ v_k^+ = \frac{A(k)}{\lambda - k + 1/2}v_{k-1}^+ - \frac{3\kappa\Lambda}{\lambda - k + 1/2}\mathbf{1}_{3\mathbb{Z}}(\ell + k + 1)v_k^- \, . \tag{4.33}$$

The actions of all elements of the algebra $\mathcal{O}St_3$ are fixed by the four constants n, λ, Λ, ℓ, where ℓ is integer and n is a positive integer. We will now examine all possible values which lead to finite irreducible representations. The conditions for the dimension to be finite, $K_+ v_0^+ = 0$ and $K_- v_n^+ = 0$ can be combined with the results (4.13) and (4.14) of Proposition 4.11 to find

$$\begin{cases} K_- K_+ v_0^+ = 0 \\ K_+ K_- v_n^+ = 0 \end{cases} \iff \begin{cases} -\big(O_{123}^2 + (O_0 + 1/2)^2\big)\big((O_0 + 1/2)^2 - \kappa^2 N_- N_+\big)v_0^+ = 0 \, , \\ -\big(O_{123}^2 + (O_0 - 1/2)^2\big)\big((O_0 - 1/2)^2 - \kappa^2 N_+ N_-\big)v_n^+ = 0 \, . \end{cases}$$

When plugging in the appropriate actions, (4.19) and (4.24), this yields the system of equations

$$\begin{cases} -\big(\Lambda^2 + (\lambda + 1/2)^2\big)\big((\lambda + 1/2)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + 1)\big) = 0 \\ -\big(\Lambda^2 + (\lambda - n - 1/2)^2\big)\big((\lambda - n - 1/2)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + n - 1)\big) = 0 \end{cases} \tag{4.34}$$

to be satisfied for every set of valid values for n, λ, Λ, ℓ. We distinguish three distinct classes of solutions of (4.34) depending on which pair of factors are zero and on the value of ℓ, which decides whether the function $\mathbf{1}_{3\mathbb{Z}}$ is 0 or 1.

(a) $\Lambda^2 + (\lambda + 1/2)^2 = 0$ and $(\lambda - n - 1/2)^2 - 9\kappa^2 = 0$, that is when $\mathbf{1}_{3\mathbb{Z}}(\ell + n - 1) = 1$

(b) $\Lambda^2 + (\lambda + 1/2)^2 = 0$ and $(\lambda - n - 1/2)^2 = 0$, that is when $\mathbf{1}_{3\mathbb{Z}}(\ell + n - 1) = 0$

(c) $\Lambda^2 + (\lambda + 1/2)^2 = 0$ and $\Lambda^2 + (\lambda - n - 1/2)^2 = 0$

Note that there are two more cases we have omitted from our classification. We briefly expand on this before continuing. First, we have the case where $\Lambda^2 + (\lambda - n - 1/2)^2 = 0$ and $(\lambda + 1/2)^2 - 9\kappa^2 = 0$, that is when $\mathbf{1}_{3\mathbb{Z}}(\ell + 1) = 1$. This will turn out to be equivalent to case (a) after renaming v_k^- to v_{n-k}^+ and vice versa, as seen from the action of O_0. Similarly, the case where $\Lambda^2 + (\lambda - n - 1/2)^2 = 0$ and $(\lambda + 1/2)^2 = 0$ will be equivalent with case (b).

We continue with the classification of all finite-dimensional irreducible representations. Note that in all three cases $\Lambda^2 + (\lambda + 1/2)^2 = 0$ which fixes the value $\Lambda = \varepsilon\, i\,(\lambda + 1/2)$, up to a sign $\varepsilon = \pm 1$, and thus the action of O_{123} in function of λ. This leaves a freedom in the choice of sign of Λ. In the algebra relations (3.5), and (4.3)–(4.4) by extension, the central element O_{123} is always accompanied by a single factor κ. We note that when one permits also negative values for κ, the sign of Λ can always be chosen such that the product κO_{123} has a positive action.

For each case we need to check whether the vectors (4.18) are independent. Since v_0 is a generating vector of V, irreducibility can be checked by verifying that for each $v_k^\pm$ there is an algebra element $X_k^\pm$ such that $X_k^\pm v_k^\pm = v_0$. Note that $(K_+)^k v_k^+ = A(k)A(k-1)\cdots A(1)v_0$ by (4.28), while $G_{12}(K_-)^k v_k^- = A(k)A(k-1)\cdots A(1)G_{12}v_0^-$ and $G_{12}v_0^- = v_0^+ = v_0$. Hence, in order to have an irreducible representation, the expression $A(k)$, given by (4.27), must be non-zero for $k \in \{1, \dots, n\}$. Plugging in $\Lambda^2 = -(\lambda + 1/2)^2$, we find

$$\begin{aligned} A(k) &= -\big(- (\lambda + 1/2)^2 + (\lambda - k + 1/2)^2 \big)\big((\lambda - k + 1/2)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + k + 1) \big) \\ &= k(2\lambda + 1 - k)\big((\lambda - k + 1/2)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + k + 1) \big)\,. \end{aligned} \tag{4.35}$$

We now work out the explicit value of λ and ℓ for the three cases.

4.1 *Case (a)*

For the case (a) we have $\mathbf{1}_{3\mathbb{Z}}(\ell + n - 1) = 1$ or thus $\ell \equiv 2n + 1 \pmod 3$ which fixes the eigenvalues of the reflections, e.g. $G_{123}v_k^\pm = \omega^{\pm(2n+1+k)} v_k^\pm$, see (4.20) and (4.21). Moreover, from $(\lambda - n - 1/2)^2 - 9\kappa^2 = 0$ we find $\lambda = n \pm 3\kappa + 1/2$. With unitary representations in mind, we first handle the case $\lambda = n + 3\kappa + 1/2$. The action of O_0 on $\mathcal{B}$ is now given by

$$O_0 v_k^\pm = \pm(n - k + 3\kappa + 1/2) v_k^\pm, \qquad k \in \{0, \dots, n\}\,. \tag{4.36}$$

We see that for a positive parameter κ every vector of the set $\mathcal{B}$ has a distinct eigenvalue for O_0, so the elements of $\mathcal{B}$ are independent. Note that $\lambda - k \pm 1/2 \neq 0$ for $\kappa > 0$ and $k \in \{0, 1, \ldots, n\}$ so the previously determined actions of the elements of $\mathcal{O}St_3$ on $\mathcal{B}$, see e.g. (4.30), are all well-defined. Moreover, the expression (4.35) is readily seen to be non-zero for all positive values κ and $k \in \{1, \ldots, n\}$. This shows that, for positive κ, the set $\mathcal{B}$ forms a basis for the $\mathcal{O}St_3$ invariant subspace containing v_0, which if V is irreducible must be all of V. The actions of the other generators of $\mathcal{O}St_3$ are given by (4.30),(4.31),(4.32),(4.33).

Next, we consider the other choice $\lambda = n - 3\kappa + 1/2$. The O_0-eigenvalues (4.19) are not necessarily all distinct when $6\kappa \in \{1, 2, \ldots, 2n + 1\}$. Moreover, the condition for irreducibility now leads to disallowed values for κ, namely $6\kappa \notin \{n + 2, n + 3, \ldots, 2n + 1\} \cup (\{1, \ldots, n\} \cap 3\mathbb{Z})$, while also $3\kappa - 1 \notin \{0, 1, \ldots, n - 1\} \cap 3\mathbb{Z}$, and $3\kappa - 2 \notin \{0, 1, \ldots, n-2\} \cap 3\mathbb{Z}$. Hence $\mathcal{B}$ would form a basis for an irreducible $\mathcal{O}St_3$ representation if and only if κ is not allowed to take on these specific values. As a consequence this choice will not lead to unitary representations for general values of κ.

Finally, note that the choice $\lambda = n - 3\kappa + 1/2$ with κ positive is equivalent to considering negative values for κ when $\lambda = n + 3\kappa + 1/2$. For a given real value of κ, the sign accompanying κ in $\lambda = n \pm 3\kappa + 1/2$ can thus always be chosen such that λ is positive. For negative κ, the disallowed values follow immediately by replacing κ by $-\kappa$ in the previously obtained conditions. These values correspond in fact to those of the S_3 Dunkl operator singular parameter set for which no intertwining operators exist [6, 8, 19].

4.2 *Case (b)*

For the case (b) we have $\mathbf{1}_{3\mathbb{Z}}(\ell + n - 1) = 0$ or thus $\ell \not\equiv 2n + 1 \pmod 3$. This gives two distinct options for the eigenvalues of G_{123} and in turn for the actions of the other reflections. The condition $(\lambda - n - 1/2)^2 = 0$ implies that $\lambda = n + 1/2$, which again yields $2n + 2$ distinct O_0 eigenvalues

$$O_0 v_k^{\pm} = \pm(n - k + 1/2) v_k^{\pm}, \qquad k \in \{0, \ldots, n\}\,.$$

For the case at hand the acquired actions (4.30),(4.31),(4.32),(4.33) do not lead to the full action of O_- or O_+, as we would have to divide by zero. Indeed, we have $O_0 v_n^+ = \frac{1}{2} v_n^+$ and $O_0 v_n^- = -\frac{1}{2} v_n^-$ so the denominator in (4.30) would become zero for $k = n$. We determine the action of O_- on v_n^+ and O_+ on v_n^- in another way. By means of relation (4.3) acting on v_n^+ we find

$$\begin{aligned}
& (O_0 O_- - O_- O_0) v_n^+ = -O_- v_n^+ + 2\kappa O_{123} N_- v_n^+ \\
\iff \quad & O_0 O_- v_n^+ - \frac{1}{2} O_- v_n^+ = -O_- v_n^+ \\
\iff \quad & O_0 O_- v_n^+ = -\frac{1}{2} O_- v_n^+\,,
\end{aligned}$$

which implies $O_- v_n^+ = \beta_- v_n^-$ for some constant β_-. In the same manner we find $O_+ v_n^- = \beta_+ v_n^+$ for some constant β_+. Using the interaction of G_{12} and $O_\pm$, see (4.10), we find

$$\beta_- v_n^- = O_- v_n^+ = O_- G_{12} v_n^- = -G_{12} O_+ v_n^- = -G_{12} \beta_+ v_n^+ = -\beta_+ v_n^- ,$$

while by (4.7) we have

$$\beta_+ \beta_- v_n^+ = O_+ O_- v_n^+ = \left(-(O_{123})^2 - \left(O_0 - \frac{1}{2}\right)^2 + \kappa^2 N_+ N_- \right) v_n^+ = -\Lambda^2 v_n^+ .$$

Hence $\beta_- = -\beta_+ = \pm\Lambda = \pm i(n+1)$. Note that we have an extra freedom in the choice of sign, besides the one present for the sign of Λ.

Finally, we check whether the expression (4.35) is non-zero for $k \in \{1, \dots, n\}$. For $\lambda = n + 1/2$, only the factor $(n-k+1)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell + k + 1)$ could become zero. Hereto, we distinguish between the two options for ℓ. For $\ell \equiv 2n \pmod 3$ this gives the conditions

$$(k+2)^2 - 9\kappa^2 \neq 0 \qquad \text{for } k \in \{0, \dots, n-1\} \cap 3\mathbb{Z} ,$$

while $\ell \equiv 2n + 2 \pmod 3$ leads to

$$(k+1)^2 - 9\kappa^2 \neq 0 \qquad \text{for } k \in \{0, \dots, n\} \cap 3\mathbb{Z} .$$

This shows that $\mathcal{B}$ forms a basis for an irreducible $\mathcal{O}St_3$ representation if and only if κ is not allowed to take on some specific values.

4.3 *Case (c)*

As n is positive, the conditions $\Lambda^2 + (\lambda + 1/2)^2 = 0$ and $\Lambda^2 + (\lambda - n - 1/2)^2 = 0$ lead to $\lambda = n/2$ and $\Lambda^2 = -(n+1)^2/4$. In this scenario, the vectors $v_k^\pm$ and $v_{n-k}^\mp$ have the same O_0 eigenvalue:

$$O_0 v_k^\pm = \pm\left(\frac{n}{2} - k\right) v_k^\pm, \qquad O_0 v_{n-k}^\mp = \mp\left(\frac{n}{2} - (n-k)\right) v_{n-k}^\mp = \pm\left(\frac{n}{2} - k\right) v_{n-k}^\mp .$$

The G_{123} eigenvalues (4.20) for $v_k^\pm$ and $v_{n-k}^\mp$ are given by

$$G_{123} v_k^\pm = \omega^{\pm(\ell+k)} v_k^\pm, \qquad G_{123} v_{n-k}^\mp = \omega^{\mp(\ell+n-k)} v_{n-k}^\mp = \omega^{\mp(n-\ell)} \omega^{\pm(\ell+k)} v_{n-k}^\mp .$$

Two different scenarios now occur depending on the value of ℓ, that is whether $\ell \equiv n \pmod 3$ or not. We distinguish in the first place with respect to the parity of n.

4.3.1 *Even n*

For n an even integer, $\lambda = n/2$ is an integer so the previously determined actions of the elements of $\mathcal{O}St_3$ on $\mathcal{B}$ are all well-defined. When $\ell \equiv n \pmod 3$, the space generated by v_0 is comprised of two irreducible components and decomposes as follows. The vectors v_λ^+ and v_λ^- both have 0 as O_0 eigenvalue and $G_{12} v_\lambda^+ = v_\lambda^-$. Hence, defining

$u_0^+ = v_\lambda^+ + v_\lambda^-$ and $u_0^- = v_\lambda^+ - v_\lambda^-$, we have $G_{12}u_0^+ = u_0^+$ and $G_{12}u_0^- = -u_0^-$, while $O_0 u_0^\pm = 0$ and furthermore $G_{123}u_0^\pm = \omega^{n+\lambda}u_0^\pm = u_0^\pm$. If we now define $u_{-k}^\pm = (K_-)^k u_0^\pm$ and $u_k^\pm = (K_+)^k u_0^\pm$ for $k \in \{1, \dots, \lambda\}$, then the sets

$$\mathcal{B}^+ = \{u_k^+ \mid k = -\lambda, \dots, 0, \dots, \lambda\} \qquad \mathcal{B}^- = \{u_k^- \mid k = -\lambda, \dots, 0, \dots, \lambda\}$$

each form the basis for an $\mathcal{O}\mathrm{St}_3$ invariant subspace of dimension $n+1$. We go over the actions on these spaces. We have $O_0 u_k^\pm = k\, u_k^\pm$ and $G_{123}u_k^\pm = \omega^{-k}u_k^\pm$. Moreover, $G_{12}u_k^\pm = \pm u_{-k}^\pm$, while $G_{23}u_k^\pm = \pm\omega^{-k}u_{-k}^\pm$ and $G_{31}u_k^\pm = \pm\omega^k u_{-k}^\pm$. For positive k, we have by definition $K_+u_k^\pm = u_{k+1}^\pm$ and $K_-u_{-k}^\pm = u_{-k-1}^\pm$. The other actions are found as follows. Note that for positive k,

$$u_k^\pm = (K_+)^k u_0^\pm = (K_+)^k(v_\lambda^+ \pm v_\lambda^-) = \prod_{l=0}^{k-1} A(\lambda - l)v_{\lambda-k}^+ \pm v_{\lambda+k}^-$$

and similarly

$$u_{-k}^\pm = (K_-)^k u_0^\pm = (K_-)^k(v_\lambda^+ \pm v_\lambda^-) = v_{\lambda+k}^+ \pm \prod_{l=0}^{k-1} A(\lambda - l)v_{\lambda-k}^- \,.$$

Again for positive k, we then find

$$K_+u_{-k}^\pm = K_+K_-u_{-k+1}^\pm = A(\lambda+k)u_{-k+1}^\pm\,, \qquad K_-u_k^\pm = K_-K_+u_{k-1}^\pm = A(\lambda+k)u_{k-1}^\pm\,.$$

Here we used $A(\lambda+k) = A(\lambda-k+1)$, which is readily verified from (4.35) with $\lambda = n/2$ and $\ell \equiv n \pmod 3$.

We check whether the expression (4.35) is non-zero for $k \in \{1, \dots, n\}$. For $\lambda = n/2$, the only factor of (4.35) with $\ell \equiv n \pmod 3$ that could become zero is

$$\left(\frac{n+1}{2} - k\right)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(n+k+1)\,.$$

This leads to the conditions

$$\left(k + \frac{3}{2}\right)^2 - 9\kappa^2 \neq 0 \qquad \text{for } k \in \{-\lambda+2, \dots, \lambda+1\} \cap 3\mathbb{Z}\,,$$

which shows that, except for specific κ values, $\mathcal{B}^+$ and $\mathcal{B}^-$ each form the basis for an $\mathcal{O}\mathrm{St}_3$ invariant space.

If $\ell \not\equiv n \pmod 3$, then $v_k^\pm$ and $v_{n-k}^\mp$ have different eigenvalues for G_{123}. We check whether the expression (4.35) is non-zero for $k \in \{1, \dots, n\}$. For $\lambda = n/2$, the only factor of (4.35) that could become zero is

$$\left(\frac{n+1}{2} - k\right)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(\ell+k+1)\,.$$

Hereto, we distinguish between the two options for ℓ. For $\ell \equiv n+1 \pmod 3$ this gives the conditions

$$\left(k+\frac{1}{2}\right)^2 - 9\kappa^2 \neq 0 \qquad \text{for } k \in \{-\lambda, \dots, \lambda-1\} \cap 3\mathbb{Z},$$

while $\ell \equiv n-1 \pmod 3$ also leads to

$$\left(k+\frac{1}{2}\right)^2 - 9\kappa^2 \neq 0 \qquad \text{for } k \in \{-\lambda, \dots, \lambda-1\} \cap 3\mathbb{Z}.$$

This shows that $\mathcal{B}$ forms a basis for an irreducible $\mathcal{O}\mathrm{St}_3$ representation if and only if κ is not allowed to take on some specific values.

4.3.2 *Odd n*

Next, we consider the case where n is an odd integer. As $\lambda = n/2$ is a half-integer now there exists an integer value $k_0 = \lambda - 1/2 = (n-1)/2$ such that

$$O_0 v_{k_0}^+ = \frac{1}{2} v_{k_0}^+, \qquad O_0 v_{k_0+1}^- = \frac{1}{2} v_{k_0+1}^-, \qquad O_0 v_{k_0+1}^+ = -\frac{1}{2} v_{k_0+1}^+, \qquad O_0 v_{k_0}^- = -\frac{1}{2} v_{k_0}^-,$$

These specific eigenvalues have as a consequence that the previously acquired actions (4.30), (4.31), (4.32), (4.33) do not lead to the full action of O_- or O_+, as we would have to divide by zero. Using (4.29), however, we find

$$v_{k_0+1}^+ = K_- v_{k_0}^+ = O_-\left(O_0 - \frac{1}{2}\right) v_{k_0}^+ + \kappa O_{123} N_- v_{k_0}^+ = 3\kappa\Lambda \mathbf{1}_{3\mathbb{Z}}(\ell + k_0 - 1) v_{k_0}^-.$$

Since $k_0 < n$, the action $K_- v_{k_0}^+$ may not result in zero by the assumption of minimality on n, so we must have $\mathbf{1}_{3\mathbb{Z}}(\ell + k_0 - 1) = 1$ or thus $\ell \equiv 2k_0 + 1 \equiv n \pmod 3$. It follows that the vectors $v_{k_0+1}^+$ and $v_{k_0}^-$, which have the same O_0 eigenvalue, are not linearly independent as now $v_{k_0+1}^+ = 3\kappa\Lambda v_{k_0}^-$. In the same way, we find $v_{k_0+1}^- = 3\kappa\Lambda v_{k_0}^+$. By means of these results and the actions (4.25) and (4.26) of $K_\pm$ we obtain that the vector v_k^- is proportional to v_{n-k}^+ for every $k \in \{0, 1, \dots, n\}$. Indeed, by (4.26) we have for instance

$$v_{k_0+2}^+ = K_- v_{k_0+1}^+ = 3\kappa\Lambda\, K_- v_{k_0}^- = -3\kappa\Lambda\left(\Lambda^2 + 1\right) v_{k_0-1}^-.$$

However, acting on $v_{k_0}^+$ with $[O_0, O_-]$, see relation (4.3), we find an equation which can never be satisfied unless $v_{k_0+1}^+ = 0$. Hence, we have no representations for odd n in case (c).

4.4 *Unitary representations*

To find irreducible unitary representations we check which of the irreducible representations admit an invariant positive definite Hermitian form. Hereto, we introduce an anti-linear anti-multiplicative involution $X \mapsto X^\dagger$ compatible with the algebraic relations (3.5) of the algebra $\mathcal{O}\mathrm{St}_3$. This involution has the properties $(aX + bY)^\dagger = \overline{a}X^\dagger + \overline{b}Y^\dagger$ and $(XY)^\dagger = Y^\dagger X^\dagger$ for $X, Y \in \mathcal{O}\mathrm{St}_3$ and $a, b \in \mathbb{C}$, where $\overline{a}$ denotes complex conjugation.

For real κ, the algebraic relations (3.5) are compatible with the star conditions

$$O_{12}^{\dagger} = -O_{12} \qquad O_{23}^{\dagger} = -O_{23} \qquad O_{31}^{\dagger} = -O_{31} \qquad O_{123}^{\dagger} = -O_{123}\,.$$

and

$$G_{12}^{\dagger} = G_{12} \qquad G_{23}^{\dagger} = G_{23} \qquad G_{31}^{\dagger} = G_{31} \qquad G_{123}^{\dagger} = G_{321}\,.$$

REMARK 4.4: Note that in the setting of [5] these conditions correspond precisely to the coordinate and momentum operators being self-adjoint, $\hat{x}_j^{\dagger} = \hat{x}_j$ and $\hat{p}_j^{\dagger} = \hat{p}_j$.

In terms of Definition 4.1, this leads to the relations (4.3)–(4.4) being compatible with the star conditions

$$O_{123}^{\dagger} = -O_{123} \qquad O_0^{\dagger} = O_0, \qquad O_{\pm}^{\dagger} = O_{\mp}, \qquad K_{\pm}^{\dagger} = K_{\mp}, \qquad N_{\pm}^{\dagger} = N_{\mp}\,. \tag{4.37}$$

We show that if the value of κ is suitably restricted, the representation V is unitary under (4.37). Hereto, we introduce a sesquilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for $X \in \mathcal{O}St_3$ and $v, w \in V$

$$\langle Xv, w \rangle = \langle v, X^{\dagger} w \rangle\,.$$

The condition $O_0^{\dagger} = O_0$ implies that vectors with different O_0 eigenvalues are orthogonal, so the previously determined bases are in fact orthogonal. Hence, we may define the form $\langle \cdot, \cdot \rangle$ by putting

$$\langle v_k^+, v_l^+ \rangle = h_k\, \delta_{k,l}\,, \qquad \langle v_k^+, v_l^- \rangle = 0\,,$$

where we can freely let $h_0 = 1$ or $\langle v_0^+, v_0^+ \rangle = 1$. Note that

$$\langle v_k^-, v_l^- \rangle = \langle G_{12} v_k^+, G_{12} v_l^+ \rangle = \langle G_{12} G_{12} v_k^+, v_l^+ \rangle = \langle v_k^+, v_l^+ \rangle = h_k\, \delta_{k,l}\,.$$

In order to be an inner product we need $h_k > 0$ for $k \geq 0$. Using the star condition $K_-^{\dagger} = K_+$ and using $K_+ v_k^+ = A(k) v_{k-1}^+$ with (4.27), we have for $k \geq 1$

$$h_k = \langle v_k^+, v_k^+ \rangle = \langle K_- v_{k-1}^+, v_k^+ \rangle = \langle v_{k-1}^+, K_+ v_k^+ \rangle = A(k) \langle v_{k-1}^+, v_{k-1}^+ \rangle = A(k) h_{k-1}\,. \tag{4.38}$$

In this way we arrive at the condition $A(k) > 0$ for $1 \leq k \leq n$, which is obviously satisfied for the case (a) with the choice $\lambda = n + 3\kappa + 1/2$. This will constitute the only class of unitary representations without further restrictions on the non-negative parameter κ. For the other choice of case (a), $\lambda = n - 3\kappa + 1/2$, this only holds when κ is restricted to $|\kappa| < 1/3$. For the case (b), we have two options for ℓ, leading to different restrictions on the value of κ in order for $A(k) > 0$ to hold for $1 \leq k \leq n$. If $\ell \equiv 2n \pmod 3$, then κ must satisfy $|\kappa| < 2/3$, while $\ell \equiv 2n + 2 \pmod 3$ implies the condition $|\kappa| < 1/3$. For the case (c) with n even we have $|\kappa| < 1/2$ if $\ell \equiv n \pmod 3$ and $|\kappa| < 1/6$ if $\ell \not\equiv n \pmod 3$.

Given an inner product we can introduce the orthonormal basis

$$w_k^{\pm} = \frac{v_{n-k}^{\pm}}{\|v_{n-k}^{\pm}\|} \qquad (k = 0, 1, \ldots, n-1, n)$$

where $\|v^{\pm}_{n-k}\| = \sqrt{\langle v^{\pm}_{n-k}, v^{\pm}_{n-k}\rangle} = \sqrt{h_{n-k}}$. We find using (4.38)

$$K_- w^+_k = K_- \frac{v^+_{n-k}}{\|v^+_{n-k}\|} = \frac{v^+_{n-k+1}}{\sqrt{h_{n-k}}} = \sqrt{A(n-k+1)}\, w^+_{k-1}$$

and by (4.28)

$$K_+ w^+_k = K_+ \frac{v^+_{n-k}}{\|v^+_{n-k}\|} = A(n-k)\frac{v^+_{n-k-1}}{\sqrt{h_{n-k}}} = \sqrt{A(n-k)}\, w^+_{k+1}\,,$$

while similarly

$$K_- w^-_k = K_- \frac{v^-_{n-k}}{\|v^-_{n-k}\|} = A(n-k)\frac{v^-_{n-k-1}}{\sqrt{h_{n-k}}} = \sqrt{A(n-k)}\, w^-_{k+1}$$

and

$$K_+ w^-_k = K_+ \frac{v^-_{n-k}}{\|v^-_{n-k}\|} = \frac{v^-_{n-k+1}}{\sqrt{h_{n-k}}} = \sqrt{A(n-k+1)}\, w^-_{k-1}\,.$$

Returning to the case (a), the right-hand side follows from

$$A(k) = k(2n+6\kappa+2-k)\big((n+3\kappa-k+1)^2 - 9\kappa^2 \mathbf{1}_{3\mathbb{Z}}(2n+k+2)\big)\,.$$

We summarize all actions for the case (a) in the following proposition.

PROPOSITION 4.5: *For a given positive parameter κ and a choice of sign $\varepsilon = \pm 1$, we have an irreducible representation of $\mathcal{O}\mathrm{St}_3$ of dimension $2n+2$ for every non-negative integer n. This representation is unitary, corresponding to the star conditions* (4.37). *The actions of the $\mathcal{O}\mathrm{St}_3$ operators on a set of basis vectors $w^+_0, w^+_1, \ldots, w^+_n$ and $w^-_0, w^-_1, \ldots, w^-_n$ are given by:*

$$O_0 w^{\pm}_k = \pm\left(k + \frac{1}{2} + 3\kappa\right) w^{\pm}_k \tag{4.39}$$

$$O_{123} w^{\pm}_k = \varepsilon\, i\,(n+1+3\kappa)\, w^{\pm}_k \tag{4.40}$$

$$K_+ w^+_k = \begin{cases} \sqrt{(k+1)(n-k)(n+k+2+6\kappa)(k+1+6\kappa)}\, w^+_{k+1} & \textit{if } k \equiv 2 \pmod 3 \\ (k+1+3\kappa)\sqrt{(n-k)(n+k+2+6\kappa)}\, w^+_{k+1} & \textit{if } k \not\equiv 2 \pmod 3 \end{cases} \tag{4.41}$$

$$K_+ w^-_k = \begin{cases} \sqrt{k(n-k+1)(n+k+1+6\kappa)(k+6\kappa)}\, w^+_{k-1} & \textit{if } k \equiv 0 \pmod 3 \\ (k+3\kappa)\sqrt{(n-k+1)(n+k+1+6\kappa)}\, w^+_{k-1} & \textit{if } k \not\equiv 0 \pmod 3 \end{cases} \tag{4.42}$$

$$K_- w^+_k = \begin{cases} \sqrt{k(n-k+1)(n+k+1+6\kappa)(k+6\kappa)}\, w^+_{k-1} & \textit{if } k \equiv 0 \pmod 3 \\ (k+3\kappa)\sqrt{(n-k+1)(n+k+1+6\kappa)}\, w^+_{k-1} & \textit{if } k \not\equiv 0 \pmod 3 \end{cases} \tag{4.43}$$

$$K_- w_k^- = \begin{cases} \sqrt{(k+1)(n-k)(n+k+2+6\kappa)(k+1+6\kappa)}\, w_{k+1}^- & \textit{if } k \equiv 2 \pmod 3 \\ (k+1+3\kappa)\sqrt{(n-k)(n+k+2+6\kappa)}\, w_{k+1}^- & \textit{if } k \not\equiv 2 \pmod 3 \end{cases} \tag{4.44}$$

while for O_+ *and* O_- *we have the following actions. If* $k \equiv 0 \pmod 3$ *then*

$$O_+ w_k^+ = \sqrt{(n-k)(n+k+2+6\kappa)}\, w_{k+1}^+ \tag{4.45}$$

$$O_+ w_k^- = -\frac{\sqrt{k(n-k+1)(n+k+1+6\kappa)(k+6\kappa)}}{k+3\kappa} w_{k-1}^- + \varepsilon\, i\, \frac{3\kappa(n+1+3\kappa)}{k+3\kappa} w_k^+ \tag{4.46}$$

$$O_- w_k^+ = \frac{\sqrt{k(n-k+1)(n+k+1+6\kappa)(k+6\kappa)}}{k+3\kappa} w_{k-1}^+ - \varepsilon\, i\, \frac{3\kappa(n+1+3\kappa)}{k+3\kappa} w_k^- \tag{4.47}$$

$$O_- w_k^- = -\sqrt{(n-k)(n+k+2+6\kappa)}\, w_{k+1}^- \tag{4.48}$$

else if $k \equiv 1 \pmod 3$ *then*

$$O_+ w_k^+ = \sqrt{(n-k)(n+k+2+6\kappa)}\, w_{k+1}^+ \tag{4.49}$$

$$O_+ w_k^- = -\sqrt{(n-k+1)(n+k+1+6\kappa)}\, w_{k-1}^- \tag{4.50}$$

$$O_- w_k^+ = \sqrt{(n-k+1)(n+k+1+6\kappa)}\, w_{k-1}^+ \tag{4.51}$$

$$O_- w_k^- = -\sqrt{(n-k)(n+k+2+6\kappa)}\, w_{k+1}^- \tag{4.52}$$

while if $k \equiv 2 \pmod 3$ *then*

$$O_+ w_k^+ = \frac{\sqrt{(k+1)(n-k)(n+k+2+6\kappa)(k+1+6\kappa)}}{k+1+3\kappa} w_{k+1}^+ - \varepsilon\, i\, \frac{3\kappa(n+1+3\kappa)}{k+1+3\kappa} w_k^- \tag{4.53}$$

$$O_+ w_k^- = -\sqrt{(n-k+1)(n+k+1+6\kappa)}\, w_{k-1}^- \tag{4.54}$$

$$O_- w_k^+ = \sqrt{(n-k+1)(n+k+1+6\kappa)}\, w_{k-1}^+ \tag{4.55}$$

$$O_- w_k^- = -\frac{\sqrt{(k+1)(n-k)(n+k+2+6\kappa)(k+1+6\kappa)}}{k+1+3\kappa} w_{k+1}^- + \varepsilon\, i\, \frac{3\kappa(n+1+3\kappa)}{k+1+3\kappa} w_k^+ . \tag{4.56}$$

For the realization of S_3 *within* $\mathcal{O}\mathrm{St}_3$ *we have the actions*

$$G_{12} w_k^\pm = w_k^\mp \qquad G_{23} w_k^\pm = \omega^{\pm(1-k)}\, w_k^\mp \qquad G_{31} w_k^\pm = \omega^{\pm(k-1)}\, w_k^\mp \tag{4.57}$$

$$G_{123} w_k^\pm = \omega^{\pm(1-k)}\, w_k^\pm \qquad G_{321} w_k^\pm = \omega^{\pm(k-1)}\, w_k^\pm . \tag{4.58}$$

We thought it to be instructive to include a diagram depicting the basis vectors and actions of Proposition 4.5 according to their eigenvalues for O_0 and G_{123}, see Figure 1. Using the notation $O_0 w_k^\pm = \pm\lambda_k w_k^\pm$, the distance between λ_0 and $-\lambda_0$ on the horizontal axis is $6\kappa + 1$, which depends on the value of the parameter κ.

5 EIGENFUNCTIONS OF THE SPHERICAL DIRAC-DUNKL OPERATOR

The irreducible unitary representations of case (a) as classified above have an explicit realization in the framework of Dunkl operators (2.1). Indeed, in the original construction of the algebra the symmetries O_{12}, O_{23}, O_{31} consist of Dunkl angular momentum operators with added reflection terms, see (3.2). When acting on an element in the kernel of the Dirac-Dunkl operator, the result is again in this kernel as the symmetries (anti-)commute with the Dirac-Dunkl operator. Furthermore, as the symmetries O_{12}, O_{23}, O_{31} are grade-preserving, it is no surprise that homogeneous polynomials of fixed degree in $\ker \underline{D}$ will form the desired representation spaces.

We set out to construct a basis for the space of Dunkl monogenics. Hereto, it is useful to emulate a setting similar to that of Definition 4.1 and Proposition 4.2 by means of a coordinate change:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{5.1}$$

The action of g_{12} on functions of (u, v, w) becomes very simple, flipping only the sign of u, $g_{12}f(u, v, w) = f(-u, v, w)$, while the other transpositions g_{23} and g_{31} act as follows

$$\begin{aligned} g_{23}f(u, v, w) &= f\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}v, \frac{\sqrt{3}}{2}u - \frac{1}{2}v, w\right), \\ g_{31}f(u, v, w) &= f\left(\frac{1}{2}u - \frac{\sqrt{3}}{2}v, -\frac{\sqrt{3}}{2}u - \frac{1}{2}v, w\right). \end{aligned}$$

For the Dunkl operators associated to this new coordinate basis we find the following explicit expressions: we have $\mathcal{D}_w = \partial_w$, while

$$\begin{aligned} \mathcal{D}_u &= \partial_u + \kappa \left(\frac{1 - g_{12}}{u} + \frac{1 - g_{23}}{u - \sqrt{3}v} + \frac{1 - g_{31}}{u + \sqrt{3}v} \right), \\ \mathcal{D}_v &= \partial_v + \kappa \left(\sqrt{3}\frac{1 - g_{23}}{-u + \sqrt{3}v} + \sqrt{3}\frac{1 - g_{31}}{u + \sqrt{3}v} \right). \end{aligned}$$

The commutation relations of $\mathcal{D}_u$, $\mathcal{D}_v$, $\mathcal{D}_w$ and u, v, w are given in Table 2. We see that in the coordinate frame of u, v, w the action of the reflection group is restricted to the (u, v)-plane.

As u, v, w form again an orthonormal basis of $\mathbb{R}^3$, the Laplace-Dunkl operator (2.3) can also be written as

$$\Delta = \mathcal{D}_u^2 + \mathcal{D}_v^2 + \mathcal{D}_w^2 .$$

By applying the same coordinate change (5.1) to the Clifford generators e_1, e_2, e_3, that is

$$e_u = \frac{1}{\sqrt{2}}(e_1 - e_2), \quad e_v = \frac{1}{\sqrt{6}}(e_1 + e_2 - 2e_3), \quad e_w = \frac{1}{\sqrt{3}}(e_1 + e_2 + e_3),$$

the Dirac-Dunkl operator can now be written as

$$\underline{D} = e_u \mathcal{D}_u + e_v \mathcal{D}_v + e_w \mathcal{D}_w \, .$$

Similarly, in these new coordinates the vector variable becomes $\underline{x} = ue_u + ve_v + we_w$ which squares to $\underline{x}^2 = u^2 + v^2 + w^2$ and the Euler operator is given by $\mathbb{E} = u\partial_u + v\partial_v + w\partial_w$. The triple e_u, e_v, e_w forms another basis of the Euclidean Clifford algebra since one readily verifies by means of the anti-commutation relations of e_1, e_2, e_3 that also

$$e_u^2 = e_v^2 = e_w^2 = 1, \qquad \{e_u, e_v\} = \{e_v, e_w\} = \{e_w, e_u\} = 0 \, .$$

For practical purposes, we will realize e_u, e_v, e_w by the Pauli matrices

$$e_u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_v = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad e_w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.2}$$

The generators of the realization of S_3 within $\mathcal{O}St_3$ in this framework become

$$G_{12} = g_{12} e_u \, , \quad G_{23} = g_{23} \frac{1}{2}(-e_u + \sqrt{3} e_v) \, , \quad G_{31} = g_{31} \frac{1}{2}(-e_u - \sqrt{3} e_v) \, ,$$

all of which anti-commute with $\underline{D}$. In terms of the Pauli matrices, we have

$$G_{12} = g_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_{23} = g_{23} \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad G_{31} = g_{31} \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}. \tag{5.3}$$

Similar to (3.2), in the u, v, w coordinates we obtain the following symmetries commuting with $\underline{D}$:

$$O_{uv} = u\mathcal{D}_v - v\mathcal{D}_u + \frac{1}{2} e_u e_v + \kappa e_u e_v (g_{12} + g_{23} + g_{31}) \, , \tag{5.4}$$

$$O_{vw} = v\mathcal{D}_w - w\mathcal{D}_v + \frac{1}{2} e_v e_w + \kappa \frac{3}{4} e_v e_w (g_{23} + g_{31}) + \kappa \frac{\sqrt{3}}{4} e_w e_u (g_{23} - g_{31}) \, , \tag{5.5}$$

$$O_{wu} = w\mathcal{D}_u - u\mathcal{D}_w + \frac{1}{2} e_w e_u + \kappa e_w e_u g_{12} + \kappa \frac{1}{4} e_u e_v (g_{23} + g_{31}) + \kappa \frac{\sqrt{3}}{4} e_v e_w (g_{23} - g_{31}) \, . \tag{5.6}$$

TABLE 2: Commutation relations $\mathcal{D}_u$, $\mathcal{D}_v$, $\mathcal{D}_w$ and u, v, w.

$[\downarrow, \rightarrow]$	u	v	w
$\mathcal{D}_u$	$1 + \kappa(2g_{12} + \frac{1}{2}g_{23} + \frac{1}{2}g_{31})$	$-\kappa\frac{\sqrt{3}}{2}(g_{23} - g_{31})$	0
$\mathcal{D}_v$	$-\kappa\frac{\sqrt{3}}{2}(g_{23} - g_{31})$	$1 + \kappa(\frac{3}{2}g_{23} + \frac{3}{2}g_{31})$	0
$\mathcal{D}_w$	0	0	1

By direct verification after applying the coordinate change (5.1), the operators of Definition 4.1 now turn out to be

$$O_0 = -iO_{uv}, \qquad O_+ = iO_{wu} + O_{vw}, \qquad O_- = iO_{wu} - O_{vw}\,,$$

and $N_\pm$ follows from the new expressions for the transpositions (5.3), while

$$O_{123} = -\frac{1}{2}e_u e_v e_w - \kappa(g_{12} + g_{23} + g_{31})e_u e_v e_w + O_{uv}e_w + O_{vw}e_u + O_{wu}e_v\,.$$

The spherical Dirac-Dunkl operator Γ is again related to O_{123}, we have $O_{123} = (\Gamma + 1)e_u e_v e_w$.

5.1 *A basis for the space of Dunkl monogenics*

Next, we construct the vectors upon which these operators act. As already alluded to, the representation space will consist of Dunkl monogenics, homogeneous polynomials in the kernel of $\underline{D}$. Finding explicit expressions for a basis of the space of Dunkl monogenics is far from trivial, except for the lowest degree or dimension. For an abelian reflection group, as in [3], one can single out coordinates and, starting from polynomials on $\mathbb{R}$, gradually work up in dimension by means of Cauchy-Kowalevsky extension maps. For a non-abelian reflection group G, however, one is not able to single out coordinates at will, as the orbits of the action, or the conjugacy classes, of G are not singleton sets. The advantage of the coordinate change (5.1) is that the coordinate w does become invariant under all reflections. This means that for the coordinate w we do in fact have a Cauchy-Kowalevsky extension map (see Proposition 5.3) which allows us to move from two-dimensional space to three dimensions. On $\mathbb{R}^2$, Dunkl monogenics follow from the expressions for the Dunkl harmonics which were determined already in [7].

When working in $\mathbb{R}^2$ spanned by the coordinates u and v, it is useful to have a separate notation for the two-dimensional analogues of the Dirac-Dunkl operator, vector variable and Laplace-Dunkl operator:

$$\underline{\tilde{D}} = e_u\mathcal{D}_u + e_v\mathcal{D}_v\,, \qquad \underline{\tilde{x}} = e_u u + e_v v\,, \qquad \tilde{\Delta} = \mathcal{D}_u^2 + \mathcal{D}_v^2 = \underline{\tilde{D}}^2, \qquad \underline{\tilde{x}}^2 = u^2 + v^2\,. \quad (5.7)$$

They satisfy the (anti-)commutation relations, readily verified by means of the relations in Table 2,

$$[\underline{\tilde{D}}, \underline{\tilde{x}}^2] = 2\underline{\tilde{x}}, \qquad \{\underline{\tilde{D}}, \underline{\tilde{x}}\} = 2(\tilde{\mathbb{E}} + 1 + 3\kappa), \qquad \tilde{\mathbb{E}} = u\partial_u + v\partial_v\,, \qquad (5.8)$$

where the Euler operator $\tilde{\mathbb{E}}$ when acting on a polynomial measures the degree in u and v.

Finally, for the following proposition, the hypergeometric series [1, 21] is defined as

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \qquad (5.9)$$

where we use the common notation for Pochhammer symbols [1, 21]: $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k = 1, 2, \ldots$.

PROPOSITION 5.1: *For a non-negative integer k, the polynomials ϕ_k^+ and ϕ_k^- defined as*

$$\phi_k^{\pm}(u,v) = (u \pm iv)^k \frac{(\kappa+1)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n,\ \kappa \\ -n-\kappa\end{matrix};\ \frac{(-u \pm iv)^3}{(u \pm iv)^3}\right), \qquad n = \lfloor k/3 \rfloor \tag{5.10}$$

form a basis for the space of Dunkl harmonics $\mathcal{H}_k(\mathbb{R}^2) = \ker \tilde{\Delta} \cap \mathcal{P}_k(\mathbb{R}^2)$.

Proof. On two dimensional space $\mathbb{R}^2$ the Laplace-Dunkl operator $\tilde{\Delta}$ can be factorized as

$$\tilde{\Delta} = \mathcal{D}_u^2 + \mathcal{D}_v^2 = (\mathcal{D}_u + i\mathcal{D}_v)(\mathcal{D}_u - i\mathcal{D}_v)\,.$$

For reflection groups on $\mathbb{R}^2$, the analogues of harmonic polynomials for the Dunkl Laplacian were determined explicitly already in [7]. The expression (5.10) is the hypergeometric form of polynomials satisfying (see [7])

$$(\mathcal{D}_u + i\mathcal{D}_v)\phi_k^+(u,v) = 0, \qquad (\mathcal{D}_u - i\mathcal{D}_v)\phi_k^-(u,v) = 0\,,$$

and hence $\tilde{\Delta}\phi_k^{\pm}(u,v) = 0$. For $k \geq 1$ the dimension of $\mathcal{H}_k(\mathbb{R}^2)$ is 2 so ϕ_k^+ and ϕ_k^- form a basis, while the dimension of $\mathcal{H}_0(\mathbb{R}^2)$ is 1 in accordance with $\phi_0^+ = 1 = \phi_0^-$. □

Note that the polynomial ϕ_k^- is simply the complex conjugate of ϕ_k^+. These polynomials can also be written in terms of the Jacobi polynomials [16], which are defined in terms of the hypergeometric series as

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1 \\ \alpha+1\end{matrix};\ \frac{1-x}{2}\right). \tag{5.11}$$

By means of the identity

$$(x+y)^n P_n^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \frac{(\alpha+1)_n}{n!} x^n\, {}_2F_1\left(\begin{matrix}-n,\ -n-\beta \\ \alpha+1\end{matrix};\ -\frac{y}{x}\right), \tag{5.12}$$

we can write (5.10), denoting $z = u + iv$ and $\overline{z} = u - iv$, as

$$\phi_k^+(u,v) = (-1)^n z^{k-3n} (z^3 + \overline{z}^3)^n P_n^{(-n-\kappa-1,-n-\kappa)}\left(\frac{z^3 - \overline{z}^3}{z^3 + \overline{z}^3}\right), \qquad n = \lfloor k/3 \rfloor\,. \tag{5.13}$$

We use the previous result to obtain spinor-valued polynomials in the kernel of the two-dimensional Dirac-Dunkl operator $\underline{\tilde{D}} = e_u\mathcal{D}_u + e_v\mathcal{D}_v$. Recall that for the three-dimensional Clifford algebra realized by the Pauli matrices, a two-dimensional Dirac spinor representation is $\mathbb{S} \cong \mathbb{C}^2$, with basis spinors $\chi_+ = (1,0)^T$ and $\chi_- = (0,1)^T$.

PROPOSITION 5.2: *For a non-negative integer k, the polynomials*

$$\varphi_k^+(u,v) = \phi_k^+(u,v)\,\chi_+ \quad \textit{and} \quad \varphi_k^-(u,v) = \phi_k^-(u,v)\,\chi_- \tag{5.14}$$

form a basis for the space $\mathcal{M}_k(\mathbb{R}^2, \mathbb{C}^2)$.

Proof. Acting with $\underline{\tilde{D}} = e_u\mathcal{D}_u + e_v\mathcal{D}_v$ on φ_k^+ we find using the Pauli matrices (5.2)

$$\begin{aligned}\underline{\tilde{D}}\varphi_k^+(u,v) &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathcal{D}_u\phi_k^+(u,v)\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\mathcal{D}_v\phi_k^+(u,v)\begin{pmatrix}1\\0\end{pmatrix}\\ &= (\mathcal{D}_u + i\mathcal{D}_v)\phi_k^+(u,v)\begin{pmatrix}0\\1\end{pmatrix}\end{aligned}$$

which vanishes by definition of ϕ_k^+. In the same way, we find $\underline{\tilde{D}}\varphi_k^-(u,v) = 0$. As the dimension of $\mathcal{M}_k(\mathbb{R}^2,\mathbb{C}^2)$ is 2, φ_k^+ and φ_k^- form a basis. □

For non-negative κ, there exists a Fischer decomposition for Dunkl monogenics in the sense of the following direct sum decomposition

$$\mathcal{P}_n(\mathbb{R}^2)\otimes\mathbb{C}^2 = \bigoplus_{k=0}^{n} \underline{\tilde{x}}^{n-k}\mathcal{M}_n(\mathbb{R}^2,\mathbb{C}^2)\,.$$

Every spinor-valued polynomial on $\mathbb{R}^2$ can thus be written in terms of Dunkl monogenics on $\mathbb{R}^2$, for which a basis is given in Proposition 5.2. The next step consists of moving from $\mathbb{R}^2$ to Dunkl monogenics on $\mathbb{R}^3$ by means of a Cauchy-Kowalevski isomorphism.

PROPOSITION 5.3: *For a non-negative integer n, a basis for the space $\mathcal{M}_n(\mathbb{R}^3,\mathbb{C}^2)$ is given by the $2n+2$ polynomials*

$$\psi_{n,k}^{\pm}(u,v,w) = \mathbf{CK}_w[\underline{\tilde{x}}^{n-k}\varphi_k^{\pm}(u,v)], \qquad k\in\{0,1,\dots,n\} \tag{5.15}$$

where the Cauchy-Kowalevski isomorphism is given by

$$\mathbf{CK}_w : \mathcal{P}_n(\mathbb{R}^2)\otimes\mathbb{C}^2 \longrightarrow \mathcal{M}_n(\mathbb{R}^3,\mathbb{C}^2) : p_n(u,v) \mapsto \exp(-we_w\underline{\tilde{D}})p_n(u,v)\,. \tag{5.16}$$

Note that as $p_n(u,v)$ is a polynomial of degree n, this reduces to the finite sum

$$\mathbf{CK}_w[p_n(u,v)] = \exp(-we_w\underline{\tilde{D}})p_n(u,v) = \sum_{a=0}^{n}\frac{(-1)^a}{a!}w^a(e_w\underline{\tilde{D}})^a p_n(u,v)\,.$$

Proof. We show that the Cauchy-Kowalevski extension $\mathbf{CK}_w$ maps $\mathcal{P}_n(\mathbb{R}^2)\otimes\mathbb{C}^2$ into $\mathcal{M}_n(\mathbb{R}^3,\mathbb{C}^2)$. Let $p_n(u,v)\in\mathcal{P}_n(\mathbb{R}^2)\otimes\mathbb{C}^2$. Using $\mathcal{D}_w = \partial_w$ and the commutation relations in Table 2 we obtain

$$\begin{aligned}\underline{D}\,\mathbf{CK}_w[p_n(u,v)] &= (\underline{\tilde{D}} + e_w\partial_w)\sum_{a=0}^{n}\frac{(-1)^a}{a!}w^a(e_w\underline{\tilde{D}})^a p_n(u,v)\\ &= \sum_{a=0}^{n-1}\frac{(-1)^a}{a!}w^a e_w(e_w\underline{\tilde{D}})^{a+1}p_n(u,v) + \sum_{a=1}^{n}\frac{(-1)^a}{(a-1)!}w^{a-1}e_w(e_w\underline{\tilde{D}})^a p_n(u,v)\end{aligned}$$

which clearly vanishes. Hence, as the map (5.16) preserves the degree of a polynomial we have $\mathbf{CK}_w[p_n(u,v)] \in \mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$.

The inverse of the isomorphism $\mathbf{CK}_w$ is given by the map which evaluates a function in $w = 0$. As the degree of a polynomial in $\mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$ is fixed, this inverse is clearly injective. □

Note that $\psi^{\pm}_{n,k}$ given by (5.15) can also be written in terms of Jacobi polynomials defined in (5.11) by working out the explicit action of the map (5.16). To achieve this, we first state a result, which follows from the commutation relations (5.8). For $M_k \in \mathcal{M}_k(\mathbb{R}^2, \mathbb{C}^2)$ and non-negative integers a, b,

$$\underline{\tilde{D}}^a \underline{\tilde{x}}^b M_k = d^k_{a,b}\, \underline{\tilde{x}}^{b-a} M_k\,, \tag{5.17}$$

where $d^k_{a,b} = 0$ for $a > b$, and otherwise distinguishing between even and odd a, b one has

$$\begin{aligned} d^k_{2\alpha,2\beta} &= 2^{2\alpha}(-\beta)_\alpha(-\beta-k-3\kappa)_\alpha\,,\\ d^k_{2\alpha+1,2\beta} &= -2^{2\alpha+1}(-\beta)_{\alpha+1}(-\beta-k-3\kappa)_\alpha\,,\\ d^k_{2\alpha,2\beta+1} &= 2^{2\alpha}(-\beta)_\alpha(-\beta-k-1-3\kappa)_\alpha\,,\\ d^k_{2\alpha+1,2\beta+1} &= -2^{2\alpha+1}(-\beta)_\alpha(-\beta-k-1-3\kappa)_{\alpha+1}\,. \end{aligned}$$

Using now in turn the identity (5.17), $\underline{\tilde{D}} e_w = -e_w \underline{\tilde{D}}$, $(2\alpha)! = 2^{2\alpha}(1)_\alpha(1/2)_\alpha$ and the identity (5.12), we obtain

$$\psi^{\pm}_{n,k}(u,v,w) = \Psi_{n-k}(\underline{\tilde{x}}, w)\varphi^{\pm}_k(u,v) \tag{5.18}$$

with $\varphi^{\pm}_k$ given by (5.14) (see also (5.13)), and

$$\Psi_{n-k}(\underline{\tilde{x}}, w) = \frac{\beta!}{(\frac{1}{2})_\beta}(u^2+v^2+w^2)^\beta \tag{5.19}$$

$$\times \begin{cases} P_\beta^{(-\frac{1}{2},k+3\kappa)}\left(\frac{u^2+v^2-w^2}{u^2+v^2+w^2}\right) - \frac{e_w w \underline{\tilde{x}}}{u^2+v^2+w^2} P_{\beta-1}^{(\frac{1}{2},k+1+3\kappa)}\left(\frac{u^2+v^2-w^2}{u^2+v^2+w^2}\right) & \text{if } n-k=2\beta\,,\\ \underline{\tilde{x}}\, P_\beta^{(-\frac{1}{2},k+1+3\kappa)}\left(\frac{u^2+v^2-w^2}{u^2+v^2+w^2}\right) - e_w w \frac{\beta+k+1+3\kappa}{\beta+\frac{1}{2}} P_\beta^{(\frac{1}{2},k+3\kappa)}\left(\frac{u^2+v^2-w^2}{u^2+v^2+w^2}\right) & \text{if } n-k=2\beta+1\,. \end{cases}$$

5.2 *Representations*

Given a non-negative integer n, we show that the basis vectors $\psi^{\pm}_{n,k}$ for $k \in \{0,1,\dots,n\}$ transform irreducibly under the action of the algebra $\mathcal{O}St_3$. As the elements of $\mathcal{O}St_3$ (anti-)commute with the Dirac-Dunkl operator, the kernel of the Dirac-Dunkl operator is invariant under the action of $\mathcal{O}St_3$. Furthermore, the elements of $\mathcal{O}St_3$ are grade-preserving so the space $\mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$ is invariant under the action of $\mathcal{O}St_3$.

The spinor $\psi^{\pm}_{n,k}$ corresponds, up to rescaling, precisely to the basis vector $w^{\pm}_k$ of Proposition 4.5. We establish this as follows. The two-dimensional vector variable and Dirac-Dunkl operator (5.7) generate another realization of the Lie superalgebra $\mathfrak{osp}(1|2)$.

When restricted to the one-sphere, similar to (2.6), the angular part $\tilde{\Gamma}$ of the Dirac-Dunkl operator $\underline{\tilde{D}}$ equals, up to an additive constant, the $\mathfrak{osp}(1|2)$ Scasimir given by

$$\tilde{\Gamma} + 1 = \frac{1}{2}[\underline{\tilde{D}}, \underline{\tilde{x}}] - \frac{1}{2} = \frac{1}{2}[\mathcal{D}_u, u] + \frac{1}{2}[\mathcal{D}_v, v] + e_u e_v (v\mathcal{D}_u - u\mathcal{D}_v + \frac{1}{2}[\mathcal{D}_u, v] - \frac{1}{2}[\mathcal{D}_v, u]) .$$

By means of the commutation relations in Table 2 we find the explicit form

$$\tilde{\Gamma} + 1 = \frac{1}{2} + \kappa(g_{12} + g_{23} + g_{31}) - e_u e_v (u\mathcal{D}_v - v\mathcal{D}_u) .$$

Comparing with expression (5.4) we observe that $\tilde{\Gamma} + 1 = -e_u e_v O_{uv}$ and hence, we also have $O_0 = -ie_u e_v(\tilde{\Gamma} + 1)$. Similar to (2.8), now using (5.8) and $\varphi_k^{\pm} \in \ker \underline{\tilde{D}}$ we find

$$\begin{aligned}(\tilde{\Gamma} + 1)\varphi_k^{\pm}(u, v) &= \frac{1}{2}([\underline{\tilde{D}}, \underline{\tilde{x}}] - 1)\varphi_k^{\pm}(u, v) = \frac{1}{2}(\underline{\tilde{D}}\,\underline{\tilde{x}} - 1)\varphi_k^{\pm}(u, v) \\ &= \frac{1}{2}(\{\underline{\tilde{D}}, \underline{\tilde{x}}\} - 1)\varphi_k^{\pm}(u, v) = \frac{1}{2}(2\tilde{\mathbb{E}} + 2 + 6\kappa - 1)\varphi_k^{\pm}(u, v)\end{aligned}$$

which gives

$$(\tilde{\Gamma} + 1)\varphi_k^{\pm}(u, v) = \left(k + \frac{1}{2} + 3\kappa\right)\varphi_k^{\pm}(u, v)$$

as the Euler operator $\tilde{\mathbb{E}} = u\partial_u + v\partial_v$ measures the degree of a polynomial in u and v. Using $-ie_u e_v \chi^{\pm} = \pm\chi^{\pm}$, which is readily verified using the Pauli matrices (5.2), the action of O_0 on $\varphi_k^{\pm}$ then follows to be

$$O_0\varphi_k^{\pm}(u, v) = \pm\left(k + \frac{1}{2} + 3\kappa\right)\varphi_k^{\pm}(u, v) .$$

Since O_0 commutes with $\underline{\tilde{D}}$, $\underline{\tilde{x}}$ and $e_w w$ we also have, by definition of $\psi_{n,k}^{\pm}$,

$$O_0\psi_{n,k}^{\pm} = \pm\left(k + \frac{1}{2} + 3\kappa\right)\psi_{n,k}^{\pm} .$$

Finally, as $O_{123} = (\Gamma + 1)e_u e_v e_w$ and $\psi_{n,k}^{\pm} \in \mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$, by (2.8) we find the action

$$O_{123}\psi_{n,k}^{\pm} = i(n + 1 + 3\kappa)\psi_{n,k}^{\pm} .$$

To conclude, we consider the action of the S_3 realization on a spinor $\psi_{n,k}^{\pm}$. Using $G_{12}\varphi_k^{\pm} = (-1)^k\varphi_k^{\pm}$, the expressions (5.3) and the fact that G_{12} anti-commutes with $\underline{\tilde{x}}$ and $\underline{\tilde{D}}$, we find

$$G_{12}\psi_{n,k}^{\pm} = (-1)^{n-k}(-1)^k\psi_{n,k}^{\mp} = (-1)^n\psi_{n,k}^{\mp} .$$

Similarly, using now $G_{23}\varphi_k^{+} = (-1)^k\omega^{\pm(1-k)}\varphi_k^{+}$ we have

$$G_{23}\psi_{n,k}^{\pm} = (-1)^n\omega^{\pm(1-k)}\psi_{n,k}^{\mp} .$$

This shows, up to rescaling, the correspondence of $\psi^{\pm}_{n,k}$ with the vector $w^{\pm}_{k}$ of Proposition 4.5.

The abstract inner product on the unitary representation (see section 4.4) can now also be realized explicitly. An integral formulation follows by combining the inner product on the spinor space $\mathbb{C}^2$ with the inner product on the unit sphere for Dunkl harmonics [9]

$$\langle \Phi_1, \Phi_2 \rangle = \int_{S^2} (\Phi_1^{\dagger} \cdot \Phi_2)\, h_{\kappa}^2(u, v, w)\, \mathrm{d}u\mathrm{d}v\mathrm{d}w\,,$$

where $h_{\kappa}(u, v, w)$ is the S_3 invariant weight function [9]

$$h_{\kappa}(u, v, w)= |u|^{\kappa}|(u^2 - 3v^2)/4|^{\kappa}\,.$$

Using this inner product, the polynomial $\psi^{\pm}_{n,k}$ given by (5.19) can be normalized to a wavefunction corresponding precisely to the normed vector $w^{\pm}_{k}$ of Proposition 4.5. The orthogonality can be verified by means of the orthogonality relation of the Jacobi polynomials [16].

6 CONCLUSION

We considered the three-dimensional case of the Dirac-Dunkl operator associated to the root system A_2 with non-abelian reflection group S_3. In previous work, the algebra generated by the symmetries of this Dirac operator was already shown to be a one-parameter extension of the classical angular momentum algebra. In the current paper, we have classified all finite-dimensional irreducible representations of this symmetry algebra and we have determined the conditions for the representations to be unitarizable. Among the obtained classes of irreducible representations of the symmetry algebra, there is one class of unitary representations for arbitrary positive value of the parameter. This last class admits a natural realization by means of Dunkl monogenics, for which we constructed an explicit basis. This basis consists of eigenfunctions of the spherical Dirac-Dunkl operator and thus form solutions to a Dirac equation on the two-sphere.

In future work we aim to elevate the setting of the current paper in two directions. On the one hand, one can consider the N-dimensional case where the reflection group associated to the Dunkl operator is the symmetric group S_N. On the other hand, it would be interesting to consider more involved root systems (as was done for the type B_3 in [13]), first in three dimensions and then also in higher dimensions. We look forward to tackle these problems using the insights obtained here.

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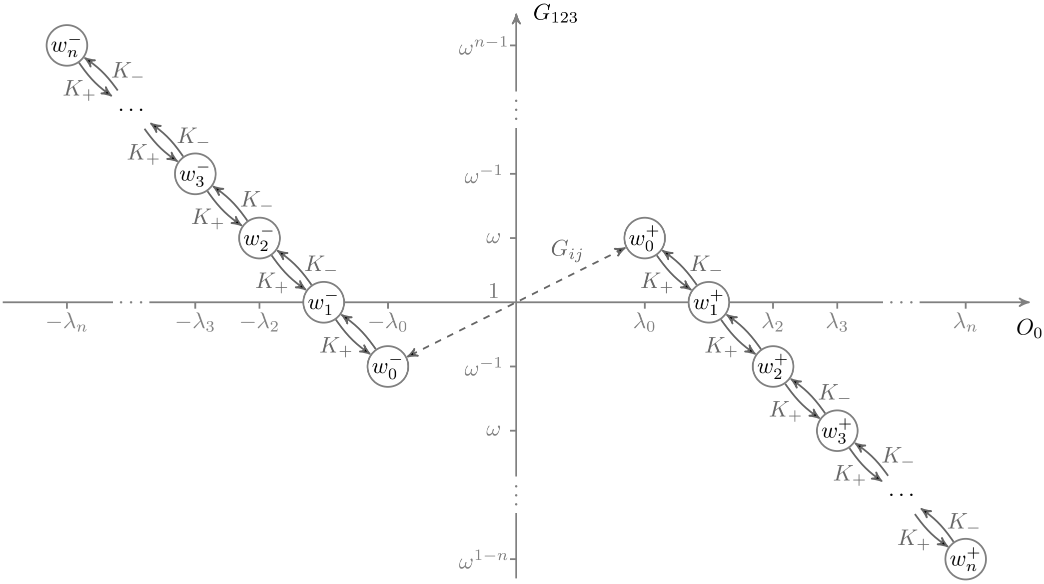

FIGURE 1: Graphical representation of the basis vectors according to their eigenvalues for O_0 and G_{123}. On the horizontal axis, the shorthand notation $O_0 w_k^{\pm} = \pm\lambda_k w_k^{\pm}$ is used, and on the vertical axis the three values 1, ω, ω^{-1} are repeated periodically. There are two main actions: 1) The arrows represent the actions of K_+ and K_- through which one moves between the vectors in one half of the vector space. 2) In this picture, the action of an odd element of S_3 corresponds to a reflection through the origin, as illustrated for w_0^+ and w_0^- by the dashed line. The action of $O_\pm$ is a combination of the two main actions.

4

Doubling (dual) Hahn polynomials: classification and applications

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abstract

We classify all pairs of recurrence relations in which two Hahn or dual Hahn polynomials with different parameters appear. Such couples are referred to as (dual) Hahn doubles. The idea and interest comes from an example appearing in a finite oscillator model by Jafarov, Stoilova and Van der Jeugt [14]. Our classification shows there exist three dual Hahn doubles and four Hahn doubles. The same technique is then applied to Racah polynomials, yielding also four doubles. Each dual Hahn (Hahn, Racah) double gives rise to an explicit new set of symmetric orthogonal polynomials related to the Christoffel and Geronimus transformations. For each case, we also have an interesting class of two-diagonal matrices with closed form expressions for the eigenvalues. This extends the class of Sylvester-Kac matrices by remarkable new test matrices. We examine also the algebraic relations underlying the dual Hahn doubles, and discuss their usefulness for the construction of new finite oscillator models.

1 introduction

The tridiagonal $(N+1)\times(N+1)$ matrix of the following form

$$C_{N+1} = \begin{pmatrix} 0 & 1 & & & & \\ N & 0 & 2 & & & \\ & N-1 & 0 & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & 0 & N \\ & & & & 1 & 0 \end{pmatrix} \tag{1.1}$$

appears in the literature under several names: the Sylvester-Kac matrix, the Kac matrix, the Clement matrix, ... It was already considered by Sylvester [28], used by M. Kac in some of his seminal work [17], by Clement as a test matrix for eigenvalue computations [9], and continues to attract attention [6, 7, 29]. The main property of the matrix C_{N+1} is that its eigenvalues are given explicitly by

$$-N, -N+2, -N+4, \cdots, N-2, N. \tag{1.2}$$

Because of this simple property, C_{N+1} is a standard test matrix for numerical eigenvalue computations, and part of some standard test matrix toolboxes (e.g. [12]).

One of the outcomes of the current paper implies that the matrix C_{N+1} has appealing two-parameter extensions. For odd dimensions, let us consider the following tridiagonal matrix:

$$C_{2N+1}(\gamma,\delta)=\begin{pmatrix} 0 & 2\gamma+2 & & & & & & & \\ 2N & 0 & 2 & & & & & & \\ & 2\delta+2N & 0 & 2\gamma+4 & & & & & \\ & & 2N-2 & 0 & 4 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 2\delta+4 & 0 & 2\gamma+2N & & \\ & & & & & 2 & 0 & 2N \\ & & & & & & 2\delta+2 & 0 \end{pmatrix}. \tag{1.3}$$

In the following, we shall sometimes use the term “two-diagonal” [10] for tridiagonal matrices with zero entries on the diagonal (not to be confused with a bidiagonal matrix, which has also two non-zero diagonals, but for a bidiagonal matrix the non-zero entries are on the main diagonal and either superdiagonal or the subdiagonal). So, just as C_{2N+1} the matrix (1.3) is two-diagonal, but the superdiagonal of C_{2N+1},

$$[1, 2, 3, 4, \ldots, 2N-1, 2N]$$

is replaced by

$$[2\gamma+2, 2, 2\gamma+4, 4, \ldots, 2\gamma+2N, 2N],$$

and in the subdiagonal of C_{2N+1},

$$[2N, 2N-1, 2N-2, \ldots, 3, 2, 1]$$

the odd entries are replaced, leading to

$$[2N, 2\delta+2N, 2N-2, \ldots, 2\delta+4, 2, 2\delta+2].$$

Clearly, for $\gamma=\delta=-\frac{1}{2}$ the matrix $C_{2N+1}(\gamma,\delta)$ just reduces to C_{2N+1}. One of our results is that $C_{2N+1}(\gamma,\delta)$ has simple eigenvalues for general γ and δ, given by

$$0, \pm 2\sqrt{1(\gamma+\delta+2)}, \pm 2\sqrt{2(\gamma+\delta+3)}, \pm 2\sqrt{3(\gamma+\delta+4)}, \ldots, \pm 2\sqrt{N(\gamma+\delta+N+1)}. \tag{1.4}$$

This spectrum simplifies even further for $\delta = -\gamma - 1$; in this case one gets back the eigenvalues (1.2).

For even dimensions, we have a similar result. Let $C_{2N}(\gamma, \delta)$ be the $(2N)\times(2N)$ tridiagonal matrix with zero diagonal, with superdiagonal

$$[2\gamma + 2, 2, 2\gamma + 4, 4, \ldots, 2N - 2, 2\gamma + 2N]$$

and with subdiagonal

$$[2\delta + 2N, 2N - 2, 2\delta + 2N - 2, \ldots, 4, 2\delta + 4, 2, 2\delta + 2].$$

Then $C_{2N}(\gamma, \delta)$ has simple eigenvalues for general γ and δ, given by

$$\pm 2\sqrt{(\gamma + 1)(\delta + 1)}, \pm 2\sqrt{(\gamma + 2)(\delta + 2)}, \ldots, \pm 2\sqrt{(\gamma + N)(\delta + N)}. \tag{1.5}$$

This spectrum simplifies for $\delta = \gamma$, and obviously for $\gamma = \delta = -\frac{1}{2}$ one gets back the eigenvalues (1.2) since in that case $C_{2N}(\gamma, \delta)$ just reduces to C_{2N}.

What is the context here for these new tridiagonal matrices with simple eigenvalue properties? Well, remember that C_{N+1} also appears as the simplest example of a family of Leonard pairs [24, 30]. In that context, this matrix is related to symmetric Krawtchouk polynomials [13, 19, 23]. Indeed, let $K_n(x) \equiv K_n(x; \frac{1}{2}, N)$, where $K_n(x; p, N)$ are the Krawtchouk polynomials [13, 19, 23]. Then their recurrence relation [19, (9.11.3)] yields

$$nK_{n-1}(x) + (N - n)K_{n+1}(x) = (N - 2x)K_n(x) \qquad (n = 0, 1, \ldots, N). \tag{1.6}$$

Writing this down for $x = 0, 1, \ldots, N$, and putting this in matrix form, shows indeed that the eigenvalues of C_{N+1} (or rather, of its transpose C_{N+1}^T) are indeed given by (1.2). Moreover, it shows that the components of the kth eigenvector of C_{N+1}^T are given by $K_n(k)$.

So we can identify the matrix C_{N+1} with the Jacobi matrix of symmetric Krawtchouk polynomials, one of the families of finite and discrete hypergeometric orthogonal polynomials. The other matrices $C_N(\gamma, \delta)$ appearing in this introduction are not directly related to Jacobi matrices of a simple set of finite orthogonal polynomials. In this paper, however, we show how two sets of distinct dual Hahn polynomials [13, 19, 23] can be combined in an appropriate way such that the eigenvalues of matrices like $C_N(\gamma, \delta)$ become apparent, and such that the eigenvector components are given in terms of these two dual Hahn polynomials. This process of combining two distinct sets is called "doubling". We examine this not only for the case related to the matrix $C_N(\gamma, \delta)$, but stronger: we classify all possible ways in which two sets of dual Hahn polynomials can be combined in order to yield a two-diagonal "Jacobi matrix". It turns out that there are exactly three ways in which dual Hahn polynomials can be "doubled" (for a precise formulation, see later). By the doubling procedure, one automatically gets the eigenvalues (and eigenvectors) of the corresponding two-diagonal matrix in explicit form.

This process of doubling and investigating the corresponding two-diagonal Jacobi matrix can be applied to other classes of orthogonal polynomials (with a finite and discrete support) as well. In this paper, we turn our attention also to Hahn and to Racah polynomials. The classification process becomes rather technical, however. Therefore, we have decided to present the proof of the complete classification only for dual Hahn polynomials (section 3). For Hahn polynomials (section 4) we give the final classification and corresponding two-diagonal matrices (but omit the proof), and for Racah polynomials we give the final classification and some examples of two-diagonal matrices in the Appendix.

We should also note that the two-diagonal matrices appearing as a result of the doubling process are symmetric. So matrices like (1.3) do not appear directly but in their symmetrized form. Of course, as far as eigenvalues are concerned, this makes no difference (see section 6).

The doubling process of the polynomials considered here also gives rise to "new" sets of orthogonal polynomials. One could argue whether the term "new" is appropriate, since they arise by combining two known sets. The peculiar property is however that the combined set has a common unique weight function. Moreover, we shall see that the support set of these doubled polynomials is interesting, see the examples in section 5. In this section, we also interpret the doubling process in the framework of Christoffel-Geronimus transforms. It will be clear that from our doubling process, one can deduce for which Christoffel parameter the Christoffel transform of a Hahn, dual Hahn or Racah polynomial is again a Hahn, dual Hahn or Racah polynomial with shifted parameters.

In section 6 we reconsider the two-diagonal matrices that have appeared in the previous sections. It should be clear that we get several classes of two-diagonal matrices (with parameters) for which the eigenvalues (and eigenvectors) have an explicit and rather simple form. This section reviews such matrices as new and potentially interesting examples of eigenvalue test matrices.

In section 7 we explore relations with other structures. Recall that in finite-dimensional representations of the Lie algebra $\mathfrak{su}(2)$, with common generators J_+, J_- and J_0, the matrix of $J_+ + J_-$ also has a symmetric two-diagonal form. The new two-diagonal matrices appearing in this paper can be seen as representation matrices of deformations or extensions of $\mathfrak{su}(2)$. We give the algebraic relations that follow from the "representation matrices" obtained here. The algebras are not studied in detail, but it is clear that they could be of interest on their own. The general algebras have two parameters, and we indicate how special cases with only one parameter are of importance for the construction of finite oscillator models.

2 INTRODUCTORY EXAMPLE

We start our analysis by the explanation of a known example taken from [27]. For this example, we first recall the definition of Hahn and dual Hahn polynomials and some of the classical notations and properties.

The Hahn polynomial $Q_n(x; \alpha, \beta, N)$ [13, 19, 23] of degree n ($n = 0, 1, \ldots, N$) in the variable x, with parameters $\alpha > -1$ and $\beta > -1$ (or $\alpha < -N$ and $\beta < -N$) is defined by [13, 19, 23]:

$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}; 1\right). \tag{2.1}$$

Herein, the function ${}_3F_2$ is the generalized hypergeometric series [4, 26]:

$${}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{z^k}{k!}. \tag{2.2}$$

In (2.1), the series is terminating because of the appearance of the negative integer $-n$ as a numerator parameter. Note that in (2.2) we use the common notation for Pochhammer symbols [4, 26] $(a)_k = a(a+1)\cdots(a+k-1)$ for $k = 1, 2, \ldots$ and $(a)_0 = 1$. Hahn polynomials satisfy a (discrete) orthogonality relation [13, 19]:

$$\sum_{x=0}^{N} w(x; \alpha, \beta, N) Q_n(x; \alpha, \beta, N) Q_{n'}(x; \alpha, \beta, N) = h_n(\alpha, \beta, N)\, \delta_{n,n'}, \tag{2.3}$$

where

$$\begin{aligned} w(x; \alpha, \beta, N) &= \binom{\alpha+x}{x}\binom{N+\beta-x}{N-x} \qquad (x = 0, 1, \ldots, N); \\ h_n(\alpha, \beta, N) &= \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n N!}. \end{aligned}$$

We denote the orthonormal Hahn functions as follows:

$$\tilde{Q}_n(x; \alpha, \beta, N) \equiv \frac{\sqrt{w(x; \alpha, \beta, N)}\, Q_n(x; \alpha, \beta, N)}{\sqrt{h_n(\alpha, \beta, N)}}. \tag{2.4}$$

The Hahn polynomials satisfy the following recurrence relation [19, (9.5.3)]

$$\Lambda(x) y_n(x) = A(n) y_{n+1}(x) - \big(A(n) + C(n)\big) y_n(x) + C(n) y_{n-1}(x) \tag{2.5}$$

with

$$\begin{aligned} &y_n(x) = Q_n(x; \alpha, \beta, N), \qquad \Lambda(x) = -x, \\ &A(n) = \frac{(n+\alpha+1)(n+\alpha+\beta+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad C(n) = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \end{aligned} \tag{2.6}$$

Related to the Hahn polynomials are the dual Hahn polynomials: $R_n(\lambda(x); \gamma, \delta, N)$ of degree n ($n = 0, 1, \ldots, N$) in the variable $\lambda(x) = x(x+\gamma+\delta+1)$, with parameters

$\gamma > -1$ and $\delta > -1$ (or $\gamma < -N$ and $\delta < -N$) which are defined similarly to (2.1) [13, 19, 23]

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\begin{matrix}-x,\, x+\gamma+\delta+1,\, -n\\ \gamma+1,\, -N\end{matrix};1\right). \tag{2.7}$$

As is well known, the (discrete) orthogonality relation of the dual Hahn polynomials is just the "dual" of (2.3):

$$\sum_{x=0}^{N} \overline{w}(x;\gamma,\delta,N) R_n(\lambda(x);\gamma,\delta,N) R_{n'}(\lambda(x);\gamma,\delta,N) = \overline{h}_n(\gamma,\delta,N)\,\delta_{n,n'}, \tag{2.8}$$

where

$$\overline{w}(x;\gamma,\delta,N) = \frac{(2x+\gamma+\delta+1)(\gamma+1)_x(-N)_x N!}{(-1)^x(x+\gamma+\delta+1)_{N+1}(\delta+1)_x x!},$$
$$\overline{h}_n(\gamma,\delta,N) = \left[\binom{\gamma+n}{n}\binom{N+\delta-n}{N-n}\right]^{-1}.$$

Orthonormal dual Hahn functions are defined by:

$$\tilde{R}_n(\lambda(x);\gamma,\delta,N) \equiv \frac{\sqrt{\overline{w}(x;\gamma,\delta,N)}\, R_n(\lambda(x);\gamma,\delta,N)}{\sqrt{\overline{h}_n(\gamma,\delta,N)}}. \tag{2.9}$$

Dual Hahn polynomials also satisfy a recurrence relation of the form (2.5), with [19, (9.6.3)]

$$\begin{aligned} y_n(x) &= R_n(\lambda(x);\gamma,\delta,N), \quad \Lambda(x) = \lambda(x) = x(x+\gamma+\delta+1),\\ A(n) &= (n+\gamma+1)(n-N), \quad C(n) = n(n-\delta-N-1). \end{aligned} \tag{2.10}$$

In [27], the following difference equations involving two sets of Hahn polynomials were derived (for convenience we use the notation $Q_n(x) \equiv Q_n(x;\alpha,\beta+1,N)$ and $\hat{Q}_n(x) \equiv Q_n(x;\alpha+1,\beta,N)$):

$$(N+\beta+1-x)Q_n(x) - (N-x)Q_n(x+1) = \frac{(n+\alpha+1)(n+\beta+1)}{\alpha+1}\hat{Q}_n(x), \tag{2.11}$$

$$(x+1)\hat{Q}_n(x) - (\alpha+x+2)\hat{Q}_n(x+1) = -(\alpha+1)Q_n(x+1). \tag{2.12}$$

Writing out these difference equations for $x = 0, 1, \ldots, N$, the resulting set of equations can easily be written in matrix form. For this matrix form, let us use the normalized version of the polynomials, and construct the following $(2N+2)\times(2N+2)$ matrix U with elements

$$U_{2x,N-n} = U_{2x,N+n+1} = \frac{(-1)^x}{\sqrt{2}}\tilde{Q}_n(x;\alpha,\beta+1,N), \tag{2.13}$$

$$U_{2x+1,N-n} = -U_{2x+1,N+n+1} = -\frac{(-1)^x}{\sqrt{2}}\tilde{Q}_n(x;\alpha+1,\beta,N), \tag{2.14}$$

where $x, n \in \{0, 1, \ldots, N\}$. By construction, this matrix is orthogonal [27]: the fact that the columns of U are orthonormal follows from the orthogonality relation of the Hahn polynomials, and from the signs in the matrix U. Thus $U^T U = UU^T = I$, the identity matrix.

The normalized difference equations (2.11)–(2.12) for $x = 0, 1, \ldots, N$ can then be cast in matrix form. The coefficients in the left hand sides of (2.11)–(2.12) give rise to a tridiagonal $(2N+2)\times(2N+2)$-matrix of the form

$$M = \begin{pmatrix} 0 & M_0 & 0 & & \\ M_0 & 0 & M_1 & \ddots & \\ 0 & M_1 & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & M_{2N} \\ & & 0 & M_{2N} & 0 \end{pmatrix}. \tag{2.15}$$

with

$$M_{2k} = \sqrt{(k+\alpha+1)(N+\beta+1-k)}, \qquad M_{2k+1} = \sqrt{(k+1)(N-k)}. \tag{2.16}$$

Suppose $\alpha > -1$, $\beta > -1$ or $\alpha < -N-1$, $\beta < -N-1$ and let U be the orthogonal matrix determined in (2.13)–(2.14). Then [27] the columns of U are the eigenvectors of M, i.e.

$$MU = UD, \tag{2.17}$$

where D is a diagonal matrix containing the eigenvalues of M:

$$D = \operatorname{diag}(-\epsilon_N, \ldots, -\epsilon_1, -\epsilon_0, \epsilon_0, \epsilon_1, \ldots, \epsilon_N),$$

$$\epsilon_k = \sqrt{(\alpha+k+1)(\beta+k+1)} \qquad (k = 0, 1, \ldots, N). \tag{2.18}$$

Note that the eigenvalues of the matrix M are (up to a factor 2) the same as those of the matrix $C_{2N+2}(\alpha, \beta)$, the two-parameter extension of the Sylvester-Kac matrix. As we will further discuss in section 6, the above result proves that the eigenvalues of $C_{2N+2}(\alpha, \beta)$ are indeed given by (1.5). Even more: the orthonormal eigenvectors of M are just the columns of U.

Another way of looking at (2.17) is in terms of the dual Hahn polynomials. Interchanging x and n in the expressions (2.13)–(2.14), we have

$$U_{2n,N-x} = U_{2n,N+x+1} = \frac{(-1)^n}{\sqrt{2}} \tilde{R}_n(\lambda(x); \alpha, \beta+1, N), \tag{2.19}$$

$$U_{2n+1,N-x} = -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}} \tilde{R}_n(\lambda(x); \alpha+1, \beta, N), \tag{2.20}$$

where $x, n \in \{0, 1, \ldots, N\}$. In this way, each row of the matrix U consists of a dual Hahn polynomial of a certain degree, having different parameter values for even and odd rows.

Now, the relation (2.17) can be interpreted as a three-term recurrence relation with M being the Jacobi matrix. Two sets of (dual) Hahn polynomials (with different parameters) are thus combined into a new set of polynomials such that the Jacobi matrix for this new set has a simple two-diagonal form, with simple eigenvalues. The pair of difference equations (2.11)–(2.12) involving two sets of Hahn polynomials then corresponds to the following relations involving the dual Hahn polynomials $R_n(x) \equiv R_n(\lambda(x); \gamma, \delta+1, N)$ and $\hat{R}_n(x) \equiv R_n(\lambda(x); \gamma+1, \delta, N)$:

$$(N+\delta+1-n)R_n(x) - (N-n)R_{n+1}(x) = \frac{(x+\gamma+1)(x+\delta+1)}{(\gamma+1)}\hat{R}_n(x), \tag{2.21}$$

$$(n+1)\hat{R}_n(x) - (n+\gamma+2)\hat{R}_{n+1}(x) = -(\gamma+1)R_{n+1}(x). \tag{2.22}$$

This is in fact a special case of the so-called Christoffel transform of a dual Hahn polynomial with its transformation parameter chosen specifically so that the result is again a dual Hahn polynomial (with different parameters). We will further elaborate on this in section 5.

This introductory example, taken from [27], opens the following question: in how many ways can two sets of (dual) Hahn polynomials be combined such that the Jacobi matrix is two-diagonal? This will be answered in the following section.

3 DOUBLING DUAL HAHN POLYNOMIALS: CLASSIFICATION

The essential relation in the previous example is the existence of a pair of "recurrence relations" (2.21)–(2.22) intertwining two types of dual Hahn polynomials (or equivalently a couple of difference equations (2.11)–(2.12) for two types of their duals, the Hahn polynomials). Let us therefore examine the existence of such relations in general. Say we have two types of dual Hahn polynomials with different parameter values for γ and δ (and possibly N) denoted by $R_n(\lambda(x); \gamma, \delta, N)$ and $R_n(\lambda(\hat{x}); \hat{\gamma}, \hat{\delta}, \hat{N})$, that are related in the following manner:

$$a(n)\, R_n(\lambda(x); \gamma, \delta, N) + b(n)\, R_{n+1}(\lambda(x); \gamma, \delta, N) = \hat{d}(x)\, R_n(\lambda(\hat{x}); \hat{\gamma}, \hat{\delta}, \hat{N}) \tag{3.1}$$

$$\hat{a}(n)\, R_n(\lambda(\hat{x}); \hat{\gamma}, \hat{\delta}, \hat{N}) + \hat{b}(n)\, R_{n+1}(\lambda(\hat{x}); \hat{\gamma}, \hat{\delta}, \hat{N}) = d(x)\, R_{n+1}(\lambda(x); \gamma, \delta, N). \tag{3.2}$$

If we want these relations to correspond to a matrix identity like (2.17), then it is indeed necessary that the (unknown) functions $a(n)$, $\hat{a}(n)$, $b(n)$ and $\hat{b}(n)$ are functions of n and not of x, and that $d(x)$ and $\hat{d}(x)$ are functions of x and not of n. Of course, the parameters $\gamma, \delta, N, \hat{\gamma}, \hat{\delta}, \hat{N}$ can appear in these functions.

In order to lift this technique also to other polynomials than just the dual Hahn polynomials, say we have the following relations between two sets of orthogonal polynomials of the same class, denoted by y_n and $\hat{y}_n$, but with different parameter values:

$$a(n)\, y_n + b(n)\, y_{n+1} = \hat{d}(x)\, \hat{y}_n, \tag{3.3}$$

$$\hat{a}(n)\, \hat{y}_n + \hat{b}(n)\, \hat{y}_{n+1} = d(x)\, y_{n+1}, \tag{3.4}$$

where a, $\hat{a}$, b, $\hat{b}$ are independent of x and d, $\hat{d}$ are independent of n. Although (3.3)–(3.4) are not actual recurrence relations since they involve both y_n and $\hat{y}_n$, we will refer to a couple of such relations intertwining two types of orthogonal polynomials as "a pair of recurrence relations".

When substituting (3.4) in (3.3), we arrive at the following recurrence relation for $\hat{y}_n$:

$$a(n)\,[\hat{a}(n-1)\,\hat{y}_{n-1} + \hat{b}(n-1)\,\hat{y}_n] + b(n)\,[\hat{a}(n)\,\hat{y}_n + \hat{b}(n)\,\hat{y}_{n+1}] = d(x)\,\hat{d}(x)\,\hat{y}_n\,. \quad (3.5)$$

In the same manner, $\hat{y}_n$ can be eliminated to find a recurrence relation for y_n:

$$\hat{a}(n-1)\,[a(n-1)\,y_{n-1}+b(n-1)\,y_n]+\hat{b}(n-1)\,[a(n)\,y_n+b(n)\,y_{n+1}] = \hat{d}(x)\,d(x)\,y_n. \quad (3.6)$$

Of course, the orthogonal polynomials y_n already satisfy a three-term recurrence relation of the form (2.5). A comparison of the coefficients of y_{n+1}, y_n, y_{n-1} in (3.5)–(3.6) with the known coefficients given in (2.10) leads to the following set of requirements for a, $\hat{a}$, b, $\hat{b}$, d, $\hat{d}$

$$a(n)\,\hat{a}(n-1)= \hat{C}(n) \quad (3.7)$$

$$a(n-1)\,\hat{a}(n-1)= C(n) \quad (3.8)$$

$$a(n)\,\hat{b}(n-1)+\hat{a}(n)\,b(n)-d(x)\,\hat{d}(x)= -[\hat{\Lambda}(x)+\hat{A}(n)+\hat{C}(n)] \quad (3.9)$$

$$a(n)\,\hat{b}(n-1)+\hat{a}(n-1)\,b(n-1)-\hat{d}(x)\,d(x)= -[\Lambda(x)+A(n)+C(n)] \quad (3.10)$$

$$b(n)\,\hat{b}(n)= \hat{A}(n) \quad (3.11)$$

$$b(n)\,\hat{b}(n-1)= A(n) \quad (3.12)$$

After a slight rearrangement of terms in the requirements (3.9) and (3.10), we arrive at two new equations where the left hand side is independent of x while the right hand side is independent of n, namely

$$a(n)\,\hat{b}(n-1)+\hat{a}(n)\,b(n)+\hat{A}(n)+\hat{C}(n)= d(x)\,\hat{d}(x)-\hat{\Lambda}(x) \quad (3.13)$$

$$a(n)\,\hat{b}(n-1)+\hat{a}(n-1)\,b(n-1)+A(n)+C(n)= \hat{d}(x)\,d(x)-\Lambda(x). \quad (3.14)$$

Hence, the two sides must be independent of both n and x. By means of (3.7)–(3.12) we can eliminate A, $\hat{A}$, C, $\hat{C}$ to find

$$a(n)[\hat{a}(n-1)+\hat{b}(n-1)] + b(n)[\hat{a}(n)+\hat{b}(n)] = d(x)\,\hat{d}(x)-\hat{\Lambda}(x) \quad (3.15)$$

$$\hat{a}(n-1)[a(n-1)+b(n-1)] + \hat{b}(n-1)[a(n)+b(n)] = \hat{d}(x)\,d(x)-\Lambda(x). \quad (3.16)$$

Moreover, subtracting one from the other yields

$$\Lambda(x)-\hat{\Lambda}(x)= \hat{a}(n-1)[a(n)-a(n-1)-b(n-1)] + b(n)[\hat{a}(n)+\hat{b}(n)-\hat{b}(n-1)]. \quad (3.17)$$

Now, for a given class of orthogonal polynomials with recurrence relation of the form (2.5), we determine all possible functions a, $\hat{a}$, b, $\hat{b}$, d, $\hat{d}$ satisfying the list of requirements (3.7)–(3.12). Hereto, we proceed as follows:

- From (3.7) and (3.8) we observe that, up to a multiplicative factor, $C(n)$ is split into two functions, $a(n-1)$ and $\hat{a}(n-1)$. When $a(n-1)$ is shifted by 1 in n and multiplied again by $\hat{a}(n-1)$ we must arrive at $\hat{C}(n)$. Hence, C and $\hat{C}$ consist of an identical part, and a part which differs by a shift of 1 in n. This observation gives a first list of possibilities for a and $\hat{a}$.
- Similarly we find a list for b and $\hat{b}$ by means of (3.11)–(3.12).
- These possibilities are then to be compared with requirements (3.9) and (3.10). From (3.13), (3.14) and (3.17) we get an expression for the product $d(x)\,\hat{d}(x)$. Finally, the set of remaining choices for a, $\hat{a}$, b, $\hat{b}$ are to be plugged in (3.4) and (3.3) in order to get d, $\hat{d}$ and to verify if these relations indeed hold.

The actual performance of the procedure just described is still quite long and tedious, when carried out for a fixed class of polynomials. In what follows we achieve this for the dual Hahn polynomials, which have the easiest recurrence relation, and it takes about three pages to present this. The reader who wishes to skip the details can advance to Theorem 7.

For dual Hahn polynomials, the data is given by (2.10):

$$y_n = R_n(\lambda(x);\gamma,\delta,N), \quad \hat{y}_n = R_n(\lambda(\hat{x});\hat{\gamma},\hat{\delta},\hat{N}), \quad \Lambda(x)= \lambda(x)= x(x+\gamma+\delta+1)$$

$$A(n)= (n+\gamma+1)(n-N), \quad C(n)= n(n-\delta-N-1),$$

with similar expressions for $\hat{\Lambda}(x)$, $\hat{A}(n)$ and $\hat{C}(n)$ (with x, γ, δ, N replaced by $\hat{x}$, $\hat{\gamma}$, $\hat{\delta}$, $\hat{N}$). From (3.17), the following expression must be independent of x:

$$\Lambda(x)-\hat{\Lambda}(x)= x(x+\gamma+\delta+1)-\hat{x}(\hat{x}+\hat{\gamma}+\hat{\delta}+1). \tag{3.18}$$

In order for the term in x^2 to disappear, we must have $\hat{x} = x+\xi$ which gives

$$x(x+\gamma+\delta+1)-(x+\xi)(x+\xi+\hat{\gamma}+\hat{\delta}+1)= (\gamma+\delta-\hat{\gamma}-\hat{\delta}-2\xi)x-\xi(\xi+\hat{\gamma}+\hat{\delta}+1)$$

and as we require the coefficient of x to be zero we find the following condition for ξ:

$$\gamma+\delta-(\hat{\gamma}+\hat{\delta})= 2\xi. \tag{3.19}$$

From (3.8) we see that we have four distinct possible combinations for $a(n-1)$ and $\hat{a}(n-1)$:

$$a(n-1) = 1\,c_a \quad \text{and} \quad \hat{a}(n-1) = n(n-\delta-N-1)\,c_a^{-1} \tag{a1}$$

$$a(n-1) = n\,c_a \quad \text{and} \quad \hat{a}(n-1) = (n-\delta-N-1)\,c_a^{-1} \tag{a2}$$

$$a(n-1) = (n-\delta-N-1)\,c_a \quad \text{and} \quad \hat{a}(n-1) = n\,c_a^{-1} \tag{a3}$$

$$a(n-1) = n(n-\delta-N-1)\,c_a \quad \text{and} \quad \hat{a}(n-1) = 1\,c_a^{-1} \tag{a4}$$

with c_a a factor. Combining this with (3.7) we must have

$$a(n)\,\hat{a}(n-1) = \hat{C}(n)= n(n-\hat{\delta}-\hat{N}-1).$$

This immediately implies that c_a is independent of n, and (a1)–(a4) yield the following possibilities:

$$n(n-\delta-N-1)= n(n-\hat{\delta}-\hat{N}-1) \implies \delta+N=\hat{\delta}+\hat{N} \tag{a1'}$$

$$(n+1)(n-\delta-N-1)= n(n-\hat{\delta}-\hat{N}-1) \implies \delta+N+1=0 \ \wedge\ \hat{\delta}+\hat{N}+2=0 \tag{a2'}$$

$$(n-\delta-N)n = n(n-\hat{\delta}-\hat{N}-1) \implies \delta+N=\hat{\delta}+\hat{N}+1 \tag{a3'}$$

$$(n+1)(n-\delta-N)= n(n-\hat{\delta}-\hat{N}-1) \implies \delta+N=0 \ \wedge\ \hat{\delta}+\hat{N}+2=0 \tag{a4'}$$

Because of the restriction on δ the option (a4') is ineligible, leaving (a1')–(a3') as only viable options.

In a similar way, from (3.12) we see that we have four possible combinations for $b(n)$ and $\hat{b}(n)$,

$$b(n) = 1\, c_b \quad \text{and} \quad \hat{b}(n-1) = (n+\gamma+1)(n-N)\, c_b^{-1} \tag{b1}$$

$$b(n) = (n+\gamma+1)\, c_b \quad \text{and} \quad \hat{b}(n-1) = (n-N)\, c_b^{-1} \tag{b2}$$

$$b(n) = (n-N)\, c_b \quad \text{and} \quad \hat{b}(n-1) = (n+\gamma+1)\, c_b^{-1} \tag{b3}$$

$$b(n) = (n+\gamma+1)(n-N)\, c_b \quad \text{and} \quad \hat{b}(n-1) = 1\, c_b^{-1} \tag{b4}$$

Combining this with (3.11) we must have

$$b(n)\,\hat{b}(n) = \hat{A}(n)= (n+\hat{\gamma}+1)(n-\hat{N})$$

This implies that c_b is independent of n and moreover for (b1)–(b4) yields:

$$(n+\gamma+2)(n-N+1)= (n+\hat{\gamma}+1)(n-\hat{N}) \implies \gamma+1=\hat{\gamma} \ \wedge\ N-1=\hat{N} \tag{b1'}$$

$$(n+\gamma+1)(n-N+1)= (n+\hat{\gamma}+1)(n-\hat{N}) \implies \gamma=\hat{\gamma} \ \wedge\ N-1=\hat{N} \tag{b2'}$$

$$(n+\gamma+2)(n-N)= (n+\hat{\gamma}+1)(n-\hat{N}) \implies \gamma+1=\hat{\gamma} \ \wedge\ N=\hat{N} \tag{b3'}$$

$$(n+\gamma+1)(n-N)= (n+\hat{\gamma}+1)(n-\hat{N}) \implies \gamma=\hat{\gamma} \ \wedge\ N=\hat{N} \tag{b4'}$$

We thus have four viable options for b, $\hat{b}$ and three for a, $\hat{a}$, giving a total of 12 possible combinations, which we will systematically consider and treat.

Case (b1). Plugging (b1) in (3.14), we get

$$a(n)(n+\gamma+1)(n-N)c_b^{-1}+\hat{a}(n-1)c_b+(n+\gamma+1)(n-N)+n(n-\delta-N-1)= \hat{d}(x)d(x)-\Lambda(x)\,.$$

As the right hand side is independent of n, so must be the left hand side. This eliminates options (a2) and (a3) for a, $\hat{a}$ as that would result in a third order term in n which cannot vanish. On the other hand, (a1) yields

$$(n+\gamma+1)(n-N)\,\frac{c_a}{c_b} + n(n-\delta-N-1)\,\frac{c_b}{c_a} + (n+\gamma+1)(n-N)+n(n-\delta-N-1)$$
$$= \hat{d}(x)\, d(x)-\Lambda(x)\,.$$

This must be independent of n, so the coefficient of n^2 in the left hand side must vanish, hence $c_a/c_b + c_b/c_a + 2 = 0$ or thus $c_a/c_b = -1$. For this value of c_a/c_b the left hand side equals zero and is indeed independent of n. Note that this leaves one degree of freedom as only the ratio c_a/c_b is fixed. This is just a global scalar factor for (3.3) and (3.4), also present in (2.5). Henceforth, for convenience, we set $c_a = 1$ and $c_b = -1$.

The combined options (b1) and (a1) thus give a valid set of equations of the form (3.3) and (3.4), and they correspond to the parameter values

$$\hat{\gamma} = \gamma + 1, \qquad \hat{\delta} = \delta + 1, \qquad \hat{N} = N - 1.$$

Moreover, by means of (3.19) we find $\xi = -1$ and so $\hat{x} = x - 1$. Finally, plugging these a, $\hat{a}$, b, $\hat{b}$ in (3.3) and (3.4), and putting $n = 0$ we find

$$R_0(\lambda(x); \gamma, \delta, N) - R_1(\lambda(x); \gamma, \delta, N) = \frac{x(x+\gamma+\delta+1)}{N(\gamma+1)} = \hat{d}(x)$$

and similarly $d(x) = N(\gamma + 1)$. Hence, for $R_n(x) \equiv R_n(\lambda(x); \gamma, \delta, N)$ and $\hat{R}_n(x) \equiv R_n(\lambda(x-1); \gamma+1, \delta+1, N-1)$ we have the relations

$$R_n(x) - R_{n+1}(x) = \frac{x(x+\gamma+\delta+1)}{N(\gamma+1)} \hat{R}_n(x),$$
$$-(n+1)(N-n+\delta)\hat{R}_n(x) + (N-n-1)(n+\gamma+2)\hat{R}_{n+1}(x) = N(\gamma+1)R_{n+1}(x).$$

Interchanging x and n, these recurrence relations for dual Hahn polynomials are precisely the known actions of the forward and backward shift operator for Hahn polynomials [19, (9.5.6),(9.5.8)].

Case (b2). Next, we consider the option (b2) for b, $\hat{b}$. Plugging (b2) in (3.14), we get

$$a(n)\,(n-N)\,c_b^{-1} + \hat{a}(n-1)\,(n+\gamma)\,c_b + (n+\gamma+1)(n-N) + n(n-\delta-N-1) = \hat{d}(x)\,d(x) - \Lambda(x).$$

Since the left hand side must be independent of n, option (a1) is ruled out. Also option (a2) is ruled out: using (a2) and $\delta + N + 1 = 0$ (from (a2')), the left hand side again cannot be independent of n. Only (a3) remains, giving

$$(n-\delta-N)\,(n-N)\,\frac{c_a}{c_b} + n\,(n+\gamma)\,\frac{c_b}{c_a} + (n+\gamma+1)(n-N) + n(n-\delta-N-1) = \hat{d}(x)\,d(x) - \Lambda(x).$$

In order for n^2 in the left hand side to vanish, we again require $c_a/c_b = -1$. This gives

$$-N(N+\gamma+\delta+1) = \hat{d}(x)\,d(x) - \Lambda(x),$$

and we see that both sides are indeed independent of n.

The combined options (b2) and (a3) also give a valid set of equations of the form (3.3) and (3.4), now corresponding to the parameter values

$$\hat{\gamma} = \gamma, \qquad \hat{\delta} = \delta, \qquad \hat{N} = N - 1.$$

Moreover, by means of (3.19) we find $\xi = 0$ and so $\hat{x} = x$. Putting again $n = 0$ in (3.3) and (3.4) for these a, $\hat{a}$, b, $\hat{b}$ we find

$$(-\delta - N)R_0(\lambda(x);\gamma,\delta,N) - (\gamma+1)R_1(\lambda(x);\gamma,\delta,N) = -\frac{(N-x)(x+\gamma+\delta+N+1)}{N} = \hat{d}(x)$$

and similarly $d(x) = N$. The relations in question are then, for $R_n(x) \equiv R_n(\lambda(x);\gamma,\delta,N)$ and $\hat{R}_n(x) \equiv R_n(\lambda(x);\gamma,\delta,N-1)$:

$$(n-\delta-N)R_n(x) - (n+\gamma+1)R_{n+1}(x) = -\frac{(N-x)(x+\gamma+\delta+N+1)}{N}\hat{R}_n(x),$$
$$(n+1)\hat{R}_n(x) - (n-N+1)\hat{R}_{n+1}(x) = NR_{n+1}(x),$$

which can be verified algebraically or by means of a computer algebra package.

Case (b3). The next option to consider is (b3), for which (3.14) becomes

$$a(n)\,(n+\gamma+1)\,c_b^{-1} + \hat{a}(n-1)\,(n-N-1)c_b + (n+\gamma+1)(n-N) + n(n-\delta-N-1) = \hat{d}(x)d(x) - \Lambda(x).$$

The independence of n in the left hand side again rules out options (a1) and (a2), while (a3) gives

$$(n-\delta-N)\,(n+\gamma+1)\frac{c_a}{c_b} + n\,(n-N-1)\frac{c_b}{c_a} + (n+\gamma+1)(n-N) + n(n-\delta-N-1) = \hat{d}(x)\,d(x) - \Lambda(x).$$

Also here, we require $c_a/c_b = -1$ to arrive at a left hand side independent of n, namely

$$(\gamma+1)\delta = \hat{d}(x)\,d(x) - \Lambda(x).$$

The combined options (b3) and (a3) thus give a valid set of equations of the form (3.3) and (3.4), and they correspond to the parameter values

$$\hat{\gamma} = \gamma+1, \qquad \hat{\delta} = \delta-1, \qquad \hat{N} = N;$$

by means of (3.19) we find $\xi = 0$ and so $\hat{x} = x$. Finally, plugging these a, $\hat{a}$, b, $\hat{b}$ in (3.3) and (3.4) and putting $n = 0$ we find

$$(-\delta-N)R_0(\lambda(x);\gamma,\delta,N) + N\,R_1(\lambda(x);\gamma,\delta,N) = -\frac{(x+\gamma+1)(x+\delta)}{(\gamma+1)} = \hat{d}(x)$$

and similarly $d(x) = \gamma+1$.

Hence we have the relations, for $R_n(x) \equiv R_n(\lambda(x);\gamma,\delta,N)$ and $\hat{R}_n(x) \equiv R_n(\lambda(x);\gamma+1,\delta-1,N)$:

$$-(n-\delta-N)R_n(x) + (n-N)R_{n+1}(x) = \frac{(x+\gamma+1)(x+\delta)}{(\gamma+1)}\hat{R}_n(x),$$

$$-(n+1)\hat{R}_n(x)+(n+\gamma+2)\hat{R}_{n+1}(x)=(\gamma+1)R_{n+1}(x).$$

These can again be verified algebraically or by means of a computer algebra package. Note that these relations coincide with (2.21)–(2.22) from the previous section (up to a shift $\delta \longrightarrow \delta+1$).

Case (b4). The final option (b4) for b, $\hat{b}$ does not correspond to a valid set of equations of the form (3.3) and (3.4) as the left hand side of (3.14) can never be independent of n for either options (a1), (a2) or (a3).

This completes the analysis in the case of dual Hahn polynomials, and we have the following result:

THEOREM 7: *The only way to double dual Hahn polynomials, i.e. to combine two sets of dual Hahn polynomials such that they satisfy a pair of recurrence relations of the form* (3.1)–(3.2) *is one of the three cases:*
dual Hahn I, $R_n(x)\equiv R_n(\lambda(x);\gamma,\delta,N)$ *and* $\hat{R}_n(x)\equiv R_n(\lambda(x-1);\gamma+1,\delta+1,N-1)$:

$$R_n(x)-R_{n+1}(x)=\frac{x(x+\gamma+\delta+1)}{N(\gamma+1)}\hat{R}_n(x),$$
$$-(n+1)(N-n+\delta)\hat{R}_n(x)+(N-n-1)(n+\gamma+2)\hat{R}_{n+1}(x)=N(\gamma+1)R_{n+1}(x).$$

dual Hahn II, $R_n(x)\equiv R_n(\lambda(x);\gamma,\delta,N)$ *and* $\hat{R}_n(x)\equiv R_n(\lambda(x);\gamma,\delta,N-1)$:

$$(n-\delta-N)R_n(x)-(n+\gamma+1)R_{n+1}(x)=-\frac{(N-x)(x+\gamma+\delta+N+1)}{N}\hat{R}_n(x),$$
$$(n+1)\hat{R}_n(x)-(n-N+1)\hat{R}_{n+1}(x)=NR_{n+1}(x).$$

dual Hahn III, $R_n(x)\equiv R_n(\lambda(x);\gamma,\delta,N)$ *and* $\hat{R}_n(x)\equiv R_n(\lambda(x);\gamma+1,\delta-1,N)$:

$$-(n-\delta-N)R_n(x)+(n-N)R_{n+1}(x)=\frac{(x+\gamma+1)(x+\delta)}{(\gamma+1)}\hat{R}_n(x),$$
$$-(n+1)\hat{R}_n(x)+(n+\gamma+2)\hat{R}_{n+1}(x)=(\gamma+1)R_{n+1}(x).$$

By interchanging x and n, each of the recurrence relations for dual Hahn polynomials in the previous theorem gives rise to a set of forward and backward shift operators for regular Hahn polynomials. The case **dual Hahn I** corresponds just to the known forward and backward shift operators for Hahn polynomials [19]: $Q_n(x)\equiv Q_n(x;\alpha,\beta,N)$ and $\hat{Q}_n(x)\equiv Q_n(x;\alpha+1,\beta+1,N-1)$:

$$Q_n(x)-Q_n(x+1)=\frac{n(n+\alpha+\beta+1)}{N(\alpha+1)}\hat{Q}_{n-1}(x),$$
$$-(x+1)(N-x+\beta)\hat{Q}_{n-1}(x)+(N-x-1)(x+\alpha+2)\hat{Q}_{n-1}(x+1)$$
$$=N(\alpha+1)Q_n(x+1).$$

The case **dual Hahn III** corresponds to our introductory example (2.11)–(2.12) (up to a shift $\beta \longrightarrow \beta+1$), and appears already in [27]. The case **dual Hahn II** yields a new set of relations (encountered recently in [16, (16)–(17)]), namely $Q_n(x) \equiv Q_n(x;\alpha,\beta,N)$ and $\hat{Q}_n(x) \equiv Q_n(x;\alpha,\beta,N-1)$:

$$(x-\beta-N)Q_n(x)-(x+\alpha+1)Q_n(x+1) = -\frac{(N-n)(n+\alpha+\beta+N+1)}{N}\hat{Q}_n(x),$$
$$(x+1)\hat{Q}_n(x)-(x-N+1)\hat{Q}_n(x+1) = NQ_n(x+1).$$

The most important thing is, however, that we have classified the possible cases.

Because the sets of recurrence relations are of the form (3.1)–(3.2), they can be cast in matrix form, like in (2.17), with a simple two-diagonal matrix. For the case **dual Hahn I**, note that the N-values of $R_n(x)$ and $\hat{R}_n(x)$ differ by 1, so the definition of the matrix U (again in terms of the normalized version of the polynomials) requires a little bit more attention. The matrix U is now of order $(2N+1)\times(2N+1)$ with matrix elements

$$\begin{aligned}
U_{2n,N-x} &= U_{2n,N+x} = \frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x);\gamma,\delta,N), \qquad (x=1,\ldots,N)\\
U_{2n+1,N-x} &= -U_{2n+1,N+x} = -\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x-1);\gamma+1,\delta+1,N-1), \quad (x=1,\ldots,N)\\
U_{2n,N} &= (-1)^n\tilde{R}_n(\lambda(0);\gamma,\delta,N), \qquad U_{2n+1,N} = 0,
\end{aligned} \tag{3.20}$$

where the row index of the matrix U (denoted here by $2n$ or $2n+1$, depending on the parity of the index) also runs over the integers from 0 up to $2N$. This matrix U is orthogonal: the orthogonality relation of the dual Hahn polynomials (2.8) and the signs in the matrix U imply that its rows are orthonormal. Thus $U^TU = UU^T = I$, the identity matrix. Then the recurrence relations for **dual Hahn I** of Theorem 7 are now reformulated in terms of a two-diagonal $(2N+1)\times(2N+1)$-matrix of the form

$$M = \begin{pmatrix}
0 & M_0 & 0 & & \\
M_0 & 0 & M_1 & \ddots & \\
0 & M_1 & 0 & \ddots & 0\\
 & \ddots & \ddots & \ddots & M_{2N-1}\\
 & & 0 & M_{2N-1} & 0
\end{pmatrix}. \tag{3.21}$$

Explicitly:

PROPOSITION 8 (dual Hahn I): *Suppose $\gamma > -1$, $\delta > -1$. Let M be the two-diagonal matrix* (3.21) *with*

$$M_{2k} = \sqrt{(k+\gamma+1)(N-k)}, \qquad M_{2k+1} = \sqrt{(k+1)(N+\delta-k)}, \tag{3.22}$$

and U the orthogonal matrix determined in (3.20). *Then the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:*

$$D = \mathrm{diag}(-\epsilon_N,\ldots,-\epsilon_1,0,\epsilon_1,\ldots,\epsilon_N); \quad \epsilon_k = \sqrt{k(k+\gamma+\delta+1)} \quad (k=1,\ldots,N). \tag{3.23}$$

Note that we have kept only the conditions under which the matrix M is real. The other conditions for which the dual Hahn polynomials in (3.20) can be normalized (namely $\gamma < -N$, $\delta < -N$) would give rise to imaginary values in (3.22). In such a case, the relation $MU = UD$ remains valid, and also D would have imaginary values.

For the case **dual Hahn II**, the matrix U is again of order $(2N + 1)\times(2N + 1)$ with matrix elements

$$\begin{aligned}
&U_{2n,x} = U_{2n,2N-x} = \frac{1}{\sqrt{2}}\tilde{R}_n(\lambda(x);\gamma,\delta,N), \qquad (x = 0,\dots,N-1)\\
&U_{2n+1,x} = -U_{2n+1,2N-x} = -\frac{1}{\sqrt{2}}\tilde{R}_n(\lambda(x);\gamma,\delta,N-1), \qquad (x = 0,\dots,N-1) \qquad (3.24)\\
&U_{2n,N} = \tilde{R}_n(\lambda(N);\gamma,\delta,N), \qquad U_{2n+1,N} = 0,
\end{aligned}$$

where the row indices are as in (3.20). The orthogonality relation of the dual Hahn polynomials and the signs in the matrix U imply that its rows are orthonormal, so $U^TU = UU^T = I$. The pair of recurrence relations for **dual Hahn II** of Theorem 7 yield:

PROPOSITION 9 (dual Hahn II): *Suppose* $\gamma > -1$, $\delta > -1$. *Let* M *be a tridiagonal* $(2N + 1)\times(2N + 1)$*-matrix of the form* (3.21) *with*

$$M_{2k} = \sqrt{(N+\delta-k)(N-k)}, \qquad M_{2k+1} = \sqrt{(k+1)(k+\gamma+1)}, \qquad (3.25)$$

and U *the orthogonal matrix determined in* (3.24). *Then the columns of* U *are the eigenvectors of* M*, i.e.* $MU = UD$*, where* D *is a diagonal matrix containing the eigenvalues of* M*:*

$$\begin{aligned}
D &= \mathrm{diag}(-\epsilon_N,\dots,-\epsilon_1,0,\epsilon_1,\dots,\epsilon_N);\\
\epsilon_k &= \sqrt{k(\gamma+\delta+1+2N-k)} \qquad (k = 1,\dots,N).
\end{aligned}$$

Note that the order in which the normalized dual Hahn polynomials appear in the matrix U is different for (3.20) and (3.24). This is related to the indices of the polynomials in the relations of Theorem 7.

Finally, for the case **dual Hahn III**, the matrix U is given by (2.19)–(2.20) and we recapitulate the results given at the end of the previous section, now in terms of the dual Hahn parameters γ and δ.

PROPOSITION 10 (dual Hahn III): *Suppose* $\gamma > -1$, $\delta > -1$ *or* $\gamma < -N-1$, $\delta < -N-1$. *Let* M *be the tridiagonal matrix* (2.15) *with*

$$M_{2k} = \sqrt{(k+\gamma+1)(N+\delta+1-k)}, \qquad M_{2k+1} = \sqrt{(k+1)(N-k)}, \qquad (3.26)$$

and U *the orthogonal matrix determined in* (2.19)–(2.20). *Then the columns of* U *are the eigenvectors of* M*, i.e.* $MU = UD$*, where* D *is a diagonal matrix containing the eigenvalues of* M*:*

$$\begin{aligned}
D &= \mathrm{diag}(-\epsilon_N,\dots,-\epsilon_1,-\epsilon_0,\epsilon_0,\epsilon_1,\dots,\epsilon_N);\\
\epsilon_k &= \sqrt{(k+\gamma+1)(k+\delta+1)} \qquad (k = 0,1,\dots,N).
\end{aligned}$$

To conclude for dual Hahn polynomials: there are three sets of recurrence relations of the form (3.1)–(3.2). Each of the three cases gives rise to a two-diagonal matrix with simple and explicit eigenvalues, and eigenvectors given in terms of two sets of dual Hahn polynomials.

4 DOUBLING HAHN POLYNOMIALS

The technique presented in the previous section can be applied to other types of discrete orthogonal polynomials with a finite spectrum. We have done this for Hahn polynomials. One level up in the hierarchy of orthogonal polynomials of hypergeometric type are the Racah polynomials. Also for Racah polynomials we have applied the technique, but here the description of the results becomes very technical. So we shall leave the results for Racah polynomials for the Appendix.

For Hahn polynomials the analysis is again straightforward but tedious, so let us skip the details of the computation and present just the final outcome here. Applying the technique described in (3.3)–(3.17), with $y_n = Q_n(x;\alpha,\beta,N)$ and $\hat{y}_n = Q_n(\hat{x};\hat{\alpha},\hat{\beta},\hat{N})$ yields the following result.

THEOREM 11: *The only way to combine two sets of Hahn polynomials such that they satisfy a pair of recurrence relations of the form* (3.3)–(3.4) *is one of the four cases:*
Hahn I, $Q_n(x)\equiv Q_n(x;\alpha,\beta,N)$ *and* $\hat{Q}_n(x)\equiv Q_n(x;\alpha+1,\beta,N)$*:*

$$\frac{(n+\alpha+\beta+N+2)}{(2n+\alpha+\beta+2)}Q_n(x)-\frac{(N-n)}{(2n+\alpha+\beta+2)}Q_{n+1}(x)=\frac{(\alpha+x+1)}{(\alpha+1)}\hat{Q}_n(x),$$
$$-\frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+3)}\hat{Q}_n(x)+\frac{(n+\alpha+\beta+2)(n+\alpha+2)}{(2n+\alpha+\beta+3)}\hat{Q}_{n+1}(x)=(\alpha+1)Q_{n+1}(x).$$

Hahn II, $Q_n(x)\equiv Q_n(x;\alpha,\beta,N)$ *and* $\hat{Q}_n(x)\equiv Q_n(x-1;\alpha+1,\beta,N-1)$*:*

$$\frac{1}{(2n+\alpha+\beta+2)}Q_n(x)-\frac{1}{(2n+\alpha+\beta+2)}Q_{n+1}(x)=\frac{x}{N(\alpha+1)}\hat{Q}_n(x),$$
$$-\frac{(n+1)(n+\beta+1)(n+\alpha+\beta+N+2)}{(2n+\alpha+\beta+3)}\hat{Q}_n(x)$$
$$+\frac{(n+\alpha+\beta+2)(N-n-1)(n+\alpha+2)}{(2n+\alpha+\beta+3)}\hat{Q}_{n+1}(x)=N(\alpha+1)Q_{n+1}(x).$$

Hahn III, $Q_n(x)\equiv Q_n(x;\alpha,\beta,N)$ *and* $\hat{Q}_n(x)\equiv Q_n(x;\alpha,\beta+1,N)$*:*

$$\frac{(n+\beta+1)(n+N+2+\alpha+\beta)}{(2n+\alpha+\beta+2)}Q_n(x)+\frac{(N-n)(n+\alpha+1)}{(2n+\alpha+\beta+2)}Q_{n+1}(x)$$
$$=(\beta+1+N-x)\hat{Q}_n(x),$$
$$\frac{(n+1)}{(2n+\alpha+\beta+3)}\hat{Q}_n(x)+\frac{(n+\alpha+\beta+2)}{(2n+\alpha+\beta+3)}\hat{Q}_{n+1}(x)=Q_{n+1}(x).$$

Hahn IV, *$Q_n(x) \equiv Q_n(x;\alpha,\beta,N)$ and $\hat{Q}_n(x) \equiv Q_n(x;\alpha,\beta+1,N-1)$:*

$$\frac{(n+\beta+1)}{(2n+\alpha+\beta+2)}Q_n(x)+\frac{(n+\alpha+1)}{(2n+\alpha+\beta+2)}Q_{n+1}(x)=\frac{(N-x)}{N}\hat{Q}_n(x),$$
$$\frac{(n+1)(n+\alpha+\beta+N+2)}{(2n+\alpha+\beta+3)}\hat{Q}_n(x)+\frac{(N-n-1)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+3)}\hat{Q}_{n+1}(x)=NQ_{n+1}(x).$$

Note that when interchanging x and n the relations in **Hahn II** coincide with the known forward and backward shift operator relations for dual Hahn polynomials [19, (9.6.6), (9.6.8)]. In the same way, the other cases yield new forward and backward shift operator relations for dual Hahn polynomials.

Since the recurrence relations are of the form (3.3)–(3.4), they can be cast in matrix form with a two-diagonal matrix. We shall write the matrix elements again in terms of normalized polynomials. For the case **Hahn I**, the matrix U of order $(2N+2)\times(2N+2)$, with elements

$$U_{2n,N-x} = U_{2n,N+x+1} = \frac{(-1)^n}{\sqrt{2}}\tilde{Q}_n(x;\alpha,\beta,N),$$
$$U_{2n+1,N-x} = -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}}\tilde{Q}_n(x;\alpha+1,\beta,N) \tag{4.1}$$

where $x, n \in \{0, 1, \ldots, N\}$, is orthogonal, and the recurrence relations yield:

PROPOSITION 12 (Hahn I): *Suppose that $\gamma, \delta > -1$. Let M be a tridiagonal $(2N+2)\times(2N+2)$-matrix of the form* (2.15) *with*

$$M_{2k} = \sqrt{\frac{(k+\alpha+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2+N)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}},$$
$$M_{2k+1} = \sqrt{\frac{(k+\beta+1)(k+1)(N-k)}{(2k+\alpha+\beta+2)(2k+\alpha+\beta+3)}}, \tag{4.2}$$

and U the orthogonal matrix determined in (4.1)*. Then the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:*

$$D = \mathrm{diag}(-\epsilon_N, \ldots, -\epsilon_1, -\epsilon_0, \epsilon_0, \epsilon_1, \ldots, \epsilon_N); \quad \epsilon_k = \sqrt{k+\alpha+1} \quad (k = 0, 1, \ldots, N). \tag{4.3}$$

For the case **Hahn II**, the orthogonal matrix U is of order $(2N+1)\times(2N+1)$, with elements

$$U_{2n,N-x} = U_{2n,N+x} = \frac{(-1)^n}{\sqrt{2}}\tilde{Q}_n(x;\alpha,\beta,N), \qquad (x = 1, \ldots, N)$$
$$U_{2n+1,N-x} = -U_{2n+1,N+x} = -\frac{(-1)^n}{\sqrt{2}}\tilde{Q}_n(x-1;\alpha+1,\beta,N-1), \qquad (x = 1, \ldots, N) \tag{4.4}$$

$$U_{2n,N} = (-1)^n \tilde{Q}_n(0; \alpha, \beta, N), \qquad U_{2n+1,N} = 0,$$

where the row indices are as in (3.20). The recurrence relations for **Hahn II** yield:

PROPOSITION 13 (Hahn II): *Suppose that $\alpha, \beta > -1$ or $\alpha, \beta < -N$. Let M be a tridiagonal $(2N+1)\times(2N+1)$-matrix of the form* (3.21) *with*

$$M_{2k} = \sqrt{\frac{(k+\alpha+1)(k+\alpha+\beta+1)(N-k)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}},$$
$$M_{2k+1} = \sqrt{\frac{(k+\beta+1)(k+\alpha+\beta+2+N)(k+1)}{(2k+\alpha+\beta+2)(2k+\alpha+\beta+3)}}, \tag{4.5}$$

and U the orthogonal matrix determined in (4.4). *Then the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:*

$$D = \mathrm{diag}(-\epsilon_N, \ldots, -\epsilon_1, 0, \epsilon_1, \ldots, \epsilon_N); \qquad \epsilon_k = \sqrt{k} \qquad (k = 1, \ldots, N). \tag{4.6}$$

Note that for both cases, the two-diagonal matrix M becomes more complicated compared to the cases for dual Hahn polynomials, but the matrix D of eigenvalues becomes simpler.

For the two remaining cases we need not give all details: the matrix M for the case **Hahn III** is equal to the matrix M for the case **Hahn I** with the replacement $\alpha \leftrightarrow \beta$, and so its eigenvalues are $\pm\sqrt{k+\beta+1}$ ($k = 0, 1, \ldots, N$). And the matrix M for the case **Hahn IV** is equal to the matrix M for the case **Hahn II** with the same replacement $\alpha \leftrightarrow \beta$, so its eigenvalues are 0 and $\pm\sqrt{k}$ ($k = 1, \ldots, N$).

5 POLYNOMIAL SYSTEMS, CHRISTOFFEL AND GERONIMUS TRANSFORMS

So far, we have only partially explained why the technique in the previous sections is referred to as "doubling" polynomials. It is indeed a fact that the combination of two sets of polynomials, each with different parameters, yields a new set of orthogonal polynomials. This can be compared to the well known situation of combining two sets of generalized Laguerre polynomials (both with different parameters α and $\alpha - 1$) into the set of "generalized Hermite polynomials" [8]. There, for $\alpha > 0$, one defines

$$P_{2n}(x) = \sqrt{\frac{n!}{(\alpha)_n}} L_n^{(\alpha-1)}(x^2), \qquad P_{2n+1}(x) = \sqrt{\frac{n!}{(\alpha)_{n+1}}}\, x L_n^{(\alpha)}(x^2). \tag{5.1}$$

Then the orthogonality relation of Laguerre polynomials leads to the orthogonality of the polynomials (5.1):

$$\int_{-\infty}^{+\infty} w(x) P_n(x) P_{n'}(x) dx = \Gamma(\alpha) \delta_{n,n'}, \tag{5.2}$$

where

$$w(x)= e^{-x^2}|x|^{2\alpha-1}. \tag{5.3}$$

Note that the even polynomials are Laguerre polynomials in x^2 (for parameter $\alpha - 1$), and the odd polynomials are Laguerre polynomials in x^2 (for parameter α) multiplied by a factor x. The weight function (5.3) is common for both types of polynomials. It is this phenomenon that appears here too in our doubling process of Hahn or dual Hahn polynomials.

From a more general point of view, this fits in the context of obtaining a new family of orthogonal polynomials starting from a set of orthogonal polynomials and its kernel partner related by a Christoffel transform [8, 22, 32]. In a way, our classification determines for which Christoffel parameter ν (see [32] for the notation) the Christoffel transform of a Hahn, dual Hahn or Racah polynomial is again a Hahn, dual Hahn or Racah polynomial with possibly different parameters. This determines moreover quite explicitly the common weight function.

For a dual Hahn polynomial $R_n(x)\equiv R_n(\lambda(x);\gamma,\delta,N)$, with data given in (2.10), and a Christoffel parameter ν the kernel partner is given by the transform

$$P_n(x)= \frac{R_{n+1}(x)-a_nR_n(x)}{\Lambda(x)-\Lambda(\nu)}, \qquad a_n = \frac{R_{n+1}(\nu)}{R_n(\nu)}. \tag{5.4}$$

Because of the recurrence relation (2.5) and what is called the Geronimus transform the original polynomials can also be expressed in terms of the kernel partners. This is usually done for monic polynomials(see [32, (3.2)-(3.3)]), but it can be extended to non-monic dual Hahn polynomials as follows

$$R_n(x)= A(n)P_n(x)-b_nP_{n-1}(x) \tag{5.5}$$

where the coefficients b_n are related to the recurrence relation (2.5) as follows

$$b_na_{n-1} = C(n), \qquad A(n)a_n + b_n = A(n)+C(n)+\Lambda(\nu). \tag{5.6}$$

Our classification now shows that only for ν equal to one of the values 0, N or $-\delta$, the kernel partner $P_n(x)$ will again be a dual Hahn polynomial. Indeed, taking for example $\nu = 0$ in (5.4) we have $R_n(0)= 1$ and

$$P_n(x)= \frac{R_{n+1}(x)-R_n(x)}{\Lambda(x)} = \frac{-1}{N(\gamma+1)}R_n(\lambda(x-1);\gamma+1,\delta+1,N-1),$$

where we used the first relation of **dual Hahn I** to obtain again a dual Hahn polynomial. The reverse transform (5.5) follows immediately from the second relation of **dual Hahn I**. Similarly, taking $\nu = N$ in (5.4) we have $R_n(N)= (-N-\delta)_n/(\gamma+1)_n$ and

$$\begin{aligned} P_n(x) &= \frac{R_{n+1}(x)-R_n(x)(n-\delta-N)/(n+\gamma+1)}{(x-N)(x+N+\gamma+\delta+1)} \\ &= \frac{-1}{N(n+\gamma+1)}R_n(\lambda(x);\gamma,\delta,N-1), \end{aligned}$$

which we obtained using the first relation of **dual Hahn II**. For the reverse transform (5.5) we find, using the second relation of **dual Hahn II** with shifted $n \mapsto n-1$,

$$\begin{aligned} A(n)P_n(x) - b_n P_{n-1}(x) &= \frac{-(n-N)}{N} R_n(\lambda(x);\gamma,\delta,N-1) + \frac{n}{N} R_{n-1}(\lambda(x);\gamma,\delta,N-1) \\ &= R_n(x). \end{aligned}$$

For the last case, taking $\nu = -\delta$ in (5.4) we have $R_n(-\delta) = (-N-\delta)_n/(-N)_n$ and

$$\begin{aligned} P_n(x) &= \frac{R_{n+1}(x) - R_n(x)(n-\delta-N)/(n-N)}{(x+\gamma+1)(x+\delta)} \\ &= \frac{1}{(\gamma+1)(n-N)} R_n(\lambda(x);\gamma+1,\delta-1,N), \end{aligned}$$

which we obtained using the first relation of **dual Hahn III**. For the transform (5.5) we have,

$$\begin{aligned} &A(n)P_n(x) - b_n P_{n-1}(x) \\ &= \frac{n+\gamma+1}{\gamma+1} R_n(\lambda(x);\gamma+1,\delta-1,N-1) - \frac{n}{\gamma+1} R_{n-1}(\lambda(x);\gamma+1,\delta-1,N), \end{aligned}$$

which equals $R_n(x)$ by the second relation of **dual Hahn III**.

In a similar way, for the Hahn polynomials, putting $Q_n(x) \equiv Q_n(x;a,b,N)$, using the data (2.6) in

$$P_n(x) = \frac{Q_{n+1}(x) - a_n Q_n(x)}{\Lambda(x) - \Lambda(\nu)}, \qquad a_n = \frac{Q_{n+1}(\nu)}{Q_n(\nu)},$$

and in (5.5)–(5.6), the cases **Hahn I, II, III, V** correspond respectively to the choices $-\alpha-1$, 0, $N+\beta+1$ and N for ν.

The task of determining for which Christoffel parameter ν the kernel partner of a dual Hahn polynomial is again of the same family is not trivial. It comes down to finding a pair of recurrence relations of the form (3.1)–(3.2) with coefficients related to ν as in (5.4). We have classified these for general coefficients, without a relation to ν, and we observe that each solution indeed corresponds to a specific choice for ν.

The transforms (5.4)–(5.5) give rise to new orthogonal systems, but in general there is no way of writing the common weight function. However, since here both sets are of the same family, we can actually do this. Let us begin with the dual Hahn polynomials, in particular the case **dual Hahn I**, for which the corresponding matrix U is given in (3.20). They give rise to a new family of discrete orthogonal polynomials with the relation $MU = UD$ corresponding to their three term recurrence relation with Jacobi matrix M (3.22). In general the support of the weight function is equal to the spectrum of the Jacobi matrix [5, 18, 20, 21]. After simplifying with the normalization factors (2.9), this leads to a discrete orthogonality of polynomials, with support equal to the eigenvalues of M (so in this case, the support follows from (3.23)). Concretely, for the case under consideration, we have:

PROPOSITION 14: *Let $\gamma > -1$, $\delta > -1$, and consider the $2N+1$ polynomials:*

$$P_{2n}(q)= \frac{(-1)^n}{\sqrt{2}} R_n(q^2; \gamma, \delta, N), \qquad (n = 0, 1, \ldots, N)$$
$$P_{2n+1}(q)= -\frac{(-1)^n}{\sqrt{2}} \frac{\sqrt{(n+\gamma+1)(N-n)}}{(\gamma+1)N} \, q \, R_n(q^2-\gamma-\delta-2; \gamma+1, \delta+1, N-1)$$
$$(n = 0, 1, \ldots, N-1). \tag{5.7}$$

These polynomials satisfy the discrete orthogonality relation

$$\sum_{q\in S} \frac{(-1)^k(2k+\gamma+\delta+1)(\gamma+1)_k(-N)_k N!}{(k+\gamma+\delta+1)_{N+1}(\delta+1)_k k!} (1+\delta_{q,0}) P_n(q) P_{n'}(q)$$
$$= \left[\binom{\gamma+\lfloor n/2\rfloor}{\lfloor n/2\rfloor} \binom{\delta+N-\lfloor n/2\rfloor}{N-\lfloor n/2\rfloor} \right]^{-1} \delta_{n,n'} \tag{5.8}$$

with

$$S = \{0, \pm\sqrt{k(k+\gamma+\delta+1)} \qquad (k = 1, 2, \ldots, N)\}.$$

Note that for $q \in S$, $q^2 = k(k+\gamma+\delta+1) \equiv \lambda(k)$, and the polynomial $P_{2n}(q)$ is of the form $R_n(\lambda(k); \gamma, \delta, N)$. In that case, $q^2-\gamma-\delta-2 = (k-1)((k-1)+(\gamma+1)+(\delta+1)+1) \equiv \lambda(k-1)$, so the polynomial $P_{2n+1}(q)$ is of the form $R_n(\hat{\lambda}(k-1); \gamma+1, \delta+1, N-1)$. The interpretation of the weight function in the left hand side of (5.8) is as follows: each q in the support S is mapped to a k-value belonging to $\{0, 1, \ldots, N\}$, and then the weight depends on this k-value.

Now we turn to the classification of section 4, where the corresponding orthogonal matrices U are given in terms of (normalized) Hahn polynomials. for the case **Hahn I**, the matrix U is given in (4.1), and the spectrum of the matrix M is given by (4.3). After simplifying the normalization factors, the orthogonality of the rows of U gives rise to:

PROPOSITION 15: *Let $\alpha > -1$, $\beta > -1$, and consider the $2N+2$ polynomials ($n = 0, 1, \ldots, N$):*

$$P_{2n}(q)= \frac{(-1)^n}{\sqrt{2}} Q_n(q^2-\alpha-1; \alpha, \beta, N),$$
$$P_{2n+1}(q)= -\frac{(-1)^n}{\sqrt{2}} \frac{1}{(\alpha+1)} \sqrt{\frac{(n+\alpha+1)(n+\alpha+\beta+1)(2n+2+\alpha+\beta)}{(n+N+\alpha+\beta+2)(2n+\alpha+\beta+1)}}$$
$$\times \, q \, Q_n(q^2-\alpha-1; \alpha+1, \beta, N). \tag{5.9}$$

These polynomials satisfy the discrete orthogonality relation

$$\sum_{q\in S} \binom{q^2-1}{q^2-\alpha-1} \binom{N-q^2+\alpha+\beta+1}{N-q^2+\alpha+1} P_n(q) P_{n'}(q)= h_{\lfloor n/2\rfloor}(\alpha, \beta, N) \beta_{n,n'}$$

with

$$S = \{-\sqrt{N+\alpha+1}, -\sqrt{N+\alpha}, \ldots, -\sqrt{\alpha+1}, \sqrt{\alpha+1}, \ldots, \sqrt{N+\alpha}, \sqrt{N+\alpha+1}\}$$

and

$$h_n(\alpha, \beta, N) = \frac{(-1)^n (n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}.$$

So $P_n(q)$ is a polynomial of degree n in the variable q, of different type (with different parameters when expressed as a Hahn polynomial) depending on whether n is even or n is odd. The support points of the discrete orthogonality are given by

$$q = \pm\sqrt{k+\alpha+1} \qquad k = 0, \ldots, N.$$

In the same way, the dual orthogonality for the case **Hahn II** gives rise to:

PROPOSITION 16: *Let $\alpha > -1$, $\beta > -1$, and consider the $2N+1$ polynomials:*

$$\begin{aligned} P_{2n}(q) &= \frac{(-1)^n}{\sqrt{2}} Q_n(q^2; \alpha, \beta, N), \qquad (n = 0, 1, \ldots, N) \\ P_{2n+1}(q) &= -\frac{(-1)^n}{\sqrt{2}} \frac{1}{(\alpha+1)N} \sqrt{\frac{(N-n)(n+\alpha+1)(n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{(2n+\alpha+\beta+1)}} \\ &\quad \times q\, Q_n(q^2-1; \alpha+1, \beta, N-1) \qquad (n = 0, 1, \ldots, N-1). \end{aligned} \tag{5.10}$$

These polynomials satisfy the discrete orthogonality relation

$$\sum_{q\in S} \binom{q^2+\alpha}{q^2}\binom{N-q^2+\beta}{N-q^2}(1+\delta_{q,0})P_n(q)P_{n'}(q) = h_{\lfloor n/2 \rfloor}(\alpha, \beta, N)\beta_{n,n'}$$

with

$$S = \{-\sqrt{N}, -\sqrt{N-1}, \ldots, -1, 0, 1, \ldots, \sqrt{N-1}, \sqrt{N}\}$$

and

$$h_n(\alpha, \beta, N) = \frac{(-1)^n (n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}.$$

The ideas described in the three propositions of this section should be clear. It would lead us too far to give also the explicit forms corresponding to the remaining cases. Let us just mention that also for these cases the support of the new polynomials coincides with the spectrum of the corresponding two-diagonal matrix M.

6 FIRST APPLICATION: EIGENVALUE TEST MATRICES

In sections 3 and 4 we have encountered a number of symmetric two-diagonal matrices M with explicit expressions for the eigenvectors and eigenvalues. In general, if one considers a two-diagonal matrix A of size $(m+2)\times(m+2)$,

$$A = \begin{pmatrix} 0 & b_0 & 0 & & \\ c_0 & 0 & b_1 & \ddots & \\ 0 & c_1 & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & b_m \\ & & 0 & c_m & 0 \end{pmatrix}, \tag{6.1}$$

then it is clear that the characteristic polynomial depends on the products $b_i c_i$ ($i = 0, \ldots, m$) only, and not on b_i and c_i separately. So the same holds for the eigenvalues. Therefore, if all matrix elements b_i and c_i are positive, the eigenvalues of A or of the related symmetric matrix

$$A' = \begin{pmatrix} 0 & \sqrt{b_0c_0} & 0 & & \\ \sqrt{b_0c_0} & 0 & \sqrt{b_1c_1} & \ddots & \\ 0 & \sqrt{b_1c_1} & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & \sqrt{b_mc_m} \\ & & 0 & \sqrt{b_mc_m} & 0 \end{pmatrix} \tag{6.2}$$

are the same. The eigenvectors of A' are those of A after multiplication by a diagonal matrix (the diagonal matrix that is used in the similarity transformation from A to A').

For matrices of type (6.1), it is sufficient to denote them by their superdiagonal $[\mathbf{b}] = [b_0, \ldots, b_m]$ and their subdiagonal $[\mathbf{c}] = [c_0, \ldots, c_m]$. So the Sylvester-Kac matrix from the introduction is denoted by

$$[\mathbf{b}] = [1, 2, \ldots, N], \qquad [\mathbf{c}] = [N, \ldots, 2, 1],$$

with eigenvalues given by (1.2).

The importance of the Sylvester-Kac matrix as a test matrix for numerical eigenvalue routines has already been emphasized in the Introduction. In this context, it is also significant that the matrix itself has integer entries only (so there is no rounding error when represented on a digital computer), and that also the eigenvalues are integers. Of course, matrices with rational numbers as entries suffice as well, since one can always multiply the matrix by an appropriate integer factor.

Let us now systematically consider the two-diagonal matrices encountered in the classification process of doubling Hahn or dual Hahn polynomials. For the matrix (3.21) of the **dual Hahn I** case, the corresponding non-symmetric form can be chosen as the two-diagonal matrix with

$$\begin{aligned}[\mathbf{b}] &= [\gamma+1, 1, \gamma+2, 2, \ldots, \gamma+N, N], \\ [\mathbf{c}] &= [N, N+\delta, N-1, N-1+\delta, \ldots, 1, \delta+1]. \end{aligned} \tag{6.3}$$

The eigenvalues are determined by Proposition 8 and given by 0, $\pm\sqrt{k(k+\gamma+\delta+1)}$ ($k = 1, \ldots, N$). This is (up to a factor 2) the matrix (1.3) mentioned in the Introduction. As test matrix, the choice $\gamma+\delta+1 = 0$ (leaving one free parameter) is interesting as it gives rise to integer eigenvalues. In Proposition 8 there is the initial condition $\gamma > -1$, $\delta > -1$. Clearly, if one is only dealing with eigenvalues, the condition for (6.3) is just $\gamma+\delta+2 \geq 0$. And when one substitutes $\delta = -\gamma-1$ in (6.3), there is no condition at all for the one-parameter family of matrices of the form (6.3).

For the **dual Hahn II** case, the matrix (3.25) is given in Proposition 9, and its non-symmetric form can be taken as

$$\begin{aligned}[\mathbf{b}] &= [\gamma+N, 1, \gamma+N-1, 2, \ldots, \gamma+1, N], \\ [\mathbf{c}] &= [N, \delta+1, N-1, \delta+2, \ldots, 1, \delta+N]. \end{aligned} \tag{6.4}$$

The eigenvalues are given by 0, $\pm\sqrt{k(\gamma+\delta+1+2N-k)}$ ($k = 1, \ldots, N$). There is no simple substitution that reduces these eigenvalues to integers.

For the **dual Hahn III** case, the matrix (2.16) is given in Proposition 10, and its simplest non-symmetric form is

$$\begin{aligned}[\mathbf{b}] &= [\gamma+1, 1, \gamma+2, 2, \ldots, \gamma+N, N, \gamma+N+1], \\ [\mathbf{c}] &= [\delta+N+1, N, \delta+N, N-1, \ldots, \delta+2, 1, \delta+1]. \end{aligned} \tag{6.5}$$

The eigenvalues are given by (2.18), i.e. $\pm\sqrt{(\gamma+k+1)(\delta+k+1)}$ ($k = 0, \ldots, N$). Up to a factor 2, this is the third matrix mentioned in the Introduction. The substitution $\delta = \gamma$ leads to a one-parameter family of two-diagonal matrices with square-free eigenvalues. And in particular when moreover γ is integer, all matrix entries and all eigenvalues are integers.

The two-diagonal matrices arising from the Hahn doubles or the Racah doubles can also be written in a square-free form of type (6.1). However, for these cases the entries in the two-diagonal matrices M are already quite involved (see e.g. Propositions 12, 13, 18 or 19), and we shall not discuss them further in this context. The three examples given here, (6.3)–(6.5), are already sufficiently interesting as extensions of the Sylvester-Kac matrix as potential eigenvalue test matrices.

7 FURTHER APPLICATIONS: RELATED ALGEBRAIC STRUCTURES AND FINITE OSCILLATOR MODELS

The original example of a (dual) Hahn double, described here in section 2, was encountered in the context of a finite oscillator model [14]. In that context, there is also a related algebraic structure. In particular, the two-diagonal matrices M of the form (2.15) or (3.21) are interpreted as representation matrices of an algebra, which can be seen as a deformation of the Lie algebra $\mathfrak{su}(2)$. Once an algebraic formulation is clear, this structure can be used to model a finite oscillator. The close relationship comes from the fact that for the corresponding finite oscillator model the spectrum of the position operator coincides with the spectrum of the matrix M.

Therefore, it is worthwhile to examine the algebraic structures behind the current matrices M. We shall do this explicitly for the three double dual Hahn cases.

For the case **dual Hahn I**, we return to the form of the matrix M given in (3.21) or (3.22). For any positive integer N, let J_+ denote the lower-triangular tridiagonal $(2N+1)\times(2N+1)$ matrix given below, and let J_- be its transpose:

$$J_+ = 2\begin{pmatrix} 0 & 0 & & & \\ M_0 & 0 & 0 & & \\ 0 & M_1 & 0 & 0 & \\ & 0 & M_2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad J_- = J_+^\dagger. \tag{7.1}$$

Let us also define the common diagonal matrix

$$J_0 = \mathrm{diag}(-N, -N+1, \ldots, N), \tag{7.2}$$

and the "parity matrix"

$$P = \mathrm{diag}(1, -1, 1, -1, \ldots). \tag{7.3}$$

Then it is easy to check that these matrices satisfy the following relations (as usual, I denotes the identity matrix):

$$\begin{aligned} &P^2 = 1, \qquad PJ_0 = J_0P, \qquad PJ_\pm = -J_\pm P, \\ &[J_0, J_\pm] = \pm J_\pm, \\ &[J_+, J_-] = 2J_0 + 2(\gamma+\delta+1)J_0P - (2N+1)(\gamma-\delta)P + (\gamma-\delta)I. \end{aligned} \tag{7.4}$$

Especially the last equation is interesting. From the algebraic point of view, it introduces some two-parameter deformation or extension of $\mathfrak{su}(2)$. When $\gamma = \delta = -1/2$, the equations coincide with the $\mathfrak{su}(2)$ relations. Another important case is when $\delta = -\gamma - 1$, leaving a one-parameter extension of $\mathfrak{su}(2)$ without quadratic terms.

For the case **dual Hahn II**, the corresponding expressions of J_+, J_-, J_0 and P are the same as above in (7.1)–(7.3), but with M_k-values given by (3.25). As far as the algebraic relations are concerned, they are also given by (7.4) but with the last relation replaced by

$$[J_+, J_-] = -2J_0 + 2(\gamma+\delta+2N+1)J_0P + (2N+1)(\gamma-\delta)P - (\gamma-\delta)I. \tag{7.5}$$

For the case **dual Hahn III**, the size of the matrices changes to $(2N+2)\times(2N+2)$. For J_+ and J_- one can use (7.1), with M_k-values given by (3.26). P has the same expression (7.3), but for J_0 we need to take

$$J_0 = \operatorname{diag}(-N-\frac{1}{2}, -N+\frac{1}{2}, \ldots, N+\frac{1}{2}). \tag{7.6}$$

With these expressions, the algebraic relations are given by (7.4) but with the last relation replaced by

$$[J_+, J_-] = 2J_0+2(\gamma-\delta)J_0P-((2N+2)(\gamma+\delta+1)+(2\gamma+1)(2\delta+1))P+(\gamma-\delta)I. \tag{7.7}$$

The structure of these algebras is related to the structure of the so-called algebra $\mathcal{H}$ of the dual -1 Hahn polynomials, see [11, 31]. It is not hard to verify that the algebra $\mathcal{H}$, determined by [11, Eqs. (3.4)–(3.6)] or [11, Eqs. (6.2)–(6.4)] , can be cast in the form (7.4) (or vice versa). Indeed, starting from the form [11, Eqs. (6.2)–(6.4)] coming from dual -1 Hahn polynomials, we can take

$$J_0 = \widetilde{K_1} - \frac{\rho}{4}, \quad J_+ = \widetilde{K_2} + \widetilde{K_3}, \quad J_- = \widetilde{K_2} - \widetilde{K_3},$$

to get the same form as (7.4)

$$\begin{aligned} &P^2 = 1, \qquad PJ_0 = J_0P, \qquad PJ_\pm = -J_\pm P, \\ &[J_0, J_\pm] = \pm J_\pm, \\ &[J_+, J_-] = 2J_0 + 2\nu J_0P + \frac{\sigma}{2}P + \frac{\rho}{2}I, \end{aligned} \tag{7.8}$$

where ν, σ, ρ depend on the parameters of the dual -1 Hahn polynomials α, β, N through [11, Eqs. (3.4)–(3.6)]. In our case, the algebraic relations are the same, but the dependence of the "structure constants" in (7.8) on the parameters γ, δ, N of the dual Hahn polynomials is different.

As far as we can see, the doubling of dual Hahn polynomials as classified in this paper gives a set of polynomials that is similar but in general not the same as a set of dual -1 Hahn polynomials [31] (except for specific values of parameters, e.g. $\delta = -\gamma - 1$ does coincide with a specific dual -1 Hahn polynomial). For general parameters, the support of the weight function is different, the recurrence relations (or difference relations) are different, and the hypergeometric series expression is different.

The algebraic structures obtained here (or special cases thereof) can be of interest for the construction of finite oscillator models [3, 1, 2, 14]. Two familiar finite oscillator models fall within this framework: the model discussed in [14] corresponds to (7.7) with $\delta = \gamma$, and the one analysed in [15] to (7.4) with $\delta = \gamma$. Observe that there are some other interesting special values. For example, the case (7.4) with $\delta = -\gamma-1$ gives rise to an interesting algebra, and in particular also to a very simple spectrum (3.23). We intend to study the finite oscillator that is modeled by this case, and study in particular the corresponding finite Fourier transform; but this will be the topic of a separate paper.

8 CONCLUSION

We have classified all pairs of recurrence relations for two types of dual Hahn polynomials (i.e. dual Hahn polynomials with different parameters), and refer to these as dual Hahn doubles. The analysis is quite straightforward, and the result is given in Theorem 7, yielding three cases. For each case, we have given the corresponding symmetric two-diagonal matrix M, its matrix of orthonormal eigenvectors U and its eigenvalues in explicit form. The same classification has been obtained for Hahn polynomials and Racah polynomials.

The orthogonality of the matrix U gives rise to new sets of orthogonal polynomials. These sets could in principle also be obtained from, for example, a set of dual Hahn polynomials and a certain Christoffel transform. In our approach, the possible cases where such a transform gives rise to a polynomial of the same type follow naturally, and also the explicit polynomials and their orthogonality relations arise automatically.

As an interesting secondary outcome, we obtain nice one-parameter and two-parameter extensions of the Sylvester-Kac matrix with explicit eigenvalue expressions. Such matrices can be of interest for testing numerical eigenvalue routines.

The first example of a (dual) Hahn double appeared in a finite oscillator model [14]. For this model, the Hahn polynomials (or their duals) describe the discrete position wavefunction of the oscillator, and the two-diagonal matrix M lies behind an underlying algebraic structure. Here, we have examined the algebraic relations corresponding to the three dual Hahn cases. It is clear that the analysis of finite oscillators for some of these cases is worth pursuing.

A APPENDIX: DOUBLING RACAH POLYNOMIALS

The technique presented in sections 3–4 is applied here for Racah polynomials.

Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ of degree n ($n = 0, 1, \ldots, N$) in the variable $\lambda(x)= x(x+\gamma+\delta+1)$ are defined by [13, 19, 23]

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)= {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix};1\right), \tag{A.1}$$

where one of the denominator parameters should be $-N$:

$$\alpha+1=-N \qquad \text{or} \qquad \beta+\delta+1=-N \qquad \text{or} \qquad \gamma+1=-N. \tag{A.2}$$

For the (discrete) orthogonality relation (depending on the choice of which parameter relates to $-N$) we refer to [19, (9.2.2)] or [25, Section 18.25]

Racah polynomials satisfy a recurrence relation of the form (2.5) with

$$y_n(x)= R_n(\lambda(x);\alpha,\beta,\gamma,\delta), \qquad \Lambda(x)= \lambda(x)= x(x+\gamma+\delta+1),$$
$$A(n)= \frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\gamma+1)(n+\beta+\delta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$

$$C(n)=\frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{A.3}$$

We have applied the technique described in (3.3)–(3.17), with $y_n = R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ and $\hat{y}_n = R_n(\lambda(\hat{x});\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta})$. The analysis is again straightforward but tedious, and the final outcome is:

THEOREM 17: *The only way to combine two sets of Racah polynomials such that they satisfy difference relations of the form* (3.3)–(3.4) *is one of the four cases:*
Racah I, $R_n(x)\equiv R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ *and* $\hat{R}_n(x)\equiv R_n(\lambda(x);\alpha,\beta+1,\gamma+1,\delta-1)$*:*

$$\frac{(n+\beta+\delta+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)}R_{n+1}(x)-\frac{(n-\delta+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+2)}R_n(x)$$
$$=\frac{(x+\delta)(x+\gamma+1)}{\gamma+1}\hat{R}_n(x),$$
$$\frac{(n+\alpha+\beta+2)(n+\gamma+2)}{(2n+\alpha+\beta+3)}\hat{R}_{n+1}(x)-\frac{(n+1)(n-\gamma+\alpha+\beta+1)}{(2n+\alpha+\beta+3)}\hat{R}_n(x)$$
$$=(\gamma+1)R_{n+1}(x).$$

Racah II, $R_n(x)\equiv R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ *and* $\hat{R}_n(x)\equiv R_n(\lambda(x);\alpha,\beta+1,\gamma,\delta)$*:*

$$\frac{(n+\gamma+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)}R_{n+1}(x)-\frac{(n-\gamma+\alpha+\beta+1)(n+\beta+1)}{(2n+\alpha+\beta+2)}R_n(x)$$
$$=\frac{(x+\beta+\delta+1)(x+\gamma-\beta)}{\beta+\delta+1}\hat{R}_n(x),$$
$$\frac{(n+\beta+\delta+2)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+3)}\hat{R}_{n+1}(x)-\frac{(n+1)(n-\delta+\alpha+1)}{(2n+\alpha+\beta+3)}\hat{R}_n(x)$$
$$=(\beta+\delta+1)R_{n+1}(x).$$

Racah III, $R_n(x)\equiv R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ *and* $\hat{R}_n(x)\equiv R_n(\lambda(x-1);\alpha+1,\beta,\gamma+1,\delta+1)$*:*

$$\frac{1}{(2n+\alpha+\beta+2)}R_{n+1}(x)-\frac{1}{(2n+\alpha+\beta+2)}R_n(x)=\frac{x(x+\gamma+\delta+1)}{(\gamma+1)(\beta+\delta+1)(\alpha+1)}\hat{R}_n(x),$$
$$\frac{(n+\gamma+2)(n+\beta+\delta+2)(n+\alpha+2)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+3)}\hat{R}_{n+1}(x)$$
$$-\frac{(n+1)(n-\gamma+\alpha+\beta+1)(n-\delta+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+3)}\hat{R}_n(x)$$
$$=(\gamma+1)(\beta+\delta+1)(\alpha+1)R_{n+1}(x).$$

Racah IV, $R_n(x)\equiv R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ *and* $\hat{R}_n(x)\equiv R_n(\lambda(x);\alpha+1,\beta,\gamma,\delta)$*:*

$$\frac{(n+\gamma+1)(n+\beta+\delta+1)}{(2n+\alpha+\beta+2)}R_{n+1}(x)-\frac{(n-\gamma+\alpha+\beta+1)(x-\delta+\alpha+1)}{(2n+\alpha+\beta+2)}R_n(x)$$
$$=\frac{(x+\gamma+\delta-\alpha)(x+\alpha+1)}{(\alpha+1)}\hat{R}_n(x),$$

$$\frac{(n+\alpha+2)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+3)}\hat{R}_{n+1}(x) - \frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+3)}\hat{R}_n(x)$$
$$= (\alpha+1)R_{n+1}(x)\,.$$

Note that after interchanging n and x, and $\alpha \leftrightarrow \gamma$ and $\beta \leftrightarrow \delta$, the relations in **Racah III** coincide with the known forward and backward shift operator relations [19, (9.2.6), (9.2.8)]. The relations in **Racah I** were already found in [16, (5)–(6)].

In the context of Section 5 it is worth noting that the above relations also correspond to Christoffel-Genonimus transforms. Taking $R_n(x) \equiv R_n(\lambda(x); \alpha, \beta, \gamma, \delta)$ in the relations (5.4)–(5.6), with data given by (A.3), the above cases **Racah I, II, III, IV** correspond respectively to the choices $\nu = -\delta$, $\nu = \beta - \gamma$, $\nu = 0$ and $\nu = -\alpha - 1$.

For each of the four cases, one can translate the set of difference relations to a matrix identity of the form $MU = UD$. In fact, for each of the four cases, there are three subcases depending on the choice of $-N$ in (A.2). We shall not give all of these cases: they should be easy to construct for the reader who needs one. Let us just give an example or two.

Consider the case **Racah I** with $\alpha + 1 = -N$. It is convenient to perform the shift $\delta \to \delta + 1$ in the two difference relations of Theorem 17. The orthogonal matrix U is of order $(2N+2)\times(2N+2)$, with elements

$$U_{2n,N-x} = U_{2n,N+x+1} = \frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta+1),$$
$$U_{2n+1,N-x} = -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha, \beta+1, \gamma+1, \delta), \qquad \text{(A.4)}$$

where $\tilde{R}_n$ is the notation for a normalized Racah polynomial. Then, one has

PROPOSITION 18: *Suppose that* $\gamma, \delta > -1$ *and* $\beta > N + \gamma$ *or* $\beta < -N - \delta - 1$. *Let* M *be a tridiagonal* $(2N+2)\times(2N+2)$*-matrix of the form* (2.15) *with*

$$M_{2k} = \sqrt{\frac{(N-\beta-k)(\gamma+1+k)(N+\delta+1-k)(k+\beta+1)}{(N-\beta-2k)(2k-N+1+\beta)}},$$
$$M_{2k+1} = \sqrt{\frac{(\gamma+N-\beta-k)(k+1)(N-k)(k+\beta+\delta+2)}{(N-\beta-2k-2)(2k-N+1+\beta)}}, \qquad \text{(A.5)}$$

and U *the orthogonal matrix determined in* (A.4). *Then the columns of* U *are the eigenvectors of* M, *i.e.* $MU = UD$, *where* D *is a diagonal matrix containing the eigenvalues of* M:

$$D = \operatorname{diag}(-\epsilon_N, \ldots, -\epsilon_1, -\epsilon_0, \epsilon_0, \epsilon_1, \ldots, \epsilon_N),$$
$$\epsilon_k = \sqrt{(k+\gamma+1)(k+\delta+1)} \qquad (k = 0, 1, \ldots, N). \qquad \text{(A.6)}$$

As a second example, consider the case **Racah III** with $\alpha + 1 = -N$. The orthogonal matrix U is now of order $(2N+1)\times(2N+1)$, with elements

$$U_{2n,N-x} = U_{2n,N+x} = \frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta), \qquad (n = 1, \ldots, N)$$

$$U_{2n+1,N-x-1} = -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha+1, \beta, \gamma+1, \delta+1),$$
$$(n = 0, \ldots, N-1)$$
$$U_{2n,N} = (-1)^n \tilde{R}_n(\lambda(0); \alpha, \beta, \gamma, \delta), \qquad U_{2n+1,N} = 0. \tag{A.7}$$

Then, one has

PROPOSITION 19: *Suppose that $\gamma, \delta > -1$ and $\beta > N+\gamma$ or $\beta < -N-\delta$. Let M be a tridiagonal* $(2N+1)\times(2N+1)$*-matrix of the form* (3.21) *with*

$$M_{2k} = \sqrt{\frac{(k+\gamma+1)(-N+\beta+k)(N-k)(k+\beta+\delta+1)}{(N-\beta-2k)(N-\beta-2k-1)}},$$
$$M_{2k+1} = \sqrt{\frac{(\gamma+N-\beta-k)(k+1)(k+\beta+1)(k-\delta-N)}{(N-\beta-2k-2)(N-\beta-2k-1)}}, \tag{A.8}$$

and U the orthogonal matrix determined in (A.7)*. Then the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:*

$$D = \operatorname{diag}(-\epsilon_N, \ldots, -\epsilon_1, 0, \epsilon_1, \ldots, \epsilon_N), \quad \epsilon_k = \sqrt{k(k+\gamma+\delta+1)} \quad (k = 1, \ldots, N). \tag{A.9}$$

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5

A finite oscillator model with equidistant position spectrum based on an extension of $\mathfrak{su}(2)$

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abstract

We consider an extension of the real Lie algebra $\mathfrak{su}(2)$ by introducing a parity operator P and a parameter c. This extended algebra is isomorphic to the Bannai-Ito algebra with two parameters equal to zero. For this algebra we classify all unitary finite-dimensional representations and show their relation with known representations of $\mathfrak{su}(2)$. Moreover, we present a model for a one-dimensional finite oscillator based on the odd-dimensional representations of this algebra. For this model, the spectrum of the position operator is equidistant and coincides with the spectrum of the known $\mathfrak{su}(2)$ oscillator. In particular the spectrum is independent of the parameter c while the discrete position wavefunctions, which are given in terms of certain dual Hahn polynomials, do depend on this parameter.

1 introduction

Finite oscillator models were introduced and investigated in a number of papers, see e.g. [5–7, 1, 19, 20]. The standard and well-recognized example is the $\mathfrak{su}(2)$ oscillator model [5, 1]. In brief, this model is based on the $\mathfrak{su}(2)$ algebra with basis elements $J_0 = J_z$, $J_\pm = J_x \pm J_y$ satisfying

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0, \tag{1.1}$$

with unitary representations of dimension $2j + 1$ (where j is integer or half-integer). Recall that the oscillator Lie algebra can be considered as an associative algebra (with unit element 1) with three generators $\hat{H}$, $\hat{q}$ and $\hat{p}$ (the Hamiltonian, the position and the momentum operator) subject to

$$[\hat{H}, \hat{q}] = -i\hat{p}, \qquad [\hat{H}, \hat{p}] = i\hat{q}, \qquad [\hat{q}, \hat{p}] = i, \tag{1.2}$$

in units with mass and frequency both equal to 1, and $\hbar = 1$. The first two are the Hamilton-Lie equations; the third the canonical commutation relation. The canonical commutation relation is not compatible with a finite-dimensional Hilbert space. Following this, one speaks of a finite oscillator model if $\hat{H}$, $\hat{q}$ and $\hat{p}$ belong to some algebra such that the Hamilton-Lie equations are satisfied and such that the spectrum of $\hat{H}$ in representations of that algebra is equidistant [5, 19].

In the $\mathfrak{su}(2)$ model, one chooses

$$\hat{H} = J_0 + j + \frac{1}{2}, \quad \hat{q} = \frac{1}{2}(J_+ + J_-), \quad \hat{p} = \frac{i}{2}(J_+ - J_-). \tag{1.3}$$

These indeed satisfy $[\hat{H}, \hat{q}] = -i\hat{p}$, $[\hat{H}, \hat{p}] = i\hat{q}$, and in the representation (j) labeled by j the spectrum of $\hat{H}$ is equidistant (and given by $n + \frac{1}{2}$; $n = 0, 1, \ldots, 2j$). Clearly, for this model the position operator $\hat{q} = \frac{1}{2}(J_+ + J_-)$ also has a finite spectrum in the representation (j) given by $q \in \{-j, -j+1, \ldots, +j\}$. In terms of the standard J_0-eigenvectors $|j, m\rangle$, the eigenvectors of $\hat{q}$ can be written as

$$|j, q) = \sum_{m=-j}^{j} \Phi_{j+m}(q)|j, m\rangle. \tag{1.4}$$

The coefficients $\Phi_n(q)$ are the position wavefunctions, and in this model [5, 1] they turn out to be (normalized) symmetric Krawtchouk polynomials, $\Phi_n(q) \sim K_n(j + q; \frac{1}{2}, 2j)$. The shape of the these wavefunctions is reminiscent of those of the canonical oscillator: under the limit $j \to \infty$ they coincide with the canonical wavefunctions in terms of Hermite polynomials.

Following the ideas of the seminal papers on the $\mathfrak{su}(2)$ oscillator model, some alternative finite oscillator models were introduced [19–21]. The interest in these different models stems from several facts: in these new models additional parameters could be introduced, leading to wavefunctions with potentially more applications; the underlying algebras have a richer structure than $\mathfrak{su}(2)$; the wavefunctions are related to other classes of discrete orthogonal polynomials, and to new properties of these polynomials. In particular, we observed that in our models the wavefunctions were related to some "doubling process" of known orthogonal polynomials. A peculiar property of the wavefunctions in the new models of Refs. [19–21], which could be considered as a disadvantage, is that the support of the discrete position wavefunctions (which is the spectrum of the position operator) is no longer equidistant.

So far, the introduction of new finite oscillator models looked rather arbitrarily. The mentioned relation to a "doubling process" for orthogonal polynomials, however, raised the question in how many ways the classical discrete orthogonal polynomials can be doubled, and whether these give rise to interesting models. In a recent paper [24], we investigated and classified all doubles for Hahn, dual Hahn and Racah polynomials, which are the standard discrete orthogonal polynomials one level up from the Krawtchouk polynomials in the Askey scheme [22]. We not only classified all possible doubles; additionally we showed that each double is essentially a Christoffel-Geronimus pair [24].

Following the classification of [24], it is worthwhile to investigate the oscillator models corresponding to Hahn or dual Hahn doubles that have not yet been studied before. We are in particular interested in models in which also the position operator spectrum is equidistant. This is how the present paper originated: from our classification [24] it is clear that there is one case (referred to as “Dual Hahn I” in [24]) giving rise to a natural equidistant position spectrum. The model related to this case is the subject here.

Rather than introducing this new model via the dual Hahn double, it is – in the finite oscillator context – more natural to start from the underlying algebra. This is the line followed here: in section 2 we introduce the algebra $\mathfrak{su}(2)_P$, an extension of $\mathfrak{su}(2)$ by a parity operator P. The extension is close to a “central extension”, with parameter c, but P is not a central element (it commutes with J_0 but anticommutes with J_+ and J_-). This algebra is interesting on its own, and we also classify all irreducible unitary finite-dimensional representations of $\mathfrak{su}(2)_P$. These representations can be understood as deformations of the common $\mathfrak{su}(2)$ representations of dimension $2j + 1$, except that not all of these can be deformed (which representations appear depends on the value of c, the parameter of the extension). In section 3 we discuss the finite oscillator model related to $\mathfrak{su}(2)_P$. In particular, we show that (for odd dimensions) the spectral problem for the position operator is of type dual Hahn I (according to the classification [24]), and we construct the orthonormal eigenvectors of the position and momentum operator. The following section deals with some properties of the corresponding position wavefunctions. The expressions of the wavefunctions are quite simple dual Hahn polynomials. We also discuss some plots of the wavefunctions, and state some natural limits (in particular to the canonical quantum oscillator). The paper ends with some concluding remarks: in particular, we clarify the connection/difference between the algebra $\mathfrak{su}(2)_P$ and previously used extended algebras $\mathfrak{u}(2)_\alpha$ [19] and $\mathfrak{su}(2)_\alpha$ [20] in the context of “Hahn oscillators”, and we discuss a reflection differential operator realization of $\mathfrak{su}(2)_P$.

2 AN EXTENSION OF SU(2) AND ITS REPRESENTATIONS

The real Lie algebra $\mathfrak{su}(2)$ [17, 29] can be defined by three basis elements J_0, J_+, J_- with commutators $[J_0, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_0$. The non-trivial unitary representations of $\mathfrak{su}(2)$, corresponding to the star relations $J_0^\dagger = J_0$, $J_\pm^\dagger = J_\mp$, are labelled [17, 29] by a positive integer or half-integer j. These representations have dimension $2j + 1$, and the action on a set of basis vectors $|j, m\rangle$ (with $m = -j, -j + 1, \ldots, +j$) is given by

$$J_0|j, m\rangle = m\,|j, m\rangle, \qquad J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\;|j, m \pm 1\rangle.$$

The Lie algebra $\mathfrak{su}(2)$ can be extended by a parity operator or involution P, whose action in these representations is given by $P|j, m\rangle = (-1)^{j+m}\;|j, m\rangle$. On the algebraic level, this means that we extend the universal enveloping algebra of $\mathfrak{su}(2)$ by an operator P that commutes with J_0, that *anticommutes* with J_+ and J_-, and for which $P^2 = 1$. Moreover, by means of this operator P the standard $\mathfrak{su}(2)$ relations can be deformed introducing a real parameter c. This gives rise to an extension of the Lie algebra of $\mathfrak{su}(2)$ which itself is not a Lie algebra (nor a Lie superalgebra). This extension will be denoted by $\mathfrak{su}(2)_P$ and is defined as follows.

DEFINITION 1: *Let c be a parameter. The algebra $\mathfrak{su}(2)_P$ is a unital algebra with basis elements J_0, J_+, J_- and P subject to the following relations:*

$$P^2 = 1, \qquad [P, J_0] = PJ_0 - J_0P = 0, \qquad \{P, J_\pm\} = PJ_\pm + J_\pm P = 0, \tag{2.1}$$

and the $\mathfrak{su}(2)$ relations which are deformed as follows:

$$[J_0, J_\pm] = \pm J_\pm \tag{2.2}$$

$$[J_+, J_-] = 2J_0 + cP\,. \tag{2.3}$$

The star relation for this algebra is determined by:

$$P^\dagger = P, \qquad J_0^\dagger = J_0, \qquad J_\pm^\dagger = J_\mp. \tag{2.4}$$

For $c = 0$ the deformed relation (2.3) reduces to the regular $\mathfrak{su}(2)$ relation. Note that this extension is very similar to a central extension; the only relation that violates this is the anticommutator in (2.1).

The appearance of both a commutator and an anticommutator in (2.1) also implies that one is not dealing with a Lie algebra nor with a Lie superalgebra. The algebraic structure defined here is not new, however. The algebra $\mathfrak{su}(2)_P$ is in fact isomorphic to a special case of the Bannai-Ito algebra where two parameters are equal to zero[10, 11]. Indeed, putting

$$K_1 = \frac{1}{2}(J_+ + J_-), \quad K_2 = -\frac{1}{2}(J_+ - J_-)P, \quad K_3 = J_0P \tag{2.5}$$

we have

$$\{K_1, K_2\} = K_3 + \frac{c}{2}, \quad \{K_2, K_3\} = K_1 \quad \{K_3, K_1\} = K_2\,.$$

The star relations (2.4) correspond to $K_i^\dagger = K_i$ for $i = 1, 2, 3$. Moreover, this algebra can also be seen as a special case of the so-called algebra $\mathcal{H}$ of the dual -1 Hahn polynomials, see [15, 28], where one of the parameters equals zero.

Using (2.2) and (2.3), one easily shows that the Casimir element of $\mathfrak{su}(2)$, given by $\Omega = 2J_0^2 + J_+J_- + J_-J_+$, remains central for the universal enveloping algebra of $\mathfrak{su}(2)_P$. By means of (2.3), the Casimir element can also be written as

$$\Omega = 2J_+J_- + 2J_0^2 - 2J_0 - cP = 2J_-J_+ + 2J_0^2 + 2J_0 + cP\,. \tag{2.6}$$

Our purpose is now to determine all finite-dimensional unitary representations of $\mathfrak{su}(2)_P$, corresponding to the star conditions (2.4).

Let (W, ρ_W) be a representation of $\mathfrak{su}(2)_P$. We consider W as an $\mathfrak{su}(2)_P$ module by setting $G \cdot v = \rho_W(G)v$ for $G \in \mathfrak{su}(2)_P$ and $v \in W$. Take $v_0 \in W$ to be an eigenvector of J_0 with eigenvalue λ. We will construct the $\mathfrak{su}(2)_P$ invariant subspace containing v_0. If W is irreducible this space must be either W or trivial. The trivial case results from v_0 being the zero vector, so from now on we assume that v_0 is not the zero vector.

From (2.1) follows that $J_0Pv_0 = PJ_0v_0 = \lambda Pv_0$, hence Pv_0 is also an eigenvector of J_0 with eigenvalue λ. We distinguish between two cases, Pv_0 is either a multiple of v_0 or not. If Pv_0 is linearly independent of v_0 and also has λ as eigenvalue for J_0, the vectors $v_0^+ = v_0 + Pv_0$ and $v_0^- = v_0 - Pv_0$ are also eigenvectors of J_0 and we have $Pv_0^+ = v_0^+$ and $Pv_0^- = -v_0^-$. The vectors v_0^+ and v_0^- will then generate two different invariant subspaces, hence the representation W is not irreducible. We may thus assume that v_0 is also an eigenvector of P.

If $J_0 v_0 = \lambda v_0$, then for a positive integer k, the vector $(J_\pm)^k v_0$ is also an eigenvector of J_0. Indeed, using $[J_0, (J_\pm)^k] = \pm k (J_\pm)^k$, which follows from (2.2), we have

$$J_0(J_\pm)^k v_0 = \big((J_\pm)^k J_0 + [J_0, (J_\pm)^k]\big) v_0 = J_\pm J_0 v_0 \pm k (J_\pm)^k J_\pm v_0 = (\lambda \pm k)(J_\pm)^k v_0 \,. \tag{2.7}$$

Moreover, the vectors $\{(J_+)^k v_0 \mid k \in \mathbb{N}\}$ must be linearly independent because they have distinct eigenvalues as eigenvectors of J_0. If we impose W to be finite-dimensional, then $(J_+)^k v_0 = 0$ for some $k \in \mathbb{N}$. Without loss of generality we may assume that $J_+ v_0 = 0$, making v_0 the highest weight vector, i.e. the eigenvector of J_0 with the highest eigenvalue, with corresponding highest weight λ.

Following the same reasoning, the sequence $\{(J_-)^k v_0 \mid k \in \mathbb{N}\}$ is also linearly independent and must terminate. We thus have $J_-(J_-^n v_0) = 0$ for some $n \in \mathbb{N}$ and we may assume without loss of generality that n is minimal in this aspect, i.e. $J_-^n v_0 \neq 0$. We will now show that the set

$$\{v_k = (J_-)^k v_0 \mid k = 0, \ldots, n\} \tag{2.8}$$

forms a basis for the $\mathfrak{su}(2)_P$ invariant subspace containing v_0. If W is irreducible this space must be all of W. So far, we have (2.8) being invariant under the action of J_0 and J_-, with $J_0 v_k = (\lambda - k) v_k$. We now look at the action of P and J_+ on (2.8).

As v_0 is an eigenvector of P and $P^2 = 1$, we necessarily have $Pv_0 = \epsilon\, v_0$ with $\epsilon = \pm 1$. Moreover, as P anti-commutes with J_- (2.1), we find the action of P on (2.8) to be

$$Pv_k = \epsilon (-1)^k v_k \,. \tag{2.9}$$

For the action of J_+, we have $J_+ v_0 = 0$, while for $k \geq 0$ we can write

$$J_+ v_{k+1} = J_+ J_- v_k = \left(\frac{1}{2}\Omega - J_0^2 + J_0 + \frac{c}{2} P\right) v_k$$

where Ω is the Casimir element (2.6) whose action is constant on W. Using $J_+ v_0 = 0$ the action of Ω on v_0 is given by

$$\Omega v_0 = (2 J_- J_+ + 2 J_0^2 + 2 J_0 + cP) v_0 = (2\lambda^2 + 2\lambda + c\epsilon) v_0$$

We thus find

$$J_+ v_{k+1} = \left((k+1)(2\lambda - k) + c\,\epsilon\, \frac{1 + (-1)^k}{2}\right) v_k \equiv A(k) v_k \,, \tag{2.10}$$

so (2.8) forms the basis for a $\mathfrak{su}(2)_P$ invariant subspace.

Now, the value of the highest weight λ follows from the action of J_0 and P on the basis (2.8). Indeed, taking the trace of both sides of (2.3) acting on W, we get

$$0 = \operatorname{tr}\big([J_+, J_-]\big) = 2 \operatorname{tr}(J_0) + \operatorname{tr}(cP) = 2(n+1)\lambda - n(n+1) + c\,\epsilon\, \frac{1 + (-1)^n}{2} \,. \tag{2.11}$$

From which we find

$$\lambda = \frac{n}{2} - \frac{c\,\epsilon}{2(n+1)} \frac{1+(-1)^n}{2} \,. \tag{2.12}$$

Note that the deformation parameter c appears in the value of the highest weight only when n is even, that is for odd-dimensional representations.

Substituting (2.12) for λ in (2.10) we arrive at

$$A(k) = (k+1)(n-k) + c\,\epsilon \left(\frac{1+(-1)^k}{2} - \frac{1+(-1)^n}{2} \frac{(k+1)}{(n+1)} \right) . \tag{2.13}$$

For n even, this reduces to

$$A(k) = \begin{cases} (k+1)\left(n-k-\frac{c\,\epsilon}{n+1}\right), & \text{if } k \text{ is odd;} \\ \left(k+1+\frac{c\,\epsilon}{n+1}\right)(n-k), & \text{if } k \text{ is even,} \end{cases} \tag{2.14}$$

while for n odd

$$A(k) = \begin{cases} (k+1)(n-k), & \text{if } k \text{ is odd;} \\ (k+1)(n-k)+c\,\epsilon, & \text{if } k \text{ is even.} \end{cases} \tag{2.15}$$

Next, we require that the representation W is unitary under the star conditions (2.4). Hereto, we introduce a sesquilinear form $\langle \cdot, \cdot \rangle : W \times W \longrightarrow \mathbb{C}$ such that

$$\langle v_k, v_\ell \rangle = h_k \, \delta_{k,\ell} \,,$$

where we can put $h_0 = 1$ or $\langle v_0, v_0 \rangle = 1$. In order to be an inner product we need $h_k > 0$ for $k \geq 0$. For $k \geq 1$ we have, imposing the star condition $J_-^\dagger = J_+$,

$$h_k = \langle v_k, v_k \rangle = \langle J_- v_{k-1}, v_k \rangle = \langle v_{k-1}, J_+ v_k \rangle = A(k-1) \langle v_{k-1}, v_{k-1} \rangle = A(k-1) h_{k-1} \,. \tag{2.16}$$

This is strictly positive if $A(k) > 0$ for $0 \leq k \leq n-1$. Distinguishing between n even and odd, we find that (2.14) is strictly positive for $-(n+1) < c\,\epsilon < n+1$, while (2.15) is strictly positive if $c\,\epsilon > -n$.

The star conditions $P^\dagger = P$, $J_0^\dagger = J_0$ are satisfied as P and J_0 have real eigenvalues on v_k. Putting $j = n/2$ and introducing the orthonormal basis

$$|j, m\rangle = \frac{v_{j-m}}{\|v_{j-m}\|} \qquad (m = -j, -j+1, \ldots, j-1, j)$$

where $\|v_k\| = \sqrt{\langle v_k, v_k \rangle} = \sqrt{h_k}$, we find using (2.16)

$$J_- |j, m\rangle = J_- \frac{v_{j-m}}{\|v_{j-m}\|} = \frac{v_{j-m+1}}{\sqrt{h_{j-m}}} = \sqrt{A(j-m)}\, |j, m-1\rangle$$

and

$$J_+ |j, m\rangle = J_+ \frac{v_{j-m}}{\|v_{j-m}\|} = A(j-m-1) \frac{v_{j-m-1}}{\sqrt{h_{j-m}}} = \sqrt{A(j-m-1)}\, |j, m+1\rangle \,.$$

We summarize this in the following result:

PROPOSITION 2: *For a given real parameter c and choice of $\epsilon = \pm 1$, we have the following irreducible unitary finite-dimensional representations of $\mathfrak{su}(2)_P$, corresponding to the star conditions* (2.4)*:*

For every positive integer j such that $2j+1 > |c|$, we have an odd-dimensional representation of dimension $2j+1$. The action of the $\mathfrak{su}(2)_P$ operators on a set of basis vectors $|j,-j\rangle$, $|j,-j+1\rangle$, ..., $|j,j\rangle$ is given by:

$$P|j,m\rangle = \epsilon(-1)^{j+m}\,|j,m\rangle, \tag{2.17}$$

$$J_0|j,m\rangle = (m-\tilde{c}/2)\,|j,m\rangle, \tag{2.18}$$

$$J_+|j,m\rangle = \begin{cases} \sqrt{(j-m+\tilde{c})(j+m+1)}\;|j,m+1\rangle, & \text{if } j+m \text{ is odd;} \\ \sqrt{(j-m)(j+m+1-\tilde{c})}\;|j,m+1\rangle, & \text{if } j+m \text{ is even.} \end{cases} \tag{2.19}$$

$$J_-|j,m\rangle = \begin{cases} \sqrt{(j+m-\tilde{c})(j-m+1)}\;|j,m-1\rangle, & \text{if } j+m \text{ is odd;} \\ \sqrt{(j+m)(j-m+1+\tilde{c})}\;|j,m-1\rangle, & \text{if } j+m \text{ is even.} \end{cases} \tag{2.20}$$

where $\tilde{c} = c\,\epsilon/(2j+1)$. Note that $2j+1 > |c|$ is equivalent to $|\tilde{c}| < 1$.

For every positive half-integer j such that $2j > -c\,\epsilon$, we have an even-dimensional representation of dimension $2j+1$. The action of the $\mathfrak{su}(2)_P$ operators on a set of basis vectors $|j,-j\rangle$, $|j,-j+1\rangle$, ..., $|j,j\rangle$ is given by:

$$P\,|j,m\rangle = \epsilon(-1)^{j+m+1}\,|j,m\rangle, \tag{2.21}$$

$$J_0\,|j,m\rangle = m\,|j,m\rangle, \tag{2.22}$$

$$J_+|j,m\rangle = \begin{cases} \sqrt{(j-m)(j+m+1)}\;|j,m+1\rangle, & \text{if } j+m \text{ is odd;} \\ \sqrt{(j-m)(j+m+1)+c\,\epsilon}\;|j,m+1\rangle, & \text{if } j+m \text{ is even.} \end{cases} \tag{2.23}$$

$$J_-|j,m\rangle = \begin{cases} \sqrt{(j+m)(j-m+1)+c\,\epsilon}\;|j,m-1\rangle, & \text{if } j+m \text{ is odd;} \\ \sqrt{(j+m)(j-m+1)}\;|j,m-1\rangle, & \text{if } j+m \text{ is even.} \end{cases} \tag{2.24}$$

We can write the actions of J_+ and J_- in the above result more compactly. For j an integer

$$J_\pm|j,m\rangle = \begin{cases} \sqrt{(j\mp m\pm\tilde{c})(j\pm m+1)}\;|j,m\pm 1\rangle, & \text{if } j+m \text{ is odd;} \\ \sqrt{(j\mp m)(j\pm m+1\mp\tilde{c})}\;|j,m\pm 1\rangle, & \text{if } j+m \text{ is even;} \end{cases}$$

while for j a half-integer

$$J_\pm|j,m\rangle = \begin{cases} \sqrt{(j\mp m)(j\pm m+1)}\;|j,m\pm 1\rangle, & \text{if } j\pm m \text{ is odd;} \\ \sqrt{(j\mp m)(j\pm m+1)+c\,\epsilon}\;|j,m\pm 1\rangle, & \text{if } j\pm m \text{ is even.} \end{cases}$$

The action of the Casimir (2.6) is indeed scalar on these representations, and given by

$$\left(2J_0^2 + J_+J_- + J_-J_+\right)|j, m\rangle = \begin{cases} 2j(j+1)+c\,\epsilon & \text{if } j \text{ is a half-integer;} \\ 2j(j+1)+\frac{\tilde{c}^2}{2} & \text{if } j \text{ is an integer,} \end{cases}$$

again with $\tilde{c} = c\,\epsilon/(2j+1)$.

REMARK 3: For j a half-integer, these representations correspond precisely to those of the unital algebra $\mathfrak{u}(2)_\alpha$ [19], which contains moreover an extra central operator C with diagonal action $C|j, m\rangle = (2j+1)|j, m\rangle$. Indeed, substituting $c\,\epsilon = (2\alpha+1)^2 + (2\alpha+1)(2j+1)$ we find the same action as in [19]:

$$(j-m)(j+m+1)+(2\alpha+1)^2+(2\alpha+1)(2j+1) = (j-m+2\alpha+1)(j+m+2\alpha+2),$$
$$(j+m)(j-m+1)+(2\alpha+1)^2+(2\alpha+1)(2j+1) = (j-m+2\alpha+2)(j+m+2\alpha+1).$$

For this reason, only the representations with j integer are new in the context of finite oscillator models. And therefore, only the odd-dimensional representations will play a role in the following sections.

REMARK 4: In [11] the finite-dimensional unitary representations of the Bannai-Ito algebra [13, 14] corresponding to a realization in terms of Dirac-Dunkl operators were determined (see also [16] for a more general approach). Since the algebra $\mathfrak{su}(2)_P$ is isomorphic to a special case of the Bannai-Ito algebra, we should observe a correspondence between the representations in Proposition 2 and those of [11]. One difference, however, is that in our case the parameter c is the basic parameter, and its value determines the existence of representations of certain dimensions.

Returning to [11], the general Bannai-Ito algebra with three parameters $\omega_1, \omega_2, \omega_3$ is characterized by three real numbers μ_1, μ_2, μ_3 appearing in the Dunkl operators, and a positive integer N (with $N+1$ the dimension of the representation). For even N, i.e. $N = 2j$ with j integer, the following choice for μ_i:

$$\mu_1 = \mu_2 = -\frac{N+1+\tilde{c}}{4}, \qquad \mu_3 = -\frac{N+1-\tilde{c}}{4} \tag{2.25}$$

in [11, eq. (48)] leads to the same (matrix) representation for K_1, K_2, K_3 as our representation (2.17)–(2.20) used in (2.5). Note that with $|\tilde{c}| < 1$, the above μ_i-values are negative. Strictly speaking, only nonnegative values for μ_i were considered in [11]. It is clear, however, that for the values (2.25) the matrix elements U_k appearing in [11, eq. (48)] are still positive and thus these values are also allowed.

For odd N, i.e. $N = 2j$ with j half-integer, the correspondence is not so simple. For that case, the basis vectors $|N, k\rangle$ of [11] are not the same as our basis vectors $|j, m\rangle$ for a particular choice of μ_1, μ_2, μ_3. So the correspondence between the matrix representations becomes complicated and we do not include it here.

3 A ONE-DIMENSIONAL OSCILLATOR MODEL

We now consider a model for a one-dimensional finite oscillator based on the odd-dimensional representations of the algebra $\mathfrak{su}(2)_P$, that is for j an integer. We will see that for this model the spectrum of the position operator is independent of the parameter c, equidistant and coincides with the spectrum of the $\mathfrak{su}(2)$ oscillator [5]. The eigenvectors (and thus also the position wavefunctions) do depend on the additional parameter c.

Following the notation and ideas of section 1, we have to choose a position, momentum and Hamiltonian operator $(\hat{q}, \hat{p}, \hat{H})$ from the algebra $\mathfrak{su}(2)_P$ such that the Hamilton-Lie equations are satisfied, and such that the spectrum of $\hat{H}$ in a representation is equidistant. Given $\mathfrak{su}(2)_P$ with parameter c, and the representation of dimension $2j + 1$ (j integer) determined in Proposition 2, with $2j + 1 > |c|$ and $\epsilon = 1$, the following choice is natural and follows [19, 20]:

$$\hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{i}{2}(J_+ - J_-), \qquad \hat{H} = J_0 + j + \frac{\tilde{c}}{2} + \frac{1}{2}, \tag{3.1}$$

where $\tilde{c} = c/(2j + 1)$ and thus the parameter $\tilde{c}$ satisfies $-1 < \tilde{c} < +1$.

It is easy to verify that the first two equations of (1.2) are satisfied and moreover from (2.18) it follows that on $|j, m\rangle$ the spectrum of $\hat{H}$ is indeed linear and given by

$$n + \frac{1}{2} \qquad (n = 0, 1, \ldots, 2j). \tag{3.2}$$

From the actions (2.19)–(2.20), one finds for even $j + m$

$$2\hat{q}|j, m\rangle = \sqrt{(j+m)(j-m+1+\tilde{c})}\ |j, m-1\rangle + \sqrt{(j-m)(j+m+1-\tilde{c})}\ |j, m+1\rangle\,,$$

while for odd $j + m$

$$2\hat{q}|j, m\rangle = \sqrt{(j+m-\tilde{c})(j-m+1)}\ |j, m-1\rangle + \sqrt{(j-m+\tilde{c})(j+m+1)}\ |j, m+1\rangle\,.$$

The action of $2i\hat{p}$ is similar. For the representation space, denoted here by W_j, we choose the following (ordered) basis:

$$\{|j, -j\rangle, |j, -j+1\rangle, \ldots, |j, j-1\rangle, |j, j\rangle\}, \tag{3.3}$$

and then the operators $2\hat{q}$, $2i\hat{p}$ take the matrix forms

$$2\hat{q} = \begin{pmatrix} 0 & M_0 & 0 & \cdots & 0 \\ M_0 & 0 & M_1 & \cdots & 0 \\ 0 & M_1 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & M_{2j-1} \\ 0 & 0 & & M_{2j-1} & 0 \end{pmatrix} \equiv M^q, \tag{3.4}$$

$$2i\hat{p} = \begin{pmatrix} 0 & M_0 & 0 & \cdots & 0 \\ -M_0 & 0 & M_1 & \cdots & 0 \\ 0 & -M_1 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & M_{2j-1} \\ 0 & 0 & & -M_{2j-1} & 0 \end{pmatrix} \equiv M^p, \tag{3.5}$$

with

$$M_k = \begin{cases} \sqrt{(k+1-\tilde{c})(2j-k)}, & \text{if } k \text{ is even;} \\ \sqrt{(k+1)(2j-k+\tilde{c})}, & \text{if } k \text{ is odd.} \end{cases} \tag{3.6}$$

For these matrices, the eigenvalues and eigenvectors are known explicitly: the system is of type "dual Hahn I" [24, Proposition 2] (with $\gamma + \delta + 1 = 0$ in the notation of [24]). The expressions of the eigenvectors involve dual Hahn polynomials, so let us first recall some notation.

For a positive integer N, the dual Hahn polynomial of degree n ($n = 0, 1, \ldots, N$) in the variable $\lambda(x)= x(x + \gamma + \delta + 1)$, with parameters $\gamma > -1$ and $\delta > -1$ (or $\gamma < -N$ and $\delta < -N$) is defined by [18, 22, 23]:

$$R_n(\lambda(x); \gamma, \delta, N)= {}_3F_2\left({-x, x+\gamma+\delta+1, -n \atop \gamma+1, -N} ; 1 \right) \tag{3.7}$$

in terms of the generalized hypergeometric series ${}_3F_2$ of unit argument [8, 25]. Dual Hahn polynomials satisfy a (discrete) orthogonality relation [22]:

$$\sum_{x=0}^{N} w(x; \gamma, \delta, N) R_n(\lambda(x); \gamma, \delta, N) R_{n'}(\lambda(x); \gamma, \delta, N)= h_n(\gamma, \delta, N)\, \delta_{n,n'}, \tag{3.8}$$

where

$$w(x; \gamma, \delta, N)= \frac{(2x+\gamma+\delta+1)(\gamma+1)_x (N-x+1)_x N!}{(x+\gamma+\delta+1)_{N+1}(\delta+1)_x x!} \qquad (x = 0, 1, \ldots, N),$$

$$h_n(\gamma, \delta, N)= \left[\binom{\gamma+n}{n} \binom{N+\delta-n}{N-n} \right]^{-1}. \tag{3.9}$$

We have used here the common notation for Pochhammer symbols [8, 25] $(a)_k= a(a + 1)\cdots(a+k-1)$ for $k = 1, 2, \ldots$ and $(a)_0= 1$. As w is the weight function and $h_n(\gamma, \delta, N)$ the "squared norm", orthonormal dual Hahn functions $\tilde{R}$ are determined by:

$$\tilde{R}_n(\lambda(x); \gamma, \delta, N) \equiv \frac{\sqrt{w(x; \gamma, \delta, N)}\, R_n(\lambda(x); \gamma, \delta, N)}{\sqrt{h_n(\gamma, \delta, N)}}. \tag{3.10}$$

From [24, Proposition 2], using the substitution $\gamma = (-\tilde{c} - 1)/2$ and $\delta = (\tilde{c} - 1)/2 = -\gamma - 1$, we have:

PROPOSITION 5: *The* $2j+1$ *eigenvalues of the position operator* $\hat{q}$ *in the representation* W_j *are given by*

$$-j, -j+1, \ldots, -1, 0, 1, \ldots, j-1, j. \tag{3.11}$$

The orthonormal eigenvector of the position operator $\hat{q}$ *in* W_j *for the eigenvalue* q*, denoted by* $|j, q)$*, is given in terms of the basis* (3.3) *by*

$$|j, q) = \sum_{m=-j}^{j} U_{j+m, j+q} |j, m\rangle. \tag{3.12}$$

Herein, $U = (U_{kl})_{0 \le k, l \le 2j}$ *is the* $(2j+1)\times(2j+1)$ *matrix with elements*

$$U_{2r,j} = (-1)^r \tilde{R}_r(\lambda(0); (-\tilde{c}-1)/2, (\tilde{c}-1)/2, j), \qquad (r \in \{0, \ldots, j\}), \tag{3.13}$$

$$U_{2r,j-s} = U_{2r,j+s} = \frac{(-1)^r}{\sqrt{2}} \tilde{R}_r(\lambda(s); (-\tilde{c}-1)/2, (\tilde{c}-1)/2, j), \qquad (s \in \{1, \ldots, j\});$$

$$U_{2r+1,j-s-1} = -U_{2r+1,j+s+1} = -\frac{(-1)^r}{\sqrt{2}} \tilde{R}_r(\lambda(s); (1-\tilde{c})/2, (\tilde{c}+1)/2, j-1),$$

$$U_{2r+1,j} = 0, \qquad (r, s \in \{0, \ldots, j-1\}) \tag{3.14}$$

where the functions $\tilde{R}$ *are normalized dual Hahn polynomials* (3.10).

The matrix U *is an orthogonal matrix,* $UU^T = U^T U = I$*, hence the* $\hat{q}$ *eigenvectors are orthonormal:*

$$(j, q|j, q') = \delta_{q,q'}.$$

Moreover,

$$M^q U = U D^q,$$

where D^q *is a diagonal matrix containing the eigenvalues* (3.11).

This is the same spectrum as that of $\hat{q}$ in the $\mathfrak{su}(2)$ oscillator model [5]. For the latter model the eigenvectors could be expressed in terms of the Krawtchouk orthogonal polynomials. Now, the eigenvectors of the position operator have components proportional to dual Hahn polynomials with parameters $(-\tilde{c}-1)/2$ and $(\tilde{c}-1)/2$ when the component has even index, and with parameters $(1-\tilde{c})/2$ and $(1+\tilde{c})/2$ when the component has odd index. With the condition $|\tilde{c}| < 1$ (see Proposition 2), the weight functions of the dual Hahn polynomials are positive.

The matrix M^p of the momentum operator $\hat{p}$ is up to signs the same as the matrix M^q. It has the same spectrum (3.11) and for the eigenvectors we have the following result.

PROPOSITION 6: *The orthonormal eigenvector of the momentum operator* $\hat{p}$ *in* W_j *for the eigenvalue* p*, denoted by* $|j, p)$*, is given in terms of the basis* (3.3) *by*

$$|j, p) = \sum_{m=-j}^{j} V_{j+m, j+p} |j, m\rangle. \tag{3.15}$$

Herein, $V = (V_{rs})_{0\le r,s\le 2j}$ is the unitary $(2j+1)\times(2j+1)$-matrix, $VV^\dagger = V^\dagger V = I$, defined by

$$V = \mathcal{J}U,$$

where $\mathcal{J} = -i\,\mathrm{diag}(i^0, i^1, i^2, \ldots, i^{2j})$ and U is the matrix determined in Proposition 5.

REMARK 7: The recurrence relation for the pair of polynomials appearing in (3.13) and (3.14) comes from $M^qU = UD^q$. By the form (3.4), this recurrence relation has zero diagonal term. This is because the corresponding polynomials can be seen as an example of Chihara's construction [9, Section 8] of symmetric orthogonal polynomials, but applied to discrete orthogonal polynomials.

4 OSCILLATOR WAVEFUNCTIONS AND THEIR PROPERTIES

The position (resp. momentum) wavefunctions are the overlaps between the normalized eigenstates of the position operator $\hat{q}$ (resp. the momentum operator $\hat{p}$) and the eigenstates of the Hamiltonian. So the wavefunctions of the $\mathfrak{su}(2)_P$ finite oscillator are the overlaps between the $\hat{q}$-eigenvectors and the $\hat{H}$-eigenvectors (or equivalently, the J_0-eigenvectors $|j, m\rangle$). We will denote the position wavefunctions by $\phi^{(c)}_{j+m}(q)$ and the momentum wavefunctions by $\phi^{(c)}_{j+m}(q)$, where m, q and p assume one of the discrete values $-j, -j+1, \ldots, +j$. Concretely, following the notation of the previous section:

$$\phi^{(c)}_{j+m}(q) = \langle j, m|j, q\rangle = U_{j+m,j+q}, \tag{4.1}$$

$$\psi^{(c)}_{j+m}(p) = \langle j, m|j, p\rangle = V_{j+m,j+p}. \tag{4.2}$$

Let us examine the explicit form of these functions in more detail, first for the position variable. The index $j+m$ ranges from 0 to $2j$. For $j+m$ even, say $j+m = 2n$, $\phi^{(c)}_{2n}(q)$ is by (3.13) an even function of the position variable q. For $q = -j, -j+1, \ldots, j$ we have

$$\phi^{(c)}_{2n}(q) = \frac{(-1)^n}{\sqrt{2-\delta_{q,0}}}\sqrt{W(n,q;\tilde{c},j)}\ {}_3F_2\left(\begin{matrix}-q,\ q,\ -n\\ (1-\tilde{c})/2,\ -j\end{matrix};1\right) \tag{4.3}$$

where

$$W(n,q;\tilde{c},j) = \frac{w(|q|; (-\tilde{c}-1)/2, (\tilde{c}-1)/2, j)}{h_n((-\tilde{c}-1)/2, (\tilde{c}-1)/2, j)},$$

with w and h_n as in (3.9). For $j+m$ odd, say $j+m = 2n+1$, it is by (3.14) an odd function of the variable q. For $q = -j, -j+1, \ldots, j$ we have

$$\phi^{(c)}_{2n+1}(q) = (-1)^n q \frac{\sqrt{(2n+1-\tilde{c})(j-n)}}{(1-\tilde{c})j}\sqrt{W(n,q;\tilde{c},j)}\ {}_3F_2\left(\begin{matrix}-q+1,\ q+1,\ -n\\ (3-\tilde{c})/2,\ -j+1\end{matrix};1\right), \tag{4.4}$$

where we used

$$\frac{w(|q|; (1-\tilde{c})/2, (\tilde{c}+1)/2, j-1)}{h_n((1-\tilde{c})/2, (\tilde{c}+1)/2, j-1)} = q^2\frac{2(2n+1-\tilde{c})(j-n)}{(1-\tilde{c})^2 j^2}W(n,q;\tilde{c},j).$$

For the momentum wavefunctions we find in exactly the same manner

$$\psi_{2n}^{(c)}(p) = \frac{-i}{\sqrt{2-\delta_{p,0}}}\sqrt{W(n,p;\tilde{c},j)}\ {}_3F_2\left(\begin{matrix}-p,\,p,\,-n\\(1-\tilde{c})/2,\,-j\end{matrix};1\right) \tag{4.5}$$

$$\psi_{2n+1}^{(c)}(p) = p\frac{\sqrt{(2n+1-\tilde{c})(j-n)}}{(1-\tilde{c})j}\sqrt{W(n,p;\tilde{c},j)}\ {}_3F_2\left(\begin{matrix}-p+1,\,p+1,\,-n\\(3-\tilde{c})/2,\,-j+1\end{matrix};1\right). \tag{4.6}$$

REMARK 8: Before we examine the behaviour of these discrete wavefunctions, let us comment on the distinction with the closely related dual -1 Hahn polynomials considered in [28]. For this purpose, let us compare the polynomial expressions in (4.3)–(4.4), i.e.

$${}_3F_2\left(\begin{matrix}-q,\,q,\,-n\\(1-\tilde{c})/2,\,-j\end{matrix};1\right), \qquad q\times\ {}_3F_2\left(\begin{matrix}-q+1,\,q+1,\,-n\\(3-\tilde{c})/2,\,-j+1\end{matrix};1\right)$$

with equations (4.6) and (4.7) from [28], in which one puts $N=2j$, i.e.

$${}_3F_2\left(\begin{matrix}-\frac{x}{4}+\eta,\,\frac{x}{4}+\eta,\,-n\\1-\frac{\alpha}{2},\,-j\end{matrix};1\right), \qquad (\frac{x}{4}-j-\eta)\ {}_3F_2\left(\begin{matrix}-\frac{x}{4}+\eta,\,\frac{x}{4}+\eta,\,-n\\1-\frac{\alpha}{2},\,-j+1\end{matrix};1\right).$$

For a particular choice of η and α, the even polynomials coincide, but the odd polynomials do not. The reason is that the dual Hahn double of this paper corresponds to a Christoffel-Geronimus pair with parameter $\nu=0$ (see [24, Section 5]) and are of type “dual Hahn I” in the terminology of [24], whereas the dual -1 Hahn polynomials seem to correspond to a Christoffel transform for dual Hahn polynomials with a different parameter $\nu=j$, and are of type “dual Hahn II” in the terminology of [24].

It is interesting to study these discrete wavefunctions for varying values of $\tilde{c}$, $-1<\tilde{c}<1$. For the special value $\tilde{c}=0$, the algebra $\mathfrak{su}(2)_P$ reduces to $\mathfrak{su}(2)$ and it is known that in this case, the wavefunctions $\phi_n^{(0)}(q)$ are in fact Krawtchouk functions. Indeed, when $\tilde{c}=0$ the dual Hahn polynomials, which are ${}_3F_2$ series appearing in (4.3)–(4.4), reduce to ${}_2F_1$ series according to

$${}_3F_2\left(\begin{matrix}-q,\,q,\,-n\\1/2,\,-j\end{matrix};1\right) = (-1)^n\frac{\binom{2j}{2n}}{\binom{j}{n}}\ {}_2F_1\left(\begin{matrix}-2n,\,-j-q\\-2j\end{matrix};2\right), \tag{4.7}$$

$${}_3F_2\left(\begin{matrix}-q+1,\,q+1,\,-n\\3/2,\,-j+1\end{matrix};1\right) = -\frac{(-1)^n}{2q}\frac{\binom{2j}{2n+1}}{\binom{j-1}{n}}\ {}_2F_1\left(\begin{matrix}-2n-1,\,-j-q\\-2j\end{matrix};2\right). \tag{4.8}$$

These reductions have been given in [26] and can be obtained, e.g., from [7, (48)]. The ${}_2F_1$ series in the right hand side correspond to symmetric Krawtchouk polynomials (i.e. Krawtchouk polynomials with $p=1/2$ [22]). When j tends to infinity, they yield the canonical oscillator wavefunctions [7] in terms of Hermite polynomials.

To investigate what happens for other values of $\tilde{c}$ we now choose a fixed value of j, namely $j=32$, and plot some of the wavefunctions $\phi_n^{(c)}(q)$ for various values of $\tilde{c}$. Recall

(Proposition 2) that $-1 < \tilde{c} < 1$ in order to have a unitary irreducible representation. In Figure 1 we take the following values for $\tilde{c}$, respectively,

$$-0.999, \quad -0.8, \quad -0.3, \quad 0, \quad 0.3, \quad 0.8, \quad 0.999\ .$$

We also plot in each case the ground state $\phi_0^{(c)}(q)$ (left column), some low energy states $\phi_1^{(c)}(q)$ and $\phi_2^{(c)}(q)$ (2nd and 3rd column), and the highest energy state $\phi_{64}^{(c)}(q)$ (4th column).

Particularly interesting behaviour is observed when $\tilde{c}$ approaches the boundary values -1 or 1. These bounds correspond to the disallowed value -1 for one of the parameters γ or δ in the dual Hahn polynomial (3.7). When $\tilde{c}$ tends to -1, the components of the highest energy state all tend to zero except for $q = 0$ which tends to 1. For all the other states, the value at $q = 0$ tends to 0. When $\tilde{c}$ tends to $+1$, it is for the lowest energy state that all components tend to zero and the component at $q = 0$ goes to 1. Similarly as for the other limit, for all the other states, the value at $q = 0$ tends to 0. It can be verified that in these limits for non-zero q the wavefunctions become up to signs those of the oscillator model based on the even-dimensional representations of $\mathfrak{u}(2)_\alpha$, see [19], for a specific parameter value in dimension $2j$. Recall that these correspond precisely to the even-dimensional representations of $\mathfrak{su}(2)_P$ obtained in Proposition 2.

The described behaviour happens according to the following relations ($q = 1, \ldots, j$):

$$\lim_{\gamma\to -1} \tilde{R}_n(\lambda(q); \gamma, 0, j) = -\tilde{R}_{n-1}(\lambda(q-1); 1, 0, j-1)$$

$$\lim_{\delta\to -1} \tilde{R}_n(\lambda(q); 0, \delta, j) = \tilde{R}_n(\lambda(q-1); 0, 1, j-1)$$

We now look what happens to $\phi_n^{(c)}(q)$ for general $\tilde{c}$ when j tends to infinity. This is done by putting $q = j^{1/2}x$ to pass from a discrete position variable q to a continuous position variable x and taking the limit $j \to \infty$ of $j^{1/4}\phi_n^{(c)}(q)$. The actual computation is similar to the one performed in [19, 20], so we shall not give all details. The limit of the ${}_3F_2$ function in (4.3) and (4.4) is quite easy:

$$\lim_{j\to\infty} {}_3F_2\left({-j^{1/2}x,\, j^{1/2}x,\, -n \atop (1-\tilde{c})/2,\, -j} ; 1 \right) = {}_1F_1\left({-n \atop (1-\tilde{c})/2} ; x^2 \right) = \frac{n!}{(a)_n} L_n^{(a-1)}(x^2), \tag{4.9}$$

$$\lim_{j\to\infty} {}_3F_2\left({-j^{1/2}x+1,\, j^{1/2}x+1,\, -n \atop (3-\tilde{c})/2,\, -j+1} ; 1 \right) = {}_1F_1\left({-n \atop (3-\tilde{c})/2} ; x^2 \right) = \frac{n!}{(a+1)_n} L_n^{(a)}(x^2), \tag{4.10}$$

where $a = (1-\tilde{c})/2$ and $L_n^{(\alpha)}$ is a Laguerre polynomial [22, 27].

The final result is:

$$\lim_{j\to\infty} j^{1/4}\phi_{2n}^{(c)}(j^{1/2}x) = (-1)^n \sqrt{\frac{n!}{\Gamma(a+n)}}\ |x|^{a-1/2} e^{-x^2/2} L_n^{(a-1)}(x^2), \tag{4.11}$$

$$\lim_{j\to\infty} j^{1/4}\phi_{2n+1}^{(c)}(j^{1/2}x) = (-1)^n \sqrt{\frac{n!}{\Gamma(a+n+1)}}\ x|x|^{a-1/2} e^{-x^2/2} L_n^{(a)}(x^2). \tag{4.12}$$

Note that for $\tilde{c} = 0$ or $a = 1/2$, one indeed finds the canonical oscillator wavefunctions

$$\lim_{j\to\infty} j^{1/4}\phi_n^{(0)}(j^{1/2}x) = \frac{1}{2^{n/2}\sqrt{n!}\pi^{1/4}} H_n(x)e^{-x^2/2}, \tag{4.13}$$

where $H_n(x)$ are the common Hermite polynomials [22, 27].

The functions in (4.11) and (4.12) are familiar: they are in fact the wavefunctions $\Psi_n^{(a)}(x)$ of the parabose oscillator with parameter $a > 0$ (see the appendix of [19] for a summary). So we have:

$$\lim_{j\to\infty} j^{1/4}\phi_n^{(c)}(j^{1/2}x) = \Psi_n^{(a)}(x) \qquad (a = \frac{1-\tilde{c}}{2}). \tag{4.14}$$

So the current model is an appealing model for a finite one-dimensional parabose oscillator with equidistant position spectrum. This also explains the shape of the discrete wavefunctions plotted in Figure 1. For $-1 < \tilde{c} < 0$, the shape typically reproduces the continuous wavefunctions of the parabose oscillator with $\frac{1}{2} < a < 1$: see the plots for $\tilde{c} = -0.8$ and those for $a = 0.9$ in Figure 2. For $0 < \tilde{c} < 1$, the shape of the wavefunctions is similar to those of the parabose oscillator with $0 < a < \frac{1}{2}$: compare the plots for $\tilde{c} = 0.8$ with those for $a = 0.1$ in Figure 2.

5 CONCLUDING REMARKS

Deformations or extensions of $\mathfrak{su}(2)$ or $\mathfrak{u}(2)$ as algebras underlying finite oscillator models have already been considered by one of us [19, 20], so let us explain the difference with the algebra $\mathfrak{su}(2)_P$ appearing here. For this, it is best to return to the classification of so-called dual Hahn doubles in [24], where it is shown that three such doubles or pairs exist. From [24, Propositions 1–3] one can see that only the cases "dual Hahn I" and "dual Hahn III" can give rise to an equidistant position spectrum when used in a finite oscillator model. The case "dual Hahn I" involves the pair of polynomials $R_n(\lambda(x);\gamma,\delta,N)$ and $R_n(\lambda(x-1);\gamma+1,\delta+1,N-1)$, and the corresponding algebra constructed from the related tridiagonal matrices was determined in [24, eq. (7.4)]. Comparing with $\mathfrak{su}(2)_P$, the relations (2.1)–(2.2) remain the same, and (2.3) is of the form

$$[J_+, J_-] = 2J_0 + 2(\gamma+\delta+1)J_0P - (2N+1)(\gamma-\delta)P + (\gamma-\delta)I. \tag{5.1}$$

Because of the appearance of N and $N-1$ in the double, the matrices (and thus also the representations) exist in odd dimension $2N+1$ only; furthermore the spectrum of the position operator consists of the values $0, \pm\sqrt{k(k+\gamma+\delta+1)}$ $(k = 1, \ldots, N)$. For $\gamma = \delta \equiv \alpha$, (5.1) coincides with [20, eq. (5)], so this is what was called the $\mathfrak{su}(2)_\alpha$ extension in [20]. The position spectrum is not equidistant. Moreover, due to the combination of terms in J_0 and J_0P in the commutator of J_+ and J_-, this algebra cannot be rewritten as a special case of the Bannai-Ito algebra.

For $\delta = -\gamma - 1$, (5.1) becomes

$$[J_+, J_-] = 2J_0 + (2\gamma+1)I - (2N+1)(2\gamma+1)P$$

and after performing a shift for J_0, this relation is of the form (2.3). So this is $\mathfrak{su}(2)_P$ (isomorphic to a special case of Bannai-Ito), the position spectrum is equidistant and this is the case (with odd dimensional representations) that was treated in the current paper.

The case “dual Hahn III” involves the pair of polynomials $R_n(\lambda(x); \gamma, \delta, N)$ and $R_n(\lambda(x); \gamma+1, \delta-1, N)$, and the corresponding algebra constructed from the related tridiagonal matrices was determined in [24, eq. (7.5)], with relation

$$[J_+, J_-] = 2J_0+2(\gamma-\delta)J_0P-((2N+2)(\gamma+\delta+1)+(2\gamma+1)(2\delta+1))P+(\gamma-\delta)I. \quad (5.2)$$

Because of the appearance of N and N in the polynomials of the double, the matrices (and representations) exist in even dimension $2N+2$ only; the spectrum of the position operator consists of the values $\pm\sqrt{(k+\gamma+1)(k+\delta+1)}$ $(k=0,\ldots,N)$. So – apart from a gap in the middle – it is equidistant for $\gamma=\delta\equiv\alpha$, and then the above relation becomes

$$[J_+, J_-] = 2J_0-((2N+2)(2\alpha+1)+(2\alpha+1)^2)P = 2J_0-(2\alpha+1)^2P-(2\alpha+1)CP,$$

for some central element C. This was called the $\mathfrak{u}(2)_\alpha$ algebra in [19]. But since C is a constant in a representation of the algebra, it can be considered as the $\mathfrak{su}(2)_P$ algebra with $-c=(2N+2)(2\alpha+1)+(2\alpha+1)^2$ in (2.3). So $\mathfrak{su}(2)_P$ and $\mathfrak{u}(2)_\alpha$ are essentially the same, and the even dimensional representations of this algebra are the ones studied in [19].

For the Lie algebra $\mathfrak{su}(2)$, there is of course the well known Schwinger boson realization. In this realization, for a positive integer or half-integer j, the $2j+1$ basis vectors can be expressed as follows

$$|j, m\rangle = \frac{x^{j+m}y^{j-m}}{\sqrt{(j+m)!(j-m)!}}, \quad (5.3)$$

and the $\mathfrak{su}(2)$ operators take the form

$$J_0 = \frac{1}{2}(x\partial_x - y\partial_y), \quad J_+ = x\partial_y, \quad J_- = y\partial_x. \quad (5.4)$$

For the algebra $\mathfrak{su}(2)_P$, there exist similar reflection/differential operator realizations, one of which follows from the Bannai-Ito algebra realization and can be found in [10]. Since our basis elements (2.1)–(2.3) of $\mathfrak{su}(2)_P$ are closely related to the standard basis of $\mathfrak{su}(2)$, it is natural to expect other operator realizations. These do indeed exist.

As a first possibility, consider the following operators, acting on functions $f(x,y)$ of two variables x and y:

$$\begin{aligned} T_x &= \partial_x + \frac{\mu}{x}(1-R_x) \\ T_y &= \partial_y - \frac{\mu}{y}(1-R_x). \end{aligned} \quad (5.5)$$

Herein, $R_x f(x, y) = f(-x, y)$. Note that T_x is a Dunkl operator, but T_y is not. Putting

$$J_0 = \frac{1}{2}(x\partial_x - y\partial_y + 2\mu), \quad J_+ = xT_y, \quad J_- = yT_x, \quad P = R_x, \tag{5.6}$$

it is easy to verify that the defining relations (2.1) and (2.2) are satisfied. For (2.3), one finds:

$$[J_+, J_-] = 2J_0 - 2\mu P(1 + x\partial_x + y\partial_y). \tag{5.7}$$

So when acting on homogeneous polynomials in x and y, like on the basis vectors (5.3), the last relation coincides with (2.3) for $\mu = -\tilde{c}/2$. For a proper action on homogeneous polynomials, one should take care of the factor $1/y$ in (5.5): the action of T_y on x^{2j} should vanish. This is the case only for integer j-values, thanks to the factor $(1 - R_x)$ in (5.5). Thus, the realization (5.5)–(5.6) is consistent with the basis realization (5.3) for j integer only. Note that on the space of homogeneous polynomials of degree $2j$, spanned by (5.3), the action of T_y does coincide with the action of a Dunkl operator $\partial_y - \frac{\mu}{y}(1 - R_y)$, where $R_y f(x, y) = f(x, -y)$.

As a second possibility, let us take

$$\begin{aligned} T_x &= \partial_x + \frac{\mu}{x}(1 - R_x) \\ T_y &= \partial_y + \frac{\mu}{y}(1 + R_x). \end{aligned} \tag{5.8}$$

and

$$J_0 = \frac{1}{2}(x\partial_x - y\partial_y), \quad J_+ = xT_y, \quad J_- = yT_x, \quad P = -R_x. \tag{5.9}$$

Once again, (2.1) and (2.2) are satisfied, and for (2.3) one finds:

$$[J_+, J_-] = 2J_0 + P\left((2\mu)^2 + 2\mu(1 + x\partial_x + y\partial_y)\right). \tag{5.10}$$

In this case, acting on homogeneous polynomials in x and y like on the basis vectors (5.3), the last relation coincides with (2.3) for $c = (2\mu)^2 + 2\mu(2j+1)$ (in agreement with Remark 3). Also here, one should take care of the factor $1/y$ in (5.8), and the action of T_y on x^{2j} should vanish. This is now the case only for half-integer j-values, due to the factor $(1+R_x)$ in (5.5). The conclusion is similar: the realization (5.8)–(5.9) is consistent with the basis realization (5.3) for j half-integer only. For more fundamental examples in which such realizations with Dunkl operators play a role, see the Schwinger-Dunkl algebra $sd(2)$ in [12].

To summarize, in this paper we have developed a new and interesting model for a finite quantum oscillator. This model preserves all the nice and essential properties of the original $\mathfrak{su}(2)$ model, in particular the equidistance of the position spectrum. It has, however, an extra parameter $\tilde{c}$ that can be used to modify the shape of the discrete position (and momentum) wavefunctions. The original interest in finite oscillator models comes mainly from optical image processing and signal analysis [7]. In signal analysis on a finite number of discrete sensors or data points, one-dimensional finite oscillator models have been used in [2–4]. For such purposes, it is an advantage if the "sensor points" of the grid are uniformly distributed, according to the equidistant position spectrum of the model. For our original Hahn oscillator in even dimensions [19] or in odd dimensions [20], this equidistance did not hold. In the current model, based on a dual Hahn double, we do recover this important property of the spectrum (in odd dimensions). We hope that the extra parameter $\tilde{c}$ opens the way to more sophisticated techniques in the analysis of signals.

The model presented here has the algebra $\mathfrak{su}(2)_P$ as underlying structure. This algebra is an extension of $\mathfrak{su}(2)$ by cP, where P which is not central but satisfies $P^2 = 1$ and either commutes or anticommutes with the standard basis elements of $\mathfrak{su}(2)$. We have shown that $\mathfrak{su}(2)_P$ is a special case of the general Bannai-Ito algebra. For $\mathfrak{su}(2)_P$, we have classified all unitary finite-dimensional irreducible representations. These depend on the central element c. Once the algebra and its representations have been analysed, the construction of the corresponding finite oscillator model is similar to that of [19]. The position wavefunctions are expressed in terms of dual Hahn polynomials (with different parameters for even and odd wavefunctions), and depend on the dimension of the representation $(2j+1)$ and the parameter $\tilde{c}$ with $|\tilde{c}| < 1$. For $\tilde{c} = 0$, the model and its wavefunctions coincide with the standard $\mathfrak{su}(2)$ finite oscillator model in terms of symmetric Krawtchouk polynomials [7]. Symmetric Krawtchouk wavefunctions can interpreted as a finite-dimensional version of the canonical Hermite wavefunctions, to which they tend when the dimension paramater j goes to infinity. There is a similar interpretation here. For $\tilde{c} \neq 0$, the wavefunctions can be seen as a finite-dimensional version (with equidistant spectrum) of the parabose wavefunctions.

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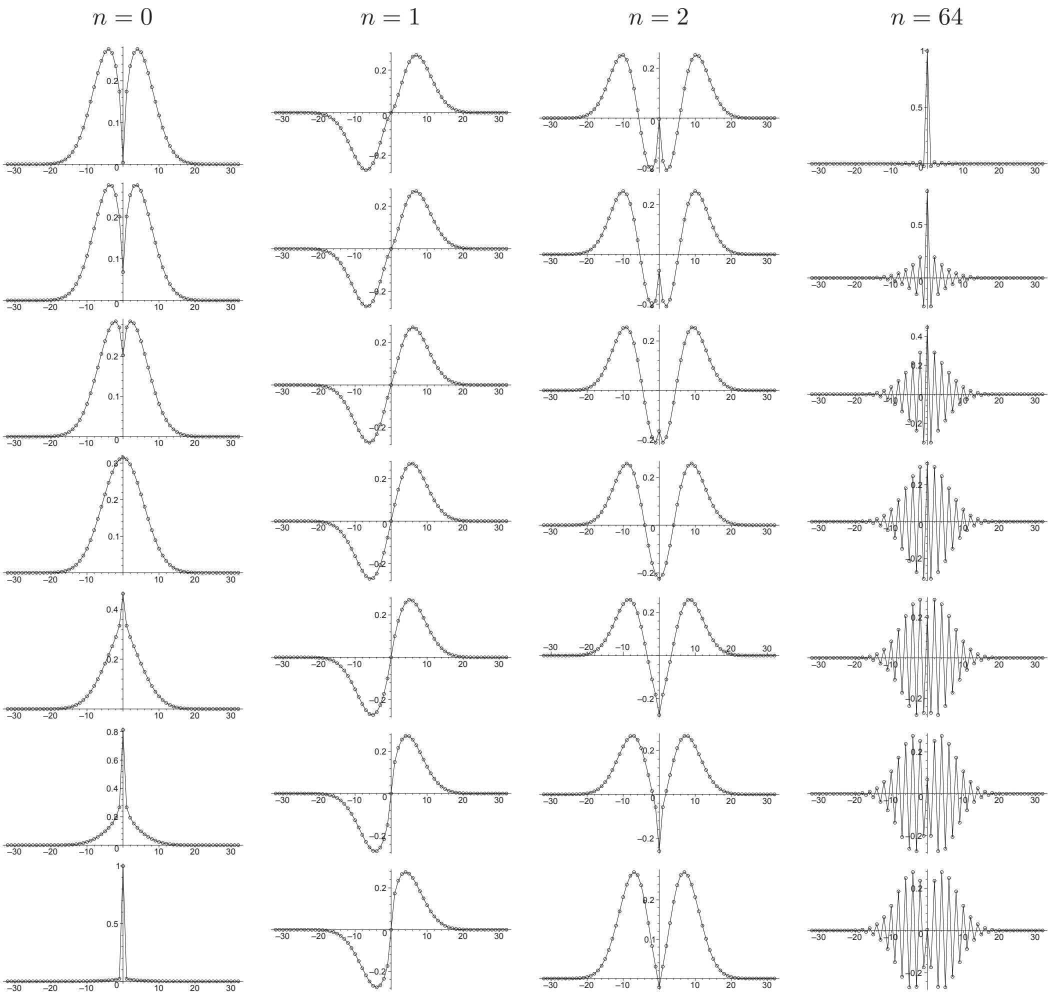

FIGURE 1: Plots of the discrete wavefunctions $\phi_n^{(c)}(q)$ in the representation with $j = 32$ for the values $\tilde{c} = -0.999$, $\tilde{c} = -0.8$, $\tilde{c} = -0.3$, $\tilde{c} = 0$, $\tilde{c} = 0.3$, $\tilde{c} = 0.8$, $\tilde{c} = 0.999$ from top to bottom. The wavefunctions are plotted for $n = 0$, 1, 2 and $n = 64$.

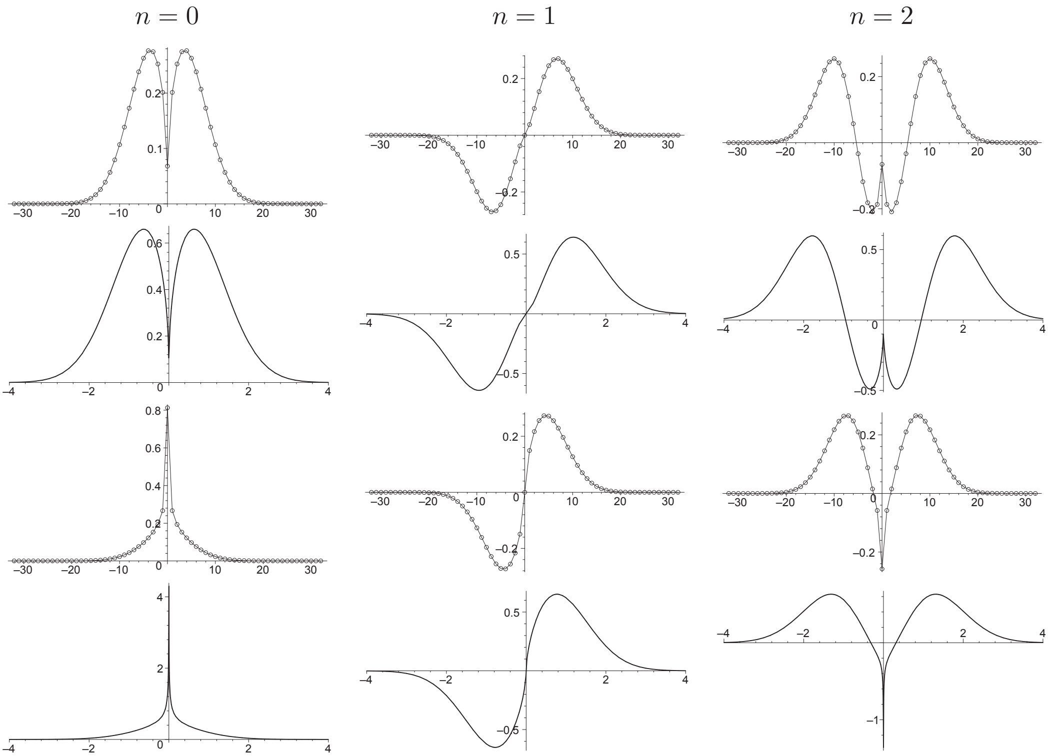

FIGURE 2: Comparing the plots of the discrete wavefunctions $\phi_n^{(c)}(q)$ with the continuous wavefunctions $\Psi_n^{(a)}(x)$ of the parabose oscillator, for $n = 0$ (left column), $n = 1$ (middle column) and $n = 2$ (right column). In the top row one finds $\phi_n^{(c)}(q)$ for $\tilde{c} = -0.8$, to be compared to the plots of $\Psi_n^{(a)}(x)$ in the second row for $a = 0.9$. In the third row one finds $\phi_n^{(c)}(q)$ for $\tilde{c} = 0.8$, to be compared to the plots of $\Psi_n^{(a)}(x)$ in the fourth row for $a = 0.1$.

6

A finite quantum oscillator model related to special sets of Racah polynomials

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abstract

In Oste and Van der Jeugt, SIGMA, 12 (2016) [12] we classified all pairs of recurrence relations in which two (dual) Hahn polynomials with different parameters appear. Such pairs are referred to as (dual) Hahn doubles, and the same technique was then applied to obtain all Racah doubles. We now consider a special case concerning the doubles related to Racah polynomials. This gives rise to an interesting class of two-diagonal matrices with closed form expressions for the eigenvalues. Just as it was the case for (dual) Hahn doubles, the resulting two-diagonal matrix can be used to construct a finite oscillator model. We discuss some properties of this oscillator model, give its (discrete) position wavefunctions explicitly, and illustrate their behaviour by means of some plots.

1 introduction

In a recent paper [12] all pairs of recurrence relations in which two Hahn, dual Hahn or Racah polynomials with different parameters appear were classified. We used the term (dual) Hahn doubles or Racah doubles for such pairs. They were shown to correspond to Christoffel-Geronimus pairs of (dual) Hahn or Racah polynomials [12].

In the present paper, we shall consider a special case of a Racah double. This special case is chosen in such a way that the related two-diagonal (Jacobi) matrix M has a very simple spectrum. The eigenvectors of M can then be written in terms of the corresponding Racah polynomials.

The main reason to study the special case considered here is because it is particularly interesting in the framework of finite oscillator models. Finite oscillator models were introduced and investigated in a number of papers, see e.g. [4, 1–3, 7, 8]. The standard example is the $\mathfrak{su}(2)$ oscillator model [1, 2]. In brief, this model is based on the $\mathfrak{su}(2)$ algebra with basis elements $J_0 = J_z$, $J_\pm = J_x \pm J_y$ satisfying

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0, \tag{1.1}$$

with unitary representations of dimension $2j + 1$ (where j is integer or half-integer). Recall that the oscillator Lie algebra can be considered as an associative algebra (with unit element 1) with three generators $\hat{H}$, $\hat{q}$ and $\hat{p}$ (the Hamiltonian, the position and the momentum operator) subject to

$$[\hat{H}, \hat{q}] = -i\hat{p}, \qquad [\hat{H}, \hat{p}] = i\hat{q}, \qquad [\hat{q}, \hat{p}] = i, \tag{1.2}$$

in units with mass and frequency both equal to 1, and $\hbar = 1$. The first two are the Hamilton-Lie equations; the third the canonical commutation relation. The canonical commutation relation is not compatible with a finite-dimensional Hilbert space. Following this, one speaks of a finite oscillator model if $\hat{H}$, $\hat{q}$ and $\hat{p}$ belong to some algebra such that the Hamilton-Lie equations are satisfied and such that the spectrum of $\hat{H}$ in representations of that algebra is equidistant [2, 7].

In the $\mathfrak{su}(2)$ model, one chooses

$$\hat{H} = J_0 + j + \frac{1}{2}, \quad \hat{q} = \frac{1}{2}(J_+ + J_-), \quad \hat{p} = \frac{i}{2}(J_+ - J_-). \tag{1.3}$$

These indeed satisfy $[\hat{H}, \hat{q}] = -i\hat{p}$, $[\hat{H}, \hat{p}] = i\hat{q}$, and the spectrum of $\hat{H}$ is equidistant in the representation (j) labeled by j (and given by $n + \frac{1}{2}$; $n = 0, 1, \ldots, 2j$). Clearly, for this model the position operator $\hat{q} = \frac{1}{2}(J_+ + J_-)$ also has a finite spectrum in the representation (j) given by $q \in \{-j, -j+1, \ldots, +j\}$. In terms of the standard J_0-eigenvectors $|j, m\rangle$, the eigenvectors of $\hat{q}$ can be written as

$$|j, q) = \sum_{m=-j}^{j} \Phi_{j+m}(q)|j, m\rangle. \tag{1.4}$$

The coefficients $\Phi_n(q)$ are the position wavefunctions, and in this model [1, 2] they turn out to be (normalized) symmetric Krawtchouk polynomials, $\Phi_n(q) \sim K_n(j + q; \frac{1}{2}, 2j)$. The shape of the these wavefunctions is reminiscent of those of the canonical oscillator: under the limit $j \to \infty$ they coincide with the canonical wavefunctions in terms of Hermite polynomials.

In the present paper we develop a related but new finite oscillator model, following the ideas of [7] where a dual Hahn double was used to extend the $\mathfrak{su}(2)$ model. The recent classification [12] of all (dual) Hahn doubles and Racah doubles opens the way to investigate such new models. The basic ingredient is a special Racah double from this classification, that is explained and analysed in Section 2. In Section 3 we study the related finite oscillator model, and in particular we focus on some properties of the discrete position wavefunctions.

2 RACAH POLYNOMIALS AND TWO RACAH DOUBLES

Racah polynomials $R_n(\lambda(x); \alpha, \beta, \gamma, \delta)$ of degree n ($n = 0, 1, \ldots, N$) in the variable $\lambda(x) = x(x + \gamma + \delta + 1)$ are defined by [6, 9, 10]

$$R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix}; 1\right), \tag{2.1}$$

where one of the denominator parameters should be $-N$:

$$\alpha + 1 = -N \qquad \text{or} \qquad \beta + \delta + 1 = -N \qquad \text{or} \qquad \gamma + 1 = -N. \tag{2.2}$$

Herein, the function ${}_4F_3$ is the generalized hypergeometric series [5, 13]:

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \tag{2.3}$$

where we use the common notation for Pochhammer symbols [5, 13] $(a)_k = a(a+1) \cdots (a+k-1)$ for $k = 1, 2, \dots$ and $(a)_0 = 1$. Note that in (2.1), the series is terminating because of the appearance of the negative integer $-n$ as a numerator parameter.

Racah polynomials satisfy a (discrete) orthogonality relation (which depends on the choice of which parameter relates to $-N$) [9, 11]. For the choice $\alpha + 1 = -N$ we have

$$\sum_{x=0}^{N} w(x; \alpha, \beta, \gamma, \delta) R_n(\lambda(x); \alpha, \beta, \gamma, \delta) R_{n'}(\lambda(x); \alpha, \beta, \gamma, \delta) = h_n(\alpha, \beta, \gamma, \delta)\, \delta_{n,n'}, \tag{2.4}$$

where

$$\begin{aligned} w(x; \alpha, \beta, \gamma, \delta) &= \frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x((\gamma+\delta+1)/2)_x(\delta+1)_x x!}, \\ h_n(\alpha, \beta, \gamma, \delta) &= \frac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N} \\ &\quad \times \frac{(n+\alpha+\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n(\beta+1)_n n!}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}. \end{aligned} \tag{2.5}$$

Under certain restrictions such as $\gamma, \delta > -1$ and $\beta > N + \gamma$ or $\beta < -N - \delta - 1$, which ensure positivity of the functions w and h, we can define orthonormal Racah functions as follows:

$$\tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta) \equiv \frac{\sqrt{w(x; \alpha, \beta, \gamma, \delta)}\, R_n(\lambda(x); \alpha, \beta, \gamma, \delta)}{\sqrt{h_n(\alpha, \beta, \gamma, \delta)}}. \tag{2.6}$$

After settling this notation, let us turn to a result from [12]. The matrices appearing here will always be of a special tridiagonal form, namely

$$M = \begin{pmatrix} 0 & M_0 & 0 & & \\ M_0 & 0 & M_1 & 0 & \\ 0 & M_1 & 0 & M_2 & \ddots \\ & 0 & M_2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.7}$$

and such matrices will be referred to as two-diagonal. The following two propositions were obtained in [12, Appendix]:

PROPOSITION 1: *Let $\alpha + 1 = -N$, and suppose that $\gamma, \delta > -1$ and $\beta > N + \gamma$ or $\beta < -N - \delta - 1$. Consider two $(2N + 2)\times(2N + 2)$ matrices U and M, defined as follows. U has elements ($n, x \in \{0, 1, \ldots, N\}$):*

$$U_{2n,N-x} = U_{2n,N+x+1} = \frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta + 1),$$
$$U_{2n+1,N-x} = -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha, \beta + 1, \gamma + 1, \delta); \tag{2.8}$$

M is the two-diagonal $(2N + 2)\times(2N + 2)$-matrix of the form (2.7) with

$$M_{2k} = 2\sqrt{\frac{(N-\beta-k)(\gamma+1+k)(N+\delta+1-k)(k+\beta+1)}{(N-\beta-2k)(2k-N+1+\beta)}},$$
$$M_{2k+1} = 2\sqrt{\frac{(\gamma+N-\beta-k)(k+1)(N-k)(k+\delta+\beta+2)}{(N-\beta-2k-2)(2k-N+1+\beta)}}. \tag{2.9}$$

Then U is orthogonal, and the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:

$$D = \mathrm{diag}(-\epsilon_N, \ldots, -\epsilon_1, -\epsilon_0, \epsilon_0, \epsilon_1, \ldots, \epsilon_N),$$
$$\epsilon_k = 2\sqrt{(k+\gamma+1)(k+\delta+1)} \qquad (k = 0, 1, \ldots, N). \tag{2.10}$$

In short, the pair of polynomials $R_n(\lambda(x); \alpha, \beta, \gamma, \delta + 1)$ and $R_n(\lambda(x); \alpha, \beta + 1, \gamma + 1, \delta)$ form a "Racah double", and the relation $MU = UD$ governs the corresponding recurrence relations with M taking the role of a Jacobi matrix [12].

PROPOSITION 2: *Let $\alpha + 1 = -N$, and suppose that $\gamma, \delta > -1$ and $\beta > N + \gamma$ or $\beta < -N - \delta$. Consider two $(2N + 1)\times(2N + 1)$ matrices U and M, defined as follows.*

$$U_{2n,N-x} = U_{2n,N+x} = \frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta), \qquad (n = 0, \ldots, N;\ x = 1, \ldots, N)$$
$$U_{2n+1,N-x-1} = -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); \alpha + 1, \beta, \gamma + 1, \delta + 1), \tag{2.11}$$
$$(n, x \in \{0, \ldots, N-1\})$$
$$U_{2n,N} = (-1)^n\tilde{R}_n(\lambda(0); \alpha, \beta, \gamma, \delta), \qquad U_{2n+1,N} = 0.$$

M is the two-diagonal $(2N + 1)\times(2N + 1)$-matrix of the form (2.7) with

$$M_{2k} = 2\sqrt{\frac{(\gamma+k+1)(-N+\beta+k)(N-k)(k+\delta+\beta+1)}{(N-\beta-2k)(N-\beta-2k-1)}},$$
$$M_{2k+1} = 2\sqrt{\frac{(\gamma+N-\beta-k)(k+1)(k+\beta+1)(k-\delta-N)}{(N-\beta-2k-2)(N-\beta-2k-1)}}. \tag{2.12}$$

Then U is orthogonal, and the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:

$$\begin{aligned} D &= \operatorname{diag}(-\epsilon_N, \ldots, -\epsilon_1, 0, \epsilon_1, \ldots, \epsilon_N), \\ \epsilon_k &= 2\sqrt{k(k+\gamma+\delta+1)} \qquad (k = 1, \ldots, N). \end{aligned} \tag{2.13}$$

The special case considered in this paper is for $\gamma = \delta = -1/2$. The reason for this will be clear in the following, but at this point one can already observe that for these values the eigenvalues of D (both in even dimensions, (2.10), as in odd dimensions, (2.13)) take a simple form. For these special values, the matrix elements of M in the case of Proposition 1 become:

$$\begin{aligned} M_{2k} &= 2\sqrt{\frac{(k-N+\beta)(k+1/2)(N-k+1/2)(k+\beta+1)}{(2k-N+\beta)(2k-N+1+\beta)}}, \\ M_{2k+1} &= 2\sqrt{\frac{(k-N+\beta+1/2)(k+1)(N-k)(k+\beta+3/2)}{(2k-N+\beta+2)(2k-N+1+\beta)}}. \end{aligned}$$

We see that in this case the expressions for coefficients with even and odd indices coincide and can be written as a single expression, namely

$$M_k = \sqrt{\frac{(k+1)(2N+1-k)(k-2N+2\beta)(k+2\beta+2)}{(2k-2N+2\beta)(2k-2N+2\beta+2)}}, \tag{2.14}$$

with $k \in \{0, \ldots, 2N\}$. Suppose we are in the case $\beta > N + \gamma$, i.e. $\beta > N - 1/2$. It will be useful to rewrite $2\beta = 2N - 1 + c$, with $c > 0$, and then the matrix elements take the form

$$M_k = \sqrt{\frac{(k+1)(2N+1-k)(k-1+c)(k+2N+1+c)}{(2k-1+c)(2k+1+c)}}, \qquad k \in \{0, \ldots, 2N\}. \tag{2.15}$$

Also in the case of Proposition 2 the matrix elements of M simplify for the special values $\gamma = \delta = -1/2$. They can also be written as a single expression, and after writing $2\beta = 2N - 1 + c$ ($c > 0$) they read:

$$M_k = \sqrt{\frac{(k+1)(2N-k)(k-1+c)(k+2N+c)}{(2k-1+c)(2k+1+c)}}, \qquad k \in \{0, \ldots, 2N-1\}. \tag{2.16}$$

Taking into account the size of the matrices in both cases, the results from Proposition 1 and Proposition 2 can be unified in the following:

PROPOSITION 3: *For d a positive integer, $k \in \{0, \ldots, d-1\}$ and a parameter $c > 0$, let*

$$M_k = \sqrt{\frac{(k+1)(d-k)(k-1+c)(k+d+c)}{(2k-1+c)(2k+1+c)}}. \tag{2.17}$$

The eigenvalues of the tridiagonal $(d+1)\times(d+1)$*-matrix of the form* (2.7) *are given by the integers*

$$-d, -d+2, -d+4, \ldots, d-4, d-2, d \tag{2.18}$$

which are equidistant, symmetric around zero, and range from $-d$ *to* d*. Hence for even* $d = 2N$*, they are* $d+1$ *consecutive even integers, while for odd* $d = 2N+1$ *they are* $d+1$ *consecutive odd integers.*
For $d = 2N$ *even, the eigenvectors of* M *are the columns of the matrix* U *given by* (2.11)*, with* $\alpha = -N-1$*,* $\beta = N - 1/2 + c/2$*,* $\gamma = -1/2$ *and* $\delta = -1/2$*.*
For $d = 2N+1$ *odd, the eigenvectors of* M *are the columns of the matrix* U *given by* (2.8)*, again with* $\alpha = -N-1$*,* $\beta = N - 1/2 + c/2$*,* $\gamma = -1/2$ *and* $\delta = -1/2$*.*

Note in particular that the eigenvalues of M are independent of the value of the parameter c, but of course c appears in the expressions of the eigenvectors.

3 A QUANTUM OSCILLATOR MODEL BASED ON RACAH POLYNOMIALS

We now consider a one-dimensional quantum oscillator model based on the findings of the previous section. Particularly interesting about this model is that it contains a parameter $c > 0$. By construction, the spectrum of the position operator in this model will be independent of the parameter c, equidistant and coincide with the spectrum of the $\mathfrak{su}(2)$ finite oscillator model [2].

Let us first return to the $\mathfrak{su}(2)$ model, briefly introduced in Section 1. Working in a representation (j) of dimension $2j+1$ (where j is integer or half-integer), and in the standard basis $|j, m\rangle$ in which J_0 is diagonal, it follows from (1.3) that the Hamiltonian is a diagonal matrix,

$$\hat{H} = \operatorname{diag}(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, 2j + \frac{1}{2}). \tag{3.1}$$

In this context, it is more common to rewrite the basis vectors $|j, m\rangle$ ($m = -j, -j+1, \ldots, j$) of this representation space as $|n\rangle \equiv |j, n-j\rangle$ ($n = 0, 1, \ldots, 2j$). Thus we can write

$$\hat{H}|n\rangle = \left(n + \frac{1}{2}\right)|n\rangle, \qquad (n = 0, 1, \ldots, 2j). \tag{3.2}$$

Also from (1.3), the matrix form of the position operator $\hat{q}$ in this basis is given by

$$\hat{q} = \frac{1}{2}\begin{pmatrix} 0 & \mu_0 & 0 & & \\ \mu_0 & 0 & \mu_1 & 0 & \\ 0 & \mu_1 & 0 & \mu_2 & \ddots \\ & 0 & \mu_2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \mu_k = \sqrt{(k+1)(2j-k)}, \tag{3.3}$$

and the momentum operator $\hat{p}$ takes the form

$$\hat{p} = \frac{i}{2}\begin{pmatrix} 0 & -\mu_0 & 0 & & \\ \mu_0 & 0 & -\mu_1 & 0 & \\ 0 & \mu_1 & 0 & -\mu_2 & \ddots \\ & 0 & \mu_2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \mu_k = \sqrt{(k+1)(2j-k)}. \tag{3.4}$$

Clearly, these operators (and matrix representations) satisfy the Hamilton-Lie equations from (1.2) (but not the canonical commutation relation).

Let us now turn to a new finite oscillator model based on the Racah polynomials introduced in the previous section. For this purpose, observe that for any dimension $d + 1 = 2j + 1$, there is a close relationship between the matrix elements of M, given by (2.17), and those of the above matrix (3.3):

$$M_k = \sqrt{(k+1)(2j-k)\frac{(c+k-1)(c+k+2j)}{(c+2k-1)(c+2k+1)}}, \qquad \mu_k = \sqrt{(k+1)(2j-k)}. \tag{3.5}$$

Indeed, the positive parameter c appearing in M_k can be seen as a "deformation" of the element μ_k. And clearly, in the limit $c \longrightarrow +\infty$ one has that $M_k \longrightarrow \mu_k$. Following this, the elements of the new finite oscillator model – in any dimension $2j + 1$ (j integer or half-integer) – are defined as follows: the Hamiltonian $\hat{H}$ is the same operator as in (3.1) or (3.2); the operators $\hat{q}$ and $\hat{p}$ are

$$\hat{q} = \frac{1}{2}M = \frac{1}{2}\begin{pmatrix} 0 & M_0 & 0 & & \\ M_0 & 0 & M_1 & 0 & \\ 0 & M_1 & 0 & M_2 & \ddots \\ & 0 & M_2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \hat{p} = \frac{i}{2}\begin{pmatrix} 0 & -M_0 & 0 & & \\ M_0 & 0 & -M_1 & 0 & \\ 0 & M_1 & 0 & -M_2 & \ddots \\ & 0 & M_2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.6}$$

with M_k given by (2.17) or equivalently (3.5).

For this new model, the Hamiltonian-Lie equations are satisfied. So let us turn our attention to the properties of the position operator $\hat{q}$ (the properties of the momentum operator $\hat{p}$ are completely similar and will not be given explicitly). Following Proposition 3, the spectrum of $\hat{q} = \frac{1}{2}M$ is simply given by

$$-j, -j+1, -j+2, \ldots, j-2, j-1, j. \tag{3.7}$$

Quite surprisingly, this spectrum is independent of the parameter c appearing in the matrix elements (3.6) of $\hat{q}$; but of course this is a consequence of Proposition 3, and in particular of the special choice of γ and δ earlier on in Section 2. So the spectrum of $\hat{q}$

in the new model is just the same as in the familiar $\mathfrak{su}(2)$ model. For the eigenvectors of $\hat{q}$, however, things are different, as follows from the last part of Proposition 3. The orthonormal eigenvector of the position operator $\hat{q}$ for the eigenvalue q, denoted by $|q)$, is given in terms of the eigenstate basis of $\hat{H}$ by

$$|q)= \sum_{n=0}^{2j} U_{n,j+q}|n\rangle, \qquad q \in \{-j, -j+1\dots, j-1, j\}. \tag{3.8}$$

Herein, $U = (U_{kl})_{0\le k,l\le 2j}$ is the $(2j+1)\times(2j+1)$ matrix with elements defined in terms of normalized Racah polynomials (2.6) as in the previous section. Explicitly, for j a half-integer, the elements of U follow from Proposition 1, with $N = j - 1/2$ and $n, x \in \{0, \dots, N\}$:

$$\begin{aligned} U_{2n,N-x} &= U_{2n,N+x+1} = \frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); -N-1, N-1/2+c/2, -1/2, 1/2),\\ U_{2n+1,N-x} &= -U_{2n+1,N+x+1} = -\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); -N-1, N+1/2+c/2, 1/2, -1/2). \end{aligned} \tag{3.9}$$

For j an integer, they follow from Proposition 2, with $N = j$:

$$\begin{aligned} U_{2n,j-x} &= U_{2n,j+x} = \frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); -j-1, j-1/2+c/2, -1/2, -1/2),\\ &\qquad (n = 0, \dots, j;\ x = 1, \dots, j)\\ U_{2n+1,j-x-1} &= -U_{2n+1,j+x+1} = -\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x); -j, j-1/2+c/2, 1/2, 1/2),\\ &\qquad (n, x \in \{0, \dots, j-1\})\\ U_{2n,j} &= (-1)^n\tilde{R}_n(\lambda(0); -j-1, j-1/2+c/2, -1/2, -1/2), \qquad U_{2n+1,j} = 0. \end{aligned} \tag{3.10}$$

These expressions deserve further attention. Remember that, just as in (1.4) for the $\mathfrak{su}(2)$ model, quite generally the position (resp. momentum) wavefunctions are the overlaps between the normalized eigenstates of the position operator $\hat{q}$ (resp. the momentum operator $\hat{p}$) and the eigenstates of the Hamiltonian. Let us denote the position wavefunctions for the new oscillator model by $\Phi_n^{(c)}(q)$, in order to emphasize the dependence upon the positive parameter c. We can thus write:

$$\Phi_n^{(c)}(q)= \langle n|q)= U_{n,j+q}, \tag{3.11}$$

where $n = 0, 1, \dots, 2j$ and $q = -j, -j+1, \dots, j-1, j$. So $\Phi_0^{(c)}(q)$ is the "ground state", $\Phi_1^{(c)}(q)$ the first excited state, and so on. All these expressions are real, and since we are dealing with a finite oscillator model they satisfy a discrete orthogonality relation:

$$\sum_{q=-j}^{j} \Phi_n^{(c)}(q)\Phi_{n'}^{(c)}(q)= \delta_{n,n'}, \qquad \sum_{n=0}^{2j} \Phi_n^{(c)}(q)\Phi_n^{(c)}(q')= \delta_{q,q'}. \tag{3.12}$$

Let us examine the explicit form of these functions in more detail, for the case j half-integer (the case j integer is similar, and will not be treated explicitly). The expressions follow essentially from (3.9). The even wavefunctions are given by

$$\Phi_{2n}^{(c)}(q)= \frac{(-1)^n}{\sqrt{2}}\sqrt{W(n,q;c,j)}\ \ {}_4F_3\left(\begin{matrix}-q+1/2,\, q+1/2,\, -n,\, n+(c-1)/2\\ 1/2,\, j+(c+1)/2,\, -j+1/2\end{matrix};1\right), \tag{3.13}$$

where

$$W(n,q;c,j)= \frac{w(|q|-1/2; -j-1/2, j-1+c/2, -1/2, 1/2)}{h_n(-j-1/2, j-1+c/2, -1/2, 1/2)}$$

is written in terms of the weight function and square norm (2.5) of the Racah polynomials. Note that $\Phi_{2n}^{(c)}(q)$ is indeed an even function of the position q, and depends on q^2 only. The odd wavefunctions are given by

$$\begin{aligned}\Phi_{2n+1}^{(c)}(q) &= (-1)^n\sqrt{\frac{(4n+c+1)(2n+c-1)(2n+1)}{(4n+c-1)(2n+c+2j)(j-n)}}\\ &\times \sqrt{W(n,q;c,j)}\ \cdot q \cdot\ {}_4F_3\left(\begin{matrix}-q+1/2,\, q+1/2,\, -n,\, n+(c+1)/2\\ 3/2,\, j+(c+1)/2,\, -j+1/2\end{matrix};1\right).\end{aligned} \tag{3.14}$$

Clearly, because of the factor q, $\Phi_{2n+1}^{(c)}(q)$ is an odd function of q. The overall factor in (3.14) arises from

$$\begin{aligned}\frac{w(|q|-1/2; -j-1/2, j+c/2, 1/2, -1/2)}{h_n(-j-1/2, j+c/2, 1/2, -1/2)} &= \frac{2q^2(4n+c+1)(2n+c-1)(2n+1)}{(4n+c-1)(2n+c+2j)(j-n)}\\ \times \frac{w(|q|-1/2; -j-1/2, j-1+c/2, -1/2, 1/2)}{h_n(-j-1/2, j-1+c/2, -1/2, 1/2)}.\end{aligned}$$

It is interesting to study these discrete wavefunctions for varying values of c. We know already that in the limit $c \longrightarrow +\infty$ the position operator $\hat{q}$ tends to the position operator of the $\mathfrak{su}(2)$ model, so also the wavefunctions should have this behavior. When c tends to infinity, the wavefunctions $\Phi_n^{(c)}(q)$ are indeed Krawtchouk functions. Clearly, the ${}_4F_3$ series in (3.13) and (3.14) reduce to ${}_3F_2$ series, which in turn reduce to ${}_2F_1$ series according to

$${}_3F_2\left(\begin{matrix}-q+1/2,\, q+1/2,\, -n\\ 1/2,\, -j+1/2\end{matrix};1\right) = (-1)^n\frac{\binom{2j}{2n}}{\binom{j-1/2}{n}}\ {}_2F_1\left(\begin{matrix}-2n,\, -j-q\\ -2j\end{matrix};2\right), \tag{3.15}$$

$${}_3F_2\left(\begin{matrix}-q+1/2,\, q+1/2,\, -n\\ 3/2,\, -j+1/2\end{matrix};1\right) = -\frac{(-1)^n}{2q}\frac{\binom{2j}{2n+1}}{\binom{j-1/2}{n}}\ {}_2F_1\left(\begin{matrix}-2n-1,\, -j-q\\ -2j\end{matrix};2\right). \tag{3.16}$$

These reductions have been given in [14] and can be obtained, e.g., from [4, (48)]. The ${}_2F_1$ series in the right hand side correspond to symmetric Krawtchouk polynomials (i.e.

Krawtchouk polynomials with $p = 1/2$ [9]). When j tends to infinity, they yield the ordinary oscillator wavefunctions [4].

For other values of c, let us examine some plots of the discrete wavefunctions. In Figure 1, we give the plots of $\Phi_n^{(c)}(q)$ for $n = 0$, $n = 1$ and $n = 2$, and for some fixed j-value $j = 33/2$. The purpose is to observe the behavior of the wavefunctions as the positive parameter c varies. With this in mind, we have plotted these functions for the following c-values:

$$c = 10^{-6}, \quad c = 0.5, \quad c = 1.5, \quad c = 2, \quad c = 4, \quad c = 8, \quad c = 32.$$

For large values of c, the discrete wavefunctions take indeed the shape of those of the $\mathfrak{su}(2)$ model (which, in turn, tend to the canonical oscillator wavefunctions when j tends to infinity). The case $c = 0$ is ruled out, but we have examined a c-value close to 0, for which the behavior is somewhat 'degenerate'. To our surprise, the value $c = 2$ is a kind of transition value for the ground state. Just looking at the ground state ($n = 0$), one observes that for $c < 2$ the shape is like a cup, whereas for $c > 2$ it is like a cap. In order to explain this transition value, recall from (3.13) that

$$\Phi_0^{(c)}(q) = \frac{1}{\sqrt{2}}\sqrt{W(0, q; c, j)} = \frac{1}{\sqrt{2}}\left(\frac{w(|q| - 1/2; -j - 1/2, j - 1 + c/2, -1/2, 1/2)}{h_0(-j - 1/2, j - 1 + c/2, -1/2, 1/2)}\right)^{1/2}. \tag{3.17}$$

Using (2.5),

$$w(|q| - 1/2; -j - 1/2, j - 1 + c/2, -1/2, 1/2) = \frac{(-j + 1/2)_{|q|-1/2}(j + c/2 + 1/2)_{|q|-1/2}}{(j + 3/2)_{|q|-1/2}(-j - c/2 + 3/2)_{|q|-1/2}}, \tag{3.18}$$

and thus for $c = 2$ one finds $w(|q| - 1/2; -j - 1/2, j - 1 + c/2, -1/2, 1/2) = 1$. In other words, for this special transition value $c = 2$, the ground state wavefunction $\Phi_0^{(c)}(q)$ is a constant function.

To conclude, in the field of finite quantum oscillators the original $\mathfrak{su}(2)$ model remains an interesting model because of two reasons: the simple equidistant spectrum of the position (and momentum) operator, and the behavior of the position wavefunctions (which really look like discrete versions of the canonical oscillator wavefunctions, and tend to them when j is sufficiently large). The new model introduced in this paper deforms the $\mathfrak{su}(2)$ model by a parameter $c > 0$. The spectrum of the position (and momentum) operator is the same and thus remains simple and equidistant. The wavefunctions are deformed by the parameter c, and tend to those of the $\mathfrak{su}(2)$ model when c goes to infinity. The wavefunctions themselves are written in terms of Racah polynomials, and originate from a Racah double [12]. The shape of the wavefunctions could open applications beyond those of the $\mathfrak{su}(2)$ model.

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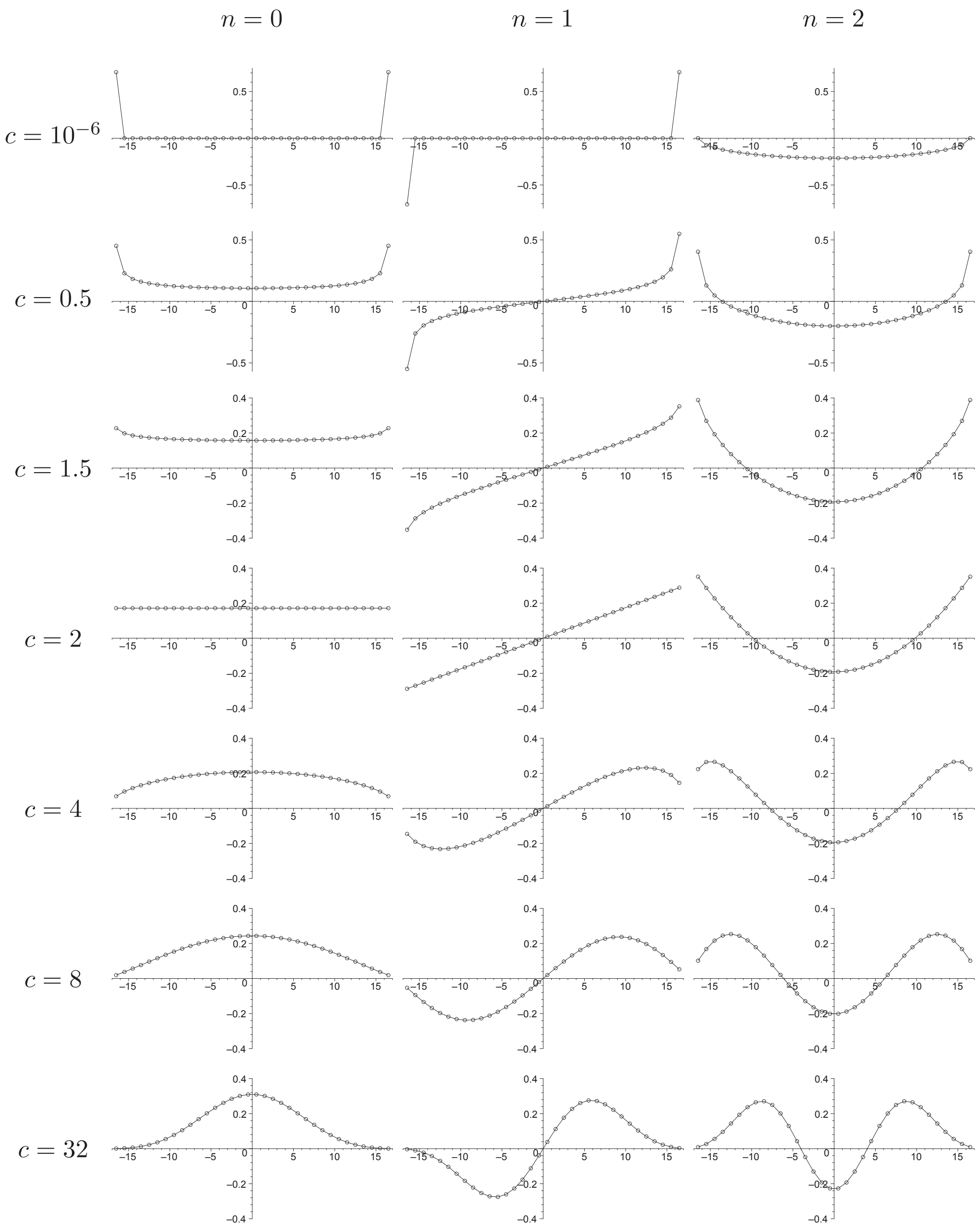

FIGURE 1: Plots of the discrete wavefunctions $\Phi_n^{(c)}(q)$ in the representation with $j = 33/2$, for $n = 0$ (left column), for $n = 1$ (middle column) and for $n = 2$ (right column). The wavefunctions are plotted for the following values of c (from top to bottom): 10^{-6}, 0.5, 1.5, 2, 4, 8, 32.

7

Tridiagonal test matrices for eigenvalue computations: two-parameter extensions of the Clement matrix

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abstract

The Clement or Sylvester-Kac matrix is a tridiagonal matrix with zero diagonal and simple integer entries. Its spectrum is known explicitly and consists of integers which makes it a useful test matrix for numerical eigenvalue computations. We consider a new class of appealing two-parameter extensions of this matrix which have the same simple structure and whose eigenvalues are also given explicitly by a simple closed form expression. The aim of this paper is to present in an accessible form these new matrices and examine some numerical results regarding the use of these extensions as test matrices for numerical eigenvalue computations.

1 introduction

For a positive integer n, consider the $(n+1)\times(n+1)$ matrix C_n whose non-zero entries are given by

$$c_{k,k+1} = c_{n+2-k,n+1-k} = k \qquad \text{for } k \in \{1, \ldots, n\}, \tag{1.1}$$

or explicitly, the matrix

$$C_n = \begin{pmatrix} 0 & 1 & & & & & \\ n & 0 & 2 & & & & \\ & n-1 & 0 & 3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3 & 0 & n-1 & \\ & & & & 2 & 0 & n \\ & & & & & 1 & 0 \end{pmatrix}. \tag{1.2}$$

This matrix appears in the literature under several names: the Sylvester-Kac matrix, the Kac matrix, the Clement matrix, etc. It was already considered by Sylvester [19], used by M. Kac in some of his seminal work [13], proposed by Clement as a test matrix for eigenvalue computations [5], and it continues to attract attention [2, 3, 7, 20].

The matrix C_n has a simple structure, it is tridiagonal with zero diagonal and has integer entries. The main property of C_n is that its spectrum is known explicitly and is remarkably simple; the eigenvalues of C_n are the integers

$$-n, -n+2, -n+4, \ldots, n-2, n. \tag{1.3}$$

The $n+1$ distinct eigenvalues are symmetric around zero, equidistant and range from $-n$ to n. Hence for even n, they are $n+1$ consecutive even integers, while for odd n they are $n+1$ consecutive odd integers.

REMARK 1: The eigenvectors of the matrix C_n are also known, they can be expressed in terms of the Krawtchouk orthogonal polynomials [16].

As the eigenvalues of (1.2) are known explicitly and because of the elegant and simple structure of both matrix and spectrum, C_n is a standard test matrix for numerical eigenvalue computations (see e.g. example 7.10 in [9]), and part of some standard test matrix toolboxes (e.g. [10]). In MATLAB, C_n can be produced from the gallery of test matrices using `gallery('clement',n+1)`. The Clement matrix also appears in several applications, e.g. as the transition matrix of a Markov chain [1], or in biogeography [11].

General tridiagonal matrices with closed form eigenvalues are rare, most examples being just variations of the tridiagonal matrix with fixed constants a, b and c on the subdiagonal, diagonal and superdiagonal respectively [6, 21]. In this paper we present appealing two-parameter extensions of C_n with closed form eigenvalues. These extensions first appeared in a different form in the paper [18] as a special case of a class of matrices related to orthogonal polynomials. Special cases of these matrices were originally encountered in the context of finite quantum oscillator models (e.g. [12]) and their classification led to the construction of new interesting models [17]. Here, we feature them in a simpler, more accessible form which immediately illustrates their relation with C_n. Moreover, we consider some specific parameter values which yield interesting special cases.

Another purpose of this paper is to demonstrate by means of some numerical experiments the use of these extensions of C_n as test matrices for numerical eigenvalue computations. Hereto, we examine how accurately the inherent MATLAB function `eig()` is able to compute the eigenvalues of our test matrices compared to the exact known eigenvalues. An interesting feature of the new class of test matrices is that they include matrices with double eigenvalues for specific parameter values.

In section 2 we present in an accessible form the two-parameter extensions of the Clement matrix. We state the explicit and rather simple form of their eigenvalues which makes them potentially interesting examples of eigenvalue test matrices. In section 3 we consider some specific parameter values for the new classes of test matrices which yield interesting special cases. In section 4 we display some numerical results regarding the use of these extensions as test matrices for numerical eigenvalue computations. This is done by looking at the relative error when the exact known eigenvalues are compared with those computed using the inherent MATLAB function `eig()`.MATLAB

2 NEW TEST MATRICES

Now, we consider the following extension of the matrix (1.2), by generalizing its entries (1.1) to:

$$h_{k,k+1} = \begin{cases} k & \text{if } k \text{ even} \\ k+a & \text{if } k \text{ odd} \end{cases} \quad \text{and} \quad h_{n+2-k,n+1-k} = \begin{cases} k & \text{if } k \text{ even} \\ k+b & \text{if } k \text{ odd} \end{cases} \tag{2.1}$$

where we introduce two parameters a and b (having a priori no restrictions). We will denote this extension by $H_n(a, b)$. For n even, the matrix $H_n(a, b)$ is given by (2.2). Note in particular that the first entry of the second row is $h_{2,1} = n$ and contains no parameter, and the same for $h_{n,n+1} = n$. For odd n, the matrix (2.4) has entries $h_{2,1} = n + b$ and $h_{n,n+1} = n + a$ which now do contain parameters, contrary to the even case.

The reason for considering this extension is that, similar to (1.3) for C_n, we also have an explicit expression for the spectrum of $H_n(a, b)$, namely:

THEOREM 2: *For n even, say $n = 2m$, the $2m + 1$ eigenvalues of*

$$H_n(a,b) = \begin{pmatrix} 0 & 1+a & & & & & \\ n & 0 & 2 & & & & \\ & n-1+b & 0 & 3+a & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3+b & 0 & n-1+a & \\ & & & & 2 & 0 & n \\ & & & & & 1+b & 0 \end{pmatrix} \tag{2.2}$$

are given by

$$0, \pm\sqrt{(2k)(2k+a+b)} \qquad \textit{for } k \in \{1, \ldots, m\}. \tag{2.3}$$

THEOREM 3: *For n odd, say $n = 2m + 1$, the $2m + 2$ eigenvalues of*

$$H_n(a,b) = \begin{pmatrix} 0 & 1+a & & & & & \\ n+b & 0 & 2 & & & & \\ & n-1 & 0 & 3+a & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3+b & 0 & n-1 & \\ & & & & 2 & 0 & n+a \\ & & & & & 1+b & 0 \end{pmatrix} \tag{2.4}$$

are given by

$$\pm\sqrt{(2k+1+a)(2k+1+b)} \qquad \textit{for } k \in \{0, \ldots, m\} \tag{2.5}$$

We will prove the results for the symmetrized form of these matrices. We briefly elaborate on this. Consider a $(n+1)\times(n+1)$ tridiagonal matrix with zero diagonal

$$A = \begin{pmatrix} 0 & b_1 & 0 & & \\ c_1 & 0 & b_2 & \ddots & \\ 0 & c_2 & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & b_n \\ & & 0 & c_n & 0 \end{pmatrix}. \tag{2.6}$$

It is clear that the characteristic polynomial of A depends on the products $b_i c_i$ ($i = 1, \ldots, n$) only, and not on b_i and c_i separately. Therefore, if all the products $b_i c_i$ are positive, the eigenvalues of A or of its symmetrized form

$$A' = \begin{pmatrix} 0 & \sqrt{b_1 c_1} & 0 & & \\ \sqrt{b_1 c_1} & 0 & \sqrt{b_2 c_2} & \ddots & \\ 0 & \sqrt{b_2 c_2} & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & \sqrt{b_n c_n} \\ & & 0 & \sqrt{b_n c_n} & 0 \end{pmatrix} \tag{2.7}$$

are the same. The eigenvectors of A' are those of A after multiplication by a diagonal matrix (the diagonal matrix that is used in the similarity transformation from A to A').

Using this procedure, the aforementioned matrices can be made symmetric. For C_n the entries (1.1) can be symmetrized to

$$\tilde{c}_{k,k+1} = \tilde{c}_{k+1,k} = \sqrt{k(n+1-k)} \qquad \text{for } k \in \{1, \ldots, n\}.$$

This matrix is also implemented in MATLAB, namely as `gallery('clement',n+1,1)`. For the extension $H_n(a, b)$, the entries of its symmetric form $\tilde{H}_n(a, b)$ are

$$\tilde{h}^e_{k,k+1} = \tilde{h}^e_{k+1,k} = \begin{cases} \sqrt{k(n+1-k+b)} & \text{if } k \text{ even} \\ \sqrt{(k+a)(n+1-k)} & \text{if } k \text{ odd} \end{cases}$$

for n even and $k \in \{1, \ldots, n\}$, while for n odd we have

$$\tilde{h}^o_{k,k+1} = \tilde{h}^o_{k+1,k} = \begin{cases} \sqrt{k(n+1-k)} & \text{if } k \text{ even} \\ \sqrt{(k+a)(n+1-k+b)} & \text{if } k \text{ odd} \end{cases}$$

for $k \in \{1, \ldots, n\}$.

The above theorems are now proved using a property of the dual Hahn polynomials, which are defined in terms of the generalized hypergeometric series as follows [14]

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\begin{matrix}-x,\, x+\gamma+\delta+1,\, -n\\ \gamma+1,\, -N\end{matrix};1\right). \tag{2.8}$$

The dual Hahn polynomials satisfy a discrete orthogonality relation, see [18, (2.7)], and we denote the related orthonormal functions as $\tilde{R}_n(\lambda(x);\gamma,\delta,N)$.

LEMMA 2.4: *The orthonormal dual Hahn functions satisfy the following pairs of recurrence relations:*

$$\begin{aligned}&\sqrt{(n+1+\gamma)(N-n)}\tilde{R}_n(\lambda(x);\gamma,\delta,N)-\sqrt{(n+1)(N-n+\delta)}\tilde{R}_{n+1}(\lambda(x);\gamma,\delta,N)\\ &= \sqrt{x(x+\gamma+\delta+1)}\tilde{R}_n(\lambda(x-1);\gamma+1,\delta+1,N-1), &(2.9)\\ &-\sqrt{(n+1)(N-n+\delta)}\tilde{R}_n(\lambda(x-1);\gamma+1,\delta+1,N-1)+\sqrt{(n+2+\gamma)(N-n-1)}\\ &\times\tilde{R}_{n+1}(\lambda(x-1);\gamma+1,\delta+1,N-1)= \sqrt{x(x+\gamma+\delta+1)}\tilde{R}_{n+1}(\lambda(x);\gamma,\delta,N)\\ && (2.10)\end{aligned}$$

and

$$\begin{aligned}&\sqrt{(n+1+\gamma)(N-n+\delta)}\tilde{R}_n(\lambda(x);\gamma,\delta,N)-\sqrt{(n+1)(N-n)}\tilde{R}_{n+1}(\lambda(x);\gamma,\delta,N)\\ &= \sqrt{(x+\gamma+1)(x+\delta)}\tilde{R}_n(\lambda(x);\gamma+1,\delta-1,N), &(2.11)\\ &-\sqrt{(n+1)(N-n)}\tilde{R}_n(\lambda(x);\gamma+1,\delta-1,N)+\sqrt{(n+2+\gamma)(N-n+\delta-1)}\\ &\times\tilde{R}_{n+1}(\lambda(x);\gamma+1,\delta-1,N)= \sqrt{(x+\gamma+1)(x+\delta)}\tilde{R}_{n+1}(\lambda(x);\gamma,\delta,N). &(2.12)\end{aligned}$$

Proof. The first two relations follow from the case dual Hahn I of [18, Theorem 1] multiplied by the square root of the weight function and norm squared, and similarly the last two from the case dual Hahn III. □

Proof of Theorem 2. Let n be an even integer, say $n = 2m$, and let a and b be real numbers greater than -1. Take $k \in \{1,\ldots,m\}$ and let $U_{\pm k} = (u_1,\ldots,u_{n+1})^T$ be the column vector with entries

$$u_l = \begin{cases}(-1)^{(l-1)/2}\tilde{R}_{(l-1)/2}(\lambda(k);\frac{a-1}{2},\frac{b-1}{2},m) & \text{if } l \text{ odd}\\ \pm(-1)^{l/2-1}\tilde{R}_{l/2-1}(\lambda(k-1);\frac{a+1}{2},\frac{b+1}{2},m-1) & \text{if } l \text{ even.}\end{cases}$$

We calculate the entries of the vector $\tilde{H}_n(a,b)U_{\pm k}$ to be

$$(\tilde{H}_n(a,b)U_{\pm k})_l = \tilde{h}^e_{l,l-1}u_{l-1} + \tilde{h}^e_{l,l+1}u_{l+1}.$$

For l even, using the recurrence relation (2.9) with the appropriate parameter values substituted in the orthonormal dual Hahn functions, this becomes

$$(\tilde{H}_n(a,b)U_{\pm k})_l = \sqrt{(l-1+a)(2m+2-l)}(-1)^{l/2-1}\tilde{R}_{l/2-1}(\lambda(k);\tfrac{a-1}{2},\tfrac{b-1}{2},m)$$

$$
\begin{aligned}
&+ \sqrt{l(2m+1-l+b)}(-1)^{l/2}\tilde{R}_{l/2}(\lambda(k); \tfrac{a-1}{2}, \tfrac{b-1}{2}, m) \\
&= 2\sqrt{k(k+\tfrac{a}{2}+\tfrac{b}{2})}(-1)^{l/2-1}\tilde{R}_{l/2-1}(\lambda(k-1); \tfrac{a+1}{2}, \tfrac{b+1}{2}, m-1) \\
&= \pm\sqrt{(2k)(2k+a+b)}\, u_l.
\end{aligned}
$$

Similarly, for l odd we have, using now the recurrence relation (2.10),

$$
\begin{aligned}
&(\tilde{H}_n(a,b)U_{\pm k})_l \\
&= \pm\sqrt{(l-1)(2m+2-l+b)}(-1)^{(l-3)/2}\tilde{R}_{(l-3)/2}(\lambda(k-1); \tfrac{a+1}{2}, \tfrac{b+1}{2}, m-1) \\
&\quad \pm\sqrt{(l+a)(2m+1-l)}(-1)^{(l-1)/2}\tilde{R}_{(l-1)/2}(\lambda(k-1); \tfrac{a+1}{2}, \tfrac{b+1}{2}, m-1) \\
&= \pm 2\sqrt{k(k+\tfrac{a}{2}+\tfrac{b}{2})}(-1)^{(l-1)/2}\tilde{R}_{(l-1)/2}(\lambda(k); \tfrac{a-1}{2}, \tfrac{b-1}{2}, m) \\
&= \pm\sqrt{(2k)(2k+a+b)}\, u_l.
\end{aligned}
$$

Finally, define $U_0 = (u_1, \ldots, u_{n+1})^T$ as the column vector with entries

$$
u_l = \begin{cases} (-1)^{(l-1)/2}\tilde{R}_{(l-1)/2}(\lambda(0); \tfrac{a-1}{2}, \tfrac{b-1}{2}, m) & \text{if } l \text{ odd} \\ 0 & \text{if } l \text{ even.} \end{cases}
$$

Putting $x = 0$ on the right-hand side of (2.9), it is clear that the entries of the vector $\tilde{H}_n(a,b)U_0$ are all zero.

This shows that the eigenvalues of $\tilde{H}_n(a,b)$ are given by (2.3), so its characteristic polynomial must be

$$
\lambda \prod_{k=1}^{m} \left(\lambda^2 - (2k)(2k+a+b)\right),
$$

which allows us to extend the result to arbitrary parameters a and b. □

Theorem 3 is proved in the same way, using now relations (2.11) and (2.12).

3 SPECIAL CASES

We now consider some particular cases where the eigenvalues as given in Theorem 2 and Theorem 3 reduce to integers or have a special form.

For the specific values $a = 0$ and $b = 0$, it is clear that $H_n(0,0)$ reduces to C_n for both even and odd values of n. As expected, the explicit formulas for the eigenvalues (2.3) and (2.5) also reduce to (1.3).

Next, we look at n even. In (2.3), we see that the square roots cancel if we take $b = -a$. For this choice of parameters, the eigenvalues of $H_n(a,-a)$ are even integers, which are precisely the same eigenvalues as those of the Clement matrix (1.3). However, the matrix

$$H_n(a,-a)=\begin{pmatrix} 0 & 1+a & & & & & \\ n & 0 & 2 & & & & \\ & n-1-a & 0 & 3+a & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3-a & 0 & n-1+a & \\ & & & & 2 & 0 & n \\ & & & & & 1-a & 0 \end{pmatrix} \tag{3.1}$$

still contains a parameter a which does not affect its eigenvalues. So for every even integer n, (3.1) gives rise to a one-parameter family of tridiagonal matrices with zero diagonal whose eigenvalues are given by (1.3). This property is used explicitly in [17] to construct a finite oscillator model with equidistant position spectrum.

For n odd, say $n = 2m + 1$, the square roots in (2.5) cancel if we take $b = a$. Substituting a for b, the matrix (2.4) becomes

$$H_n(a,a)=\begin{pmatrix} 0 & 1+a & & & & & \\ n+a & 0 & 2 & & & & \\ & n-1 & 0 & 3+a & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3+a & 0 & n-1 & \\ & & & & 2 & 0 & n+a \\ & & & & & 1+a & 0 \end{pmatrix}, \tag{3.2}$$

while for the eigenvalues we get

$$\pm\,|2k+1+a| \qquad \text{for } k \in \{0,\ldots,m\}. \tag{3.3}$$

These are integers for integer a and real numbers for real a. We see that for $a = 0$, (3.3) reduces to the eigenvalues (1.3), but then the matrix is precisely C_n. Non-zero values for a induce a shift in the eigenvalues, away from zero for positive a and towards zero for $-1 < a < 0$. However, for $-n < a < -1$ the positive and negative (when $a > -1$) eigenvalues get mingled. Moreover, for a equal to a negative integer ranging from -1 to $-n$, we see that there are double eigenvalues. A maximum number of double eigenvalues occurs for $a = -m - 1$, then each of the values

$$2k-m \qquad \text{for } k \in \{0,\ldots,m\},$$

is a double eigenvalue. By choosing a nearly equal to a negative integer, we can produce a matrix with nearly, but not exactly, equal eigenvalues. For $a < -n$, all positive eigenvalues (when $a > -1$) become negative and vice versa. Finally, for the special value $a = -n - 1$, the eigenvalues (3.3) also reduce to (1.3) while the matrix becomes

$$H_n(-n-1,-n-1)=\begin{pmatrix} 0 & -n & & & & & & & \\ -1 & 0 & 2 & & & & & & \\ & n-1 & 0 & 2-n & & & & & \\ & & -3 & 0 & 4 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 4 & 0 & -3 & & \\ & & & & & 2-n & 0 & n-1 & \\ & & & & & & 2 & 0 & -1 \\ & & & & & & & -n & 0 \end{pmatrix}.$$

This is up to a similarity transformation, as explained at the end of section 2, the matrix C_n.

Also for n odd, another peculiar case occurs when $b = a = 1$. Scaling by one half we then have the matrix

$$\frac{1}{2}H_{2m+1}(1,1)=\begin{pmatrix} 0 & 1 & & & & & & & \\ m+1 & 0 & 1 & & & & & & \\ & m & 0 & 2 & & & & & \\ & & m & 0 & 2 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 2 & 0 & m & & \\ & & & & & 2 & 0 & m & \\ & & & & & & 1 & 0 & m+1 \\ & & & & & & & 1 & 0 \end{pmatrix}. \tag{3.4}$$

with eigenvalues, by (2.5),

$$\pm(k+1) \qquad \text{for } k \in \{0,\ldots,m\}.$$

The even equivalent of this matrix,

$$\frac{1}{2}H_{2m}(1,1)=\begin{pmatrix} 0 & 1 & & & & & & & \\ m & 0 & 1 & & & & & & \\ & m & 0 & 2 & & & & & \\ & & m-1 & 0 & 2 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 2 & 0 & m-1 & & \\ & & & & & 2 & 0 & m & \\ & & & & & & 1 & 0 & m \\ & & & & & & & 1 & 0 \end{pmatrix}, \tag{3.5}$$

does not have integer spectrum, but instead, using the expression (2.3), has as eigenvalues

$$0, \pm\sqrt{k(k+1)} \qquad \text{for } k \in \{1, \ldots, m\}.$$

We have reviewed the special cases where the explicit formulas for the eigenvalues (2.3) and (2.5) which generally contain square roots, reduce to integers. In the following, we will use the notation

$$H_n(a)=\begin{cases} H_n(a,-a) & \text{if } n \text{ even} \\ H_n(a,a) & \text{if } n \text{ odd} \end{cases} \tag{3.6}$$

to denote these special cases.

4 NUMERICAL RESULTS

We now examine some numerical results regarding the use of the extensions of the previous sections as test matrices for numerical eigenvalue computations. This is done by comparing the exact known eigenvalues of $H_n(a, b)$ with those computed using the inherent MATLAB function `eig()`. These numerical experiments are included merely to illustrate the potential use of the matrices $H_n(a, b)$ as eigenvalue test matrices, and to examine the sensitivity of the computed eigenvalues on the new parameters.

A measure for the accuracy of the computed eigenvalues is the relative error

$$\frac{\|x - x^*\|_\infty}{\|x\|_\infty},$$

where x is the vector of eigenvalues as ordered list (by real part) and x^* its approximation.

Recall that both for n odd and n even, for the special case $H_n(a)$ the square roots in the expressions for the eigenvalues cancel, yielding real eigenvalues for every real value of the parameter a. In the general case, the eigenvalues (2.3) are real when $a+b > -2$ and those in (2.5) are real when $a > -1$ and $b > -1$ or $a < -n$ and $b < -n$. A first remark is that when we compute the spectrum of C_n using `eig()` in MATLAB, eigenvalues with imaginary parts are found when n exceeds 116, but not for lower values of n. Therefore, for the extensions, we have chosen $n = 100$ and $n = 101$ (for the even and the odd case respectively) for most of our tests, as this gives reasonably large matrices but is below the bound of 116. We will see that in this case for the extensions, the `eig()` function in MATLAB does find eigenvalues with imaginary part for certain parameter values.

We first consider the special case (3.6). For n even, $H_n(a)$ has the eigenvalues (1.3), which are integers independent of the parameter a. In figure 1, we have depicted the largest imaginary part of the computed eigenvalues for the matrix $H_n(a)$ for $n = 10$ and $n = 100$ at different values for the parameter a. We see that outside a central region imaginary parts are found. For example, for $H_{100}(a)$, MATLAB finds eigenvalues with imaginary parts when $a > 21$ or $a < -2.5$. Moreover, the relative error for the computed eigenvalues, shown in figures 2, increases as a approaches the region where eigenvalues with imaginary parts are found. In this latter region, the size of the relative error is of course due to the presence of imaginary parts which do not occur in the theoretical exact expression for the eigenvalues. As a reference, the relative error for C_{100} is 3.6612×10^{-5}, while that for $H_{100}(20)$ is 1.1471×10^{-3} and 4.9444×10^{-3} for $H_{100}(20.97)$.

For n odd, $H_n(a)$ has the eigenvalues (3.3), which dependent on the parameter a but are real for every real number a. Nevertheless, even for a small dimension such as $n = 11$, eigenvalues with (small) imaginary parts are found when a equals -2, -4, -6 or -8. This produces a relative error of order 10^{-8}, while for other values of a (and for the Clement matrix) the relative error is of order 10^{-15}, near machine precision. For $H_{101}(a)$, the largest imaginary part of the computed eigenvalues is portrayed in figure 3, together with the relative error. MATLAB finds eigenvalues with imaginary parts when $-100 \le a < -1.5$. These findings correspond to the region where double eigenvalues occur as mentioned in the previous section. The relative error is largest around this region and is several orders smaller when moving away from this region. As a reference, the relative error for C_{101} is 3.6881×10^{-5}, while that for $H_{101}(-1.75)$ is 1.4840×10^{-3}. Finally, we note that eigenvalues with imaginary parts also appear when a is extremely large, i.e. $a > 10^{10}$ or $a < -10^{10}$.

Next, we consider the general setting where we have two parameters a and b, starting with the case where n is even. Although the two parameters a and b occur symmetric in (2.3) and in the matrix (2.2) itself, there are some disparities in the numerical results. From the expression for the eigenvalues (2.3) we see that they are real when the parameters satisfy $a + b > -2$. However,

- when a is a negative number less than -2, MATLAB finds eigenvalues with imaginary parts for almost all values of b.
- for negative values of b, no imaginary parts are found as long as $a + b > -2$.
- For positive values of a, eigenvalues with imaginary parts are found when b gets sufficiently large, as illustrated in figure 4.

- For positive values of b the opposite holds: eigenvalues with imaginary parts are found when a is comparatively small, see figure 5.

In the case where n is odd, similar results hold for positive parameter values for a and b, as shown in figure 6 for example. For negative parameter values we have a different situation, as the eigenvalues (2.5) can become imaginary if the two factors have opposite sign. When $a < -n$ and $b < -n$, the eigenvalues (2.5) are real again and the behavior mimics that of the positive values of a and b. The picture we get is a mirror image of figure 6.

The reason for this disparity between the seemingly symmetric parameters a and b is that the QR algorithm wants to get rid of subdiagonal entries in the process of creating an upper triangular matrix. As a consequence, the numerical computations are much more sensitive to large values of b as it resides on the subdiagonal. This is showcased in figures 4, 5 and 6. Most important is the sensitivity on the extra parameters (a or a and b) which makes them appealing as test matrices.

It would be interesting future work if these new eigenvalue test matrices were to be used to test also numerical algorithms for computing eigenvalues designed specifically for matrices having multiple eigenvalues [8], being tridiagonal [15], or symmetric and tridiagonal [4].

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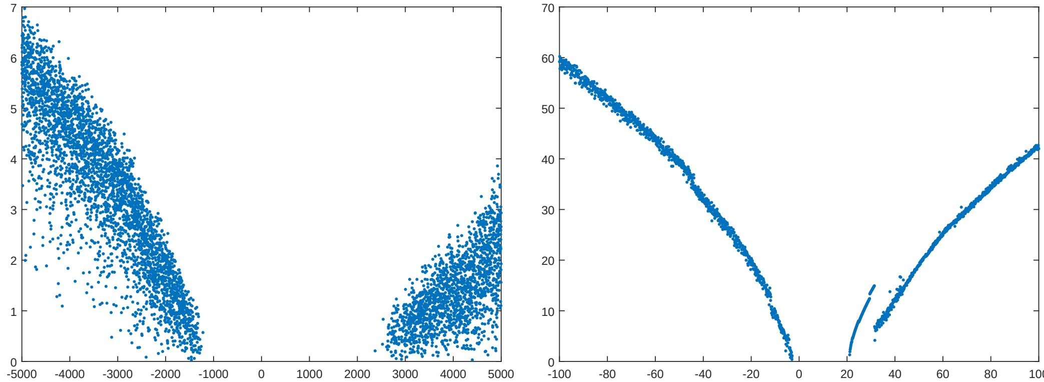

FIGURE 1: Plots of the largest imaginary part of the computed eigenvalues of $H_n(a)$ (n even and $a = -b$) for different values of a on horizontal axis. Left for $n = 10$ and right $n = 100$.

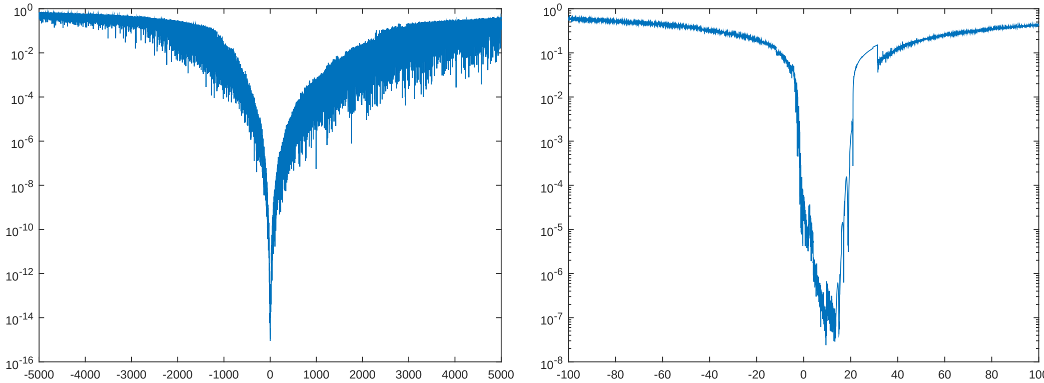

FIGURE 2: Plots of the relative error in the computed eigenvalues of $H_n(a)$ (n even and $a = -b$) for different values of a on horizontal axis. Left for $n = 10$ and right $n = 100$.

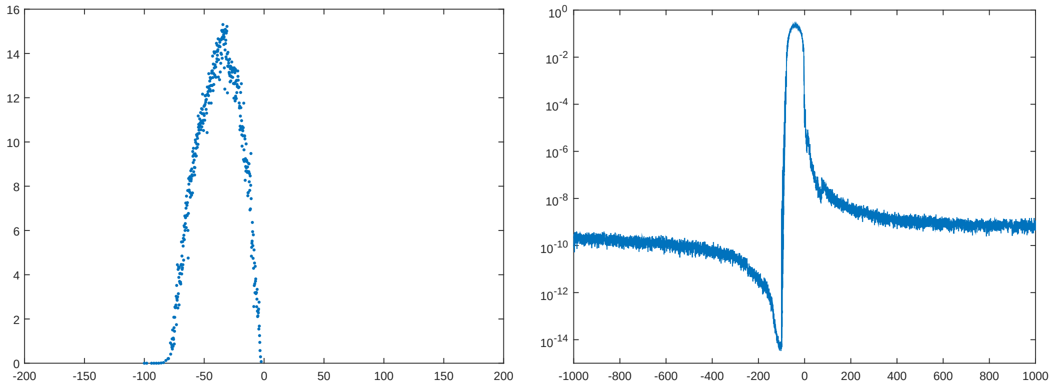

FIGURE 3: For different values of a as denoted on the horizontal axis, left a plot of the largest imaginary part and right a plot of the relative error of the computed eigenvalues of $H_{101}(a)$.

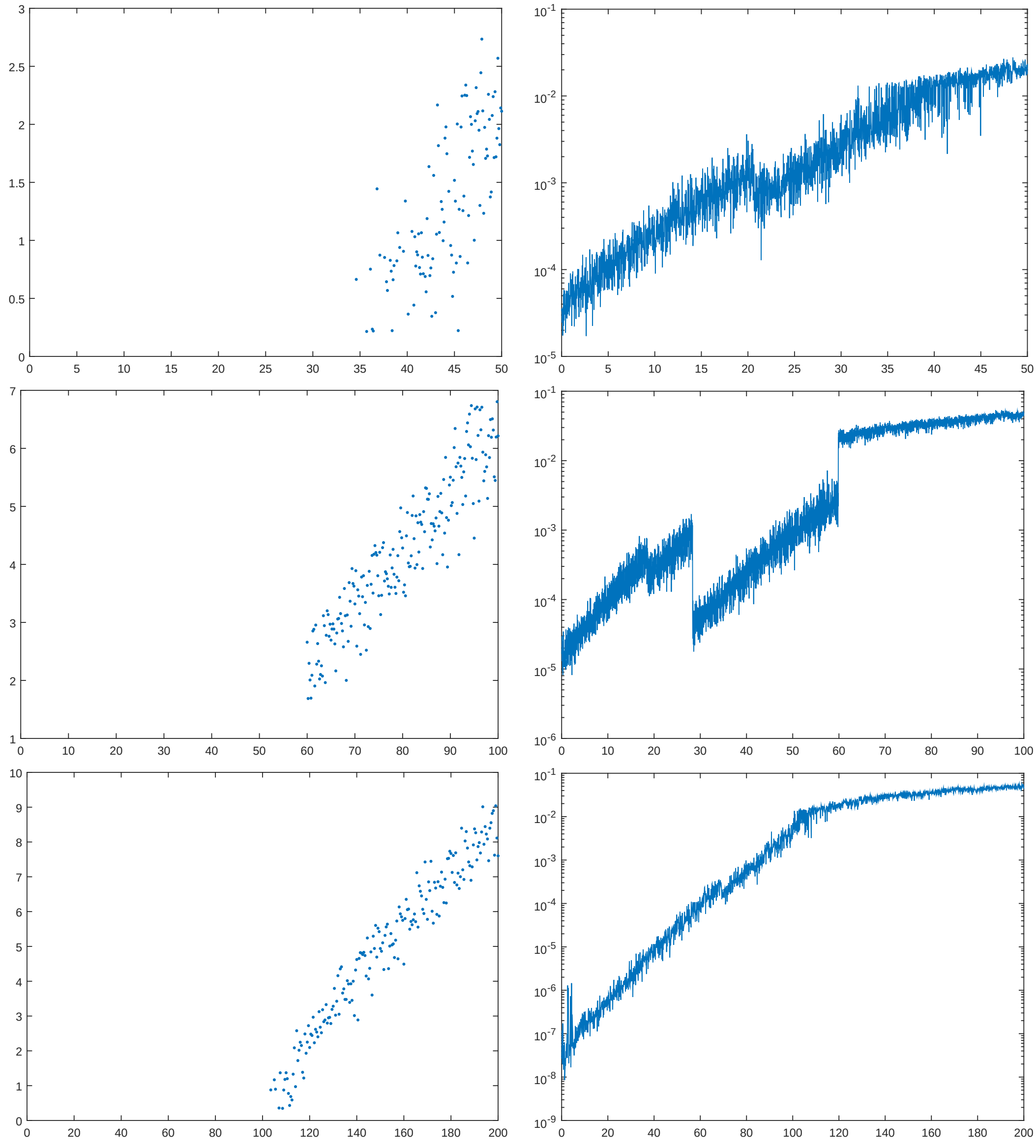

FIGURE 4: Plots of the largest imaginary part (left) and the relative error (right) in the computed eigenvalues of $H_{100}(a, b)$, horizontal axis varying values of b. Top row $a = 0$, middle row $a = 1$, bottom row $a = 20$.

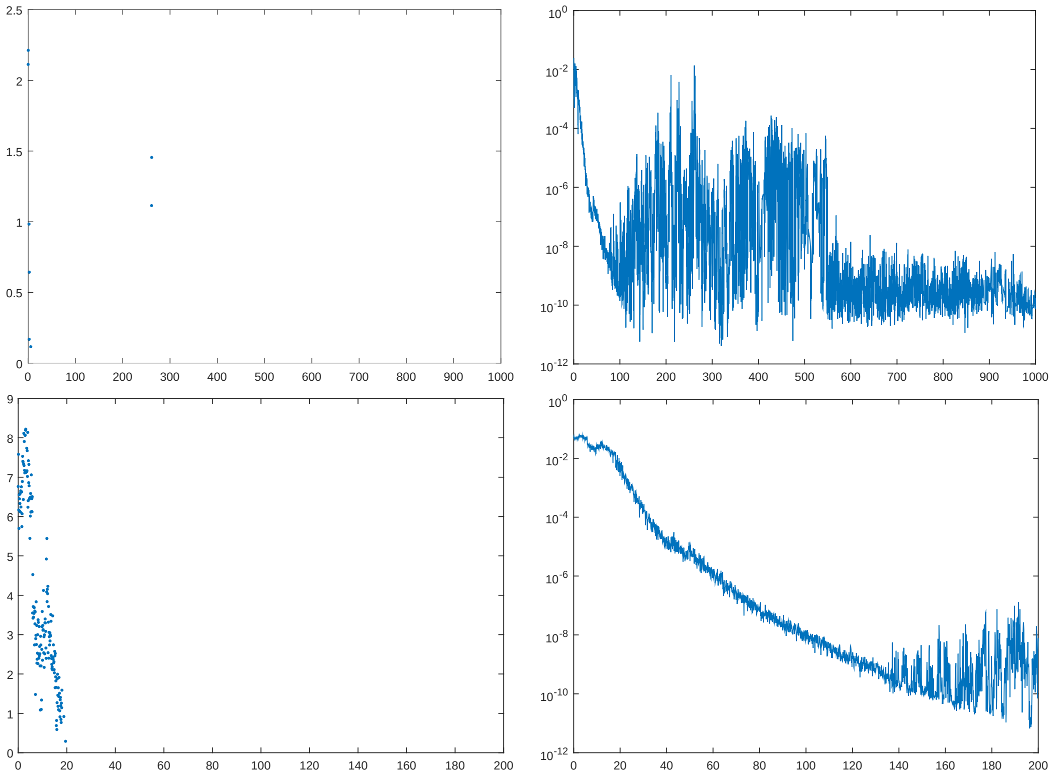

FIGURE 5: Plots of the largest imaginary part (left) and the relative error (right) in the computed eigenvalues of $H_{100}(a, b)$, horizontal axis varying values of a. Top row $b = 50$, bottom row $b = 100$.

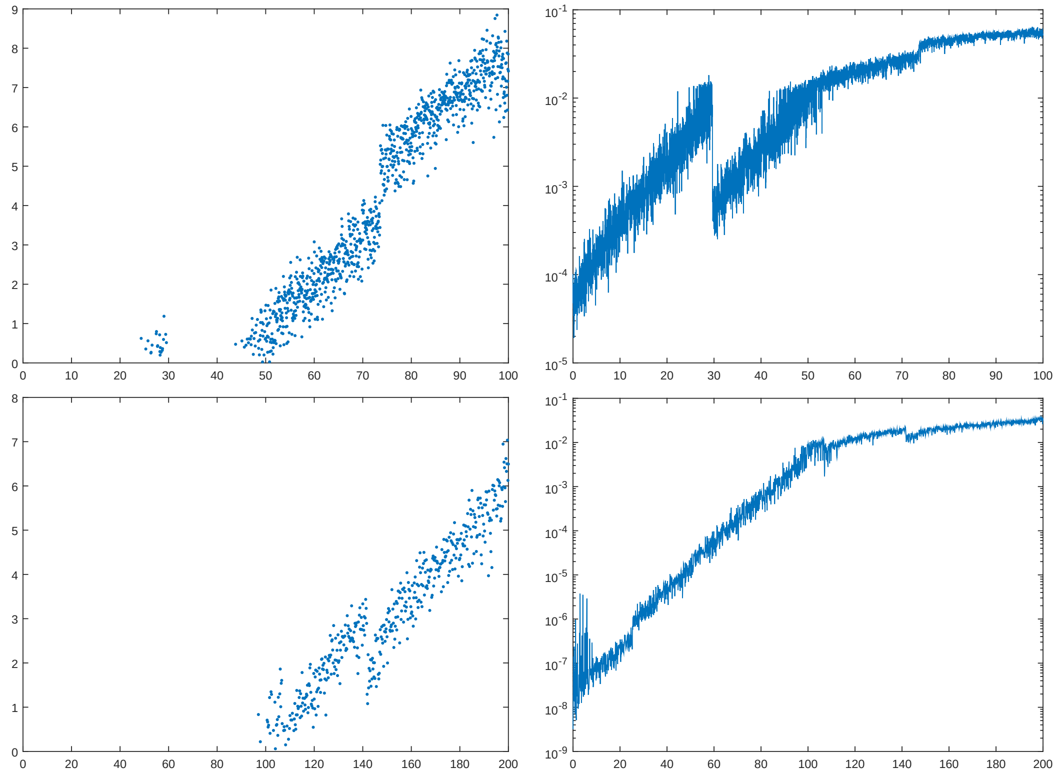

FIGURE 6: Plots of the largest imaginary part (left) and the relative error (right) in the computed eigenvalues of $H_{101}(a, b)$, horizontal axis varying values of b. Top row $a = 0$, bottom row $a = 25$.

Part III

EPILOGUE

The end is in( )sight.

CONCLUSION AND OUTLOOK

In Chapter 1, we obtained a class of Fourier-like transforms on the space of rapidly decreasing functions which intertwine the Laplace operator and the squared norm on $\mathbb{R}^n$. These transforms are described in two ways. First, as exponentials of elements in the enveloping algebra of the related operator realization of $\mathfrak{sl}(2)$. This first description makes use of the Casimir element of $\mathfrak{sl}(2)$ and integer-valued polynomials on squares of integers or of half-integers, depending on the parity of the dimension. Second, for each operator an equivalent formulation is given as an integral transform with a plane wave decomposition of the integral kernel in terms of Bessel functions and Gegenbauer polynomials. For a finite subset of transforms which closely resemble the classical Fourier transform, the integral kernels could be reduced to closed formulas which are polynomially bounded. Furthermore, we established uncertainty principles for these Fourier-like transforms.

In the setting of functions taking values in a Clifford algebra, the same approach was repeated to determine Fourier-like transforms intertwining the Dirac operator and its dual, the vector variable. Similar results are obtained in this case. We also have equivalent descriptions as operator exponentials, using now the $\mathfrak{osp}(1|2)$ Casimir element, and as integral transform, now with a Clifford algebra valued kernel. For a select set of transforms we again find polynomially bounded formula for the kernel, which are of similar nature as the kernel of the Clifford-Fourier transform [6].

In the second chapter, we considered generalizations of the Laplace and Dirac operator in the framework of an n-dimensional Wigner quantum system, involving n position operators $x_1, \ldots, x_n$ and n momentum operators $p_1, \ldots, p_n$. Here, we have a Laplace-like operator Δ which arises by replacing partial derivatives by (a prior abstract) momentum operators $p_1, \ldots, p_n$ in the expression of the regular n dimensional Laplace. In the same way, a Dirac-like operator $\underline{D}$ is defined, which squares to Δ. For these two abstract operators, our aim was to study the symmetry algebras generated by operators commuting or anti-commuting with Δ or $\underline{D}$. A specific case included in this study is that of the Laplace-Dunkl and Dirac-Dunkl operator (for arbitrary root system), by identifying the momentum operators with Dunkl operators.

For our Laplace-like operator Δ, we first obtained an embedding in an $\mathfrak{sl}(2)$ realization, a result known already for the classical as well as the Dunkl case. In the framework of the Wigner quantum system, we described all symmetries commuting with Δ and determined the algebraic relations satisfied by these elements. In general, we have n^2 basic symmetries indexed by two coordinate labels. These basic symmetries are linear in both position and momentum operators and form an analog of linear differential operators preserving the degree of homogeneous polynomials. The resulting symmetry algebra was seen to be an extension of the symmetry algbra in the classical case, namely the Lie algebra $\mathfrak{so}(n)$. In the Dunkl case, this gives rise to a deformation of $\mathfrak{so}(n)$ incorporating reflections of the associated reflection group accompanied by parameters present also in the Dunkl operators. This algebraic structure is in accordance with results on Dunkl operators specifically for root systems of type A_{n-1} [7].

For the Dirac operator $\underline{D}$, we work in an extended framework including now also Clifford algebra elements, akin to the gamma matrices in the original Dirac equation. Herein, we obtained a realization of $\mathfrak{osp}(1|2)$ and determined all operators that commute or anti-commute with $\underline{D}$. Working in this bigger framework, the class of basic symmetries, indexed now by up to n coordinate labels, is of order 2^n as opposed to just n^2 for the Laplace case. In particular, these symmetries consist of combinations of the previously obtained Laplace symmetries accompanied by Clifford algebra elements. The quadratic relations satisfied by these symmetries of the Dirac operator $\underline{D}$ were then determined in abstract fashion. For the Dunkl case, these abstract relations reduce to explicit expressions involving the associated reflection group. The relations obtained in this way agree with other work [4, 5] where the underlying reflection group was taken to be $\mathbb{Z}_2^n$, the simplest non-trivial case. Here, the symmetries and their algebraic relations give rise to the so-called (higher rank) Bannai-Ito algebra. Our findings can be considered as an extension of these relations to an arbitrary underlying reflection group, in fact in an even more general context.

The results obtained here open the way to several new investigations. In particular, they can be combined with the results of Chapter 1. On the one hand, the strategy developed there can now be employed for these generalized Laplace and Dirac operators. On the other hand, the classical case can be revisited using the symmetries obtained here. For the Laplace operator we do not expect any substantial differences, as exponentials of these symmetries generate rotations, or thus orthogonal transformations, which were dismissed as trivial variations. However, it would be interesting to see what the effect is of the symmetries containing Clifford algebra elements in the Dirac case.

In a different direction, one can examine whether other types of operators, besides regular partial derivatives and Dunkl operators, pose valid candidates for the momentum operators in a Wigner quantum systems. Possible examples are trigonometric Dunkl operators, or discrete difference operators.

An immediate follow-up consists of studying in detail the symmetry algebras of the Laplace-Dunkl and the Dirac-Dunkl operators for specific cases of root systems and associated reflection groups. The abstract symmetries and algebraic relations then take on an explicit form determined by this choice of momentum operators. The area of particular interest is then the representation theory of this symmetry algebra. These could lead to superintegrable models and relations with families of orthogonal polynomials, as was already observed for the case of the reflection group $(\mathbb{Z}_2)^n$.

In Chapter 3, we considered in detail such a specific type of Dirac-Dunkl operator, namely for the root system A_2 with Weyl group S_3, the symmetric group on three elements. For this three-dimensional case the symmetry algebra was seen to be a one-parameter deformation of the classical angular momentum algebra, the Lie algebra $\mathfrak{so}(3)$, incorporating elements of S_3. We classified all finite-dimensional irreducible representations of this algebra in abstract fashion, by constructing a form of ladder operators. Among the obtained classes of irreducible representations of the symmetry algebra, there is one class of unitary representations for arbitrary positive value of the parameter. This last class admits a natural realization by means of Dunkl monogenics, for which we constructed an explicit basis. This basis consists of eigenfunctions of the spherical Dirac-Dunkl operator and thus form solutions to a Dirac equation on the two-sphere.

Besides the finite-dimensional representations classified here, we can attempt to do the same for infinite-dimensional representations. Moreover, in future work we aim to elevate the setting of Chapter 3 in two other directions. On the one hand, one can consider the n-dimensional case where the reflection group associated to the Dunkl operator is the symmetric group S_n. On the other hand, it would be interesting to consider more involved root systems (as was done for the type B_3 [9]), first in three dimensions and then also in higher dimensions.

In Chapter 4, we classified all pairs of recurrence relations that connect two sets of discrete orthogonal polynomials of the same family, having different parameters. These families include the Hahn, dual Hahn and Racah polynomials in the discrete side of the Askey-scheme of hypergeometric orthogonal polynomials. In turn, this classification gives all Christoffel-Geronimus transforms where the kernel partner of a given Hahn, dual Hahn or Racah polynomial is again of this same family, with different parameters. For each pair of recurrence relations, there is a corresponding symmetric tridiagonal matrix M with zero diagonal, obtained by normalizing the discrete orthogonal polynomials using the weight function. Explicit expressions for the eigenvalues of M and their orthonormal eigenvectors follow immediately from the recurrence relations and are given in terms of the discrete orthogonal polynomials.

These new tridiagonal matrices could be seen as representation matrices of deformations or extensions of $\mathfrak{su}(2)$. For the dual Hahn case, we determined all algebraic relations for these new algebras, which in general contain two parameters. Special cases of these algebras, including still one parameter, could be used as the underlying algebraic structure for the construction of finite oscillator models. In such a model, the discrete position wavefunction of the oscillator is then described by the related pair of orthogonal polynomials. In Chapter 5, we considered in detail one of the arising special cases. The algebra in question was seen to be an extension of $\mathfrak{su}(2)$ by means of a parity operator P and a parameter and was called $\mathfrak{su}(2)_P$. For this algebra $\mathfrak{su}(2)_P$, we have classified all unitary finite-dimensional irreducible representations.

The rest of Chapter 5 and also Chapter 6 was devoted to the development of new models for a finite quantum oscillator based on the previous results. These new models preserve all the nice and essential properties of the original $\mathfrak{su}(2)$ model, but include an extra parameter which determines the shape of the discrete position (and momentum) wavefunctions. With applications in mind, the most important property is the equidistance of the position spectrum, independent of the value of the parameter.

The original Hahn oscillator models [11, 12] also included a parameter, which appeared as well in the expressions for the position values and thus influenced the distance between them. In those models, the position spectrum was only seen to be equidistant for a specific parameter value, in which case the model reduces to the standard $\mathfrak{su}(2)$

one. The latter model has been used in signal analysis on a finite number of discrete sensors or data points [1–3], where it is an advantage if the sensor points of the grid are uniformly distributed, according to the equidistant position spectrum of the model. The presence of an extra parameter in our new models, while retaining the equidistant position spectrum, could open up additional options in the analysis of signals.

In particular, image analysis using discrete orthogonal polynomials has received considerable attention, with orthogonal moment functions being used in image reconstruction, pattern recognition and object identification [15–18]. The use of discrete as opposed to continuous orthogonal polynomials has the advantage that the computation of the moments involves no numerical approximation of the orthogonality relation. Furthermore, there is no need for a spatial domain renormalization as the polynomials are defined on a discrete domain, as is an image. The currently proposed image analysis by dual Hahn or Racah polynomials uses an intermediate, non-uniform (quadratic) lattice instead of working directly on the image grid. The reason for this is that the orthogonality of both dual Hahn and Racah polynomials holds on a set of points which are not equidistant. This intermediate, non-uniform lattice can be omitted using the developed models with equidistant position values.

Another path worth considering is that of q-analogues of hypergeometric orthogonal polynomials and the related quantum algebras. These contain an additional parameter q and reduce to their classical variants for $q \longrightarrow 1^-$. We refer to Ref. [8, 10, 13, 14] for precise definitions and properties. It could prove interesting to investigate the associated algebraic structures appearing when the current results are repeated in the q-setting.

In the final chapter, we reviewed an interesting secondary outcome of the classification above. We obtained matrices, containing parameters, with a well-defined tridiagonal structure and explicit expressions for both eigenvalues and eigenvectors. Such matrices are useful for testing the accuracy of numerical eigenvalue routines. For the dual Hahn case, these matrices were shown to be two-parameter extensions of the Sylvester-Kac matrix, a standard test matrix with simple integer entries and eigenvalues. In this regard, the matrices are cast in a form with integer entries offset by a parameter, present also in the expressions for eigenvalues. We showcased special cases where the parameters are chosen such that the eigenvalues consist of simple equidistant integers, or double eigenvalues occur. Furthermore, we gave some numerical results regarding the use of these extensions as test matrices for eigenvalue computations, using the inherent MATLAB function `eig()`.

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SUMMARY

A characteristic property of the Fourier transform is that it interchanges differentiation and multiplication when moving between the time and frequency domain. As a consequence, on n-dimensional Euclidean space, the Fourier transform intertwines the Laplace operator and the squared norm. In the first chapter, we investigate which operators on the space of rapidly decreasing functions on $\mathbb{R}^n$ portray this same behavior. In this way, we obtain a non-trivial class of such Fourier-like transforms which are described in two ways.

The first description relies on a realization of the Lie algebra $\mathfrak{sl}(2)$ generated by the Laplace operator and the squared norm. This algebraic structure is naturally connected with the classical Fourier transform. The desired operators are constructed as exponentials of elements in the enveloping algebra of $\mathfrak{sl}(2)$, using the Casimir element which commutes with all elements of the algebra. The explicit exponents are given in terms of integer-valued polynomials on squares of integers or of half-integers, depending on the parity of the dimension.

Second, for each operator an equivalent formulation is given as an integral transform with a plane wave decomposition of the integral kernel in terms of Bessel functions and Gegenbauer polynomials. For a finite subset of transforms, which closely resemble the classical Fourier transform, the integral kernels can be reduced to closed formulas which are polynomially bounded. Furthermore, we establish uncertainty principles for these Fourier-like transforms.

In the setting of functions taking values in a Clifford algebra, the Dirac operator is defined as a square root of the Laplace operator. The underlying algebraic structure is the Lie superalgebra $\mathfrak{osp}(1|2)$. We repeat the same procedure and determine Fourier-like transforms intertwining the Dirac operator and its dual, the vector variable. We also obtain two equivalent descriptions for each such transform: as an operator exponential, using now the $\mathfrak{osp}(1|2)$ Casimir element, and as an integral transform with a Clifford algebra-valued kernel. For a select set of transforms we again find polynomially bounded formula for the kernel, which are of similar nature as those of other transforms in the context of Clifford analysis.

In Chapter 2, we consider a generalization of the classical Laplace operator, where we replace partial derivatives by abstract momentum operators. In particular, this generalization includes the Laplace-Dunkl operator when we identify the momentum operators with the so-called Dunkl operators. The latter are differential-difference operators associated with a root system and the corresponding finite reflection group. For this Laplace-like operator, we determine a set of symmetries commuting with it, and we present the algebraic relations for the symmetry algebra. In the classical case, this forms a realization of the Lie algebra $\mathfrak{so}(n)$. We obtain novel extensions or deformations of $\mathfrak{so}(n)$ as the symmetry algebra of more general cases.

In this context, the generalized Dirac operator is then defined as a square root of our Laplace-like operator. We explicitly determine a family of graded operators which commute or anti-commute with our Dirac-like operator depending on their degree. The algebra generated by these symmetry operators is shown to be a generalization of the standard angular momentum algebra in the three-dimensional case, and of the recently defined higher rank Bannai-Ito algebra in higher dimensions.

In Chapter 3, we study a specific case of the abstract results obtained in the previous chapter. We consider the three-dimensional case of the Dirac-Dunkl operator associated to the root system A_2, and the associated Dirac equation. The corresponding Weyl group is the symmetric group on three elements, denoted by S_3. The explicit form of the symmetry algebra in this case is a one-parameter deformation of the classical angular momentum algebra $\mathfrak{so}(3)$ incorporating elements of S_3. For this algebraic structure in abstract form, we classify all finite-dimensional, irreducible representations and we determine the conditions for the representations to be unitarizable. A realization of the natural class of unitary irreducible representations is given by the action of the symmetries on polynomials in the kernel of the Dirac-Dunkl operator. The representation space consists of eigenfunctions of the Dirac-Dunkl operator on the two-sphere. Using a Cauchy-Kowalevsky extension theorem we obtain explicit expressions for these eigenfunctions in terms of Jacobi polynomials.

The next part originated from the context of finite oscillator models. In earlier models, the discrete position and momentum wavefunctions were seen to consist of a combination of dual Hahn polynomials with different parameters. The action of the position operator on these wavefunctions boiled down to a pair of recurrence relations for these sets of dual Hahn polynomials. To construct new models, in Chapter 4, we thus classify all pairs of recurrence relations that connect two sets of discrete orthogonal polynomials of the same family, having different parameters. These families include the Hahn, dual Hahn and Racah polynomials in the discrete side of the Askey-scheme of hypergeometric orthogonal polynomials. In turn, this classification gives all Christoffel-Geronimus transforms where the kernel partner of a dual Hahn polynomial is again of this same family, with different parameters.

For each pair of recurrence relations, there is a corresponding symmetric tridiagonal matrix M with zero diagonal, obtained by normalizing the discrete orthogonal polynomials using the weight function. Explicit expressions for the eigenvalues of M and their orthonormal eigenvectors follow immediately from the recurrence relations and are given in terms of the discrete orthogonal polynomials. We examine also the underlying algebraic relations, and discuss their usefulness for the construction of new finite oscillator models.

In Chapter 5, we investigate in particular a finite oscillator model which could have equidistant position values. The latter is of interest for applications in optics and signal analysis. The related algebraic structure, in this case, is an extension of the Lie algebra $\mathfrak{su}(2)$ by means of a parity operator P and a parameter, which we refer to as the algebra $\mathfrak{su}(2)_P$. Before getting to the oscillator model, we classify all irreducible unitary finite-dimensional representations of this algebra.

Next, the obtained odd-dimensional representations are used to construct a finite oscillator model with equidistant position values related to the algebra $\mathfrak{su}(2)_P$. The orthonormal eigenvectors of the position and momentum operator form the corresponding wavefunctions, which are given in terms of the previously determined pair of dual Hahn polynomials. In particular the position spectrum is independent of the parameter included in the model while the discrete position wavefunctions do depend on this parameter, as pictured in plots pertaining to the lowest energy states.

Subsequently, in Chapter 6, we develop a finite oscillator model pertaining to a pair of Racah polynomials. This pair was also obtained in the previous classification, where their Jacobi matrix was observed to have equidistant eigenvalues for specific values of the Racah polynomial parameters. We discuss some properties of this oscillator model, give its (discrete) position wavefunctions explicitly, and illustrate their behavior by means of some plots.

In the final chapter, as a spin-off of the previous results, the newly obtained matrices are seen as two-parameter extensions of the Clement or Sylvester-Kac matrix. The latter is a tridiagonal matrix with zero diagonal and simple integer entries, whose spectrum is known explicitly and consists of integers. This makes it a useful test matrix for numerical eigenvalue computations. Our new class of appealing two-parameter extensions of this matrix have the same simple structure and their eigenvalues are also given explicitly by a simple closed form expression. We consider special cases for specific parameter values having integer eigenvalues or double eigenvalues, and we examine some numerical results regarding the use of these extensions as test matrices for eigenvalue computations.

NEDERLANDSTALIGE SAMENVATTING

Een belangrijke eigenschap van de Fouriertransformatie is dat differentiëren in het tijdsdomein overeenkomt met vermenigvuldigen in het frequentiedomein en omgekeerd. Bijgevolg zal de Fouriertransformatie op een n-dimensionale Euclidische ruimte de Laplace-operator Δ omzetten in een vermenigvuldiging met de norm in het kwadraat. In het eerste hoofdstuk onderzoeken we of er nog andere operatoren zijn die ook dit gedrag vertonen. Het antwoord hierop is positief en we bekomen een niet-triviale klasse van zulke transformaties die op twee manieren beschreven kunnen worden.

De eerste voorstelling steunt op een realisatie van de klassieke Lie-algebra $\mathfrak{sl}(2)$ door de Laplace-operator en de gekwadrateerde norm. Deze algebraïsche structuur hangt op natuurlijke wijze samen met de klassieke Fouriertransformatie. Elk van de nieuwe transformaties komt overeen met de exponent van een operator in de omhullende algebra van de Lie-algebra $\mathfrak{sl}(2)$. Het Casimir-element dat met alle andere elementen van de algebra commuteert, speelt hierin een sleutelrol. Verder steunen we op de theorie van veeltermen die enkel gehele waarden aannemen om de operatoren in de exponent op te bouwen.

Als tweede equivalente voorstelling hebben we voor elke operator een formulering als integraaltransformatie met een kern bestaande uit Besselfuncties en Gegenbauerpolynomen. Voor een eindige deelverzameling van transformaties die sterk lijken op de klassieke Fouriertransformatie, kunnen we de kernen reduceren tot gesloten formules die polynomiaal begrensd zijn. Bovendien bepalen we onzekerheidsrelaties voor deze transformaties.

In de context van de Cliffordanalyse, waar functies waarden aannemen in een Cliffordalgebra, herhalen we bovenstaande procedure. De Dirac-operator is hier zodanig gedefinieerd dat het kwadraat ervan de Laplace-operator is. De focus ligt nu op operatoren die de Dirac-operator en diens Fouriergetransformeerde — de vectorvariabele — in elkaar omzetten. We verkrijgen opnieuw twee equivalente formuleringen voor transformaties die hieraan voldoen. De relevante algebraïsche structuur is nu de Lie-superalgebra $\mathfrak{osp}(1|2)$. Voor een bijzondere klasse van transformaties vinden we weer een polynomiaal begrensde formule voor de kern. Deze is gelijkaardig aan andere veralgemeende Fouriertransformaties in het kader van Cliffordanalyse.

In hoofdstuk 2 beschouwen we een veralgemening van de klassieke Laplace-operator, waar we de partiële afgeleiden vervangen door abstracte impulsoperatoren. In het bijzonder omvat deze veralgemening de Laplace-Dunkl-operator, wanneer we de impulsoperatoren identificeren met Dunkl-operatoren. Laatstgenoemde zijn deformaties van partiële afgeleiden door middel van differentietermen geassocieerd aan een eindige reflectiegroep. Voor deze veralgemeende Laplace-operator bepalen we een reeks symmetrieën die ermee commuteren, en we geven de algebraïsche relaties voor de symmetriealgebra. In het klassieke geval is dit de Lie-algebra $\mathfrak{so}(n)$, waarvoor we op deze manier extensies of deformaties bekomen.

In deze context definiëren we ook een veralgemeende Dirac-operator als vierkantswortel van onze Laplace-operator. We bepalen expliciet een familie van gegradeerde symmetrieën die, afhankelijk van hun graad, commuteren of anti-commuteren met onze Dirac-operator. De algebra die door deze symmetrieën wordt gegenereerd, blijkt in het driedimensionale geval een veralgemening te zijn van de Lie-algebra $\mathfrak{so}(3)$ en in hogere dimensies van de recent gedefinieerde hogere rang Bannai-Ito-algebra.

In hoofdstuk 3 bestuderen we een speciaal voorbeeld uit de abstracte resultaten van het vorige hoofdstuk. We beschouwen de driedimensionale Dirac-Dunkl-operator, en de bijhorende Diracvergelijking, voor het wortelsysteem A_2 met reflectiegroep S_3, de symmetrische groep op drie elementen. De expliciete vorm van de symmetriealgebra is in dit geval een één-parameter deformatie van de Lie-algebra $\mathfrak{so}(3)$ die elementen van S_3 bevat. Voor deze algebra classificeren we alle eindig-dimensionale, irreduciebele representaties. Verder bepalen we de voorwaarden opdat de representaties unitair zijn. De eigenfuncties van de sferische Dirac-Dunkl-operator vormen een realisatie van de unitaire, irreduciebele representatie van de symmetriealgebra. Met behulp van een Cauchy-Kowalevsky extensie krijgen we expliciete uitdrukkingen voor deze eigenfuncties die we kunnen schrijven als Jacobi-polynomen.

Vervolgens gaan we over naar de discrete wereld. We gaan op zoek naar nieuwe discrete en eindige modellen voor de kwantum harmonische oscillator. In eerder beschouwde modellen bestonden de discrete golffuncties uit een combinatie van twee discrete orthogonale veeltermen met verschillende parameterwaarden. Dat deze veeltermen samenhoren, komt neer op het bestaan van twee recursierelaties. De vraag is nu of we deze structuur voor de golffuncties kunnen gebruiken om nieuwe modellen te bouwen. Hiertoe classificeren we in hoofdstuk 4 alle paren van zulke recursierelaties voor twee discrete orthogonale veeltermen van dezelfde familie, met verschillende parameters. Deze families omvatten de Hahn-, duale Hahn- en Racah-polynomen uit het Askey-schema van hypergeometrische orthogonale polynomen. Verder geeft deze classificatie ook alle Christoffel-Geronimus transformaties waar de verkregen partner opnieuw van dezelfde familie is als de oorspronkelijke veelterm.

Bij elk paar recursierelaties hoort een symmetrische tridiagonale matrix M met nullen op de diagonaal. Deze verkrijgen we door het normeren van de discrete orthogonale veeltermen met behulp van de gewichtsfunctie. Expliciete uitdrukkingen voor de eigenwaarden van M en hun eigenvectoren volgen onmiddellijk uit de recursierelaties. We onderzoeken verder ook de onderliggende algebraïsche relaties en bespreken hun nut voor de constructie van nieuwe eindige modellen.

Van de nieuwe kandidaat-modellen beschouwen we in hoofdstuk 5 een specifiek model met equidistante positiewaarden. Dit is van belang voor mogelijke toepassingen in de optica en signaalanalyse. De bijhorende algebraïsche structuur van dit model is een extensie van de Lie-algebra $\mathfrak{su}(2)$ met een pariteitsoperator P en een parameter. We noemen deze extensie verder de algebra $\mathfrak{su}(2)_P$. Voordat we het oscillatormodel zelf bespreken, classificeren we alle irreduciebele unitaire eindige-dimensionale representaties van de algebra $\mathfrak{su}(2)_P$.

Vervolgens gebruiken we de verkregen oneven-dimensionale representaties om een eindig oscillatormodel op te stellen met equidistante positiewaarden. De orthonormale eigenvectoren van de positie- en impulsoperator zijn dan de respectievelijke golffuncties, gegeven in termen van de eerder bepaalde duale Hahn-polynomen. In het bijzonder is het positiespectrum equidistant en onafhankelijk van de parameter in het model, terwijl de discrete positiegolffuncties wel deze parameter bevatten, zoals duidelijk te zien is in de gemaakte grafieken.

Ten slotte ontwikkelen we in Hoofdstuk 6 een eindig oscillatormodel met behulp van een paar recursierelaties voor Racah-polynomen. Dit paar werd ook verkregen in de classificatie van hoofdstuk 4. We constateerden dat voor specifieke waarden van de parameters in de Racah-polynomen de bijhorende Jacobi-matrix equidistante eigenwaarden heeft. Dit leidt bijgevolg tot equidistante positiewaarden voor een oscillatormodel. We bespreken de voornaamste eigenschappen van dit model, geven expliciete uitdrukkingen voor de (discrete) positiegolffuncties en illustreren hun gedrag met enkele figuren.

Als een spin-off van de vorige resultaten, bekijken we in het laatste hoofdstuk de verkregen matrices als twee-parameter extensies van de Clement- of Sylvester-Kac-matrix. Deze laatste is een tridiagonale matrix met nullen op de diagonaal en eenvoudige natuurlijke getallen als elementen, waarvan het spectrum expliciet gekend is en ook bestaat uit natuurlijke getallen. Deze eigenschappen zijn nuttig als testmatrix voor numerieke berekeningen van eigenwaarden. Onze nieuwe klasse van uitbreidingen van deze matrix hebben dezelfde eenvoudige structuur en hun eigenwaarden zijn ook expliciet gekend maar bevatten twee parameters. We beschouwen bijzondere gevallen voor specifieke parameterwaarden die leiden tot natuurlijke getallen als eigenwaarden of ontaarde eigenwaarden. Verder onderzoeken we enkele numerieke resultaten bij het gebruik van deze extensies als testmatrices voor berekeningen van eigenwaarden.

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