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Fractional Calabi–Yau categories from Landau–Ginzburg models

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ABSTRACT

We give criteria for the existence of a Serre functor on the derived category of a gauged Landau–Ginzburg model. This is used to provide a general theorem on the existence of an admissible (fractional) Calabi–Yau subcategory of a gauged Landau–Ginzburg model and a geometric context for crepant categorical resolutions. We explicitly describe our framework in the toric setting. As a consequence, we generalize several theorems and examples of Orlov and Kuznetsov, ending with new examples of semi-orthogonal decompositions containing (fractional) Calabi–Yau categories.

1. Introduction

In [BK90], Bondal and Kapranov generalized Serre duality to triangulated categories by providing the following definition.

DEFINITION 1.1. Let κ be an algebraically closed field of characteristic 0. A Serre functor on a κ-linear triangulated category $\mathcal{T}$ is an exact auto-equivalence $S\colon \mathcal{T} \to \mathcal{T}$ such that there exist bifunctorial isomorphisms

$$\operatorname{Hom}(A, B) \cong \operatorname{Hom}(B, S(A))^\vee\,.$$

The category $\mathcal{T}$ is called *Calabi–Yau* (CY) of dimension d if $S = [d]$; it is called *fractional Calabi–Yau* (FCY) of dimension a/b if $S^b = [a]$.

The term Serre functor is inspired by the case where $\mathcal{T}$ is the bounded derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ for a smooth projective variety X of dimension n. In this case, the Serre functor is a rephrasing of Serre duality; hence,

$$S = - \otimes \omega_X[n]\,.$$

In particular, the derived category of a Calabi–Yau variety of dimension n is a Calabi–Yau category of dimension n as the canonical bundle is trivial. Similarly, if the canonical bundle of X is torsion, then $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ is fractional Calabi–Yau.

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Kuznetsov showed that fractional Calabi–Yau categories also occur as admissible subcategories of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ when $X \subseteq \mathbf{P}^n$ is a smooth hypersurface of degree $d \leqslant n+1$ (see, for example, [Kuz04, Corollary 4.4]). He first defined the admissible subcategory

$$\mathcal{A}_X := \left\{ C \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \mathrm{H}^j(C(i)) = 0 \text{ for } 0 \leqslant i \leqslant n-d \text{ and all } j \right\}$$

and then proved directly that $\mathcal{A}_X$ is FCY of dimension $(n+1)(d-2)/d$ and CY when d divides $n+1$.

In the special case of cubic fourfolds ($n = 5$, $d = 3$), we get a 2-dimensional Calabi–Yau category. Kuznetsov went on to show that for most of the known rational cubic fourfolds, $\mathcal{A}_X$ is equivalent to the derived category of a $K3$ surface. He conjectured that a smooth cubic fourfold X is rational if and only if there is a $K3$ surface Y and an equivalence of categories

$$\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)\,.$$

This conjecture has steered the study of rational cubic fourfolds ever since.

Orlov later provided a beautiful description of $\mathcal{A}_X$ in terms of the categorical analogue of the Landau–Ginzburg model corresponding to the hypersurface. Let X be a smooth projective hypersurface defined by a function w. There is an equivalence of categories

$$\mathcal{A}_X \cong \mathrm{D}^{\mathrm{abs}}\left[\mathbb{A}^{n+1}, \mathbb{G}_m, w\right].$$

The category $\mathrm{D}^{\mathrm{abs}}[\mathbb{A}^{n+1}, \mathbb{G}_m, w]$ can be loosely defined as the derived category associated to the gauged Landau–Ginzburg model $(\mathbb{A}^{n+1}, \mathbb{G}_m, w)$. Here, w is a section of the equivariant bundle $\mathcal{O}(\chi)$ for the dth-power character χ (see Subsection 2.1 for a precise definition). This description has the advantage of being a geometric description of $\mathcal{A}_X$.

Orlov's description of $\mathcal{A}_X$ gives rise to two leading questions:

- When is the derived category of a Landau–Ginzburg model (fractional) Calabi–Yau?
- When do derived categories of Landau–Ginzburg models that are fractional Calabi–Yau appear as admissible subcategories of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$?

In this paper, we give sufficient criteria for these questions.

By studying the derived category of Landau–Ginzburg models, we give an alternate view of identifying (fractional) Calabi–Yau categories to that given by Kuznetsov in [Kuz17]. There, Kuznetsov provides examples of (fractional) Calabi–Yau categories for a smooth variety X by finding a spherical functor $\Phi\colon \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} M)$ to another variety M whose derived category comes equipped with a Lefschetz fibration. He provides a list of examples in [Kuz17, Subsection 4.5]. Many of his examples come from complete intersections in homogeneous varieties.

In our viewpoint, we pass to the Landau–Ginzburg model and use geometric invariant theory (GIT) to find a GIT chamber that is associated to a Calabi–Yau category instead of using a spherical functor. Due to this difference, our theories work in different contexts. For example, our framework is quite concrete for complete intersections in toric varieties, while [Kuz17] naturally recovers many of the examples given in [IM15] of complete intersections in homogeneous spaces.

1.1 General results

First, we establish the Serre functor on the derived category of a Landau–Ginzburg model.

THEOREM 1.2 (Theorem 2.18). *Let X be a smooth algebraic variety and G a linearly reductive algebraic group acting on X. Let $\chi\colon G \to \mathbb{G}_m$ be a character and $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$. Assume* *that* $[X/\ker\chi]$ *has finite diagonal. In addition, assume* $\partial w \subseteq Z(w)$ *and that* $[\partial w/\ker\chi]$ *is proper. Then,* $\mathrm{D}^{\mathrm{abs}}[X,G,w]$ *admits a Serre functor given by*

$$S := (-\otimes\omega_X)[\dim X - \dim G + 1]\,.$$

The relevance of the theorem above in the context of the literature is the presence of a G-action. For example, Serre–Grothendieck duality without the presence of a G-action was proven by Efimov–Positselski [EP15]. In addition, the existence of a Serre functor follows from a differential graded (dg) enhancement which is smooth and proper [Shk07] (which we will use). The existence of such a dg-enhancement in the case where X is affine space, $G = 1$, and ∂w is isolated was proven by Dyckerhoff [Dyc11]. Lin–Pomerleano [LP13] subsequently exhibited a smooth and proper dg-enhancement for any smooth variety X in the case $G = 1$ and ∂w proper. Furthermore, when X is Calabi–Yau, they demonstrated that the category is as well. Preygel independently proved similar results for matrix factorizations on derived schemes, using a different set of tools from derived algebraic geometry [Pre11]. The G-equivariant case was first studied independently by Polishchuk–Vaintrob and by Ballard, Katzarkov, and the first-named author [PV16, BFK14]. These provide a suitable dg-enhancement in the G-invariant case which we will rely heavily on.

The theorem above also has the following corollary, which provides sufficient criteria for when the derived category of a Landau–Ginzburg model is (fractional) Calabi–Yau.

Theorem 1.3 (Corollary 2.19). *Let* X *be a smooth algebraic variety and* G *a linearly reductive algebraic group acting on* X*. Let* $\chi\colon G\to\mathbb{G}_m$ *be a character and* $w\in\Gamma(X,\mathcal{O}_X(\chi))^G$*. Assume that* $[X/\ker\chi]$ *has finite diagonal and torsion canonical bundle. In addition, assume that the critical locus* ∂w *is contained in* $Z(w)$ *and that* $[\partial w/\ker\chi]$ *is proper. Then* $\mathrm{D}^{\mathrm{abs}}[X,G,w]$ *is fractional Calabi–Yau.*

Then, we can use the birational geometry of Landau–Ginzburg (LG) models to attack the second problem.

Definition 1.4. Let (Y_1,w_1) and (Y_2,w_2) be two gauged LG models with Y_i smooth, G acting on Y_i, and w_i a section of $\mathcal{O}(\chi)$ for a character $\chi\colon G\to\mathbb{G}_m$. We say that (Y_1,w_1) *K-dominates* (Y_2,w_2) if there exist a smooth G-variety Z and proper equivariant birational morphisms $f_1\colon Z\to Y_1$ and $f_2\colon Z\to Y_2$ such that

- $f_1^*w_1 = f_2^*w_2$,
- $f_1^*K_{Y_1} - f_2^*K_{Y_2} \geqslant 0$.

In the context of finding FCY admissible subcategories, Kawamata's LG model conjecture (see [BFK14, Conjecture 4.3.7]) specializes to the following.

Conjecture 1.5. *If* (Y_1,w_1) *K-dominates* (Y_2,w_2) *and* $[Y_2/\ker\chi_2]$ *has torsion canonical bundle, then* $\mathrm{D}^{\mathrm{abs}}[Y_2,G,w_2]$ *is a FCY admissible subcategory of* $\mathrm{D}^{\mathrm{abs}}[Y_1,G,w_1]$*.*

While general birational relationships like K-dominance are more difficult to analyze, Wlodarcykz's weak factorization theorem [Wło03] shows that all birational transformations can be broken up into a sequence of simpler ones called elementary wall crossings (see Definition 3.10). These transformations were shown to yield fully faithful functors between derived categories of gauged LG models in [BFK17]. An immediate consequence of Theorem 2.19 and [BFK17] is the following.

Corollary 1.6 (Corollary 2.20). *Conjecture 1.5 holds for elementary wall crossings.*

1.2 Toric results

In Section 4, we specialize to the toric situation, and the description we find is quite pleasing. Indeed, we obtain comparisons between FCY categories and derived categories of toric gauged Landau–Ginzburg models very similar to those found in [Orl06].

Let us begin by describing the toric backdrop. Let M and N be dual lattices. Let $\nu = \{v_1, \dots, v_n\} \subset N$ be a collection of distinct primitive lattice points. Consider the cone $\sigma := \mathrm{Cone}(\nu)$. We say that σ is $\mathbb{Q}$-Gorenstein (respectively, almost Gorenstein) if there exists an element $\mathfrak{m} \in M_{\mathbb{Q}}$ (respectively, $\mathfrak{m} \in M$) such that the cone σ is generated over $\mathbb{Q}$ by finitely many lattice points $\{n \in N \mid \langle \mathfrak{m}, n\rangle = 1\}$. We partition the set ν as $\nu_{=1} \cup A$, where $\nu_{=1} = \{v_i \in \nu \mid \langle \mathfrak{m}, n\rangle = 1\}$ and A is its complement in ν.

Associate a group S_ν to the point collection ν in the following way. Consider the right-exact sequence

$$\begin{aligned} M &\xrightarrow{f_\nu} \mathbb{Z}^n \xrightarrow{\pi} \mathrm{coker}(f_\nu) \longrightarrow 0\,, \\ m &\longmapsto \sum_{i=1}^{n} \langle m, v_i\rangle e_i\,. \end{aligned} \tag{1.1}$$

Set $S_\nu := \mathrm{Hom}(\mathrm{coker}(f_\nu), \mathbb{G}_m)$. If we apply $\mathrm{Hom}(-, \mathbb{G}_m)$ to the above sequence, we obtain

$$1 \longrightarrow S_\nu \xrightarrow{\hat{\pi}} \mathbb{G}_m^n \xrightarrow{\hat{f}_\nu} M \otimes \mathbb{G}_m\,.$$

This defines an action of S_ν on $\mathbb{A}^n$ by first taking the inclusion of S_ν into the maximal torus $\mathbb{G}_m^n$ given by the map $\hat{\pi}$ and then extending the action naturally.

Given a subset $R \subset \nu$, we can also define a $\mathbb{G}_m$-action called *R-charge* to act on $\mathbb{A}^\nu$ by extending the action

$$\lambda \cdot (x_1, \dots, x_n) = (y_1, \dots, y_n)\,, \quad \text{where} \quad y_i := \begin{cases} \lambda x_i & \text{if } v_i \in R\,, \\ x_i & \text{if } v_i \notin R\,. \end{cases}$$

Given the cone $\sigma := \mathrm{Cone}(\nu)$, we can consider the dual cone

$$\sigma^\vee := \{m \in M_{\mathbb{R}} \mid \langle m, n\rangle \geqslant 0 \text{ for all } n \in \sigma\}\,.$$

Define a superpotential w on $\mathbb{A}^n$ given by taking a finite set $\Xi \subset \sigma^\vee \cap M$ and defining w to be

$$w = \sum_{m \in \Xi} c_m x^m\,, \quad \text{where} \quad x^m := \prod_{i=1}^{n} x_i^{\langle m, v_i\rangle}\,.$$

Let $\tilde{\Sigma}$ be any simplicial fan such that $\tilde{\Sigma}(1) = \nu$. The quotient construction of a toric variety determines an open set $U_{\tilde{\Sigma}}$ of $\mathbb{A}^\nu$ (for a precise treatment, see equation (4.4) for the fan associated to this open set). The triple of data

$$(U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w)$$

constitutes a toric gauged Landau–Ginzburg model.

Define another gauged Landau–Ginzburg model associated to ν, as follows. Take any simplicial fan Σ such that $\Sigma(1) \subseteq \nu_{=1}$ and $\mathrm{Cone}(\Sigma(1)) = \sigma$. We can define a group H which depends on ν, $\Sigma(1)$, and R (see equation (5.3) below) that acts on U_Σ. Consider the action of H on the open affine set $U_\Sigma \subset \mathbb{A}^{\Sigma(1)}$. Finally, construct a potential $\bar{w}$ by just taking

$$\bar{w} = \sum_{m \in \Xi} c_m \bar{x}^m\,, \quad \text{where} \quad \bar{x}^m := \prod_{i, \mathrm{Cone}(v_i) \subseteq \Sigma(1)}^{n} x_i^{\langle m, v_i\rangle}\,.$$

There is another gauged Landau–Ginzburg model that comes from the triple of data $(U_\Sigma, H, \bar{w})$.

We prove an Orlov-type theorem that compares the derived categories associated to these two gauged Landau–Ginzburg models.

THEOREM 1.7 (Theorem 5.8). *Let $\tilde{\Sigma}$ be any simplicial fan such that $\tilde{\Sigma}(1) = \nu$ and $X_{\tilde{\Sigma}}$ is semiprojective. Similarly, let Σ be any simplicial fan such that $\Sigma(1) \subseteq \nu_{=1}$, X_Σ is semiprojective, and* $\operatorname{Cone}(\Sigma(1)) = \sigma$. *We have the following:*

(i) *If $\langle \mathfrak{m}, a\rangle > 1$ for all $a \in \nu_{\neq 1}$, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \to \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w]\,.$$

(ii) *If $\langle \mathfrak{m}, a\rangle < 1$ for all $a \in \nu_{\neq 1}$, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w] \to \mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]\,.$$

(iii) *If $\nu_{\neq 1} = \emptyset$, then there is an equivalence*

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \cong \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w]\,.$$

Furthermore, if $\partial\bar{w} \subseteq Z(\bar{w})$ and $[\partial\bar{w}/S_{\Sigma(1)}]$ is proper, then $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ is fractional Calabi–Yau. If, in addition, σ is almost Gorenstein, then $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ is Calabi–Yau.

These types of relationships are intimately related to Aspinwall and Plesser's formulation of mirror pairs [AP15]. In particular, a corollary of this theorem is that if one considers a gauged linear σ-model in their setting that is nonsingular, then it has an associated Calabi–Yau category.

For certain simplicial fans Σ and $\tilde{\Sigma}$, the categories $\mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w]$ and $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ may be geometric, that is, equivalent to the derived category of some stack. One way to realize these equivalences is via the following setup. Suppose that

- $\tilde{\Sigma}$ is a fan where the toric stack $\mathcal{X}_{\tilde{\Sigma}}$ is the total space of a vector bundle

$$\mathcal{X}_\Sigma = \operatorname{tot}\left(\bigoplus_{i=1}^{t} \mathcal{O}_{\mathcal{X}_\Psi}(\chi_{D_i})\right),$$

  where Ψ is some fan corresponding to a semiprojective toric stack $\mathcal{X}_\Psi$ and the D_i are $\mathbb{Q}$-Cartier anti-nef divisors on $\mathcal{X}_\Psi$;
- the $\mathbb{G}_m$-action is by fiberwise dilation on the total space of the vector bundle; and
- the potential w is of the form

$$w = u_1 f_1 + \cdots + u_t f_t\,,$$

  where $f_i \in \Gamma(\mathcal{X}_\Psi, \mathcal{O}_{\mathcal{X}_\Psi}(D_i))$ and u_i is the coordinate corresponding to the ray in Ψ associated to the construction of the line bundle $\mathcal{O}_{\mathcal{X}_\Psi}(\chi_{D_i})$.

In this case, one can consider the complete intersection

$$\mathcal{Z} = Z(f_1, \ldots, f_t) \subset \mathcal{X}_\Psi\,.$$

A result of Hirano ([Hir17, Proposition 4.8], repeated here as Theorem 3.5 for convenience) provides an equivalence between the derived category of coherent sheaves of $\mathcal{Z}$ and the factorization category $\mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w]$. By requiring the data Σ, H, and $\bar{w}$ to satisfy the analogous criteria, one has a different complete intersection $\mathcal{Z}' \subseteq \mathcal{X}_\Upsilon$ in some other toric stack $\mathcal{X}_\Upsilon$ associated to a fan Υ. Thus under appropriate conditions, one or both of the relevant categories can be made geometric. In this case we get Corollary 5.15, which relates the derived categories of the stacks $\mathcal{Z}$ and $\mathcal{Z}'$. For a precise explanation of these conditions and results, we refer to Subsection 5.2.

Almost immediately, we start to recover many theorems and examples as corollaries to our framework. For example, the Batyrev–Nill conjecture [BN08, Conjecture 5.3] is just case (iii) of Corollary 5.15. This recovers the main result of [FK17]. As an instructive example, we specialize to the case of Orlov's theorem on the fan associated to the line bundle $\mathrm{tot}(\mathcal{O}_{\mathbf{P}^n}(-d))$, which we do as an example in Subsection 6.1.

1.3 Crepant categorical resolutions

In [Kuz08], Kuznetsov studies the derived category of coherent sheaves on a singular variety Y. He constructs a subcategory $\tilde{\mathcal{D}}$ of the derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\tilde{Y})$ of coherent sheaves on a resolution $\tilde{Y}$ of Y that he views as a categorical resolution of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,Y)$. In Section 3, we provide an interpretation of crepant categorical resolutions in terms of Landau–Ginzburg models.

In sum, crepant categorical resolutions have a simple geometric interpretation as partial compactifications of LG models. Roughly speaking, if the singular locus of w is not proper, we can make it proper by partially compactifying. This, in turn, provides a crepant categorical resolution.

Specifically, one finds a G-equivariant variety U such that V is openly immersed in U and the function w extends to V so that $\mathrm{D}^{\mathrm{abs}}[U, w, G]$ is smooth and proper. In other words, $[U/G]$ is a partial compactification of $[V/G]$ which has the benefit of satisfying the criteria of Corollary 1.3. Hence, to obtain crepant categorical resolutions for singular complete intersections $\mathcal{Z}$ in X, first, apply Hirano's result to replace $\mathcal{Z}$ by an LG model (V, w, G):

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}) = \mathrm{D}^{\mathrm{abs}}[V, w, G]\,.$$

Second, find a G-equivariant compactification (U, w, G) satisfying the conditions of Corollary 1.3. In the examples below, such a G-equivariant compactification can be found by performing birational operations on the total space of a vector bundle on a blow-up on X.

Furthermore, we show that this geometric interpretation of crepant categorical resolutions behaves well with respect to full subcategories coming from variation of GIT. Namely, if $\mathcal{A}$ is a full subcategory of $\mathrm{D}^{\mathrm{abs}}[U, w, G]$ obtained from an elementary wall crossing, then there is a corresponding elementary wall crossing of $\mathrm{D}^{\mathrm{abs}}[V, w, G]$ and the corresponding subcategory $\tilde{\mathcal{A}}$ is a crepant categorical resolution of $\mathcal{A}$ (Theorem 3.14).

1.4 Examples

Our results on fractional CY subcategories and crepant categorical resolutions can be used to generalize Kuznetsov's work on singular cubic fourfolds [Kuz10].

First, we generalize the example of singular cubic fourfolds outlined by Kuznetsov in [Kuz10] to higher dimension.

EXAMPLE 1.8. Let X be a singular cubic hypersurface in $\mathbf{P}^{3n+2}$ defined by the polynomial

$$f(x_1, \ldots, x_{3n+3}) = \sum_{i=1}^{n} x_i f_i(x_{n+1}, \ldots, x_{3n+3}) + f_0(x_{n+1}, \ldots, x_{3n+3})\,,$$

where f_0 is a generic cubic with the given variables and $f_1, \ldots, f_n$ are generic quadrics. There is a semi-orthogonal decomposition for X in the case where $n = 1$:

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X) = \langle \mathcal{A}, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle\,.$$

Kuznetsov proves that while the category $\mathcal{A}$ is not Calabi–Yau, it has a crepant categorical resolution $\tilde{\mathcal{A}}$ that is a Calabi–Yau category [Kuz10, Theorem 5.2]. Moreover, $\tilde{\mathcal{A}}$ is the derived category of a K3 surface. Here, when $n > 1$, we can generalize the story. Analogously, there is a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) = \langle \mathcal{A}, \mathcal{O}, \ldots, \mathcal{O}(3n-1)\rangle \, .$$

We find that $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ is a crepant categorical resolution of $\mathcal{A}$, where Y is the $(n+1)$-dimensional Calabi–Yau complete intersection Y in $\mathbf{P}^{2n+2}$ given by the zero locus of $f_0, \ldots, f_n$.

Second, we generalize the example of cubic fourfolds containing two planes given in [Has00].

Example 1.9. Let X be a generic degree d hypersurface in $\mathbf{P}^{2d-1}$ that contains a 2-dimensional plane P_1 and a $(2d-4)$-dimensional plane P_2 such that $P_1 \cap P_2 = \emptyset$. While smooth when $d = 3$, this example becomes singular when $d > 3$. By Orlov's theorem, there is a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) = \langle \mathcal{A}, \mathcal{O}, \ldots, \mathcal{O}(d-1)\rangle \, .$$

If $d = 3$, then $\mathcal{A}$ is the derived category of a K3 surface [Kuz10, Proposition 4.7]. We prove that when $d > 4$, there exists a Calabi–Yau $(2d-4)$-fold Y defined by the complete intersection of two hypersurfaces of bidegrees $(d-1, 1)$ and $(d-2, 2)$ in $\mathbf{P}^{2d-4} \times \mathbf{P}^2$ such that $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ is a crepant categorical resolution of $\mathcal{A}$.

1.5 Plan of the paper

The plan for this paper is as follows. Section 2 introduces the factorization category, the main object of study for the paper. After providing its proper definition, we give criteria for showing that it admits a Serre functor and then compute it explicitly, proving Theorem 1.2. We end with the proofs of Theorem 1.3 (Corollary 2.19) and Corollary 1.6 (Corollary 2.20).

Section 3 explains the relationship between crepant categorical resolutions, LG models, and variation of GIT.

Section 4 provides the required toric geometry to study the factorization category for toric complete intersections, setting up the next section. Here we recall the necessary definitions of cones associated to certain total spaces of invertible sheaves over toric stacks. We also recall the relevant machinery for studying variation of GIT on affine spaces and its relation to the secondary fan.

Section 5 provides sufficient criteria for when a factorization category associated to a toric Landau–Ginzburg model is FCY and explicitly computes its dimension in terms of the fan and the R-charge. We then prove a comparison theorem, Theorem 1.7, for two birational toric gauged Landau–Ginzburg models. We finish the section by considering the case where one or more of the Landau–Ginzburg models have a geometric interpretation as a complete intersection in a toric variety.

We end the paper with Section 6, where we provide a set of examples of our theorems, including a re-proving of Orlov's theorem, a semi-orthogonal decomposition with a geometric FCY category, and the generalizations of the cases of singular cubic fourfolds and a cubic fourfold containing two planes outlined by Examples 1.8 and 1.9 above.

2. Serre functors for Landau–Ginzburg models

In this section, we will prove that a certain class of triangulated categories associated to Landau–Ginzburg models admit an explicit Serre functor.

2.1 Background on factorizations

In order to keep the paper as self-contained as possible, we provide a summary of the language of factorizations; see [BFK14] for more details.

Let κ be an algebraically closed field of characteristic 0. Let X be a smooth variety over κ and G an affine algebraic group that acts on it via the map $\sigma\colon G \times X \to X$. Take w to be a G-invariant section of an invertible G-equivariant sheaf $\mathcal{L}$; that is, $w \in \Gamma(X, \mathcal{L})^G$.

DEFINITION 2.1. A *factorization* is the data $\mathcal{E} = \left(\mathcal{E}_{-1}, \mathcal{E}_0, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0\right)$, where $\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are G-equivariant quasi-coherent sheaves and

$$\mathcal{E}_{-1} \xrightarrow{\phi^{\mathcal{E}}_0} \mathcal{E}_0 \xrightarrow{\phi^{\mathcal{E}}_{-1}} \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}$$

are morphisms such that $\phi^{\mathcal{E}}_{-1} \circ \phi^{\mathcal{E}}_0 = w$ and $\left(\phi^{\mathcal{E}}_0 \otimes \mathcal{L}\right) \circ \phi^{\mathcal{E}}_{-1} = w$.

A morphism $f\colon \mathcal{E} \to \mathcal{F}[2k]$ of even degree between two factorizations is a pair $f = (f_0, f_{-1})$ defined by

$$\mathrm{Hom}^{2k}_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathrm{Qcoh}_G X}\left(\mathcal{E}_{-1}, \mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{k}\right) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_G X}\left(\mathcal{E}_0, \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{k}\right)$$

and, similarly, a morphism $f\colon \mathcal{E} \to \mathcal{F}[2k+1]$ of odd degree is a pair $f = (f_0, f_{-1})$ defined by

$$\mathrm{Hom}^{2k+1}_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathrm{Qcoh}_G X}\left(\mathcal{E}_0, \mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{k+1}\right) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_G X}\left(\mathcal{E}_{-1}, \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{k}\right).$$

You can equip these Hom sets with a differential coming from the graded commutator with the morphisms defining E and F. This yields a dg-category $\mathsf{Fact}(X, G, w)$. Also, denote by $\mathsf{fact}(X, G, w)$ the full dg-subcategory of $\mathsf{Fact}(X, G, w)$ whose components are coherent.

We now take a subcategory of $\mathsf{Fact}(X, G, w)$ with the same objects but with only the closed degree 0 morphisms between any two objects $\mathcal{E}$ and $\mathcal{F}$. Denote this subcategory by $Z^0\mathsf{Fact}(X, G, w)$. The category $Z^0\mathsf{Fact}(X, G, w)$ is abelian. Hence, the notion of a complex of objects in $Z^0\mathsf{Fact}(X, G, w)$ makes sense.

Given a complex of objects from $Z^0\mathsf{Fact}(X, G, w)$

$$\cdots \longrightarrow \mathcal{E}^b \xrightarrow{f^b} \mathcal{E}^{b+1} \xrightarrow{f^{b+1}} \cdots,$$

the *totalization* of the complex is the factorization $\mathcal{T} \in \mathsf{Fact}(X, G, w)$ given by the data

$$\begin{aligned}
\mathcal{T}_{-1} &:= \bigoplus_{i=2k} \mathcal{E}^i_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{-k} \oplus \bigoplus_{i=2k-1} \mathcal{E}^i_{0} \otimes_{\mathcal{O}_X} \mathcal{L}^{-k}, \\
\mathcal{T}_{0} &:= \bigoplus_{i=2k} \mathcal{E}^i_{0} \otimes_{\mathcal{O}_X} \mathcal{L}^{-k} \oplus \bigoplus_{i=2k+1} \mathcal{E}^i_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{-k}, \\
\phi^{\mathcal{T}}_0 &:= \bigoplus_{i=2k} f^i_0 \otimes \mathcal{L}^{-k} \oplus \bigoplus_{i=2k-1} f^i_{-1} \otimes \mathcal{L}^{-k}, \\
\phi^{\mathcal{T}}_1 &:= \bigoplus_{i=2k} f^i_{-1} \otimes \mathcal{L}^{-k} \oplus \bigoplus_{i=2k-1} f^i_{0} \otimes \mathcal{L}^{-k}.
\end{aligned} \tag{2.1}$$

Now, let $\mathsf{Acyc}(X, G, w)$ be the full subcategory of objects of $\mathsf{Fact}(X, G, w)$ consisting of totalizations of bounded exact complexes of $Z^0\mathsf{Fact}(X, G, w)$. Similarly, let $\mathsf{acyc}(X, G, w) = \mathsf{Acyc}(X, G, w) \cap \mathsf{fact}(X, G, w)$. Finally, we denote the homotopy category of a dg-category $\mathcal{C}$ by $[\mathcal{C}]$.

We have the following general definition.

Definition 2.2. The *absolute derived category* $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(X,G,w)]$ of $[\mathsf{Fact}(X,G,w)]$ is the Verdier quotient of $[\mathsf{Fact}(X,G,w)]$ by $[\mathsf{Acyc}(X,G,w)]$.

However, the category we focus on in this paper uses only coherent sheaves as objects. For this, we use the following slightly abbreviated notation.

Definition 2.3. The *absolute derived category* $\mathrm{D}^{\mathrm{abs}}[X,G,w]$ of $[\mathsf{fact}(X,G,w)]$ is the idempotent completion of the Verdier quotient of $[\mathsf{fact}(X,G,w)]$ by $[\mathsf{acyc}(X,G,w)]$. Equivalently, this is the full subcategory of $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(X,G,w)]$ split-generated by objects in $\mathsf{fact}(X,G,w)$.

The category $\mathrm{D}^{\mathrm{abs}}[X,G,w]$ can be thought of as the derived category of the gauged Landau–Ginzburg model (X,G,w).

Remark 2.4. The category $\mathrm{D}^{\mathrm{abs}}[X,G,w]$ is triangulated with shift functor

$$\mathcal{E}[1] := \left(\mathcal{E}_0, \mathcal{E}_{-1}\otimes\mathcal{L}, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0\otimes\mathcal{L}\right),$$

where $\mathcal{E} = \left(\mathcal{E}_{-1},\mathcal{E}_0,\phi^{\mathcal{E}}_{-1},\phi^{\mathcal{E}}_0\right)$. Note that, in particular,

$$[2] = -\otimes\mathcal{L}\,. \tag{2.2}$$

2.2 Serre functors on $\mathrm{D}^{\mathrm{abs}}[X,G,w]$

In this section, our goal is to calculate the Serre functor on $\mathrm{D}^{\mathrm{abs}}[X,G,w]$. We do this by first proving that, under certain assumptions, a certain dg-enhancement of $\mathrm{D}^{\mathrm{abs}}[X,G,w]$ is homologically smooth and proper. This implies that it admits a Serre functor, by a result of [Shk07].

Following [BFK14], we take our enhancement to be $\mathsf{Inj}_{\mathrm{coh}}(X,G,w)$, which is defined to be the full subcategory of $\mathsf{Fact}(X,G,w)$ consisting of objects with injective components which are isomorphic in $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(X,G,w)]$ to objects with coherent components.

Proposition 2.5 ([BFK14, Proposition 5.11]). *The dg-category* $\mathsf{Inj}_{\mathrm{coh}}(X,G,w)$ *is a dg-enhancement of* $\mathrm{D}^{\mathrm{abs}}[X,G,w]$.

We can then describe the Serre functor starting from the formal definition in [Shk07]. This requires a sequence of lemmas. Many of the technical aspects of these can be outsourced to [BFK14], which we cite often in this section. Hence, we follow the notation and conventions of ibid.

Let us start by collecting some notation and definitions. Suppose that G is an algebraic group acting on two algebraic varieties X and Y. We define the following shorthand for the global quotient stack:

$$X \overset{G}{\times} Y := [X\times Y/G]\,.$$

If H is a closed subgroup of G, we let H act on G by inverse multiplication on the right, $g\cdot h := hg^{-1}$, to define $G\overset{H}{\times} X$. Then we define an inclusion

$$\begin{aligned}\iota\colon X &\to G\overset{H}{\times} X\\ x &\mapsto (e,x)\,.\end{aligned}$$

By [Tho87, Lemma 1.3], the pullback functor ι^* induces the equivalences of equivariant categories of sheaves

$$\mathrm{Qcoh}_G\left(G\overset{H}{\times} X\right) \cong \mathrm{Qcoh}_H(X)\,.$$

DEFINITION 2.6. Let H be a closed subgroup of G, and assume that we have an action $\sigma\colon G\times X \to X$. Consider the inclusion map $\iota\colon X \to G \overset{H}{\times} X$ and the G-equivariant morphism $\alpha\colon G \overset{H}{\times} X \to X$ descending from the action $\sigma\colon G\times X \to X$. The *induction functor* is the composition

$$\operatorname{Ind}_H^G := \alpha_* \circ (\iota^*)^{-1} .$$

The induction functor allows us to remind the reader of the following notation from [BFK14]:

$$\nabla(\mathcal{F}) := \operatorname{Ind}_G^{G\times_{\mathbb{G}_m} G} \Delta_* \mathcal{F} ,$$

where Δ is the diagonal map.

Remark 2.7. The functors Ind_H^G and Δ_* are exact as Δ and α are both affine morphisms and ι^* is an equivalence of abelian categories. Hence, functors appearing in the definition of ∇ can be viewed in both the abelian and derived setting.

LEMMA 2.8. *Let $s, p\colon \ker\chi \times X \to X$ denote the action and the projection, respectively, and consider the map*

$$\begin{aligned}(s,p)\colon \ker\chi \times X &\to X \times X ,\\ (g,x) &\mapsto (gx, x) .\end{aligned}$$

Let $\mathcal{F}$ be a quasi-coherent sheaf. There is an isomorphism of quasi-coherent sheaves

$$\nabla(\mathcal{F}) \cong (s,p)_* s^* \mathcal{F} .$$

Proof. Let $G \times_{\mathbb{G}_m} G$ be the fiber product using the character $\chi\colon G \to \mathbb{G}_m$ for both factors. We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "\Delta"'] & (G \times_{\mathbb{G}_m} G) \overset{G}{\times} X \arrow[r, "\hat{\Phi}"] \arrow[d, "\tilde{\Delta}"'] & \ker\chi \times X \arrow[d, "\hat{\Delta}"'] \\
X \times X \arrow[r, "\iota"] & (G \times_{\mathbb{G}_m} G) \overset{G}{\times} X \times X \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & \ker\chi \times X \times X \arrow[dl, "(p,p)"] \\
 & X \times X ,&
\end{tikzcd}$$

where

$$\Delta(x) = (x,x) , \qquad j(x) = (e,e,x) ,$$
$$\tilde{\Delta}(g_1,g_2,x) = (g_1,g_2,x,x) , \qquad \hat{\Delta}(g,x) = (g,gx,x) ,$$
$$\Phi(g_1,g_2,x,y) = \big(g_1g_2^{-1}, g_1x, g_2y\big) , \qquad \hat{\Phi}(g_1,g_2,x) = \big(g_1g_2^{-1}, g_1x\big) .$$

We compute

$$\begin{aligned}\nabla(\mathcal{F}) &= \alpha_* \circ (\iota^*)^{-1}\Delta_*\mathcal{F} \;=\; \alpha_* \circ (\iota^*)^{-1}\Delta_* j^* \hat{\Phi}^* s^* \mathcal{F} \;=\; \alpha_* \circ \tilde{\Delta}_* \hat{\Phi}^* s^* \mathcal{F} \\ &= (p,p)_* \hat{\Delta}_* s^* \mathcal{F} \;=\; (s,p)_* s^* \mathcal{F} .\end{aligned}$$

□

DEFINITION 2.9. A dg-category A is called *homologically smooth* if A is a compact object of $D(A \otimes A^{\mathrm{op}} - \mathrm{Mod})$, that is, $A \in D_{\mathrm{perf}}(A \otimes A^{\mathrm{op}})$.

Definition 2.10. Consider a group G acting on a space X, and let w be a global function defined on X. We say that w is *semi-invariant* with respect to a character χ of G if, for any $g \in G$,

$$w(g \cdot x) = \chi(g) w(x)\,.$$

The global function w is semi-invariant if and only if w is a section of the equivariant line bundle $\mathcal{O}(\chi)$ on the global quotient stack $[X/G]$. This can also be written $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$.

For the rest of the paper, we restrict our attention to the case where w is a semi-invariant function.

Lemma 2.11. *Let X be a smooth algebraic variety and G an algebraic group acting on X. Let $\chi\colon G \to \mathbb{G}_m$ be a character and $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$ a semi-invariant function. Denote by ∂w the critical locus (with its reduced scheme structure). Assume that $[X/\ker\chi]$ has finite diagonal and that we have $\partial w \subseteq Z(w)$. Then, the dg-category $\mathsf{Inj}_{\mathrm{coh}}(X, G, w)$ is homologically smooth.*

Proof. The diagonal map for $[X/\ker\chi]$ is realized as

$$(s, p)\colon\ \ker\chi \times X \longrightarrow X \times X\,.$$

This is finite by assumption, hence proper. Therefore, $(s,p)_* \mathcal{O}_{\ker\chi \times X}$ is coherent.

By Lemma 2.8, we have $\nabla(\mathcal{O}_X) = (s,p)_* \mathcal{O}_{\ker\chi\times X}$. So, renotating, $\nabla(\mathcal{O}_X)$ is coherent. It follows from [BFK14, Proposition 3.15] that since $\nabla(\mathcal{O}_X)$ is an object of $\mathrm{fact}(X \times X, G \times_{\mathbb{G}_m} G, w \boxplus w)$, it is a compact object.

By [BFK14, Theorem 5.15], we have a dg-functor

$$\lambda_{w\boxplus w}\colon\ \mathsf{Inj}_{\mathrm{coh}}\big(X \times X, G \overset{\mathbb{G}_m}{\times} G, w \boxplus w\big) \longrightarrow \big(\mathsf{Inj}_{\mathrm{coh}}(X,G,w) \otimes \mathsf{Inj}_{\mathrm{coh}}(X,G,w)^{\mathrm{op}}\big) - \mathrm{Mod}$$

(here, we have implicitly used the assumption $\partial w \subseteq Z(w)$ to remove the support condition in the statement of Theorem 5.15 of ibid).

This induces an equivalence

$$\mathrm{D}^{\mathrm{abs}}\big[X \times X, G \overset{\mathbb{G}_m}{\times} G, w \boxplus w\big] \to D\big(\big(\mathsf{Inj}_{\mathrm{coh}}(X,G,w) \otimes \mathsf{Inj}_{\mathrm{coh}}(X,G,w)^{\mathrm{op}}\big) - \mathrm{Mod}\big)\,.$$

This equivalence takes ∇ to the bimodule $\mathsf{Inj}_{\mathrm{coh}}(X,G,w)$ by [BFK14, Lemma 3.54].

In conclusion, when viewed as a bimodule, $\mathsf{Inj}_{\mathrm{coh}}(X,G,w)$ is a compact object of

$$D\big(\mathsf{Inj}_{\mathrm{coh}}(X,G,w) \otimes \mathsf{Inj}_{\mathrm{coh}}(X,G,w)^{\mathrm{op}} - \mathrm{Mod}\big)\,;$$

that is, $\mathsf{Inj}_{\mathrm{coh}}(X,G,w)$ is cohomologically smooth. □

Remark 2.12. Any separated Deligne–Mumford stack has finite diagonal. Conversely, over $\mathbb{C}$, any stack with finite diagonal is separated and Deligne–Mumford.

Definition 2.13. A dg-category A is called *proper* if there exists a strong generator E of the homotopy category of A such that $\oplus_r \mathrm{H}^r(\mathrm{Hom}_A(E,E))$ is finite-dimensional.

Recall from [BFK14] that given a G-equivariant sheaf $\mathcal{F}$ supported on $Z(w)$, we can define a factorization

$$\Upsilon\mathcal{F} := (0, \mathcal{F}, 0, 0)\,,$$

using the notation given in Definition 2.1.

Lemma 2.14. *Let X be a smooth algebraic variety and G a linearly reductive algebraic group acting on X. Let $\chi\colon G \to \mathbb{G}_m$ be a nontrivial character. Assume that $[X/\ker\chi]$ has finite diagonal* *and is proper over* $\operatorname{Spec}\kappa$. *In addition, assume that we have the containment* $\partial w \subseteq Z(w)$. *Then,* $\mathsf{Inj}_{\mathrm{coh}}(X, G, w)$ *is a proper dg-category.*

Proof. By Lemma 2.11, the category $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ is homologically smooth. Hence, by [BFK14, Lemma 4.23], the diagonal object $\operatorname{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_* \mathcal{O}_X$ is generated by exterior products. Now, if $\operatorname{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_* \mathcal{O}_X$ is a summand of a finite sequence of cones of exterior products $\mathcal{E}_i \boxtimes \mathcal{F}_i$, then thinking of these exterior products as integral transforms expresses any object as a summand of a finite sequence of cones of some graded vector spaces tensored with the $\mathcal{F}_i$. Therefore, $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ admits a strong generator.

Now, we show that the category is Ext-finite so that, in particular, the cohomologies of the endomorphism algebra of a strong generator must be finite-dimensional. By [BFK14, Proposition 3.64 and Lemma 4.13], the category $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ is generated by objects of the form ΥE, where $E \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh}[\partial w/G])$. Since $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}[\partial w/G])$ is generated by sheaves, it suffices to show that

$$\bigoplus_r \operatorname{Hom}_{\mathrm{D}^{\mathrm{abs}}(\operatorname{fact} w)}(\Upsilon E, \Upsilon F[r])$$

is finite-dimensional for any $E, F \in \operatorname{coh}[\partial w/G]$.

That is, let E, F be G-equivariant coherent sheaves on ∂w. By [BDFIK16, Lemma 3.11], there is a spectral sequence whose E_1-page is

$$E_1^{p,q} = \begin{cases} \operatorname{Ext}^{p+q}_{[X/G]}(E, F \otimes \mathcal{O}_X(-s\chi))\,, & p = 2s\,, \\ 0\,, & p = 2s+1\,, \end{cases}$$

which strongly converges to $\bigoplus_r \operatorname{Hom}_{\mathrm{D}^{\mathrm{abs}}(\operatorname{fact} w)}(E, F[r])$.

Since X is smooth, we have $E_1^{p,q} = 0$ unless $0 \leqslant p + q \leqslant \dim X$. Now, since G and $\ker\chi$ are linearly reductive,

$$\begin{aligned} \bigoplus_{s\in\mathbb{Z}} \operatorname{Ext}^i_{[X/G]}(E, F \otimes \mathcal{O}_X(-s\chi)) &= \bigoplus_{s\in\mathbb{Z}} \operatorname{Ext}^i_X(E, F \otimes \mathcal{O}_X(-s\chi))^G \\ &\subseteq \operatorname{Ext}^i_X(E, F)^{\ker\chi} \\ &= \operatorname{Ext}^i_{[X/\ker\chi]}(E, F)\,. \end{aligned}$$

The right-hand side is finite-dimensional by assumption.

Therefore, there are finitely many pairs (s, i) where $\operatorname{Ext}^i_{[X/G]}(E, F \otimes \mathcal{O}_X(-s\chi))$ is nonzero. It follows that $E_1^{2s,q}$ is nonzero for finitely many q. Furthermore, since $E_1^{2s,q} = 0$ unless $0 \leqslant 2s+q \leqslant \dim X$, it follows that there are also finitely many values of s for which $E_1^{2s,q}$ is nonzero.

In conclusion, the spectral sequence is bounded and its terms are finite-dimensional. Hence, $\bigoplus_r \operatorname{Hom}_{\mathrm{D}^{\mathrm{abs}}(\operatorname{fact} w)}(E, F[r])$ is finite-dimensional, as desired. ☐

Remark 2.15. It is enough to assume that $[\partial w/\ker\chi]$ is only cohomologically proper. This means, essentially by definition, that $\operatorname{Ext}^i_{[X/\ker\chi]}(E, F)$ is finite-dimensional for any two coherent sheaves E and F, which is all that is used in the proof. The assumption that $[\partial w/\ker\chi]$ is proper propagates to other results in this section, which could also be replaced by cohomologically proper. For later applications in this paper, we will always have that $[\partial w/\ker\chi]$ is proper.

LEMMA 2.16. *Let* G *be an algebraic group acting on a smooth variety* X. *There is a* $G \times_{\mathbb{G}_m} G$*-equivariant isomorphism*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{X\times X}\big(\operatorname{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_* \mathcal{O}_X, \mathcal{O}_{X\times X}\big) \cong \operatorname{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_* \omega_X^{-1}[\dim G - 1 - \dim X]\,.$$

Proof. We formally compute

$$\begin{aligned}\mathbf{R}\mathcal{H}\mathrm{om}_{X\times X}\big(\mathrm{Ind}_G^{G\times_{\mathbb{G}_m}G}\Delta_*\mathcal{O}_X,\mathcal{O}_{X\times X}\big) &= \mathbf{R}\mathcal{H}\mathrm{om}_{X\times X}((p,s)_*\mathcal{O}_{\ker\chi\times X},\mathcal{O}_{X\times X})\\ &= (p,s)_*(p,s)^!\mathcal{O}_{X\times X}\\ &= (p,s)_*s^*\omega_X^{-1}[\dim G-1-\dim X]\\ &= \mathrm{Ind}_G^{G\times_{\mathbb{G}_m}G}\Delta_*\omega_X^{-1}[\dim G-1-\dim X]\,.\end{aligned}$$

The first line is Lemma 2.8. The second line is equivariant Grothendieck duality [Has09]. The third line is just a computation of the relative canonical bundle since $\ker\chi\times X$ and $X\times X$ are smooth. The fourth line is Lemma 2.8 again. □

Lemma 2.17. *Let $\mathcal{F}$ be a G-equivariant sheaf on $Z(w)$ such that*

$$\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{O}_X)\cong\mathcal{G}[-t]\,,$$

where $\mathcal{G}$ is also a G-equivariant sheaf on $Z(w)$. Then there is an isomorphism in $\mathrm{D}^{\mathrm{abs}}(\mathrm{fact}\,w)$,

$$\mathbf{R}\mathcal{H}\mathrm{om}_X(\Upsilon\mathcal{F},\mathcal{O}_X)\cong\Upsilon\mathcal{G}[-t]\,.$$

Proof. By [BFK14, Proposition 3.14], there exists an exact sequence of factorizations in the abelian category of factorizations

$$0\to\mathcal{V}_s\xrightarrow{d_s}\cdots\xrightarrow{d_1}\mathcal{V}_0\to\Upsilon\mathcal{F}\to 0$$

such that $\Upsilon\mathcal{F}$ is isomorphic to the totalization of the complex $\mathcal{V}_s\to\cdots\to\mathcal{V}_0$ in $\mathrm{D}^{\mathrm{abs}}[X,G,w]$. Therefore, $\mathbf{R}\mathcal{H}\mathrm{om}_X(\Upsilon\mathcal{F},\mathcal{O}_X)$ is isomorphic to the totalization of the complex

$$\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_0,\mathcal{O}_X)\xrightarrow{d_1^\vee}\cdots\xrightarrow{d_s^\vee}\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_s,\mathcal{O}_X)[-s]\,.$$

Now, notice that there are also exact sequences

$$\begin{aligned}&0\to\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_0,\mathcal{O}_X)\to\cdots\to\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_{t-1},\mathcal{O}_X)\to\mathrm{im}\left(d_t^\vee\right)\to 0\,,\\ &0\to\ker\left(d_{t+1}^\vee\right)\to\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_t,\mathcal{O}_X)\to\cdots\to\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_s,\mathcal{O}_X)\to 0\,,\end{aligned}$$

and

$$0\to\mathrm{im}\left(d_t^\vee\right)\to\ker\left(d_{t+1}^\vee\right)\to\Upsilon\mathcal{G}\to 0$$

in the abelian category of factorizations.

Hence, we have a distinguished triangle

$$\mathrm{im}\left(d_t^\vee\right)\to\ker\left(d_{t+1}^\vee\right)\to\Upsilon\mathcal{G}\to\mathrm{im}\left(d_t^\vee\right)[1] \tag{2.3}$$

in $\mathrm{D}^{\mathrm{abs}}[X,G,w]$. Denoting the totalization of

$$\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_0,\mathcal{O}_X)\to\cdots\to\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_{t-1},\mathcal{O}_X)$$

by A and the totalization of

$$\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_t,\mathcal{O}_X)\to\cdots\to\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_s,\mathcal{O}_X)$$

by B, we can replace the terms in the distinguished triangle (2.3) to obtain

$$A\to B[t-s]\to\Upsilon\mathcal{G}\to A[1]\,.$$

Hence, $\Upsilon\mathcal{G}[-t]$ is the cone of $A[-t]\to B[-s]$ in $\mathrm{D}^{\mathrm{abs}}[X,G,w]$. The totalization of

$$\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_s,\mathcal{O}_X)\to\cdots\to\mathbf{R}\mathcal{H}\mathrm{om}_X(\mathcal{V}_0,\mathcal{O}_X)[-s]$$

can also be described as the cone of $A[-t] \to B[-s]$. Hence, $\Upsilon G[-t]$ agrees with the derived dual of ΥF, as desired. □

Theorem 2.18. *Let X be a smooth algebraic variety and G a linearly reductive algebraic group acting on X. Let $\chi\colon G \to \mathbb{G}_m$ be a character and $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$. Assume that $[X/\ker\chi]$ has finite diagonal. In addition, assume $\partial w \subseteq Z(w)$ and that $[\partial w/\ker\chi]$ is proper. Then, $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ admits a Serre functor given by*

$$S := (- \otimes \omega_X)[\dim X - \dim G + 1]\,.$$

Proof. By Lemma 2.11, the category $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ is homologically smooth. Hence, by [BFK14, Lemma 4.23], the diagonal object $\mathrm{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_* \mathcal{O}_X$ is generated by exterior products. It follows that $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ admits a strong generator. Hence,

$$\mathrm{D}^{\mathrm{abs}}[X, G, w] \cong D_{\mathrm{perf}}(A)$$

for a dg-algebra A.

Since $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ is homologically smooth (Lemma 2.11) and proper (Lemma 2.14), so is A. Hence, by [Shk07, Theorems 4.2 and 4.4], it admits a Serre functor whose inverse is given formally by

$$A^! := \mathrm{RHom}_{A^{\mathrm{op}}\otimes A}\left(A, A^{\mathrm{op}} \otimes A\right).$$

By [BFK14, Lemma 3.30], we have $A^{\mathrm{op}} \cong \mathrm{D}^{\mathrm{abs}}[X, G, -w]$. Now, by [BFK14, Theorem 5.15], the dg-category A^e is quasi-equivalent to $\mathsf{fact}[X \times X, G \times_{\mathbb{G}_m} G, w \boxplus -w]$ and $A^!$ is identified with $\mathbf{R}\mathcal{H}\mathrm{om}_{X\times X}(\nabla(\mathcal{O}_X), \mathcal{O}_{X\times X})$. Now,

$$\begin{aligned} S^{-1} &= \mathbf{R}\mathcal{H}\mathrm{om}_{X\times X}(\nabla(\mathcal{O}_X), \mathcal{O}_{X\times X}) \\ &= \mathbf{R}\mathcal{H}\mathrm{om}_{X\times X}\big(\Upsilon\,\mathrm{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_* \mathcal{O}_X, \mathcal{O}_{X\times X}\big) \\ &\cong \Upsilon\mathbf{R}\mathcal{H}\mathrm{om}_{X\times X}\big(\mathrm{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_* \mathcal{O}_X, \mathcal{O}_{X\times X}\big) \\ &\cong \Upsilon\,\mathrm{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_*\omega_X^{-1}[\dim G - 1 - \dim X]\,. \end{aligned}$$

The second line is by definition. The third line is Lemma 2.17. The fourth line is Lemma 2.16. Finally, as an integral kernel, $\Upsilon\,\mathrm{Ind}_G^{G\times_{\mathbb{G}_m}G} \Delta_*\omega_X^{-1}[\dim G - 1 - \dim X]$ is just

$$\big(- \otimes \omega_X^{-1}\big)[\dim G - 1 - \dim X]\,,$$

by [BFK14, Lemma 3.54]. The inverse to this functor is $S = (- \otimes \omega_X)[\dim X - \dim G + 1]$. □

Corollary 2.19. *Let X be a smooth algebraic variety and G a linearly reductive algebraic group acting on X. Let $\chi\colon G \to \mathbb{G}_m$ be a character and $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$. Assume that $[X/\ker\chi]$ has finite diagonal and ω_X is torsion as a $\ker\chi$-equivariant line bundle. In addition, assume $\partial w \subseteq Z(w)$ and that $[\partial w/\ker\chi]$ is proper. Then, $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ is fractional Calabi–Yau. If the canonical bundle of $[X/\ker\chi]$ is trivial, then this category is Calabi–Yau.*

Proof. By Theorem 2.18, the Serre functor on $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ is given by

$$S = (- \otimes \omega_X)[\dim X - \dim G + 1]\,,$$

where ω_X has the natural G-equivariant structure. Applying $\mathrm{Hom}(-, \mathbb{G}_m)$ to the exact sequence

$$0 \to \ker\chi \to G \to \mathbb{G}_m \to 0\,,$$

we get

$$0 \to \mathbb{Z} \to \widehat{G} \to \widehat{\ker\chi} \to 0\,, \tag{2.4}$$

where $\mathbb{Z}$ is generated by the character χ.

Now, by assumption, we have $\omega_X^{\otimes l} = \mathcal{O}_X$ with its natural $\ker\chi$-equivariant structure; that is, $\omega_X^{\otimes l}$ is in the kernel of the map $\widehat{G} \to \widehat{\ker\chi}$. Therefore, from the exact sequence (2.4), we have

$$\omega_X^{\otimes l} = \mathcal{O}_X(\chi)^{\otimes m}$$

as G-equivariant sheaves, for some $m \in \mathbb{Z}$.

Using equation (2.2), there is a natural isomorphism of functors $- \otimes \mathcal{O}_X(\chi) = [2]$. Hence,

$$\begin{aligned} S^l &= \big(- \otimes \omega_X^{\otimes l}\big)[l(\dim X - \dim G + 1)] \\ &= \big(- \otimes \mathcal{O}_X(\chi)^{\otimes m}\big)[l(\dim X - \dim G + 1)] \\ &= [l(\dim X - \dim G + 1) + 2m]\,. \end{aligned}$$

The Calabi–Yau case is when $l = 1$. ☐

Corollary 2.20. *Conjecture 1.5 holds for elementary wall crossings (see Definition 3.10).*

Proof. This follows immediately from [BFK17, Proposition 4.3.8] and Corollary 2.19. ☐

3. Crepant categorical resolutions via LG models

Let Z be a variety with a G-action and $\mathcal{D}$ an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G])$. We denote by $\mathcal{D}^{\mathrm{perf}}$ the full subcategory of $\mathcal{D}$ consisting of G-equivariant perfect complexes on Z.

Definition 3.1. Let $\tilde{\mathcal{D}}$ be the homotopy category of a homologically smooth and proper pretriangulated dg-category. A pair of exact functors

$$F\colon \tilde{\mathcal{D}} \to \mathcal{D}\,, \quad G\colon \mathcal{D}^{\mathrm{perf}} \to \tilde{\mathcal{D}}$$

is a *categorical resolution of singularities* if G is left adjoint to F and the natural morphism of functors $\mathrm{Id}_{\mathcal{D}^{\mathrm{perf}}} \to FG$ is an isomorphism. We say that the categorical resolution of singularities is *crepant* if G is also right adjoint to F.

Remark 3.2. The definition presented here is slightly different than that of [Kuz10]. There, $\tilde{\mathcal{D}}$ is required to be an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$, where X is a smooth variety. This definition is in lieu of requiring $\mathcal{D}$ to be a homologically smooth and proper triangulated dg-category. All examples in this paper will be crepant categorical resolutions in both senses.

Let U be a variety with the action of a linearly reductive group G, χ a character of G, and w a section of $\mathcal{O}_U(\chi)$. Let

$$i\colon V \to U$$

be a G-equivariant open immersion. We have a (both left- and right-)adjoint pair of functors between categories of factorizations with quasi-coherent components

$$\begin{aligned} i_*\colon\ & \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}\ V, G, w] \to \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}\ U, G, w]\,, \\ i^*\colon\ & \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}\ U, G, w] \to \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}\ V, G, w]\,. \end{aligned}$$

Note that since i is an open immersion, i_* is both left and right adjoint to i^*.

Definition 3.3. Let $\mathrm{D}^{\mathrm{abs}}[V, G, w]_{\mathrm{rel}\,U}$ denote the full subcategory of $\mathrm{D}^{\mathrm{abs}}[V, G, w]$ consisting of factorizations $\mathcal{F}$ such that the closure of the support of $\mathcal{F}$ as a subset of U does not intersect $U \backslash V$.

Then, the adjunction between i_* and i^* restricts to

$$\begin{aligned} i_* \colon\ \mathrm{D}^{\mathrm{abs}}[V, G, w]_{\mathrm{rel}\, U} &\to \mathrm{D}^{\mathrm{abs}}[U, G, w]\,, \\ i^* \colon\ \mathrm{D}^{\mathrm{abs}}[U, G, w] &\to \mathrm{D}^{\mathrm{abs}}[V, G, w]\,. \end{aligned}$$

Let Y be a smooth quasi-projective variety with a G-action. Suppose that s is a regular section of a G-equivariant vector bundle $\mathcal{E}$ on Y with vanishing locus $Z := Z(s)$. Let $\mathbb{G}_m$ act on $\operatorname{tot}(\mathcal{E}^\vee)$ by fiberwise dilation, and consider the pairing $w = \langle s, - \rangle$ as a section of $\mathcal{O}_{\operatorname{tot}(\mathcal{E}^\vee)}(\chi)$, where χ is the projection character.

DEFINITION 3.4. We define the gauged Landau–Ginzburg model associated to the complete intersection Z to be the data $\big(\operatorname{tot}(\mathcal{E}^\vee), G \times \mathbb{G}_m, w\big)$.

The following theorem is originally due to Isik [Isi13] and Shipman [Shi12] and due to Hirano [Hir17] in the G-equivariant case, which is the case we will use.

THEOREM 3.5 ([Hir17, Proposition 4.8]). *Assume that w is a regular section of $\mathcal{E}$. There is an equivalence of categories*

$$\Omega \colon\ \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G]) \to \mathrm{D}^{\mathrm{abs}}\left[\operatorname{tot}\left(\mathcal{E}^\vee\right), G \times \mathbb{G}_m, \langle w, - \rangle\right].$$

The following lemma is the G-equivariant case of [Shi12, Remark 3.7].

LEMMA 3.6. *Assume that Y admits a G-ample line bundle. The equivalence of categories*

$$\Omega \colon\ \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G]) \to \mathrm{D}^{\mathrm{abs}}[\operatorname{tot}(\mathcal{E}^\vee), G \times \mathbb{G}_m, w]$$

restricts to an equivalence between the full subcategory of perfect objects $\mathrm{Perf}[Z/G]$ *and the full subcategory of* $\mathrm{D}^{\mathrm{abs}}[\operatorname{tot}(\mathcal{E}^\vee), G \times \mathbb{G}_m, w]$ *with objects supported on the zero section of $\mathcal{E}^\vee$.*

Proof. Recall that the functor Ω equals $j_*(\pi|_Z)^*$, where $\pi|_Z \colon \operatorname{tot}(\mathcal{E}^\vee)|_Z \to Z$ is the projection and $j \colon \operatorname{tot}(\mathcal{E}^\vee)|_Z \to \operatorname{tot}\mathcal{E}^\vee$ is the inclusion. To clarify the notation, there is also a map $\pi \colon \operatorname{tot}(\mathcal{E}^\vee) \to Y$. Let $h \colon Z \to Y$ be the inclusion. Since Y is quasi-projective with a G-ample line bundle $\mathcal{L}$, the category $\mathrm{Perf}[Z/G]$ is generated by objects of the form $h^*\mathcal{L}^{\otimes n}$.

Since $h^*\mathcal{L}^{\otimes n}$ is a generator of $\mathrm{Perf}[Z/G]$, it is enough to check that $\Omega(h^*\mathcal{L}^{\otimes n})$ is supported on the zero section of $\mathcal{E}^\vee$ and that the objects $\Omega(h^*\mathcal{L}^{\otimes n})$ generate the full subcategory of $\mathrm{D}^{\mathrm{abs}}[\operatorname{tot}(\mathcal{E}^\vee), G \times \mathbb{G}_m, w]$ with objects supported on the zero section of $\mathcal{E}^\vee$. Now, we have

$$\begin{aligned} \Omega(h^*\mathcal{L}^{\otimes n}) &= j_*(\pi|_Z)^* h^*\mathcal{L}^{\otimes n} \ = \ j_* j^* \pi^* \mathcal{L}^{\otimes n} \\ &\cong (0, \mathcal{O}_{Z(\pi^* s)}, 0, 0) \otimes \pi^*\mathcal{L}^{\otimes n} \\ &\cong (\mathcal{O}_{Z(\mathrm{taut})}, 0, 0, 0) \otimes \det(\mathcal{E}) \otimes \pi^*\mathcal{L}^{\otimes n}[-\operatorname{rk}\mathcal{E}] \\ &\cong (\det(\mathcal{E}) \otimes \pi^*\mathcal{L}^{\otimes n}|_{Z(\mathrm{taut})}, 0, 0, 0)[-\operatorname{rk}\mathcal{E}]\,. \end{aligned}$$

Line three is [BFK14, Proposition 3.20].

First, this shows, in particular, that $\Omega(h^*\mathcal{L}^{\otimes n})$ is supported on $Z(\mathrm{taut})$, the zero section of $\mathcal{E}^\vee$. Second, let $\mathcal{F} = (\mathcal{F}_{-1}, \mathcal{F}_0, \phi^{\mathcal{F}}_{-1}, \phi^{\mathcal{F}}_0)$ be an object of $\mathrm{D}^{\mathrm{abs}}[\operatorname{tot}(\mathcal{E}^\vee), G \times \mathbb{G}_m, w]$ supported on the zero section of $\mathcal{E}^\vee$. We aim to show that $\mathcal{F}$ is generated by objects of the form $(\det(\mathcal{E}) \otimes \mathcal{L}^{\otimes n}|_{Z(\mathrm{taut})}, 0, 0, 0)$. For this, notice that

$$Z(w) = Z(\mathrm{taut}) \cup Z(\pi^* s)\,.$$

Now, the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z(w)/G \times \mathbb{G}_m])$ consisting of objects supported on $Z(\mathrm{taut})$ is generated by the essential image of the pushforward. Since $\det(\mathcal{E}) \otimes \mathcal{L}^{\otimes n}$ generates $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Y/G])$, we may just use objects of the form $\det(\mathcal{E}) \otimes \mathcal{L}^{\otimes n}|_{Z(\mathrm{taut})}$. Finally, under the equivalence

$$\mathrm{D}_{sg}[Z(w)/G \times \mathbb{G}_m] \to \mathrm{D}^{\mathrm{abs}}[\mathrm{tot}(\mathcal{E}^\vee), G \times \mathbb{G}_m, w]$$

(see [Hir17, Theorem 3.6]), these objects go precisely to $(\det(\mathcal{E}) \otimes \mathcal{L}^{\otimes n}|_{Z(\mathrm{taut})}, 0, 0, 0)$ and objects supported on $Z(\mathrm{taut})$ go to objects supported on $Z(\mathrm{taut})$, as desired. □

Theorem 3.7. *With the setup as above, assume that Y admits a G-ample line bundle. Let U be a $G \times \mathbb{G}_m$-equivariant partial compactification of* $\mathrm{tot}(\mathcal{E}^\vee)$. *Assume that*

- *w extends to U as a section of $\mathcal{O}(\chi)$,*
- *$[U/G]$ has finite diagonal, and*
- *$[\partial w/G] \subseteq [U/G]$ is proper over* $\operatorname{Spec} \kappa$ *and $\partial w \subseteq Z(w)$ in U.*

Then, the functors

$$i_* \circ \Omega \colon \ \mathrm{Perf}([Z/G]) \to \mathrm{D}^{\mathrm{abs}}[U, G, w]\,,$$
$$\Omega^{-1} \circ i^* \colon \ \mathrm{D}^{\mathrm{abs}}[U, G, w] \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G])$$

form a crepant categorical resolution.

Proof. The assumptions ensure that $\mathrm{D}^{\mathrm{abs}}[U, G, w]$ is the homotopy category of a homologically smooth and proper dg-category, by Lemmas 2.11 and 2.14 and [BFK14, Proposition 5.11]. Since i is an open immersion, the functors are both left and right adjoint. Furthermore, the adjunction morphism factors via the following natural isomorphisms:

$$\Omega^{-1} \circ i^* \circ i_* \circ \Omega \cong \Omega^{-1} \circ \mathrm{Id}_{\mathrm{D}^{\mathrm{abs}}[\mathrm{tot}(\mathcal{E}^\vee), G \times \mathbb{G}_m, w]} \circ \Omega \cong \mathrm{Id}_{\mathrm{Perf}([Z/G])}\,. \qquad \square$$

Remark 3.8. An extension of a general w need not exist. We will give two examples of such extensions in the toric case in Subsections 6.3 and 6.4.

We now give the general framework for identifying crepant categorical resolutions for factorization categories using variations of geometric invariant theory quotients. We finish with a general theorem. In the following sections, we will specialize to the toric case. The reader more interested in toric applications can refer to Section 5 for a specialization of Theorem 3.12 to toric varieties, namely Theorem 5.7.

Definition 3.9. Let $\lambda \colon \mathbb{G}_m \to G$ be a 1-parameter subgroup of G. We shall denote a connected component of U^λ by Z_λ^0. To Z_λ^0, we can associate another subvariety

$$Z_\lambda := \left\{ u \in U \mid \lim_{\alpha \to 0} \sigma(\lambda(\alpha), u) \in Z_\lambda^0 \right\}.$$

We call Z_λ the *contracting variety* associated to Z_λ^0.

We will also close up these varieties under the action of G. We set

$$S_\lambda^0 := G \cdot Z_\lambda^0 \quad \text{and} \quad S_\lambda := G \cdot Z_\lambda\,.$$

Also, set

$$P(\lambda) := \left\{ g \in G \mid \lim_{\alpha \to 0} \lambda(\alpha) g \lambda(\alpha)^{-1} \text{ exists} \right\},$$

and

$$U_+ := U \setminus S_\lambda\,, \quad U_- := U \setminus S_{-\lambda}\,.$$

Definition 3.10. Let U be a smooth, quasi-projective variety equipped with a G-action and $\lambda\colon \mathbb{G}_m \to G$ a 1-parameter subgroup. Fix a connected component Z_λ^0 of the fixed locus. Assume that

- the morphisms $\tau_{\pm\lambda}\colon G \overset{P(\pm\lambda)}{\times} Z_{\pm\lambda} \to S_{\pm\lambda}$ are isomorphisms;
- the subsets $S_{\pm\lambda}$ are closed.

Under these assumptions, the pair of stratifications

$$U = U_+ \sqcup S_\lambda\,, \quad U = U_- \sqcup S_{-\lambda}$$

is called an *elementary wall crossing.*

Definition 3.11. Let V be a smooth variety with a G-action. Fix a 1-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ and a connected component Z_λ^0 of the fixed locus for the G action on V such that $V = V_\pm \sqcup S_{\pm\lambda}$ is an elementary wall crossing. Let U be a smooth variety with a G-action. We say that a G-equivariant open immersion $V \to U$ is *compatible* with an elementary wall crossing if $Z_{\pm\lambda}$ remains closed in U.

Given an elementary wall crossing $U = U_+ \sqcup S_\lambda$, $U = U_- \sqcup S_{-\lambda}$, we let

$$t(\mathfrak{K}^\pm) := \mu(\omega_{S_{\pm\lambda}|U}, \pm\lambda, u)$$

for $u \in Z_\lambda^0$, where μ is the Hilbert–Mumford numerical function. Here,

$$\omega_{S_{\pm\lambda}|U} = \bigwedge^{\operatorname{codim} S_{\pm\lambda}} \mathcal{N}^\vee_{S_{\pm\lambda}|U}$$

is the relative canonical sheaf of the embedding, $S_{\pm\lambda} \to U$.

Theorem 3.12 ([BFK17, Theorem 3.5.2]). *Let U be a smooth, quasi-projective variety equipped with the action of a reductive linear algebraic group G. Let $w \in \mathrm{H}^0(U, \mathcal{L})^G$ be a G-invariant section of a G-line bundle $\mathcal{L}$, and assume $\mu(\mathcal{L}, \lambda, u) = 0$ for $u \in Z_\lambda^0$.*

Assume that we have an elementary wall crossing

$$U = U_+ \sqcup S_\lambda\,, \quad U = U_- \sqcup S_{-\lambda}$$

and that S_λ^0 admits a G-invariant affine open cover. Fix $d \in \mathbb{Z}$. For the following functors, abuse the notation to also let them represent their essential image. Then the following hold:

(i) *If $t(\mathfrak{K}^+) < t(\mathfrak{K}^-)$, then there are fully faithful functors*

$$\Phi_d^+\colon\ \mathrm{D}^{\mathrm{abs}}[U_-, G, w|_{U_-}] \to \mathrm{D}^{\mathrm{abs}}[U_+, G, w|_{U_+}]$$

and, for $-t(\mathfrak{K}^-) + d \leqslant j \leqslant -t(\mathfrak{K}^+) + d - 1$,

$$\Upsilon_j^+\colon\ \mathrm{D}^{\mathrm{abs}}[w|_{Z_\lambda^0}, C(\lambda), w|_{Z_\lambda^0}]_j T \to \mathrm{D}^{\mathrm{abs}}[U_+, G, w|_{U_+}]\,,$$

and a semi-orthogonal decomposition,

$$\mathrm{D}^{\mathrm{abs}}[U_+, G, w|_{U_+}] = \langle \Upsilon^+_{-t(\mathfrak{K}^-)+d}, \dots, \Upsilon^+_{-t(\mathfrak{K}^+)+d-1}, \Phi_d^+ \rangle\,.$$

(ii) *If $t(\mathfrak{K}^+) = t(\mathfrak{K}^-)$, then there is an exact equivalence*

$$\Phi_d^+\colon\ \mathrm{D}^{\mathrm{abs}}[U_-, G, w|_{U_-}] \to \mathrm{D}^{\mathrm{abs}}[U_+, G, w|_{U_+}]\,.$$

(iii) *If $t(\mathfrak{K}^+) > t(\mathfrak{K}^-)$, then there are fully faithful functors*

$$\Phi_d^-\colon\ \mathrm{D}^{\mathrm{abs}}[U_+, G, w|_{U_+}] \to \mathrm{D}^{\mathrm{abs}}[U_-, G, w|_{U_-}]$$

and, for $-t(\mathfrak{K}^+)+d \leqslant j \leqslant -t(\mathfrak{K}^-)+d-1$,

$$\Upsilon_j^-\colon \mathrm{D}^{\mathrm{abs}}[Z_\lambda^0, C(\lambda), w|_{Z_\lambda^0}]_j \to \mathrm{D}^{\mathrm{abs}}[U_-, G, w|_{U_-}]\,,$$

and a semi-orthogonal decomposition,

$$\mathrm{D}^{\mathrm{abs}}[U_-, G, w|_{U_-}] = \left\langle \Upsilon^-_{-t(\mathfrak{K}^+)+d}, \ldots, \Upsilon^-_{-t(\mathfrak{K}^-)+d-1}, \Phi_d^- \right\rangle.$$

Lemma 3.13. *Suppose that* $V \to U$ *is a* G*-equivariant open immersion which is compatible with an elementary wall crossing* $V = V_\pm \sqcup S_\pm$*. Then, the fully faithful functor*

$$\Phi_d^\pm\colon \mathrm{D}^{\mathrm{abs}}[U_\mp, G, w] \to \mathrm{D}^{\mathrm{abs}}[U_\pm, G, w]$$

restricts to a functor

$$\Phi_d^\pm\colon \mathrm{D}^{\mathrm{abs}}[V_\mp, G, w]_{\mathrm{rel}\, U_\mp} \to \mathrm{D}^{\mathrm{abs}}[V_\pm, G, w]_{\mathrm{rel}\, U_\pm}\,.$$

Proof. Without loss of generality, we consider just the case

$$\Phi_d^+\colon \mathrm{D}^{\mathrm{abs}}[U_-, G, w] \to \mathrm{D}^{\mathrm{abs}}[U_+, G, w]\,.$$

Let $\mathcal{F}$ be an object of $\mathrm{D}^{\mathrm{abs}}[U_-, G, w]$ whose support does not intersect $U_-\backslash V_-$. The functor Φ_d^+ is constructed in the proof of [BFK17, Theorem 3.5.2]. By definition, it is constructed precisely so that there is an object $\tilde{F} \in \mathrm{D}^{\mathrm{abs}}[U, G, w]$ whose restriction to U_- is $\mathcal{F}$ and whose restriction to U_+ is $\Phi_d^+(\mathcal{F})$. This means that the support of $\tilde{F}$ is contained in $\operatorname{Supp}\mathcal{F} \cup S_\lambda$. Now,

$$(\operatorname{Supp}\mathcal{F} \cup S_\lambda) \cap U\backslash V = \emptyset$$

by the assumption that the wall crossing is compatible. Hence,

$$\operatorname{Supp}\Phi_d^+(\mathcal{F}) \cap U_+\backslash V_+ \subseteq (\operatorname{Supp}\mathcal{F} \cup S_\lambda \cap U_+) \cap U\backslash V = \emptyset\,,$$

as desired. □

Now, suppose that $V \to U$ is a G-equivariant open immersion which is compatible with an elementary wall crossing $V = V_\pm \sqcup S_\pm$ such that $[V_+/G]$ is isomorphic to $\operatorname{tot}(\mathcal{E}^\vee)$ over $[Y/G]$. Denote by $\mathrm{D}^{\mathrm{abs}}[V_-, G, w]^{\mathrm{perf}}$ the full subcategory of $\mathrm{D}^{\mathrm{abs}}[V_-, G, w]$ whose image under $\Omega^{-1} \circ \Phi_d^+$ lies in $\operatorname{Perf}[Z/G]$.

Assume further that the zero section of $[V_+/G]$ does not intersect $[U_+\backslash V_+/G]$. Then, by Lemmas 3.6 and 3.13, the category $\mathrm{D}^{\mathrm{abs}}[V_-, G, w]^{\mathrm{perf}}$ is a full subcategory of $\mathrm{D}^{\mathrm{abs}}[V_-, G, w]_{\mathrm{rel}\, U_-}$.

Finally, recall that we have a pair of functors

$$i_*\colon \mathrm{D}^{\mathrm{abs}}[V_-, G, w]_{\mathrm{rel}\, U_-} \to \mathrm{D}^{\mathrm{abs}}[U_-, G, w]\,, \quad i^*\colon \mathrm{D}^{\mathrm{abs}}[U_-, G, w] \to \mathrm{D}^{\mathrm{abs}}[V_-, G, w]\,.$$

These restrict to a pair of functors

$$i_*\colon \mathrm{D}^{\mathrm{abs}}[V_-, G, w]^{\mathrm{perf}} \to \mathrm{D}^{\mathrm{abs}}[U_-, G, w]\,, \quad i^*\colon \mathrm{D}^{\mathrm{abs}}[U_-, G, w] \to \mathrm{D}^{\mathrm{abs}}[V_-, G, w]\,.$$

Theorem 3.14. *Suppose that* $V \to U$ *is a* G*-equivariant open immersion which is compatible with an elementary wall crossing* $V = V_\pm \sqcup S_\pm$ *such that*

- $[V_+/G]$ *is isomorphic to* $\operatorname{tot}(\mathcal{E}^\vee)$ *over* $[Y/G]$,
- *the zero section of* $[V_+/G]$ *does not intersect* $[U_+\backslash V_+/G]$,
- $t(\mathfrak{K}^+) \leqslant t(\mathfrak{K}^-)$ *and* w *extends to* U *as a section of* $\mathcal{O}(\chi)$,
- $[U_-/G]$ *has finite diagonal,* $[\partial w/G] \subseteq [U_-/G]$ *is proper over* $\operatorname{Spec}\kappa$*, and*
- $\partial w \subseteq Z(w)$ *in* V_-.

Then, the functors

$$i_*\colon \mathrm{D}^{\mathrm{abs}}[V_-, G, w]^{\mathrm{perf}} \to \mathrm{D}^{\mathrm{abs}}[U_-, G, w]\,, \quad i^*\colon \mathrm{D}^{\mathrm{abs}}[U_-, G, w] \to \mathrm{D}^{\mathrm{abs}}[V_-, G, w]$$

form a crepant categorical resolution.

Proof. The assumptions ensure that $\mathrm{D}^{\mathrm{abs}}[U_-, G, w]$ is the homotopy category of a homologically smooth and proper dg-category by Lemmas 2.11 and 2.14 and [BFK14, Proposition 5.11]. Since i is an open immersion, the functors are both left and right adjoint and $i^* \circ i_*$ is the identity. ☐

Remark 3.15. Theorem 3.14 has a natural context for resolving factorization categories associated to Landau–Ginzburg models that correspond to nonsmooth toric complete intersections. We see such a natural context for using Theorem 3.14 in the proof of Proposition 6.11 in the section Examples.

4. Toric Landau–Ginzburg models: Their cones and phases

4.1 Polytopes and Gorenstein cones

In this subsection, we will review standard definitions in order to set notation. Good references are [CLS11, BN08]. Let M and N be dual lattices of dimension d, and set $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let σ be a strictly convex cone in $N_{\mathbb{R}}$ of dimension d. Recall that the dual cone $\sigma^\vee$ in $M_{\mathbb{R}}$ is defined to be $\sigma^\vee := \big\{m \in M_{\mathbb{R}} \,|\, \langle m, n\rangle \geqslant 0 \text{ for all } n \in \sigma\big\}$.

Definition 4.1. A full-dimensional strictly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is called

(i) *Gorenstein* if there exists an element $\mathfrak{m} \in M$ such that the semigroup $\sigma \cap N$ is generated by finitely many lattice points $n \in N$ that are contained in the affine hyperplane $\{n \in N_{\mathbb{R}} \,|\, \langle \mathfrak{m}, n\rangle = 1\}$;

(ii) *almost Gorenstein* if there exists an element $\mathfrak{m} \in M$ such that the cone is generated over $\mathbb{Q}$ by finitely many lattice points in $\{n \in N \,|\, \langle \mathfrak{m}, n\rangle = 1\}$; and

(iii) $\mathbb{Q}$-*Gorenstein* if there exists an element $\mathfrak{m} \in M_{\mathbb{Q}}$ such that the cone is generated over $\mathbb{Q}$ by finitely many lattice points in $\{n \in N_{\mathbb{R}} \,|\, \langle \mathfrak{m}, n\rangle = 1\}$.

Example 4.2. (1) With respect to the lattice $N = \mathbb{Z}^2$, the cone $\sigma = \mathrm{Cone}((1,1), (-1,1))$ is Gorenstein with $\mathfrak{m} = (0,1)$.

(2) With respect to the lattice $N = \mathbb{Z}^4$, the cone $\sigma = \mathrm{Cone}((1,0,0,0), (0,1,0,0), (0,0,1,0), (-1,-1,-1,2))$ is almost Gorenstein with $\mathfrak{m} = (1,1,1,2)$, but not Gorenstein. Indeed, $n = (0,0,0,1)$ is a generator of the semigroup $\sigma \cap N$, but $\langle \mathfrak{m}, n\rangle = 2$.

(3) With respect to the lattice $N = \mathbb{Z}^2$, the cone $\sigma = \mathrm{Cone}((1,2), (-1,2))$ is $\mathbb{Q}$-Gorenstein with $\mathfrak{m} = (0, \frac{1}{2}) \in M_{\mathbb{Q}}$, but not almost Gorenstein.

As the cone σ is full-dimensional, the lattice element $\mathfrak{m}$ is unique. Moreover, $\mathfrak{m}$ is in the interior of the dual cone $\sigma^\vee$, since it does not pair to 0 with any nonzero element of the cone σ. We define the *kth slice* of the cone σ to be the polytope

$$\sigma_{(k)} := \{n \in \sigma \,|\, \langle \mathfrak{m}, n\rangle = k\}\,.$$

If, in addition, the dual cone $\sigma^\vee$ is a $\mathbb{Q}$-Gorenstein cone with respect to an element $\mathfrak{n} \in N_{\mathbb{Q}}$, we can define the *index* r of σ to be the pairing $\langle \mathfrak{m}, \mathfrak{n}\rangle$. Since $\mathfrak{m} \in M_{\mathbb{Q}}$ and $\mathfrak{n} \in N_{\mathbb{Q}}$ defined above are unique, the index is well defined. Note that the index may be a rational number.

Let us now take a minor detour to the realm of polytopes in order to set up the definition of t-split $\mathbb{Q}$-Gorenstein cones. Take M to be a lattice. Consider t lattice polytopes $\Delta_1, \dots, \Delta_t$ that are positive-dimensional in a real vector space $M_{\mathbb{R}}$. We define the *Cayley polytope* $\Delta_1 * \cdots * \Delta_t$ associated to the polytopes $\Delta_1, \dots, \Delta_t$ to be the polytope in the vector space $M_{\mathbb{R}} \oplus \mathbb{R}^t$ defined by

$$\Delta_1 * \cdots * \Delta_t := \mathrm{Conv}((\Delta_1, e_1), \dots, (\Delta_t, e_t)) ,$$

where the e_i are the elementary basis vectors for the vector space $\mathbb{R}^t$.

Definition 4.3. A polytope Δ is called a *Cayley polytope of length t* if $\Delta = \Delta_1 * \cdots * \Delta_t$ for some $\Delta_1, \dots, \Delta_t$.

4.2 Toric vector bundles

In this subsection, we will give examples of algebro-geometric manifestations of the cones described in the previous subsection. They show up as supports of fans associated to certain toric vector bundles.

Recall the following construction of a split toric vector bundle over a toric variety. Start with a toric variety X_Σ associated to a fan $\Sigma \subseteq N_{\mathbb{R}}$. Any torus-invariant Weil divisor D can be written as a linear combination of torus-invariant divisors associated to rays, that is,

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$$

for some $a_\rho \in \mathbb{Z}$. Take r such torus-invariant Weil divisors $D_i = \sum_{\rho \in \Sigma} a_{i\rho} D_\rho$ for some $a_{i\rho} \in \mathbb{Z}$. Let u_ρ be the primitive generator of the ray $\rho \in \Sigma(1)$. For all $\sigma \in \Sigma$, define the cone

$$\sigma_{D_1,\dots,D_r} := \mathrm{Cone}(\{u_\rho - a_{1\rho} e_1 - \cdots - a_{r\rho} e_r \mid \rho \in \sigma(1)\} \cup \{e_i \mid i \in \{1, \dots, r\}\}) \subset N_{\mathbb{R}} \oplus \mathbb{R}^r .$$

Take the fan $\Sigma_{D_1,\dots,D_r}$ to be the fan generated by the cones $\sigma_{D_1,\dots,D_r}$ and their proper faces. Recall that if the D_i are Cartier, then by iterating [CLS11, Proposition 7.3.1], we can see that the toric variety $X_{\Sigma_{D_1,\dots,D_r}}$ is the vector bundle $\bigoplus_{i=1}^r \mathcal{O}(D_i)$ over X_Σ.

Now, we describe when the support $|\Sigma_{D_1,\dots,D_r}|$ of the fan $\Sigma_{D_1,\dots,D_r}$ associated to the toric vector bundle is one of the special cones described in Section 4.1. Assume that X_Σ is semiprojective and the divisors D_i are $\mathbb{Q}$-Cartier and anti-nef.

Lemma 4.4 ([FK17, Lemma 5.19]). *Let Σ be a fan, and suppose that X_Σ is semiprojective. If $-D$ is nef and $\mathbb{Q}$-Cartier, then X_{Σ_D} is semiprojective.*

Corollary 4.5. *Let Σ be a fan, and suppose that X_Σ is semiprojective. If $-D_1, \dots, -D_r$ are nef and $\mathbb{Q}$-Cartier, then $X_{\Sigma_{D_1,\dots,D_r}}$ is semiprojective.*

Proof. This follows immediately by induction on i. □

We can describe the dual cone $|\Sigma_{D_1,\dots,D_r}|^\vee$ explicitly. Such a description was given by Mavlyutov [Mav09, Lemma 1.6] for the case where $\sum_i D_i = -K_{X_\Sigma}$. In [FK17], this hypothesis is dropped.

Lemma 4.6 ([FK17, Lemma 5.17]). *Let Σ be a complete fan, and let*

$$D_i = \sum_\rho a_{i\rho} D_\rho \quad \text{for } 1 \leqslant i \leqslant r$$

be nef and $\mathbb{Q}$-Cartier divisors. The dual cone to $|\Sigma_{-D_1,\dots,-D_r}|$ is equal to the Cayley cone on the set of polytopes

$$\Delta_i := \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geqslant -a_{i\rho} \text{ for all } \rho \in \Sigma(1)\} ;$$

that is,

$$|\Sigma_{-D_1,\dots,-D_r}|^\vee = \mathbb{R}_{\geqslant 0}(\Delta_1 * \cdots * \Delta_r) = \mathbb{R}_{\geqslant 0}(\Delta_1 + e_1^*) + \cdots + \mathbb{R}_{\geqslant 0}(\Delta_r + e_r^*)\,.$$

Moreover, if the divisors D_i are all Cartier, then the Δ_i are lattice polytopes.

The cone $|\Sigma_{D_1,\dots,D_r}|$ can be any of the four types of strictly convex cones: Gorenstein, almost Gorenstein, $\mathbb{Q}$-Gorenstein, or not $\mathbb{Q}$-Gorenstein. We give examples of all four.

EXAMPLE 4.7. Given a toric Fano variety X_Σ, take a nef partition $D_1,\dots,D_r$ of Cartier divisors of its anti-canonical bundle $-K_{X_\Sigma}$, that is, nef divisors D_i such that

$$\sum D_i = -K_{X_\Sigma}\,.$$

We get a vector bundle $\operatorname{tot}(\bigoplus_{i=1}^r \mathcal{O}(-D_i))$ with anti-canonical determinant. The corresponding cone $|\Sigma_{D_1,\dots,D_r}|$ is a completely split Gorenstein cone. See [BN08] for details.

EXAMPLE 4.8. Let u_i be the standard basis for $\mathbb{Z}^n$, and set $u_0 = -\sum_i u_i$. Let $\Sigma \subseteq \mathbb{R}^n$ be the complete fan on the rays ρ_i generated by the u_i. The corresponding toric variety is $\mathbf{P}^n$. The fan $\Sigma_{-2D_{\rho_0},-2D_{\rho_0}}$ gives the vector bundle $\operatorname{tot}(\mathcal{O}(-2)^{\oplus 2})$.

The cone $|\Sigma_{-2H,-2H}|$ is generated by $u_1,\dots,u_n,e_1,e_2,u_0+2e_1+2e_2$. Note that $u_1,\dots,u_n$, e_1,e_2, and $u_0+2e_1+2e_2$ are all extremal generators of the cone $|\Sigma_{-2D_{\rho_0},-2D_{\rho_0}}|$. If $|\Sigma_{-2D_{\rho_0},-2D_{\rho_0}}|$ is $\mathbb{Q}$-Gorenstein, then we must have $\mathfrak{n} = u_1^* + \cdots + u_n^* + e_1^* + e_2^*$. But

$$\langle u_1^* + \cdots + u_n^* + e_1^* + e_2^*, u_0 + 2e_1 + 2e_2\rangle = 4 - n\,.$$

Hence, $|\Sigma_{-2D_{\rho_0},-2D_{\rho_0}}|$ is almost Gorenstein if and only if $n = 3$. If $n > 3$, then $|\Sigma_{-2D_{\rho_0},-2D_{\rho_0}}|$ is not $\mathbb{Q}$-Gorenstein.

PROPOSITION 4.9. *Let X_Σ be a projective simplicial toric variety, and let D be a Weil divisor linearly equivalent to $-qK_{X_\Sigma}$ for some positive rational number q. If D is nef and $\mathbb{Q}$-Cartier, then the cone $|\Sigma_{-D}|$ is $\mathbb{Q}$-Gorenstein. Moreover, if $q = 1/r$ for some positive integer r and D is Cartier, then $|\Sigma_{-D}|$ is almost Gorenstein.*

Proof. Write $D = \sum_\rho a_\rho D_\rho$ for some $a_\rho \in \mathbb{Z}$. Since D is nef, it suffices to find an element $(m,t) \in (M \times \mathbb{Z})_{\mathbb{Q}}$ such that $\langle (m,t),(u_\rho,a_\rho)\rangle = 1$ for all $\rho \in \Sigma(1)$.

Consider the projection $\pi\colon N \times \mathbb{Z} \to N$ that induces the projection $\pi\colon X_{\Sigma_{-D}} \to X_\Sigma$. Let ρ_b be the ray in Σ_{-D} given by the 1-dimensional cone $\operatorname{Cone}(0,1)$. Consider the exact sequence

$$M \times \mathbb{Z} \xrightarrow{f_{\Sigma_{-D}(1)}} \mathbb{Z}^{\Sigma_{-D}(1)} \longrightarrow \operatorname{Cl}(X_{\Sigma_{-D}}) \longrightarrow 0\,.$$

The first map is defined by $(m,t) \mapsto \sum_{\rho\in\Sigma(1)} \langle (m,t),(u_\rho,a_\rho)\rangle e_{\bar\rho} + t e_{\rho_b}$, where $\bar\rho := \operatorname{Cone}(u_\rho,a_\rho)$ is the ray in $\Sigma_{-D}(1)$ that corresponds to $\rho \in \Sigma(1)$. The image of $(0,1)$ under the map $f_{\Sigma_{-D}}(1)$ is

$$f_{\Sigma_{-D}(1)}(0,1) = \sum_{\rho\in\Sigma(1)} a_\rho e_{\bar\rho} + e_{\rho_b}\,.$$

Thus, in $\operatorname{Cl}(X_{\Sigma_{-D}})$, we have the equality

$$-\sum_{\rho\in\Sigma(1)} a_\rho D_{\bar\rho} = D_{\rho_b}\,. \tag{4.1}$$

By [CLS11, Proposition 4.2.7], since X_Σ is simplicial, we know that for any $\rho_i \in \Sigma(1)$, there is a $d_i \in \mathbb{N}$ such that $d_i D_{\rho_i}$ is Cartier. Note that by [CLS11, Proposition 6.2.7], we know that the support function for the pullback $\pi^*d_iD_{\rho_i}$ is given by the composition $|\Sigma_{-D}| \xrightarrow{\pi} |\Sigma| \xrightarrow{\varphi_{d_iD_{\rho_i}}} \longrightarrow \mathbb{R}$, where $\varphi_D(u_\rho) = -d_i$ if $\rho = \rho_i$ and $\varphi_D(u_\rho) = i$ otherwise. Moreover, since $\pi(\rho_b) = 0$, the support function of any pullback of any divisor on X_Σ will map ρ_b to 0. Hence, by the support function description of the pullback, we can see that $\pi^*(d_iD_{\rho_i}) = d_iD_{\bar{\rho}_i}$ in $\mathrm{Cl}(X_{\Sigma_{-D}})$. Let $d := \prod d_i$. Plugging this into (4.1), we obtain

$$d\pi^*D = dD_{\rho_b} \tag{4.2}$$

in $\mathrm{Cl}(X_{\Sigma_{-D}})$.

Now, note that $D = qK_{X_\Sigma}$; hence, $-q\sum_{\rho\in\Sigma(1)} D_{\bar{\rho}} = q\pi^*K_{X_\Sigma} = \pi^*D = D_{\rho_b}$. Thus, we have the equality

$$\sum_{\rho\in\Sigma(1)} D_{\bar{\rho}} + \frac{1}{q}D_{\rho_b} = 0 \tag{4.3}$$

in $\mathrm{Cl}(X_{\Sigma_{-D}}) \otimes \mathbb{Q}$. By applying $- \otimes \mathbb{Q}$ to the exact sequence we started with, we have

$$(M \oplus \mathbb{Z})_{\mathbb{Q}} \overset{f_{\Sigma_{-D}(1)}}{\longrightarrow} \mathbb{Q}^{\Sigma_{-D}(1)} \longrightarrow \mathrm{Cl}(X_{\Sigma_{-D}}) \otimes \mathbb{Q} \longrightarrow 0\,.$$

Since $\sum_{\rho\in\Sigma(1)} e_{\bar{\rho}} + (1/q)e_{\rho_b}$ is in the kernel of the second map, there exists an element $(m,t) \in (M \oplus \mathbb{Z})_{\mathbb{Q}}$ such that $f_{\Sigma_{-D}(1)}(m,t) = \sum_{\rho\in\Sigma(1)} e_{\bar{\rho}} + (1/q)e_{\rho_b}$. By the definition of $f_{\Sigma_{-D}(1)}$, we then have

$$\langle (m,t),(u_\rho,a_\rho)\rangle = 1 \quad \text{for all} \quad \rho\in\Sigma(1) \quad \text{and} \quad \langle (m,t),(0,1)\rangle = \frac{1}{q}\,.$$

In the case where D is Cartier, we obtain (4.2) in the Picard group. Since the Picard group has no torsion by [CLS11, Proposition 4.2.5], this yields an equality $\pi^*D = D_{\rho_b}$ in $\mathrm{Pic}(X_{\Sigma_{-D}})$. Furthermore, if $q = 1/r$ for some positive integer r, then (4.3) holds in $\mathrm{Pic}(X_{\Sigma_{-D}})$. Thus, by the same logic as above, there exists an $(m,t) \in M \oplus \mathbb{Z}$ such that $\langle (m,t),(u_\rho,a_\rho)\rangle = 1$ for all $\rho \in \Sigma(1)$ and $\langle (m,t),(0,1)\rangle = r$. $\square$

Remark 4.10. We do not know the appropriate generalization for complete intersections except when $q = 1$. In this case, $|\Sigma_{-D_1,\ldots,-D_r}|$ and $|\Sigma_{-D_1,\ldots,-D_r}|^\vee$ are Gorenstein of index r if and only if $\sum D_i = -K$; see [BB97, Proposition 3.6].

4.3 Toric stacks associated to fans

We now define a quotient stack $\mathcal{X}_\Sigma$ that is associated to the fan Σ that is the quotient of an open subset of affine space by an abelian group. This quotient stack will be isomorphic to the toric variety X_Σ when the toric variety is smooth. Let n be the number of rays in the fan Σ. We can associate a new fan $\mathrm{Cox}(\Sigma) \subseteq \mathbb{R}^{\Sigma(1)}$ to Σ that is defined to be

$$\mathrm{Cox}(\Sigma) := \{\mathrm{Cone}(e_\rho|\ \rho\in\sigma)|\ \sigma\in\Sigma\}\,. \tag{4.4}$$

By enumerating the rays, we see that this fan is a subfan of the standard fan for $\mathbb{A}^n$,

$$\Sigma_n := \{\mathrm{Cone}(e_i\,|\, i\in I)\,|\, I\subseteq\{1,\ldots,n\}\}\,.$$

Hence, the toric variety $U_\Sigma := X_{\mathrm{Cox}(\Sigma)}$ is an open subset of $\mathbb{A}^n$. We now define the group $S_{\Sigma(1)}$, which acts on U_Σ.

We first describe a quotient associated to a set of lattice elements $\nu = (v_1,\ldots,v_n) \subseteq N$, where N is a lattice of dimension d. We will focus on the case where $\nu = \{u_\rho\,|\,\rho\in\Sigma(1)\} \subseteq N$, where u_ρ is the primitive lattice generator of the ray ρ. Let M be the dual lattice to N. We get a right-exact sequence

$$\begin{aligned} M &\xrightarrow{f_\nu} \mathbb{Z}^n \xrightarrow{\pi} \operatorname{coker}(f_\nu) \to 0\,, \\ m &\longmapsto \sum_{i=1}^n \langle v_i, m\rangle e_i\,. \end{aligned} \tag{4.5}$$

Applying $\operatorname{Hom}(-,\mathbb{G}_m)$, we get a left-exact sequence

$$0 \longrightarrow \operatorname{Hom}(\operatorname{coker}(f_\nu),\mathbb{G}_m) \xrightarrow{\widehat{\pi}} \mathbb{G}_m^n \xrightarrow{\widehat{f_\nu}} \mathbb{G}_m^m\,. \tag{4.6}$$

We set

$$S_\nu := \operatorname{Hom}(\operatorname{coker}(f_\nu),\mathbb{G}_m)\,. \tag{4.7}$$

We write $S_{\Sigma(1)}$ for S_ν when $\nu = \{u_\rho \mid \rho \in \Sigma(1)\}$.

DEFINITION 4.11. We call U_Σ the *Cox open set* associated to Σ. We define the *Cox stack* associated to Σ to be $\mathcal{X}_\Sigma := [U_\Sigma/S_{\Sigma(1)}]$.

The Cox stack is called the canonical toric stack in the previous literature (see, for example, [FMN10]).

THEOREM 4.12. *If Σ is simplicial, then $\mathcal{X}_\Sigma$ is a smooth Deligne–Mumford stack with coarse moduli space X_Σ. When Σ is smooth (or, equivalently, X_Σ is smooth), $\mathcal{X}_\Sigma \cong X_\Sigma$.*

Proof. The first statement is Theorem 4.11 of [FMN10]. It also follows from a combination of Proposition 5.1.9 and Theorem 5.1.11 in [CLS11], which also gives the second statement. ☐

Note that $S_{\Sigma(1)} \subseteq \mathbb{G}_m^{|\Sigma(1)|}$. Note that any element in $\chi \in \operatorname{coker}(f_\nu)$ gives a map $\chi\colon S_{\Sigma(1)} \to \mathbb{G}_m$. Consequently, each ray $\rho \in \Sigma(1)$ gives a character χ_ρ of $S_{\Sigma(1)}$ given by the element $\pi(e_\rho) \in \operatorname{coker}(f_\nu)$. Hence, to a divisor $D = \sum a_\rho D_\rho$ on X_Σ, we associate a character $\chi_D := \prod_\rho \chi_\rho^{a_\rho}$ of $S_{\Sigma(1)}$, defined by the element $\pi(\sum_\rho a_\rho e_\rho)$. The total space $\operatorname{tot}(\mathcal{O}_{\mathcal{X}_\Sigma}(\chi_D))$ is a quotient stack given by $S_{\Sigma(1)}$ whose action on $U_\Sigma \times \mathbb{C}$ is induced by the standard action on U_Σ and the character on $\mathbb{C}$. This can be done iteratively for a split vector bundle.

We can use this dictionary to move between split vector bundles over toric varieties and quotient stacks. Namely, we have the following proposition.

PROPOSITION 4.13 ([FK17, Proposition 5.16]). *Let $D_1, \dots, D_r$ be divisors on X_Σ. There is an isomorphism of stacks*

$$\mathcal{X}_{\Sigma_{D_1,\dots,D_r}} \cong \operatorname{tot}\left(\bigoplus_{i=1}^r \mathcal{O}_{\mathcal{X}_\Sigma}(\chi_{D_i})\right). \tag{4.8}$$

We can break the above proposition into the following two lemmas. These are already implicit in the proof of [FK17, Proposition 5.16], but we include them here for completeness.

LEMMA 4.14. *There is a group isomorphism $S_{\Sigma(1)} \cong S_{\Sigma_{D_1,\dots,D_r}(1)}$.*

Proof. First, note that $\mathbb{Z}^{\Sigma_{D_1,\dots,D_t}(1)} = \mathbb{Z}^{\Sigma(1)} \times \mathbb{Z}^t$. We write the generators of the direct summands of this decomposition as e_ρ and e_i, respectively. Using equation (4.5), construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \oplus \mathbb{Z}^t \arrow[r, "f_{\Sigma_{D_1,\dots,D_t}(1)}"] \arrow[d, "\mathrm{proj}_M"'] & \mathbb{Z}^{\Sigma(1)+t} \arrow[r, "\pi"] \arrow[d, "g"'] & \mathrm{coker}(f_{\Sigma_{D_1,\dots,D_t}(1)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f_{\Sigma(1)}"] & \mathbb{Z}^{\Sigma(1)} \arrow[r, "\pi"] & \mathrm{coker}(f_{\Sigma(1)}) \arrow[r] & 0\,,
\end{tikzcd} \tag{4.9}$$

where $\mathrm{proj}_M \colon M \oplus \mathbb{Z}^t \to M$ is the standard projection and $g\colon \mathbb{Z}^{\Sigma(1)+t} \to \mathbb{Z}^{\Sigma(1)}$ is defined by

$$e_\rho \mapsto e_\rho\,, \quad e_i \mapsto \sum_\rho a_{i\rho} e_\rho\,.$$

The final vertical map is induced by the first two. The kernel of proj_M is $\mathbb{Z}^t$ and the cokernel of proj_M is trivial, and the kernel of g is $\mathbb{Z}^t$ and the cokernel of g is trivial. This gives an equality of cokernels. As $S_{\Sigma_{D_1,\dots,D_r}(1)} := \mathrm{Hom}(\mathrm{coker}(f_{\Sigma_{D_1,\dots,D_t}(1)}), \mathbb{G}_m)$ and $S_{\Sigma(1)} := \mathrm{Hom}(\mathrm{coker}(f_{\Sigma(1)}), \mathbb{G}_m)$, this commutative diagram induces an equality of groups. $\square$

Lemma 4.15. *We have an isomorphism of quasi-affine varieties*

$$U_{\Sigma_{D_1,\dots,D_t}} = U_\Sigma \times \mathbb{A}^t\,,$$

which induces the isomorphism of stacks (4.8).

Proof. Consider the fan Σ above. Let $\Sigma_t := \{\mathrm{Cone}(e_i) \mid i \in I) \mid I \subset \{1, \dots, t\}\}$. Note that

$$\begin{aligned}
\mathrm{Cox}(\Sigma_{D_1,\dots,D_t}) &= \{\mathrm{Cone}(e_\rho \mid \rho \in \sigma) \mid \sigma \in \Sigma_{D_1,\dots,D_t}\} \\
&= \{\mathrm{Cone}(e_\rho \mid \rho \in \sigma) \mid \sigma \in \Sigma\} \times \Sigma_t \\
&= \mathrm{Cox}(\Sigma) \times \Sigma_t\,.
\end{aligned} \tag{4.10}$$

The first line is by definition, the second line comes from all maximal cones in $\Sigma_{D_1,\dots,D_t}$ containing the rays generated by e_i for all i, and the third line is by definition.

Now, the action of $S_{\Sigma(1)} = S_{\Sigma_{D_1,\dots,D_t}(1)}$ on $U_{\Sigma_{D_1,\dots,D_t}} = U_\Sigma \times \mathbb{A}^t$ is described by (4.9). This shows that $S_{\Sigma_{D_1,\dots,D_t}(1)}$ acts on U_Σ via the isomorphism with $S_{\Sigma(1)}$. Moreover, on the ith coordinate u_i of $\mathbb{A}^t$, it acts via the character χ_{D_i}. This gives the isomorphism of stacks (4.8). $\square$

4.4 Variation of geometric invariant theory on affine space

Take an affine space

$$X := \mathbb{A}^{n+r} = \mathrm{Spec}\, \kappa[x_1, \dots, x_n, u_1, \dots, u_r]\,.$$

Consider the open dense torus $\mathbb{G}_m^{n+r}$ with the standard embedding and action on X. Take a subgroup $S \subseteq \mathbb{G}_m^{n+r}$. For $1 \leqslant i \leqslant n+r$, we get a character χ_i from the composition of the inclusion and the projection onto the ith summand.

Definition 4.16. Let $\times\colon \mathbb{G}_m^{n+r} \to \mathbb{G}_m$ be the multiplication map. We say that S satisfies the *quasi-Calabi–Yau condition* if $\times|_S$ is torsion.

Remark 4.17. The quasi-Calabi–Yau condition is equivalent to the sum $\sum_{i=1}^{n+r} \chi_i$ being torsion.

The reason for the distinction between the variables x_i and u_i is that we equip $\mathbb{A}^{n+r}$ with an additional $\mathbb{G}_m$-action. Namely, for $\lambda \in \mathbb{G}_m$, we define

$$\lambda \cdot x_i := x_i\,, \quad \lambda \cdot u_i := \lambda u_i$$

and call this $\mathbb{G}_m$-action *R-charge*.

This gives an action $S \times \mathbb{G}_m$ on $\mathbb{A}^{n+r}$ and defines a distinguished character χ_R coming from projection onto the second factor. A *superpotential* is a semi-invariant function w with respect to χ_R. The data $(\mathbb{A}^{n+r}, S \times \mathbb{G}_m, w, \chi_R)$ is a *gauged Landau–Ginzburg model.* We can restrict this data to any invariant open subset of $\mathbb{A}^{n+r}$ to get various new gauged Landau–Ginzburg models.

In this paper, we choose such open sets using geometric invariant theory (GIT) for the action of S on $\mathbb{A}^{n+r}$. Let us now review this story, which is called variation of geometric invariant theory quotients (VGIT). The possible GIT quotients of $\mathbb{A}^{n+r}$ by S correspond to a choice of a (rational) character

$$\chi \in \hat{S}_{\mathbb{Q}} := \operatorname{Hom}(S, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

That is, given a $\chi \in \hat{S}_{\mathbb{Q}}$, rationalize the denominator to get a G-equivariant line bundle $\mathcal{O}(d\chi)$ for some $d > 0$. Geometric invariant theory determines an open subset U_χ of $\mathbb{A}^{n+r}$ called the semistable locus.

Partition $\hat{S}_{\mathbb{Q}}$ into the subsets

$$\sigma_\chi := \left\{\tau \in \hat{S}_{\mathbb{Q}} \,|\, U_\tau = U_\chi\right\}.$$

It turns out that each σ_χ is a cone and the set of all such cones forms a fan Σ_{GKZ} in $\hat{S}_{\mathbb{Q}}$ called the Gel'fand–Kapranov–Zelevinskii fan (or *GKZ-fan* for short) or *secondary fan.* The maximal cones of this fan are called *chambers*, and the codimension 1 cones are called *walls.* There is only a finite number of chambers in the fan Σ_{GKZ}. For any character χ_p in the interior of a chamber σ_p, we denote by U_p the open set of $\mathbb{A}^{n+r}$ that consists of the semistable points with respect to the line bundle associated to χ_p.

Theorem 4.18. *For any two chambers σ_p and σ_q, if S satisfies the quasi-Calabi–Yau condition, then there is an equivalence of categories*

$$\mathrm{D}^{\mathrm{abs}}[U_p, S \times \mathbb{G}_m, w] \cong \mathrm{D}^{\mathrm{abs}}[U_q, S \times \mathbb{G}_m, w]\,.$$

Proof. This is a consequence of Theorem 3.12. As for why this works for all chambers, one can use [CLS11, Theorem 14.4.7]. Namely, one can get from any chamber to any other chamber by a sequence of elementary wall crossings as the GKZ-fan has convex support.

As we are in the toric setting, the result, in fact, goes back to [HW12, Theorem 3]. Another version of this result can be found in [Hal15, Corollary 4.8 and Proposition 5.5]. □

4.5 Variation of geometric invariant theory for toric stacks and the secondary fan

Here, we consider the geometric invariant theory associated to the quotient of the affine space $X := \mathbb{A}^n$ by the abelian group $S := S_\nu$, as defined in equation (4.7). As it turns out, the different GIT quotients of X by S have an interpretation both in terms of the secondary fan and in terms of fans whose rays have primitive generators in the point collection ν.

Following [CLS11, § 15.2], take $\nu = (v_1, \ldots, v_n)$ to be a collection of distinct, nonzero points in N.

Definition 4.19. We say that ν is *geometric* if each $v_i \in \nu$ is nonzero and generates a distinct ray in $N_{\mathbb{Q}}$.

Proposition 4.20 ([CLS11, Exercise 15.1.8]). *Suppose that ν is geometric. Then, there is a bijective correspondence between chambers of the secondary fan and simplicial fans Σ such that $\Sigma(1) \subseteq \{\mathrm{Cone}(v_i) \,|\, 1 \leqslant i \leqslant n\}$, $|\Sigma| = \mathrm{Cone}(\nu)$, and X_Σ is semiprojective.*

Allow us to describe the cones in the secondary fan in a bit more detail. Tensoring the short exact sequence in (4.5) with $\mathbb{Q}$, we get the sequence

$$M_{\mathbb{Q}} \xrightarrow{f_\nu} \mathbb{Q}^n \xrightarrow{\pi} \operatorname{coker}(f_\nu) \otimes \mathbb{Q} \to 0\,.$$

Here, the vector space $(\hat{S}_\nu)_{\mathbb{Q}} = \operatorname{coker}(f_\nu) \otimes \mathbb{Q}$ is the space in which the secondary fan Σ_{GKZ} lives.

Note that if we take the standard basis vectors e_i for $\mathbb{Q}^n$, then the support of the secondary fan $|\Sigma_{\mathrm{GKZ}}|$ is the cone $\operatorname{Cone}(\pi(e_i))$. We now will give the structure of the fan that comes from a decomposition, but first let us give a definition in order to give a general setup.

Definition 4.21 ([CLS11, Definition 6.2.2]). A generalized fan Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that

(i) every $\sigma \in \Sigma$ is a rational polyhedral cone,
(ii) for any $\sigma \in \Sigma$, each face of σ is also in Σ, and
(iii) for any $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Start with a generalized fan Σ in $N_{\mathbb{R}}$ and a set $I_\emptyset \subset \nu$ such that the support $|\Sigma|$ is $\operatorname{Cone}(\nu)$, the toric variety X_Σ is semiprojective, and we can write any cone $\sigma \in \Sigma$ as $\operatorname{Cone}(v_i \,|\, v_i \in \sigma, v_i \notin I_\emptyset)$. Given such a pair $(\Sigma, I_\emptyset)$, we define a cone in $\mathbb{Q}^n$:

$$\tilde{\Gamma}_{\Sigma,I_\emptyset} := \left\{ (a_i) \in \mathbb{Q}^n \;\middle|\; \begin{array}{l} \text{there exists a convex support function } \varphi \text{ such that} \\ \varphi(v_i) = -a_i \text{ if } v_i \notin I_\emptyset \text{ and } \varphi(v_i) \geqslant -a_i \text{ if } v_i \in I_\emptyset \end{array} \right\}.$$

Take the cone $\Gamma_{\Sigma,I_\emptyset}$ to be the image of $\tilde{\Gamma}_{\Sigma,I_\emptyset}$ by π. The set of all such $\Gamma_{\Sigma,I_\emptyset}$ gives the fan Σ_{GKZ}. By [CLS11, Proposition 15.2.1], we have a tractable description of the GKZ-cone as an intersection of cones:

$$\Gamma_{\Sigma,I_\emptyset} = \bigcap_{\sigma \in \Sigma_{\max}} \operatorname{Cone}(\pi(e_\rho) \,|\, u_\rho \in I_\emptyset \text{ or } \rho \notin \sigma)\,. \tag{4.11}$$

Fix a fan $\Sigma \in N_{\mathbb{R}}$ with $\Sigma(1) \subseteq \nu$. A priori, we get two different stacks. One is associated to $\Gamma_{\Sigma,\nu\setminus\Sigma(1)}$. This comes from the GIT problem belonging to the S_ν-action on $\mathbb{A}^\nu$. The other is associated to $\Gamma_{\Sigma,\emptyset}$, which comes from the GIT problem associated to the $S_{\Sigma(1)}$-action on $\mathbb{A}^{\Sigma(1)}$. Fortunately, the two stacks are isomorphic.

Lemma 4.22. *Suppose that we have an exact sequence of algebraic groups*

$$0 \to H \xrightarrow{i} G \xrightarrow{\pi} Q \to 0\,.$$

Let G act on X and hence on $X \times Q$ via π. Then, we have an isomorphism of stacks

$$[X \times Q/G] \cong [X/H]\,. \tag{4.12}$$

Proof. The stack $[X \times Q/G]$ is the functor that assigns to a scheme Y the groupoid $Y \leftarrow \mathcal{G} \to X \times Q$ of G-torsors over Y with G-equivariant maps to $X \times Q$. Similarly, $[X/H]$ assigns to a scheme Y the groupoid $Y \leftarrow \mathcal{H} \to X$ of H-torsors over Y with H-equivariant maps to X. Finally, one straightforwardly yet tediously checks that there is an equivalence of groupoids

$$\begin{gathered} \{Y \leftarrow \mathcal{G} \to X \times Q\} \Longleftrightarrow \{Y \leftarrow \mathcal{H} \to X\}\,, \\ [Y \leftarrow \mathcal{G} \to X \times Q] \Longrightarrow [Y \leftarrow \mathcal{G} \times_{X \times Q} X \to X]\,, \\ [Y \leftarrow \mathcal{H} \overset{H}{\times} G \to X \times Q] \Longleftarrow [Y \leftarrow \mathcal{H} \to X]\,. \end{gathered}$$

□

COROLLARY 4.23. *There is a natural isomorphism of stacks* $[U_\Sigma \times \mathbb{G}_m^{I_\emptyset}/S_\nu] \cong [U_\Sigma/S_{\nu\backslash I_\emptyset}]$.

Proof. This is a special case of Lemma 4.22. Since $\nu, \nu\backslash I_\emptyset$ span $N_\mathbb{R}$, the maps f_ν, $f_{\nu\backslash I_\emptyset}$ are injective. Starting with the second and third row, the snake lemma gives the top isomorphism in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}^{I_\emptyset} \arrow[r, equal] \arrow[d] & \mathbb{Z}^{I_\emptyset} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f_\nu"] \arrow[d, equal] & \mathbb{Z}^\nu \arrow[r] \arrow[d] & \widehat{S_\nu} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f_{\nu\backslash I_\emptyset}"] & \mathbb{Z}^{\nu\backslash I_\emptyset} \arrow[r] & \widehat{S_{\nu\backslash I_\emptyset}} \arrow[r] & 0\,.
\end{tikzcd}$$

Applying $\operatorname{Hom}(-, \mathbb{G}_m)$ to the right vertical exact sequence gives

$$0 \to S_{\nu\backslash I_\emptyset} \to S_\nu \to \mathbb{G}_m^{I_\emptyset} \to 0\,.$$

Hence, we can specialize equation (4.12) to $H = S_{\nu\backslash I_\emptyset}$, $G = S_\nu$, $Q = \mathbb{G}_m^{I_\emptyset}$, $X = U_\Sigma$. □

Now, let $\sigma \subseteq N_\mathbb{R}$ be a $\mathbb{Q}$-Gorenstein cone. Let $\nu \subseteq \sigma \cap N$ be a finite set which contains the ray generators of σ. Define

$$\nu_{=1} := \{v \in \nu \,|\, \langle \mathfrak{m}, v\rangle = 1\}\,, \quad \nu_{\neq 1} := \{a \in \nu \,|\, \langle \mathfrak{m}, a\rangle \neq 1\}\,.$$

Let χ_K be the character $-\sum_{v\in\nu} \chi_{D_\nu}$.

LEMMA 4.24. *Take* $\nu = \{v_1, \ldots, v_n\}$ *to be a geometric collection of nonzero lattice points in* N. *Let* Σ *be a simplicial fan in* $N_\mathbb{R}$ *and* $I_\emptyset \subset \nu$ *such that the support* $|\Sigma|$ *is the cone* $\operatorname{Cone}(\nu)$, *the toric variety* X_Σ *is semiprojective, and* $\Sigma(1) \coprod I_\emptyset = \nu$. *If the cone* σ *is* $\mathbb{Q}$*-Gorenstein, then we have the following:*

(i) *If* $\langle \mathfrak{m}, u_\rho\rangle > 1$ *for all* $\rho \in \nu_{\neq 1}$ *and* $\nu_{\neq 1} \subseteq I_\emptyset$, *then* $\chi_K \in \Gamma_{\Sigma, I_\emptyset}$.

(ii) *If* $\langle \mathfrak{m}, u_\rho\rangle < 1$ *for all* $\rho \in \nu_{\neq 1}$ *and* $\nu_{\neq 1} \subseteq I_\emptyset$, *then* $-\chi_K \in \Gamma_{\Sigma, I_\emptyset}$.

(iii) *If* $\nu_{\neq 1} = \emptyset$, *then* χ_K *is* 0 *in* $(\hat{S}_\nu)_\mathbb{Q}$, *hence in every chamber of the secondary fan.*

Proof. Using the description of $\Gamma_{\Sigma, I_\emptyset}$ given in equation (4.11), we know that

$$\operatorname{Cone}(\pi(e_\rho) \,|\, u_\rho \in \nu_{\neq 1}) \subset \Gamma_{\Sigma, I_\emptyset}\,.$$

Now, in $\widehat{S_\nu} \otimes_\mathbb{Z} \mathbb{Q}$, we have

$$0 = (\pi \circ f_\nu)(\mathfrak{m}) \;=\; \sum_{v\in\nu} \langle \mathfrak{m}, v\rangle \chi_{D_i} \;=\; \sum_{v\in\nu_{=1}} \chi_{D_v} + \sum_{v\in\nu_{\neq 1}} \langle \mathfrak{m}, v\rangle \chi_{D_a} \tag{4.13}$$

as $\langle \mathfrak{m}, v\rangle = 1$ for all $v \in \nu_{=1}$. This implies

$$\chi_K = -\sum_{v\in\nu} \chi_{D_v} \;=\; \sum_{u_\rho\in\nu_{\neq 1}} (\langle \mathfrak{m}, u_\rho\rangle - 1)\chi_{D_\rho}\,.$$

Hence, if $\langle \mathfrak{m}, u_\rho\rangle > 1$ for all $u_\rho \in \nu_{\neq 1}$, we have $\chi_K \in \operatorname{Cone}(\pi(e_\rho) \,|\, u_\rho \in \nu_{\neq 1}) \subseteq \Gamma_{\Sigma, I_\emptyset}$. Similarly, if $\langle \mathfrak{m}, u_\rho\rangle < 1$ for all $u_\rho \in \nu_{\neq 1}$, we have $-\chi_K \in \operatorname{Cone}(\pi(e_\rho) \,|\, u_\rho \in \nu_{\neq 1}) \subseteq \Gamma_{\Sigma, I_\emptyset}$. Finally, if $\nu_{\neq 1} = \emptyset$, then $\chi_K = \sum_{\rho\in\Sigma(1)} \chi_{D_\rho} = (\pi \circ f_\nu)(\mathfrak{m}) = 0$. □

5. Derived categories of toric LG models and fractional CY categories

5.1 Serre functors of factorization categories

As before, let $\sigma \subseteq N_{\mathbb{R}}$ be a $\mathbb{Q}$-Gorenstein cone and $\nu \subseteq \sigma \cap N$ a finite, geometric collection of lattice points which contains the ray generators of σ. Partition the set ν into two subsets

$$\nu_{=1} = \{v \in \nu \,|\, \langle \mathfrak{m}, v\rangle = 1\} \quad \text{and} \quad \nu_{\neq 1} = \{v \in \nu \,|\, \langle \mathfrak{m}, v\rangle \neq 1\}\,. \tag{5.1}$$

Since σ is $\mathbb{Q}$-Gorenstein, the ray generators of σ are contained in $\nu_{=1}$. Choose any subset $R \subseteq \nu$. The set R gives an action of $\mathbb{G}_m$ on $\mathbb{A}^\nu$ by

$$\lambda_R \cdot x_i := \begin{cases} \lambda x_i & \text{if } v_i \in R\,, \\ x_i & \text{if } v_i \notin R\,. \end{cases} \tag{5.2}$$

All together, this gives an action of $S_\nu \times \mathbb{G}_m$ on $\mathbb{A}^\nu$.

Let $\Sigma \subseteq N_{\mathbb{R}}$ be a simplicial fan such that $\Sigma(1) \subseteq \nu_{=1}$, X_Σ is semiprojective, and $\mathrm{Cone}(\Sigma(1)) = \sigma$. This gives an open subset $U_\Sigma \times \mathbb{G}_m^{\nu\backslash\Sigma(1)} \subseteq \mathbb{A}^\nu$, and we restrict the action of $S_\nu \times \mathbb{G}_m$ to this open subset.

Let $\chi\colon S_\nu \times \mathbb{G}_m \to \mathbb{G}_m$ be the projection character onto the $\mathbb{G}_m$-factor. Finally, let w be a function on $\mathbb{A}^\nu$ which is semi-invariant with respect to χ, that is, $w \in \Gamma(U_\Sigma \times \mathbb{G}_m^{\nu\backslash\Sigma(1)}, \mathcal{O}(\chi))^{S_\nu \times \mathbb{G}_m}$.

Remark 5.1. We have restricted our additional $\mathbb{G}_m$-action and choice of w with geometric applications in mind, namely, so that we can apply Theorem 3.5 to obtain Proposition 5.11. These restrictions force w to be of the form $w = \sum x_i f_i$, where $v_i \in R$ and f_i is a function of the variables not in R. Therefore, $\partial w \subseteq Z(w)$. We implicitly use this in applying Theorem 2.18 below. In fact, this condition holds for any quasi-homogeneous function of nonzero degree by Euler's homogeneous function theorem.

Recall that for each $v_i \in \nu$, we also have characters χ_{D_i} defined by the composition

$$S_\nu \hookrightarrow \mathbb{G}_m^\nu \xrightarrow{\pi_i} \mathbb{G}_m\,,$$

where π_i is the projection onto the factor corresponding to v_i.

Denote by χ_K the inverse of the character corresponding to the composition

$$\begin{array}{c} S_\nu \times \mathbb{G}_m \xhookrightarrow{\hat{\pi}\times\lambda_R} \mathbb{G}_m^\nu \xrightarrow{\times} \mathbb{G}_m\,, \\ (g, \lambda) \longmapsto \hat{\pi}(g)\cdot \lambda_R\,, \end{array}$$

where $\hat{\pi}$ is from equation (4.6) and

$$(\lambda_R)_i := \begin{cases} \lambda & \text{if } v_i \in R\,, \\ 1 & \text{if } v_i \notin R\,. \end{cases}$$

In other words, if we identify the character group $\widehat{S_\nu \times \mathbb{G}_m} = \widehat{S_\nu} \oplus \mathbb{Z}$, then

$$\chi_K := -\sum_{v_i \in R} (\chi_{D_i}, 1) - \sum_{v_i \notin R} (\chi_{D_i}, 0) \;=\; -\sum_{v_i \in \nu} (\chi_{D_i}, 0) - (0, |R|)\,.$$

Notice that for any $v_j \in \nu\backslash\Sigma(1)$, we have $\chi_{D_j} \neq 0$. Indeed, if $\chi_{D_j} = 0$, then there exists a lattice element $m \in M$ such that $\langle m, v_i\rangle = \delta_{ij}$. This, in turn, means that v_j is a ray generator of $\mathrm{Cone}(\nu)$, which is ruled out by the assumption that $v_j \in \nu\backslash\Sigma(1)$ as $\mathrm{Cone}(\nu) = \mathrm{Cone}(\Sigma(1))$.

Hence, as each χ_{D_j} is nontrivial, we have a surjective map

$$\begin{aligned} S_\nu \times \mathbb{G}_m &\overset{p}{\twoheadrightarrow} \mathbb{G}_m^{\nu\setminus\Sigma(1)}\,, \\ (g,\lambda) &\mapsto \prod_{v_j \in \nu\setminus\Sigma(1)} \chi_{D_j}(g)\cdot \lambda_{R\cap(\nu\setminus\Sigma(1))}\,. \end{aligned} \tag{5.3}$$

Let $H_{\Sigma,R}$ be the kernel of this map, so that there is an exact sequence

$$0 \to H_{\Sigma,R} \to S_\nu \times \mathbb{G}_m \to \mathbb{G}_m^{\nu\setminus\Sigma(1)} \to 0\,. \tag{5.4}$$

We will write H when Σ and R are understood in the immediate context.

Lemma 5.2. *When χ_K is viewed as a character of H, the equivariant canonical bundle $\omega_{[U_\Sigma/H]}$ is isomorphic to $\mathcal{O}(\chi_K)$.*

Proof. Recall, from Lemma 4.22, that there is a natural isomorphism of stacks

$$\left[U_\Sigma \times \mathbb{G}_m^{\nu\setminus\Sigma(1)}/S_\nu \times \mathbb{G}_m\right] \cong [U_\Sigma/H]\,. \tag{5.5}$$

Hence, the statement is equivalent to showing that $\mathcal{O}(\chi_K)$ is isomorphic to the equivariant canonical bundle of $[U_\Sigma \times \mathbb{G}_m^{\nu\setminus\Sigma(1)}/S_\nu \times \mathbb{G}_m]$ when we view χ_K as a character of $S_\nu \times \mathbb{G}_m$.

The canonical bundle on $[U_\Sigma \times \mathbb{G}_m^{\nu\setminus\Sigma(1)}/S_\nu \times \mathbb{G}_m]$ is the restriction of the canonical bundle on $[\mathbb{A}^\nu/S_\nu \times \mathbb{G}_m]$, so we can reduce to checking the statement on affine space. Now, we have the standard fact that the cotangent bundle on affine space is just the dual vector space (with its natural equivariant structure, which in this case is just the dual grading on the dual vector space), that is,

$$\Omega_{[\mathbb{A}^\nu/S_\nu\times\mathbb{G}_m]} = \bigoplus_{v_i\in R} \mathcal{O}(-D_i,1) \oplus \bigoplus_{v_i\notin R} \mathcal{O}(-D_i,0)\,.$$

Therefore,

$$\omega_{[U_\Sigma/H]} = \Omega^{|\nu|}_{[U_\Sigma/S_\nu\times\mathbb{G}_m]} = \mathcal{O}\Big(-\sum_{v_i\in R}(D_i,1) - \sum_{v_i\notin R}(D_i,0)\Big) = \mathcal{O}(\chi_K)\,. \qquad \square$$

Convention 5.3. Identifying $\widehat{H}$ as a quotient of $\widehat{S_\nu}\oplus\mathbb{Z}$, we write elements of $\widehat{H}$ in the form (a,b) with $a\in\widehat{S_\nu}$ and $b\in\mathbb{Z}$ and view them as equivalence classes.

Lemma 5.4. *We have the following equality in $\widehat{H}\otimes_{\mathbb{Z}}\mathbb{Q}$:*

$$\chi_K = \Big(0, -\sum_{v_i\in\nu_{\neq 1}} \langle \mathfrak{m}, v_i\rangle + |\nu_{\neq 1}| - |R|\Big)\,.$$

If σ is almost Gorenstein, then the equality holds in $\widehat{H}$.

Proof. As $\mathrm{Hom}(-,\mathbb{G}_m)$ is exact, we may apply it to equation (5.4) to obtain an exact sequence

$$0 \longrightarrow \mathbb{Z}^{\nu\setminus\Sigma(1)} \xrightarrow{\hat{p}} \widehat{S_\nu}\oplus\mathbb{Z} \longrightarrow \widehat{H} \longrightarrow 0\,.$$

Notice that we have

$$\hat{p}(e_i) = \begin{cases} (\chi_{D_i},1) & \text{if } v_i\in R\,,\\ (\chi_{D_i},0) & \text{if } v_i\notin R\,. \end{cases}$$

Hence, in $\widehat{H}$,

$$\begin{aligned} (\chi_{D_i},1) &= 0 \quad \text{if } v_i\in R\,,\\ (\chi_{D_i},0) &= 0 \quad \text{if } v_i\notin R\,. \end{aligned} \tag{5.6}$$

Thus,

$$\begin{aligned}\chi_K &= -\sum_{v_i\in\nu}(\chi_{D_i},0)-(0,|R|)\\ &= \Big(-\sum_{v_i\in\nu_{=1}}\chi_{D_i}-\sum_{v_i\in\nu_{\neq 1}}\chi_{D_i},|R|\Big)\\ &= \Big(-\sum_{v_i\in\nu_{\neq 1}}(\langle\mathfrak{m},v_i\rangle-1)\chi_{D_i},|R|\Big)\\ &= \Big(0,-\sum_{v_i\in\nu_{\neq 1}}\langle\mathfrak{m},v_i\rangle+|\nu_{\neq 1}|-|R|\Big)\,.\end{aligned}$$

The first line is by definition. The second line is because ν is a disjoint union of $\nu_{=1}$ and $\nu_{\neq 1}$. The third line follows from (4.13) (notice that this holds over $\mathbb{Z}$ when $\mathfrak{m}\in M$ and over $\mathbb{Q}$ when $\mathfrak{m}\in M_{\mathbb{Q}}$). The fourth line is (5.6). ☐

We can restrict w by defining a function $\bar{w}$ on $\mathbb{A}^{\Sigma(1)}$ by setting all variables associated to points in the set $\nu\setminus\Sigma(1)$ to 1. When we restrict $\bar{w}$ to U_Σ, we have $\bar{w}\in\Gamma(U_\Sigma,\mathcal{O}_{U_\Sigma}((0,1)))^H$.

Remark 5.5. Under the isomorphism of stacks (5.5), the function w corresponds to $\bar{w}$.

We can now state the following special case of Theorem 2.18.

Corollary 5.6. *Assume that $[\partial\bar{w}/S_{\Sigma(1)}]$ is proper. The category $\mathrm{D}^{\mathrm{abs}}[U_\Sigma,H,\bar{w}]$ is fractional Calabi–Yau of dimension*

$$-2\sum_{v_i\in\nu_{\neq 1}}\langle\mathfrak{m},v_i\rangle+2|\nu_{\neq 1}|-2|R|+\dim N_{\mathbb{R}}\,.$$

If σ is almost Gorenstein, then the category is Calabi–Yau.

Proof. Let d be the smallest natural number such that $d\cdot\mathfrak{m}\in M$ (notice that $d=1$ when σ is almost Gorenstein). We have

$$\begin{aligned}S^d &= \big(-\otimes\omega^d_{[U_\Sigma/H]}\big)[d(\dim U_\Sigma-\dim H-1)]\\ &= (-\otimes\mathcal{O}(d\chi_K))[d(\dim U_\Sigma-\dim H-1)]\\ &= \Big(-\otimes\mathcal{O}\Big(d\Big(0,-\sum_{v_i\in\nu_{\neq 1}}\langle\mathfrak{m},v_i\rangle+|\nu_{\neq 1}|-|R|\Big)\Big)\Big)[d(\dim U_\Sigma-\dim H-1)]\\ &= \Big[d\Big(-2\sum_{v_i\in\nu_{\neq 1}}\langle\mathfrak{m},v_i\rangle+2|\nu_{\neq 1}|-2|R|+\dim U_\Sigma-\dim H-1\Big)\Big]\\ &= \Big[d\Big(-2\sum_{v_i\in\nu_{\neq 1}}\langle\mathfrak{m},v_i\rangle+2|\nu_{\neq 1}|-2|R|+\dim N_{\mathbb{R}}\Big)\Big]\,.\end{aligned}$$

The first line is Theorem 2.18. The second line follows from the fact that $\omega_{[U_\Sigma/H]}$ is the restriction of the canonical bundle on $[\mathbb{A}^{\Sigma(1)}/H]$. The third line is Lemma 5.4. The fourth line follows from the isomorphism of functors $[2]=(-\otimes\mathcal{O}(0,1))$ in $\mathrm{D}^{\mathrm{abs}}[U_\Sigma,H,\bar{w}]$ since $\bar{w}$ is a section of the equivariant line bundle $\mathcal{O}(0,1)$. The last line follows from the definition of H. ☐

We now provide a toric specialization of Theorem 3.12, found in the unabridged version [BFK12] of the paper [BFK17]. First, let us provide some notation and the assumptions for the setup of this theorem. Let

- Σ_+ and Σ_- be two adjacent chambers in the secondary fan sharing a wall τ;
- $\lambda\colon \mathbb{G}_m \to S_\nu$ be the primitive 1-parameter subgroup defining hyperplane containing the wall τ separating Σ_+ and Σ_-;
- $\chi_+ \in \Sigma_+$ and $\chi_- \in \Sigma_-$ be two characters in the relative interior of adjacent chambers and $\chi_0 \in \tau$ a character in the relative interior of the wall τ separating the two chambers;
- U_+, U_- be the open semistable loci of $\mathbb{A}^\nu$ corresponding to the characters χ_+, χ_-, respectively, and U_0 be the intersection of the fixed locus of λ with the open semistable locus of χ_0;
- S_0 be the quotient group $S_\nu/\lambda(\mathbb{G}_m)$;
- G be a group whose action commutes with S_ν (in our application, this will be R-charge);
- $w \in \kappa[x_i \,|\, v_i \in \nu]$ be an S_ν-invariant and G-semi-invariant function with respect to a character η of G; and
- w_+, w_-, w_0 be induced sections of the line bundles determined by η on $[U_+/S_\nu \times G]$, $[U_-/S_\nu \times G]$, and $[U_0/S_0 \times G]$, respectively.

THEOREM 5.7 ([BFK12, Theorem 5.1.2]). *Let $\mu = -\sum_{v_i \in \nu} \langle \lambda, v_i \rangle$ and $d \in \mathbb{Z}$. The following statements hold:*

(i) *If $\mu > 0$, there exist fully faithful functors*

$$\Phi_d\colon \mathrm{D}^{\mathrm{abs}}[U_-, S_\nu \times G, w_-] \to \mathrm{D}^{\mathrm{abs}}[U_+, S_\nu \times G, w_+]\,,$$
$$\Upsilon_-\colon \mathrm{D}^{\mathrm{abs}}[U_0, S_0 \times G, w_0] \to \mathrm{D}^{\mathrm{abs}}[U_+, S_\nu \times G, w_+]$$

and a semi-orthogonal decomposition, with respect to Φ_d and Υ_-,

$$\mathrm{D}^{\mathrm{abs}}[U_-, S_\nu \times G, w_-] = \left\langle \mathcal{D}_-(-\mu-d+1), \ldots, \mathcal{D}_-(-d), \mathrm{D}^{\mathrm{abs}}[U_-, S_\nu \times G, w_-] \right\rangle,$$

where $\mathcal{D}_-(\ell)$ is the image of Υ_- twisted by a character of λ-weight ℓ.

(ii) *If $\mu = 0$, there is an equivalence*

$$\Phi_d\colon \mathrm{D}^{\mathrm{abs}}[U_-, S_\nu \times G, w_-] \to \mathrm{D}^{\mathrm{abs}}[U_+, S_\nu \times G, w_+]\,.$$

(iii) *If $\mu < 0$, there exist fully faithful functors,*

$$\Phi_d\colon \mathrm{D}^{\mathrm{abs}}[U_+, S_\nu \times G, w_+] \to \mathrm{D}^{\mathrm{abs}}[U_-, S_\nu \times G, w_-]\,,$$
$$\Upsilon_+\colon \mathrm{D}^{\mathrm{abs}}[U_0, S_0 \times G, w_0] \to \mathrm{D}^{\mathrm{abs}}[U_-, S_\nu \times G, w_-]$$

and a semi-orthogonal decomposition, with respect to Φ_d and Υ_+,

$$\mathrm{D}^{\mathrm{abs}}[U_-, S_\nu \times G, w_-] = \left\langle \mathcal{D}_+(-d), \ldots, \mathcal{D}_+(\mu-d+1), \mathrm{D}^{\mathrm{abs}}[U_+, S_\nu \times G, w_+] \right\rangle,$$

where $\mathcal{D}_+(\ell)$ is the image of Υ_+ twisted by a character of λ-weight ℓ.

We now use this theorem iteratively to provide a comparison theorem to a GIT chamber whose absolute derived category corresponds to a (fractional) Calabi–Yau category.

THEOREM 5.8. *Let $\tilde{\Sigma}$ be any simplicial fan such that $\tilde{\Sigma}(1) = \nu$ and $X_{\tilde{\Sigma}}$ is semiprojective. Similarly, let Σ be any simplicial fan such that $\Sigma(1) \subseteq \nu_{=1}$, X_Σ is semiprojective, and $\mathrm{Cone}(\Sigma(1)) = \sigma$. We have the following:*

(i) *If $\langle \mathfrak{m}, a \rangle > 1$ for all $a \in \nu_{\neq 1}$, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \longrightarrow \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w]\,.$$

(ii) *If* $\langle \mathfrak{m}, a\rangle < 1$ *for all* $a \in \nu_{\neq 1}$*, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w] \longrightarrow \mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]\,.$$

(iii) *If* $A = \emptyset$*, then there is an equivalence*

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \cong \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w]\,.$$

Furthermore, if $[\partial\bar{w}/S_{\Sigma(1)}]$ *is proper, then* $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ *is fractional Calabi–Yau. If, in addition,* σ *is almost Gorenstein, then* $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ *is Calabi–Yau.*

Proof. We prove statement (i) and later state the necessary adjustments for statements (ii) and (iii). Proposition 4.20 says that $\Gamma_{\tilde{\Sigma},\emptyset}$ and $\Gamma_{\Sigma,\nu\setminus\Sigma(1)}$ are both chambers of the GKZ-fan of ν. Suppose $\langle \mathfrak{m}, a\rangle > 1$ for all $a \in \nu_{\neq 1}$. Then, by Lemma 4.24, we have $-\chi_K \in \Gamma_{\Sigma,\nu\setminus\Sigma(1)}$.

Choose a straight line path $\gamma_1\colon [0,1] \to (\widehat{S_\nu})_{\mathbb{R}}$ such that

- $\gamma_1(0)$ lies in the interior of $\Gamma_{\tilde{\Sigma},\emptyset}$;
- $\gamma_1(1) = -\chi_K$;
- for any $\epsilon > 0$, the set $\gamma_1((0, 1-\epsilon))$ does not intersect any cone of codimension 2.

The existence of such a path is easily justified. Namely, a generic choice will do; see, for example, the proof of [BFK17, Theorem 5.2.3].

Since there are finitely many chambers, the union of the chambers not containing $-\chi_K$ is closed. Hence, we may choose ϵ sufficiently small such that $B_\epsilon(-\chi_K)$, the ball of radius ϵ centered at $-\chi_K$, only intersects chambers containing $-\chi_K$. Then, choose a second straight line path such that

- $\gamma_2(0) = \gamma_1(1-\epsilon)$;
- $\gamma_2(1)$ lies in rel $\mathrm{int}(\Gamma_{\Sigma,\nu\setminus\Sigma(1)}) \cap B_\epsilon(-\chi_K)$;
- $\gamma_2([0,1])$ does not intersect any cone of codimension 2.

Similarly, the existence of such a path follows from the convexity of $B_\epsilon(-\chi_K) \cap |\Sigma_{\mathrm{GKZ}}|$ and, as before, the fact that you can generically avoid codimension 2 cones.

The concatenation of γ_1 and γ_2 defines a sequence of wall crossings in the GKZ-fan of ν which begins in $\Gamma_{\tilde{\Sigma},\emptyset}$ and ends in $\Gamma_{\Sigma,\nu\setminus\Sigma(1)}$. The fans corresponding to $\Gamma_{\tilde{\Sigma},\emptyset}$ and $\Gamma_{\Sigma,\nu\setminus\Sigma(1)}$ are, by definition, $\tilde{\Sigma}$ and Σ, respectively.

Notice that for each wall τ which intersects $\gamma_1([0, 1-\epsilon])$, the character $-\chi_K$ either lies on τ or is in the direction of γ. Furthermore, each τ which intersects $\gamma_2([0,1])$ must also intersect $B_\epsilon(-\chi_K)$ and, hence, $-\chi_K$ lies in τ.

Hence, by Theorem 5.7, each wall crossing induces a fully faithful functor or equivalence between categories of singularities corresponding to successive chambers. Concatenating gives the desired fully faithful functor,

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \longrightarrow \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{G}_m, w]\,.$$

This finishes the proof of statement (i). To prove statement (ii), replace $-\chi_K$ by χ_K, which switches the direction of the fully faithful functor. To prove statement (iii), only use the path γ_2. The last part of the statement of the theorem is just a repetition of Theorem 5.6. □

Remark 5.9. A choice of Σ as in Theorem 5.8 always exists; for example, one can apply [CLS11, Proposition 15.1.6] to the set $\nu_{=1}$.

Remark 5.10. The fully faithful functors appearing in Theorem 5.8 actually give rise to a semi-orthogonal decomposition. This can be described explicitly in terms of the wall crossings which occur in the path γ by iteratively using the semi-orthogonal decompositions described in Theorem 5.7. The description of the orthogonal can be rather complicated and cumbersome. We describe the orthogonal in the example appearing in Subsection 6.2 to illustrate how the orthogonal is always computable in practice.

5.2 Application to toric complete intersections

We now unpack the geometric consequences of this theorem. Let $\Psi \subseteq N_{\mathbb{R}}$ be a complete fan such that X_Ψ is projective, and let $D_1, \dots, D_t$ be nef divisors. Write these nef divisors as linear combinations $D_i = \sum_{\rho \in \Psi(1)} a_{i\rho} D_\rho$. Assume that $|\Psi_{-D_1,\dots,-D_t}| \subseteq (N \times \mathbb{Z}^t)_{\mathbb{R}}$ is $\mathbb{Q}$-Gorenstein. We refer the reader back to Proposition 4.9 for a class of examples of such cones. Let e_i be the standard basis of the sublattice $\mathbb{Z}^t$ in $(N \times \mathbb{Z}^t)$, and set $\mathfrak{n} := \sum_{i=1}^t e_i$. In the case where the D_i are all Cartier, this definition is aligned with the definition of $\mathfrak{n}$ in Section 4.1, by Lemma 4.6, but here we do not assume that the cone $(\mathrm{Cone}(\nu))^\vee$ is $\mathbb{Q}$-Gorenstein.

We restrict to the case where $\mathrm{Cone}(\nu) = |\Psi_{-D_1,\dots,-D_t}|$ and $\{u_\rho \,|\, \rho \in \Psi_{-D_1,\dots,-D_t}(1)\} \subseteq \nu$. The first condition amounts to $\mathrm{Cone}(\nu)^\vee$ being a Cayley cone of length t. Set R to be the subset $\{e_i \,|\, 1 \leqslant i \leqslant t\}$. Suppose that Σ is any fan such that

- X_Σ is semiprojective;
- $|\Sigma| = |\Psi_{-D_1,\dots,-D_t}|$; and
- for any $\delta \in \Sigma(1)$, we have $u_\delta \in \nu$ and $\langle \mathfrak{m}, u_\delta \rangle = 1$.

Consider the set of lattice points

$$\Xi := \left\{ m \in |\Psi_{-D_1,\dots,-D_t}|^\vee \cap \left(M \times \mathbb{Z}^t \right) \,|\, \langle m, \mathfrak{n} \rangle = 1 \right\}.$$

We also specialize to the case where the superpotential w is of the form

$$w = \sum_{m \in \Xi} c_m x^m$$

for some $c_m \in \kappa$. Let $|\nu| = n$. Enumerate the rays $\rho_1, \dots, \rho_{n-t}$ corresponding to the ray generators in $\nu \backslash R$, and introduce the variable x_k for the ray ρ_k, for $1 \leqslant k \leqslant n-t$. Similarly, introduce the variables u_j for $1 \leqslant j \leqslant t$ corresponding to the ray generators e_j in R.

If $m \in \Xi$, then there exists a unique j_0 such that

$$\langle m, e_j \rangle = \begin{cases} 1 & \text{if } j = j_0\,, \\ 0 & \text{if } j \neq j_0\,. \end{cases}$$

Hence, we can partition the set Ξ into subsets

$$\Xi_j := \{ m \in \Xi \,|\, \langle m, e_j \rangle = 1 \}.$$

Note that $\Xi = (\Delta_1 * \cdots * \Delta_t) \cap (M \times \mathbb{Z}^t)$ and $\Xi_j = (\Delta_j, e_j^*) \cap (M \times \mathbb{Z}^t)$, using the polytopes defined in Lemma 4.6.

We can decompose w as

$$w = \sum_{m \in \Xi} c_m x^m = \sum_{i=1}^t u_i f_i\,, \quad \text{where} \quad f_i := \sum_{m \in \Xi_j} c_m \prod_{k=1}^{n-t} x_k^{\langle m, u_{\rho_k} \rangle}.$$

If $m \in \Xi_j$, then the corresponding function $\prod_{k=1}^{n-t} x_k^{\langle m, u_{\rho_k}\rangle}$ is a global section of the nef divisor D_k. Hence, the function f_j is a global section of D_j. The common zero locus of all f_j is a global quotient substack

$$\mathcal{Z} := Z(f_1, \dots, f_t) \subseteq \mathcal{X}_\Psi$$

of $\mathcal{X}_\Psi$. When $f_1, \dots, f_t$ define a complete intersection, we can relate $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})$ to the factorization category associated to the fan Σ.

Proposition 5.11. *Assume that $f_1, \dots, f_t$ define a complete intersection. Then, there is an equivalence of categories*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}) \cong \mathrm{D}^{\mathrm{abs}}[U_\Psi, S_\nu \times \mathbb{G}_m, w]\,.$$

Proof. This is really a corollary of Theorem 3.5 due to Isik, Shipman, and Hirano. We describe the specifics of our setup below.

First, by Proposition 4.23, we can reduce to the case where $\nu = \{u_\rho \,|\, \rho \in \Psi_{-D_1,\dots,-D_t}(1)\}$. Then, by Lemma 4.15, the map $\pi\colon U_{\Psi_{-D_1,\dots,-D_t}} \to U_\Psi \times \mathbb{A}^t$ induces the isomorphism of stacks

$$\mathcal{X}_{\Psi_{-D_1,\dots,-D_t}} \cong \operatorname{tot}\left(\bigoplus_{i=1}^{t} \mathcal{O}_{\mathcal{X}_\Psi}(\chi_{-D_i})\right).$$

Since we chose $R = \{e_i \,|\, 1 \leqslant i \leqslant t\}$, the subgroup $\mathbb{G}_m$ in $S_\nu \times \mathbb{G}_m$ acts by scaling the coordinates u_i of $\mathbb{A}^t$. This is precisely fiberwise dilation of the vector bundle $\operatorname{tot}(\oplus_{i=1}^r \mathcal{O}_{\mathcal{X}_\Psi}(\chi_{-D_i}))$. Hence, we may apply Theorem 3.5 to get the result. □

Corollary 5.12. *Assume that $f_1, \dots, f_t$ define a complete intersection. Let Σ be any simplicial fan such that $\Sigma(1) \subseteq \nu_{=1}$, X_Σ is semiprojective, and* $\operatorname{Cone}(\Sigma(1)) = \sigma$. *We have the following:*

(i) *If $\langle u_\rho + \sum_{i=1}^t a_{i\rho} e_i, \mathfrak{m}\rangle \geqslant 1$ for all i, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})\,.$$

(ii) *If $\langle u_\rho + \sum_{i=1}^t a_{i\rho} e_i, \mathfrak{m}\rangle \leqslant 1$ for all i, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}) \longrightarrow \mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]\,.$$

(iii) *If $\langle u_\rho + \sum_{i=1}^t a_{i\rho} e_i, \mathfrak{m}\rangle = 1$ for all i, then there is an equivalence*

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})\,.$$

Furthermore, if $\mathcal{Z}$ is smooth, then $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ is fractional Calabi–Yau. If, in addition, σ is almost Gorenstein, then $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ is Calabi–Yau.

Proof. This is a direct corollary of combining Theorem 5.8 and Proposition 5.11. Note that since X_Ψ is projective, Corollary 4.5 implies that $X_{\Psi_{-D_1,\dots,-D_t}}$ is semiprojective. The hypotheses in Theorem 5.8 are then satisfied. □

This corollary is quite general. In particular, we can relate it to the examples in Subsection 4.2.

Example 5.13. Let X_Σ be a projective toric variety in $N_\mathbb{R}$, and let $D = -qK_{X_\Sigma}$ for some positive rational number q. Suppose that D is nef and that there exists a global section $f \in \Gamma(X_\Sigma, D)$. We consider the hypersurface $\mathcal{Z} = Z(f) \subset \mathcal{X}_\Sigma$. By Proposition 4.9, the toric variety $X_{\Sigma_{-D}}$ is $\mathbb{Q}$-Gorenstein. Let $\mathfrak{m} \in M_\mathbb{Q}$ be the element such that the cone $|\Sigma_{-D}|$ is generated by $\{n \in N_\mathbb{R} \,|\, \langle \mathfrak{m}, n\rangle = 1\}$. In the proof of Proposition 4.9, we show that $\langle \mathfrak{m}, (0,1)\rangle = 1/q$, hence we have the following:

(1) If $q < 1$, there is a fully faithful functor $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})$

(2) If $q > 1$, there is a fully faithful functor $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}) \longrightarrow \mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$.

(3) If $q = 1$, there is an equivalence $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})$.

If $\mathcal{Z}$ is smooth, then $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ is fractional Calabi–Yau. If $q = 1/r$, then $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}]$ is Calabi–Yau. In Subsection 6.1, we go through this example in detail in the case where $X_\Sigma = \mathbf{P}^n$.

We can specialize even further to the case where both categories are geometric. Suppose that there exist elements $e_i' \in N \times \mathbb{Z}^t$ for $1 \leqslant i \leqslant s$ such that

$$\sum_{i=1}^{s} e_i' = \sum_{i=1}^{t} e_i = \mathfrak{n}$$

and that there exists a $\mathbb{Z}$-basis for $N \times \mathbb{Z}^t$ which contains the set $\{e_i'\}$. Assume also that under the projection $p\colon N \times \mathbb{Z}^t \to N \times \mathbb{Z}^t / \langle e_i' \rangle$, the lattice points $p(u_\rho)$ for all $\rho \in \Psi_{-D_1,\dots,-D_t}(1)$ are primitive, so that the cones over each $p(u_\rho)$ can become the rays of a new fan Υ we construct below. We assume $\langle \mathfrak{m}, e_i' \rangle = 1$ for all i, that is, $\{e_i'\} \subset \nu_{=1}$. This is automatically implied if either $\mathfrak{m} \in M \times \mathbb{Z}^t$ or $\langle \mathfrak{m}, e_i' \rangle \in \mathbb{Z}$ for all i.

In this case, set

$$\nu = \{u_\rho \mid \rho \in \Psi_{-D_1,\dots,-D_t}(1)\} \cup \{e_1', \dots, e_s'\}\,.$$

The e_i' define a new collection of polytopes

$$\Delta_i' := \left\{a \in M \times \mathbb{Z}^t \mid \langle a, e_j' \rangle = \delta_{ij}\right\}$$

such that $\Delta_1' * \cdots * \Delta_s' = |\Psi_{-D_1,\dots,-D_t}|^\vee$.

Let $L := (N \times \mathbb{Z}^t)/\mathbb{Z}^r$, and let p be the projection with dual projection $p^*\colon (N \times \mathbb{Z}^t)^* \to L^*$. Consider the Minkowski sum

$$\Delta' := \sum_{i=1}^{s} p^*(\Delta_i') \subseteq L_\mathbb{R}\,.$$

Then, we can let $\Upsilon \subseteq L_\mathbb{R}$ be a simplicial refinement of the normal fan to Δ'. Each Minkowski summand Δ_i' defines a nef divisor E_i on X_Υ. Furthermore,

$$|\Psi_{-D_1,\dots,-D_t}|^\vee = \mathbb{R}_{\geqslant 0}(\Delta_1' * \cdots * \Delta_s') \;=\; |\Upsilon_{-E_1,\dots,-E_s}|^\vee\,,$$

where the second equality is Proposition 4.6. Hence,

$$|\Psi_{-D_1,\dots,-D_t}| = |\Upsilon_{-E_1,\dots,-E_s}|\,.$$

Now, we can add two additional $\mathbb{G}_m$-actions to the S_ν-action on $\mathbb{A}^\nu$. The first action $(\mathbb{G}_m)_1$ is determined by $R_1 = \{e_1, \dots, e_t\}$, and the second action $(\mathbb{G}_m)_2$ is determined by $R_2 = \{e_1', \dots, e_s'\}$.

Lemma 5.14. *There is an isomorphism of stacks* $\left[\mathbb{A}^\nu / S_\nu \times (\mathbb{G}_m)_1\right] \cong \left[\mathbb{A}^\nu / S_\nu \times (\mathbb{G}_m)_2\right]$.

Proof. Consider the 1-parameter subgroup $\beta\colon \mathbb{G}_m \hookrightarrow \mathbb{G}_m^\nu$ acting on $\mathbb{A}^\nu$ so that for $v \in \nu$, the element $s \in \mathbb{G}_m$ acts by

$$\beta(s) \cdot x_v := \begin{cases} \frac{1}{s} x_v & \text{if } v \in R_1 \backslash R_2\,, \\ s x_v & \text{if } v \in R_2 \backslash R_1\,, \\ x_v & \text{otherwise}\,. \end{cases} \tag{5.7}$$

We claim that $\beta(\mathbb{G}_m) \subseteq S_\nu$. Indeed, by definition, S_ν lies in an exact sequence

$$0 \longrightarrow S_\nu \xrightarrow{\widehat{\pi}} \mathbb{G}_m^\nu \xrightarrow{\widehat{f_\nu}} \mathbb{G}_m^{\dim N + t}\,;$$

hence, to show that $\beta(\mathbb{G}_m) \subseteq S_\nu$, we can simply check that $\widehat{f_\nu} \circ \beta = 0$. Since the functor $\widehat{(-)} := \operatorname{Hom}(-, \mathbb{G}_m)$ is exact, this is equivalent to $\widehat{\beta} \circ f_\nu = 0$. The latter is a morphism between free $\mathbb{Z}$-modules, hence vanishes if and only if the dual morphism vanishes. The vanishing of the dual morphism goes as follows:

$$\begin{aligned} f_\nu^\vee\big(\widehat{\beta}^\vee(n)\big) &= f_\nu^\vee\Bigg(n\bigg(\sum_{\rho \in R_2 \backslash R_1} e_\rho - \sum_{\rho \in R_1 \backslash R_2} e_\rho\bigg)\Bigg) \\ &= n\bigg(\sum_{i=1}^{s} e'_i - \sum_{i=1}^{t} e_i\bigg) = 0\,. \end{aligned}$$

Notice that β splits as $\chi_{D_i} \circ \beta = \mathrm{Id}$ for $i \in R_2 \backslash R_1$ (without loss of generality, we may assume $R_2 \backslash R_1 \neq \emptyset$, as otherwise $R_1 = R_2$ and the statement of the lemma is empty).

Now, add the $(\mathbb{G}_m)_1$-action. This is a $\mathbb{G}_m$-action on $\mathbb{A}^{|\nu|}$ which is given explicitly as

$$s \cdot x_v := \begin{cases} s x_v & \text{if } v \in \{e_1, \dots, e_t\}\,, \\ x_v & \text{if } v \notin \{e_1, \dots, e_t\}\,. \end{cases} \tag{5.8}$$

Let $\overline{S_\nu}$ be the subgroup induced by the splitting, so that $S_\nu = \overline{S_\nu} \times \beta(\mathbb{G}_m) \subset \mathbb{G}_m^\nu$. There is an automorphism

$$\begin{aligned} F \colon (\overline{S_\nu} \times \mathbb{G}_m) \times \mathbb{G}_m &\to (\overline{S_\nu} \times \mathbb{G}_m) \times \mathbb{G}_m\,, \\ (\bar{s}, s_1, s_2) &\mapsto (\bar{s}, s_2 s_1, s_2)\,. \end{aligned}$$

Now, the global quotient stack $[\mathbb{A}^\nu / S_\nu \times \mathbb{G}_m]$ can be considered with the action of $S_\nu \times \mathbb{G}_m$ given by precomposition with F. Under F, the action of $S_\nu \times 1$ on $\mathbb{A}^{|\nu|}$ is the same as $F(S_\nu \times 1)$ $= S_\nu \times 1$. However, the projection action of $1 \times \mathbb{G}_m$ becomes the action of the element $F(1, 1, s) =$ $(1, s, s)$.

To determine the action of the element $(1, s, s)$, notice that the action of $(1, s, 1)$ is given by equation (5.7) and the action of $(1, 1, s)$ is given by equation (5.8). Combining these two equations, we get

$$(1, s, s) \cdot x_v := \begin{cases} s x_v & \text{if } v \in \{e'_1, \dots, e'_s\}\,, \\ x_v & \text{if } v \neq \{e'_1, \dots, e'_s\}\,. \end{cases} \tag{5.9}$$

This is the $\mathbb{G}_m$-action determined by R_2. Hence, we have

$$\big[\mathbb{A}^\nu / S_\nu \times (\mathbb{G}_m)_1\big] \cong \big[\mathbb{A}^\nu / F^{-1}(S_\nu \times \mathbb{G}_m)\big] \;=\; \big[\mathbb{A}^\nu / S_\nu \times (\mathbb{G}_m)_2\big]\,, \tag{5.10}$$

as desired. □

Our new decomposition of $\mathfrak{n}$ gives a new decomposition of Ξ. Namely, if $m \in \Xi$, then there exists a unique j_0 such that

$$\langle m, e'_j \rangle = \begin{cases} 1 & \text{if } j = j_0\,, \\ 0 & \text{if } j \neq j_0\,. \end{cases}$$

This gives a new partition of Ξ into subsets

$$\Xi'_j := \{m \in \Xi \mid \langle m, e'_j \rangle = 1\}\,.$$

We again enumerate the rays as $\rho_1, \dots, \rho_{n-s}$, corresponding to the ray generators in $\nu \backslash R_2$, and introduce the variable x_k for the ray ρ_k, for $1 \leqslant k \leqslant n - s$. Similarly, we introduce the variables

u_j for $1 \leqslant j \leqslant s$ corresponding to the ray generators e'_i in R_2. We get a decomposition of w as

$$w = \sum_{m\in\Xi} c_m x^m = \sum_{i=1}^{t} u'_j g_j\,, \quad \text{where} \quad g_i := \sum_{m\in\Xi'_j} c_m \prod_{k\in 1}^{n-t} x_k^{\langle m, u_{\rho_k}\rangle}\,.$$

As above, the functions g_i can be interpreted as global sections of $\mathcal{O}(E_i)$ on X_Υ and we have a closed substack $\mathcal{Z}' := Z(g_1, \dots, g_s) \subseteq \mathcal{X}_\Upsilon$ of $\mathcal{X}_\Upsilon$.

COROLLARY 5.15. *Assume that $f_1, \dots, f_t$ and $g_1, \dots, g_s$ define complete intersections. Assume further that $s = \langle \mathfrak{m}, \mathfrak{n}\rangle$. We have the following:*

(i) *If $\langle u_\rho + \sum a_{i\rho} e_\rho, \mathfrak{m}\rangle \geqslant 1$ for all i, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}') \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})\,.$$

(ii) *If $\langle u_\rho + \sum a_{i\rho} e_\rho, \mathfrak{m}\rangle \leqslant 1$ for all i, then there is a fully faithful functor*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}) \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}')\,.$$

(iii) *If $\langle u_\rho + \sum a_{i\rho} e_\rho, \mathfrak{m}\rangle = 1$ for all i, then there is an equivalence*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}') \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})\,.$$

Furthermore, if $\mathcal{Z}'$ is smooth, then it has torsion canonical bundle. If, in addition, σ is almost Gorenstein, then $\mathcal{Z}'$ is Calabi–Yau.

Proof. We apply Theorem 5.8 to the case $\nu = \{u_\rho \mid \rho \in \Psi_{-D_1,\dots,-D_t}(1)\} \cup \{e'_1, \dots, e'_s\}$ and $R = R_1$. Proposition 5.11 give us the equivalence

$$\mathrm{D}^{\mathrm{abs}}[U_{\Psi_{-D_1,\dots,-D_t}}, S_{\Psi(1)} \times (\mathbb{G}_m)_1, w] \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})\,.$$

Similarly, Lemma 5.14 and Proposition 5.11 gives us the equivalences

$$\begin{aligned}\mathrm{D}^{\mathrm{abs}}[U_{\Upsilon_{-E_1,\dots,-E_t}}, S_{\Upsilon(1)} \times (\mathbb{G}_m)_1, w] &\cong \mathrm{D}^{\mathrm{abs}}[U_{\Upsilon_{-E_1,\dots,-E_t}}, S_{\Upsilon(1)} \times (\mathbb{G}_m)_2, w] \\ &\cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}')\,.\end{aligned}$$

□

Remark 5.16. Case (iii) of Corollary 5.15 proves the Batyrev–Nill conjecture, as in [FK17].

Remark 5.17. When $t = 1$, case (i) of Corollary 5.15 relates a Calabi–Yau complete intersection to a hypersurface in a projective bundle. It proves the fully faithfulness of the semi-orthogonal decomposition in [Orl06, Proposition 2.10] for the case of Calabi–Yau complete intersections in toric varieties.

A new case where Corollary 5.15 applies is the following.

EXAMPLE 5.18. Let $N = \mathbb{Z}^6$, and let M be its dual lattice, with $\{e_i\}$ the standard elementary basis for N. Consider the point collection $\nu = \{v_1, \dots, v_7, a_2, a_2\}$ in N, where

$$\begin{aligned} v_1 &= (1,0,0,0,0,0)\,, & v_2 &= (0,1,2,0,0,0)\,, & v_3 &= (0,0,0,2,1,0)\,,\\ v_4 &= (0,0,0,0,0,1)\,, & v_5 &= (1,1,0,0,0,0)\,, & v_6 &= (0,0,1,1,0,0)\,,\\ v_7 &= (0,0,0,0,1,1)\,, & a_1 &= (1,1,1,0,0,0)\,, & a_2 &= (0,0,0,1,1,1)\,. \end{aligned} \tag{5.11}$$

Here, $\mathfrak{m} = (1, 0, \frac{1}{2}, \frac{1}{2}, 0, 1)$, $\nu_{=1} = \{v_i\}$, and $\nu_{\neq 1} = \{a_i\}$. Note that $\langle \mathfrak{m}, a_i\rangle > 1$ for both a_i. In this example, there are multiple vector bundle structures. Here, note that we have

$$a_1 + a_2 = v_5 + v_6 + v_7\,,$$

which correspond to the sets $\{e_i\}$ and $\{e'_i\}$, respectively, in the notation above and $\mathfrak{n} = (1, 1, 1, 1, 1, 1)$. Moreover, we have $\langle \mathfrak{m}, \mathfrak{n} \rangle = 3$.

The first vector bundle structure can be seen via looking at the projection $\pi \colon N \to N/\langle a_1, a_2\rangle$. Here, we can see a fan in $N/\langle a_1, a_2\rangle \cong \mathbb{Z}^4$ (we use the isomorphism given by changing to the basis $e_1, e_2, e_4, e_5, e_1+e_2+e_3, e_4+e_5+e_6$ and then projecting to the first four dimensions). Set $\bar{v}_i$ to be $\pi(v_i)$. There is a fan Σ, where X_Σ is semiprojective and $\Sigma(1)$ is generated by

$$\begin{aligned} &\bar{v}_1 = (1,0,0,0)\,, && \bar{v}_2 = (-2,-1,0,0)\,, && \bar{v}_3 = (0,0,2,1)\,, \\ &\bar{v}_4 = (0,0,-1,-1)\,, && \bar{v}_5 = (1,1,0,0)\,, && \bar{v}_6 = (-1,-1,1,0) \\ &\bar{v}_7 = (0,0,-1,0)\,, \end{aligned} \tag{5.12}$$

Let $D_{\bar{\rho}_i}$ correspond to the divisor associated to the ray $\bar{\rho}_i = \operatorname{Cone}(\bar{v}_i)$. Here, we identify two divisors $D_1 = 2D_{\bar{\rho}_2} + D_{\bar{\rho}_6}$ and $D_2 = D_{\bar{\rho}_4} + D_{\bar{\rho}_7}$ such that $\Sigma_{-D_1,-D_2}$ is a semiprojective fan. Moreover, we can see that $\Sigma_{-D_1,-D_2}(1) = \nu$.

The second vector bundle structure can be seen by looking at the projection $\pi \colon N \to N/\langle v_5, v_6, v_7\rangle$. Here, we have a fan in $N/\langle v_5, v_6, v_7\rangle \cong \mathbb{Z}^3$ (using the isomorphism given by changing to the basis $e_1, e_3, e_5, e_1+e_2, e_3+e_4, e_5+e_6$ and then projecting to the first three dimensions). Set $\bar{v}'_i$ to be $\pi'(v_i)$. There is a fan Υ, where X_Υ is semiprojective and $\Upsilon(1)$ is generated by the cones over each of the following lattice points:

$$\bar{v}'_1 = (1,0,0)\,, \quad \bar{v}'_2 = (-1,2,0)\,, \quad \bar{v}'_3 = (0,-2,1)\,, \quad \bar{v}'_4 = (0,0,-1)\,. \tag{5.13}$$

Let $D_{\bar{\rho}'_i}$ be the divisor associated to $\operatorname{Cone}(v'_i) \in \Sigma'(1)$. We define three divisors:

$$E_1 = D_{\bar{\rho}'_2}\,, \quad E_2 = 2D_{\bar{\rho}'_3}\,, \quad E_3 = D_{\bar{\rho}'_4}\,. \tag{5.14}$$

Here, $\Upsilon_{-E_1,-E_2,-E_3}(1) \subset \nu_{=1}$.

Define a global function on the affine space $\mathbb{A}^\nu$ by taking the finite set

$$\Xi = \{m \in M \mid m \in \operatorname{Cone}(\nu), \langle m, v_i\rangle = 1 \text{ for } i = 5,6,7, \langle m, a_i\rangle = 1 \text{ for } i = 1,2\}\,.$$

Take a generic potential

$$W = \sum_{m \in \Xi} c_m x_i^{\langle m, v_i\rangle}\,,$$

which expands as

$$W = c_1x_1x_5x_8 + c_2x_2x_5x_8 + c_3x_1^2x_6x_8 + c_4x_2^2x_6x_8 + c_5x_3^2x_6x_9 + c_6x_4^2x_6x_9 + c_7x_3x_7x_9 + c_8x_4x_7x_9\,.$$

The global sections associated to each of the divisors above are

$$\begin{aligned} &f_1 = c_1x_1x_5 + c_2x_2x_5 + c_3x_1^2x_6 + c_4x_2^2x_6\,, \quad f_2 = c_5x_3^2x_6 + c_6x_4^2x_6 + c_7x_3x_7 + c_8x_4x_7\,, \\ &g_1 = c_1x_1 + c_2x_2\,, \quad g_2 = c_3x_1^2 + c_4x_2^2 + c_5x_3^2 + c_6x_4^2\,, \quad g_3 = c_7x_3 + c_8x_4\,. \end{aligned}$$

Now, $\mathcal{Z}' = Z(g_1, g_2, g_3) \subset \mathcal{X}_\Upsilon$ and $\mathcal{Z}'$ is a 0-dimensional stack with 2-torsion canonical bundle. On the other hand, $\mathcal{Z} = Z(f_1, f_2) \subset \mathcal{X}_\Sigma$ is a 2-dimensional stack. Since $\langle \mathfrak{m}, a_i\rangle > 1$ for both a_i, by Corollary 5.15, we have a fully faithful functor $\mathrm{D^b}(\operatorname{coh}\mathcal{Z}') \longrightarrow \mathrm{D^b}(\operatorname{coh}\mathcal{Z})$.

Remark 5.19. In Example 5.18, both of the decompositions of $\mathfrak{n}$ into sums of elements in $\operatorname{Cone}(\nu) \cap N$ are maximal in that there does not exist a set of elements $I \subset \operatorname{Cone}(\nu) \cap N$ such that $\sum_{n\in I} n = v_i$ or $\sum_{n \in I} n = a_i$. This is implied by the fact that ν is a Hilbert basis for the semigroup $\operatorname{Cone}(\nu) \cap N$. This differentiates our results from those in [Kuz17] as, in this example, the two vector bundle structures are, at least, not related by a toric projective bundle construction. This is the most basic example we have found. There are higher-dimensional examples as well.

6. Examples

6.1 Smooth degree d hypersurfaces in projective space

Let $N = \mathbb{Z}^{n+1}$, with elementary basis vectors e_i. Let M be the dual lattice of N. Take the geometric point collection $\nu = \{v_1, \ldots, v_n, v_{n+1}, a\}$, where

$$v_i = e_i \quad \text{for } 1 \leqslant i \leqslant n\,, \quad v_{n+1} \; = -e_1 - \cdots - e_n + de_{n+1}\,, \quad a = e_{n+1}\,. \tag{6.1}$$

The cone $\sigma := \operatorname{Cone}(\nu)$ is $\mathbb{Q}$-Gorenstein and $\mathfrak{m} := \big(1, \ldots, 1, (n+1)/d\big)$. Note that $\mathfrak{m} \in M$ if and only if d divides $n+1$. We have $\langle \mathfrak{m}, a \rangle = (n+1)/d$, so

(1) $\langle \mathfrak{m}, a \rangle > 1$ if $d < n+1$,

(2) $\langle \mathfrak{m}, a \rangle < 1$ if $d > n+1$, and

(3) $\langle \mathfrak{m}, a \rangle = 1$ if $d = n+1$.

Now, one easily computes that S_ν equals $\mathbb{G}_m$ acting on $X := \mathbb{A}^{n+2}$ with weights $1, \ldots, 1, -d$. We denote the coordinates of $\mathbb{A}^{n+2}$ by $x_1, \ldots, x_{n+1}$ for the lattice points $v_1, \ldots, v_{n+1}$ and the final coordinate by u for the lattice point a.

The secondary fan/GIT fan for this action of S_ν is 1-dimensional and pictured in Figure 6.1. The irrelevant ideal and corresponding GIT quotients are also included in the figure.

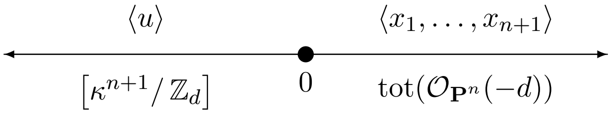

FIGURE 6.1. GIT fan for $\mathbb{G}_m$-action

Consider the set of lattice points

$$\Xi := \{m \in M \cap \sigma^\vee \mid \langle m, \mathfrak{n} \rangle = 1\}\,.$$

Note that $\operatorname{Conv}(\Xi)$ is a regular simplex with side lengths d. Also, we then have a superpotential

$$w = \sum_{m \in \Xi} c_m u x^m$$

for some $c_m \in \kappa$. The sum $f = \sum_{m \in \Xi} c_m x^m$ is a homogeneous degree d polynomial in the variables x_i. Choose the coefficients c_m in such a way that $Z(f)$ is a smooth hypersurface in $\operatorname{Proj}(\kappa[x_1, \ldots, x_{n+1}])$.

We have two fans Σ and $\tilde{\Sigma}$ that correspond to the two chambers of the secondary fan. The fan Σ corresponding to the negative direction is the collection of cones consisting of σ and its proper faces. Note that Σ is simplicial, $X_\Sigma = \mathbb{A}^{n+1} / \mathbb{Z}_d$ is semiprojective, $\Sigma(1) \subseteq \nu_{=1}$, and $\operatorname{Cone}(\Sigma(1)) = \sigma$. The corresponding potential on U_Σ is $\bar{w} = f$.

The fan $\tilde{\Sigma}$ corresponding to the positive direction is the star subdivision of Σ along e_{n+1}. Hence, $\tilde{\Sigma}$ is a simplicial fan with $\tilde{\Sigma}(1) = \nu$ and $X_{\tilde{\Sigma}} = \operatorname{tot}(\mathcal{O}_{\mathbf{P}^n}(-d))$ is semiprojective. We therefore are in the context of Proposition 5.11 and can study the derived category of the hypersurface $\mathcal{Z} := Z(f) \subset \mathbf{P}^n$. Another way to say this is that $\tilde{\Sigma}$ equals Ψ_{-dD_1}, where Ψ is the fan for $\mathbf{P}^n$ and D_1 is the coordinate hyperplane defined by x_1.

By Corollary 5.12,

(1) if $d < n+1$, then we have a fully faithful functor $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, \mathbb{G}_m, f] \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})$;

(2) if $d > n+1$, then we have a fully faithful functor $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathcal{Z}) \longrightarrow \mathrm{D}^{\mathrm{abs}}[U_\Sigma, \mathbb{G}_m, f]$;

(3) if $d = n+1$, then we have an equivalence $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, \mathbb{G}_m, f] \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathcal{Z})$.

Moreover, since f cuts out a smooth hypersurface, by Theorem 5.6, the category $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, \mathbb{G}_m, f]$ is fractional Calabi–Yau of dimension $(n+1)(d-2)/d$. If d divides $n+1$, then $\mathfrak{m} \in M$ and the category $\mathrm{D}^{\mathrm{abs}}[U_\Sigma, \mathbb{G}_m, f]$ is Calabi–Yau of the given dimension.

The path γ crosses a single wall determined by the "identity" 1-parameter subgroup. The fixed locus for the action of $\mathbb{G}_m$ is just the origin of $\mathbb{A}^\nu$. Using the description of the right orthogonal in [BFK17, Theorem 5.2.1], we get [Orl09, Theorem 3.11] (with the possible addition of a finite group action). Without the final group action, details of the explicit comparison were already provided in [BFK17, Section 7]. The statement can also be derived from a minor generalization of Orlov's proof.

6.2 A semi-orthogonal decomposition with a geometric FCY category

We start by defining a set $\nu \subset N := \mathbb{Z}^6$ consisting of eight lattice points

$$\begin{aligned} &v_1 = (1,0,0,0,0,0)\,, && v_2 = (0,1,0,0,0,0)\,, && v_3 = (0,0,1,0,0,0)\,, && v_4 = (0,1,0,2,1,2)\,,\\ &v_5 = (-1,-2,-1,-2,0,0)\,, && v_6 = (0,0,0,0,1,0)\,, && v_7 = (0,0,0,0,-1,1)\,, && v_8 = (0,0,0,0,0,1)\,. \end{aligned}$$

We can see that the cone $\operatorname{Cone}(\nu)$ is $\mathbb{Q}$-Gorenstein. Here, $\mathfrak{m} = (1,1,1,-\frac{5}{2},1,2)$ and $\nu_{=1} = \{v_1, \ldots, v_7\}$. In this example, we have $\mathfrak{n} = (0,0,0,0,0,1)$, so $\langle \mathfrak{m}, \mathfrak{n} \rangle = 2$. We define a superpotential on $\mathbb{A}^\nu = \mathbb{A}^8$ with variables $x_1, \ldots, x_8$:

$$\begin{aligned} w = {} & x_8x_6x_1^2x_2 + x_8x_6x_2^3 + x_8x_6x_3^2x_4 + x_8x_6x_4^3 + x_8x_6x_4x_5^2 \\ & + x_8x_7x_1^2 + x_8x_7x_2^2 + x_8x_7x_3^2 + x_8x_7x_4^2 + x_8x_7x_5^2\,. \end{aligned}$$

There are two vector bundle structures such that their rays are generated by the elements in ν. First, consider the projection $\pi\colon N \to N/\langle e_6 \rangle$ and a complete fan Υ with rays

$$\begin{aligned} &\rho_1 = (1,0,0,0,0)\,, && \rho_2 = (0,1,0,0,0)\,, && \rho_3 = (0,0,1,0,0)\,, && \rho_4 = (0,1,0,2,1)\,,\\ &\rho_5 = (-1,-2,-1,-2,0)\,, && \rho_6 = (0,0,0,0,1)\,, && \rho_7 = (0,0,0,0,-1) \end{aligned}$$

such that X_Υ is semiprojective. Consider the line bundle associated to the toric divisor $D = 2D_{\rho_4} + D_{\rho_7}$. Here, $\Upsilon_{-D}(1) = \{\operatorname{Cone}(v_i) \mid v_i \in \nu\}$. Here, x_8 is the bundle coordinate. Note that in this example, $\mathrm{D}^{\mathrm{abs}}\,[U_{\Upsilon_{-D}}, S_\nu \times \mathbb{G}_m, w] \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathcal{Z})$, where $\mathcal{Z}$ is the zero set of the global section

$$f_1 = x_6x_1^2x_2 + x_6x_2^3 + x_6x_3^2x_4 + x_6x_4^3 + x_6x_4x_5^2 + x_7x_1^2 + x_7x_2^2 + x_7x_3^2 + x_7x_4^2 + x_7x_5^2$$

of the divisor D. By a routine check, we can see that the zero locus $\mathcal{Z} := Z(f_1) \subset X_\Upsilon$ is a smooth stack.

Alternatively, consider the projection $\pi'\colon N \to N/\langle v_6, v_7 \rangle \cong \mathbb{Z}^4$. We can define a complete fan Υ with rays

$$\rho_1 = (1,0,0,0)\,, \quad \rho_2 = (0,1,0,0)\,, \quad \rho_3 = (0,0,1,0)\,, \quad \rho_4 = (0,1,0,2)\,, \quad \rho_5 = (-1,-2,-1,-2)$$

such that X_Υ is semiprojective. Namely, $\mathcal{X}_\Upsilon$ is the quotient stack $[\mathbf{P}^4 \,/\, \mathbb{Z}_2]$, where the $\mathbb{Z}_2$ acts by

$$g \cdot (y_0 : y_1 : y_2 : y_3 : y_4) = (y_0 : -y_1 : y_2 : -y_3 : y_4)\,.$$

Define two line bundles associated to the toric divisors $E_1 = 3D_{\rho_4}$ and $D_2 = 2E_{\rho_4}$. Here, we can write the split vector bundle $\Upsilon_{-D_1,-D_2}$ with rays generated by $\nu_{=1}$. We can compute from w that the functions

$$g_1 := y_1^2 y_2 + y_2^3 + y_3^2 y_4 + y_4^3 + y_4 y_5^2 \,, \quad g_2 := y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2$$

are global sections of E_1 and E_2, respectively.

Let $H_{\Upsilon,R}$ be the subgroup of S_ν corresponding to $\Upsilon(1) \subset \nu$ and $R = \{v_6, v_7\}$, and $\bar{w}$ the potential corresponding to setting x_8 to 1. We can see that

$$\mathrm{D}^{\mathrm{abs}} \left[U_{\Upsilon_{-E_1,-E_2}}, H_{\Upsilon,R} \times \mathbb{G}_m, \bar{w} \right] \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}') \,,$$

where $\mathcal{Z}'$ is the smooth stacky complete intersection

$$\mathcal{Z}' := Z(g_1, g_2) \subseteq \left[\mathbf{P}^4 / \mathbb{Z}_2 \right] .$$

Here, $\mathcal{Z}'$ is a 2-dimensional stack with a 2-torsion canonical bundle. By Corollary 5.15(i), there is a fully faithful functor $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}') \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})$.

To compute the semi-orthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z})$, we first must state the GIT problem associated to ν. We have $X := \mathbb{A}^8$ with variables x_i and can show that $S_\nu = \mathbb{G}_m^2 \times \mathbb{Z}_2$ through a computation. We summarize the weight of each variable with the following table:

Coordinates	Weight in $\mathbb{G}_m^2 \times \mathbb{Z}_2$
x_1, x_3, x_5	$(1, 1, 1)$
x_2, x_4	$(1, 1, 0)$
x_6	$(-1, 0, 0)$
x_7	$(0, 1, 0)$
x_8	$(-2, -3, 0)$

The secondary fan for this action of S_ν is 2-dimensional and is pictured in below:

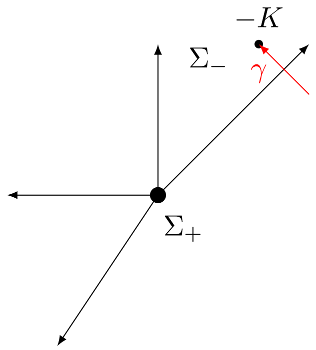

Figure 6.2. GIT fan for the $\mathbb{G}_m^2$-action

In Figure 6.2, the chambers Σ_- and Σ_+ correspond to the categories $\mathrm{D}^{\mathrm{abs}} [U_{\Upsilon_{-D}}, S_\nu \times \mathbb{G}_m, w]$ and $\mathrm{D}^{\mathrm{abs}} [U_{\Upsilon_{-E_1,-E_2}}, H_{\Upsilon,R} \times \mathbb{G}_m, \bar{w}]$, respectively. The wall that the chambers share corresponds to the 1-parameter subgroup $\lambda \colon \mathbb{G}_m \to S_\nu$ corresponding to the element $(1, -1)$. The fixed locus of λ is $Z(x_6, x_7, x_8)$, where the semistable locus is the open set $\mathbb{A}^8 \setminus Z(x_1, \ldots, x_5)$; hence, $U_0 = Z(x_6, x_7, x_8) \setminus Z(x_1, \ldots, x_5)$. One can compute that we have $S_0 = S_\nu / \lambda(\mathbb{G}_m) \cong \mathbb{G}_m \times \mathbb{Z}_2$ acting with weight $(1, 1)$ on x_1, x_3, and x_5 and weight $(1, 0)$ on x_2 and x_4. The induced section w_0 is zero as x_8 divides w.

We can compute that $\mu = -\sum_{v_i \in \nu} \langle (1,-1), v_i \rangle = 1$. By Theorem 5.7, we then have

$$\mathrm{D}^{\mathrm{abs}}\left[U_{\Upsilon_{-D}}, S_\nu \times \mathbb{G}_m, w\right] = \langle \mathrm{D}^{\mathrm{abs}}\left[U_0, S_0 \times \mathbb{G}_m, w_0\right], \mathrm{D}^{\mathrm{abs}}\left[U_{\Upsilon_{-E_1,-E_2}}, H_{\Upsilon,R} \times \mathbb{G}_m, \bar{w}\right] \rangle .$$

Using Theorem 5.11, this simplifies to

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}) = \langle \mathrm{D}^{\mathrm{abs}}\left[U_0, S_0 \times \mathbb{G}_m, 0\right], \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}') \rangle .$$

Since the $\mathbb{G}_m$-factor of $S_0 \times \mathbb{G}_m$ acts trivially on U_0, by [BDFIK17, Proposition 2.1.6] or [PV16, Proposition 1.2.2], we have

$$\begin{aligned}\mathrm{D}^{\mathrm{abs}}\left[U_0, S_0 \times \mathbb{G}_m, 0\right] = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathbf{P}^4 / \mathbb{Z}_2]) = \langle & \mathcal{O}(0,0), \mathcal{O}(1,0), \mathcal{O}(2,0), \mathcal{O}(3,0), \mathcal{O}(4,0), \\ & \mathcal{O}(0,1), \mathcal{O}(1,1), \mathcal{O}(2,1), \mathcal{O}(3,1), \mathcal{O}(4,1) \rangle .\end{aligned}$$

In conclusion, we can combine the last two displayed formulas and use a mutation to say that there is a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}) = \left\langle \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\left[\mathbf{P}^4 / \mathbb{Z}_2\right]), \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}') \right\rangle = \langle \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}'), E_1, \ldots, E_{10} \rangle ,$$

where $E_1, \ldots, E_{10}$ are exceptional objects.

6.3 Singular cubic $(3n+1)$-folds

In this section, we apply our results to demonstrate Example 1.8 from the introduction. Take n to be a positive integer. Consider the cubic $(3n+1)$-fold $\mathcal{Z}_{\mathrm{sing}}$ given by the polynomial

$$\sum_{i=1}^{n} x_i f_i(x_{n+1}, \ldots, x_{3n+3}) + x_{3n+4} f_0(x_{n+1}, \ldots, x_{3n+3}) .$$

In the case $n = 1$, the cubic fourfold is singular at a point, namely at $P = (1,0,0,0,0,0) \in \mathbf{P}^5$. This case was studied by Kuznetsov in [Kuz10]. In our generalization, the cubic $(3n+1)$-fold is singular in an $(n-1)$-dimensional hyperplane $\{x_{n+1} = \cdots = x_{3n+3} = 0\}$.

Recall that by Orlov's theorem, we have a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_{\mathrm{sing}}) = \langle \mathcal{A}, \mathcal{O}, \ldots, \mathcal{O}(3n-1) \rangle . \tag{6.2}$$

Here, the subcategory $\mathcal{A}$ is not homologically smooth, hence is not a Calabi–Yau category, but has a crepant categorical resolution.

We prove that a crepant categorical resolution of $\mathcal{A}$ is geometric. This is achieved by interpreting $\mathcal{A}$ as the absolute derived category of a Landau–Ginzburg model. We can also find a Landau–Ginzburg model interpretation of the crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_{\mathrm{sing}})$. The details of this will be provided in the exposition and proofs below. For now, we have the following summary.

Proposition 6.1. *There is a chain of fully faithful functors*

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_{\mathrm{CY}}) \longrightarrow \mathrm{D}^{\mathrm{b}}(\widetilde{\mathrm{coh}\, \mathcal{Z}_{\mathrm{sing}}}) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \widetilde{\mathcal{Z}_{\mathrm{sing}}}) ,$$

where

(i) *$\mathcal{Z}_{\mathrm{CY}}$ is the $(n+1)$-dimensional Calabi–Yau complete intersection in $\mathbf{P}^{2n+2}$ given by one generic cubic f_0 and n generic quadrics $f_1, \ldots, f_n$, and its derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_{\mathrm{CY}})$ is a crepant categorical resolution of the category $\mathcal{A}$ in equation (6.2);*

(ii) *$\mathrm{D}^{\mathrm{b}}(\widetilde{\mathrm{coh}\, \mathcal{Z}_{\mathrm{sing}}})$ is a crepant categorical resolution of the derived category of a singular cubic*

$(3n+1)$-fold $\mathcal{Z}_{\mathrm{sing}}$ given by the equation

$$\sum_{i=1}^{n} x_i f_i(x_{n+1},\dots,x_{3n+3}) + x_{3n+4} f_0(x_{n+1},\dots,x_{3n+3})\,;$$

(iii) *$\widetilde{\mathcal{Z}_{\mathrm{sing}}}$ is the blow-up of $\mathcal{Z}_{\mathrm{sing}}$ along the hyperplane $\{x_{n+1} = \dots = x_{3n+3} = 0\}$.*

Remark 6.2. Remark 3.2 points out a difference between our definition of a categorical resolution of singularities and the one in [Kuz08]. The final fully faithful functor in Proposition 6.1 guarantees that $\mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh}\mathcal{Z}_{\mathrm{sing}}})$ is a crepant categorical resolution in the sense of ibid. as well.

First, we will describe the three distinct factorization categories in the same toric GIT problem. Then, we will show that they all correspond to the categories in Proposition 6.1.

We follow the notation set up in Sections 4 and 5. Let $N = \mathbb{Z}^{3n+3}$, with elementary basis vectors e_i and dual lattice M. Now, consider the geometric point collection $\nu = \{v_1,\dots,v_{3n+4},a\}$ in N, where

$$\begin{aligned} v_i &= e_i \quad \text{for } 1 \leqslant i \leqslant 3n+2\,,\\ v_{3n+3} &= -\sum_{i=1}^{3n+2} e_i + 3e_{3n+3}\,,\\ v_{3n+4} &= -\sum_{i=1}^{n} e_i + e_{3n+3}\,,\\ a &= e_{3n+3}\,. \end{aligned} \tag{6.3}$$

The cone $\sigma := \operatorname{Cone}(\nu)$ is almost Gorenstein with $\mathfrak{m} = (1,\dots,1,n+1)$. Here, $\nu_{=1} = \{v_i\}$ and $A = \{a\}$. We have $\langle \mathfrak{m}, a\rangle = n+1 > 1$. We compute that $S_\nu = \mathbb{G}_m^2$ acts on $X := \mathbb{A}^{3n+5}$ by the weights in the following table:

Coordinates	Weight of $\mathbb{G}_m^2$
$x_1,\dots,x_n$	$(1,1)$
$x_{n+1},\dots,x_{3n+3}$	$(1,0)$
x_{3n+4}	$(0,1)$
u	$(-3,-1)$

Let $R_1 = \{a\}$, and let $R_2 = \{v_1,\dots v_n, v_{3n+4}\}$. That is, the R-charge $\mathbb{G}_m$-action associated to the subset R_1 denoted by $(\mathbb{G}_m)_{R_1}$ acts with weight 0 on the x_i and weight 1 on u. Analogously, $(\mathbb{G}_m)_{R_2}$ acts with weight 0 on $u, x_{n+1},\dots,x_{3n+3}$ and weight 1 on $x_1,\dots,x_n,x_{3n+4}$. Recall that, by Lemma 5.14, there is a stack isomorphism between different choices of R-charge.

The secondary fan for this action of S_ν is 2-dimensional and is pictured in Figure 6.3.

We can compute the relevant irrelevant ideals

$$\begin{aligned} \mathcal{I}_{\Gamma_1} &= \langle ux_1,\dots,ux_n,ux_{3n+4}\rangle\,,\\ \mathcal{I}_{\Gamma_2} &= \langle ux_1,\dots,ux_n,x_{3n+4}x_1,\dots,x_{3n+4}x_{3n+3}\rangle\,,\\ \mathcal{I}_{\Gamma_3} &= \langle x_{n+1},\dots,x_{3n+3}\rangle\langle x_{3n+4},x_1,\dots,x_n\rangle\,,\\ \mathcal{I}_{\Gamma_4} &= \langle ux_{n+1},\dots,ux_{3n+3}\rangle\,. \end{aligned} \tag{6.4}$$

A generic superpotential w is of the form

$$w = u\left(\sum_{i=1}^{n} x_i f_i(x_{n+1},\dots,x_{3n+3}) + x_{3n+4} f_0(x_{n+1},\dots,x_{3n+3})\right),$$

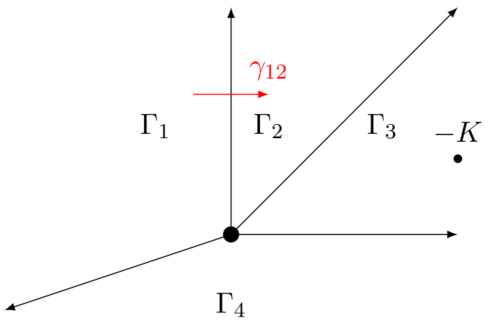

Figure 6.3. GIT fan for the $\mathbb{G}_m^2$-action

where f_0 is a cubic and $f_1, \dots, f_n$ are quadrics.

For each chamber, there is an open set $U_i = \mathbb{A}^{3n+5} \setminus Z(\mathcal{I}_{\Gamma_i})$ such that there is a factorization category $\mathrm{D}^{\mathrm{abs}}(U_i, S_\nu \times (\mathbb{G}_m)_{R_1}, w)$ associated to each chamber Γ_i.

Proof of Proposition 6.1. By Theorem 3.12, we know that there is a poset structure for which the factorization category has a fully faithful functor into another. Namely, we have

$$\begin{gathered}\mathrm{D}^{\mathrm{abs}}(U_1, S_\nu \times (\mathbb{G}_m)_{R_1}, w) \cong \mathrm{D}^{\mathrm{abs}}(U_4, S_\nu \times (\mathbb{G}_m)_{R_2}, w) \\ \downarrow \\ \mathrm{D}^{\mathrm{abs}}(U_2, S_\nu \times (\mathbb{G}_m)_{R_1}, w) \\ \downarrow \\ \mathrm{D}^{\mathrm{abs}}(U_3, S_\nu \times (\mathbb{G}_m)_{R_1}, w)\,. \end{gathered} \tag{6.5}$$

By providing equivalences to the geometric categories specified in the proposition, part (i) is proven by combining Propositions 6.3 and 6.4, part (ii) by Proposition 6.4 and part (iii) by Proposition 6.7 below. □

We will go through each chamber systematically, explaining their geometric content. Note that the R-charge for the Γ_4-chamber has changed as the bundle coordinates will change in our geometric interpretation. Here, we can show that $\mathrm{D}^{\mathrm{abs}}(U_1, S_\nu \times (\mathbb{G}_m), w) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{CY}})$ via the chamber Γ_4.

Proposition 6.3. *The category $\mathrm{D}^{\mathrm{abs}}(U_1, S_\nu \times (\mathbb{G}_m), w)$ is equivalent to the derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{CY}})$, where $\mathcal{Z}_{\mathrm{CY}}$ is the $(n+1)$-dimensional Calabi–Yau complete intersection $Z(f_0, \dots, f_n)$ in $\mathbf{P}^{2n+2}$ defined by one cubic f_0 and n quadrics $f_1, \dots, f_n$.*

Proof. First, recall that

$$\mathrm{D}^{\mathrm{abs}}(U_1, S_\nu \times (\mathbb{G}_m), w) \cong \mathrm{D}^{\mathrm{abs}}(U_4, S_\nu \times (\mathbb{G}_m), w)\,.$$

In the chamber Γ_4, note that the fan Σ_4 corresponding to this chamber has the rays generated by $v_1, \dots, v_n, v_{3n+4}$ as generators for all maximal cones. We can then take the projection $\pi\colon \mathbb{Z}^{3n+3} \to \mathbb{Z}^{3n+3} / \langle e_1, \dots, e_n, e_{3n+4} - e_1 - \cdots - e_n \rangle \cong \mathbb{Z}^{2n+2}$. Denote by Ψ_4 the fan generated by the image under π of the cones in Σ_4. Then, Ψ_4 is the standard fan for $\mathbf{P}^{2n+2}$. One can check that

$$X_{\Sigma_4} = \operatorname{tot}\left(\mathcal{O}_{\mathbf{P}^{2n+2}}(-3) \oplus \mathcal{O}_{\mathbf{P}^{2n+2}}(-2)^{\oplus n} \right).$$

Let $\mathcal{Z}_{\mathrm{CY}}$ denote the complete intersection

$$\mathcal{Z}_{\mathrm{CY}} = Z(f_0, f_1, \ldots, f_n) \subseteq \mathbf{P}^{2n+2}\,.$$

Since the f_i are generic, $\mathcal{Z}_{\mathrm{CY}}$ is a smooth stack. We have the equivalence

$$\mathrm{D}^{\mathrm{abs}}(U_4, S_\nu \times (\mathbb{G}_m)_{R_2}, w) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{CY}})\,. \tag{6.6}$$

Moreover, by Corollary 5.6, since $\Sigma_4(1) = \nu_{=1}$, the category $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{CY}})$ is Calabi–Yau of dimension

$$-2 \sum_{a \in \nu_{\neq 1}} \langle \mathfrak{m}, a\rangle + 2|\nu_{\neq 1}| - 2|R_2| + \dim N_{\mathbb{R}} = -2(0) + 0 - 2(n+1) + (3n+3) = n+1\,. \qquad \square$$

PROPOSITION 6.4. (i) *The category* $\mathrm{D}^{\mathrm{abs}}(U_1, S_\nu \times (\mathbb{G}_m)_{R_1}, w)$ *is a crepant categorical resolution of the Calabi–Yau category* $\mathcal{A}$ *in equation* (6.2).

(ii) *The category* $\mathrm{D}^{\mathrm{abs}}(U_2, S_\nu \times (\mathbb{G}_m)_{R_1}, w)$ *is a crepant categorical resolution of the category* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}})$ *in equation* (6.2).

Proof. This follows from Lemmas 6.5 and 6.6 below. $\square$

The idea of the proof of this proposition is to show that U_1 and U_2 correspond to partial compactifications of gauged Landau–Ginzburg models corresponding to the Orlov theorem described in Subsection 6.1. We will first recall the necessary data from that subsection and will then use the machinery created in Section 3 to prove the lemma.

We define subsets of U_1 and U_2. Consider the subideals

$$\mathcal{J}_{\Gamma_1} := \langle u x_{3n+4}\rangle \subset \mathcal{I}_{\Gamma_1}\,, \quad \mathcal{J}_{\Gamma_2} := \langle x_{3n+4}x_1, \ldots, x_{3n+4}x_n\rangle \subset \mathcal{I}_{\Gamma_2}\,. \tag{6.7}$$

Now, we have two new open subsets $V_i := \mathbb{A}^{3n+5} \setminus Z(\mathcal{J}_{\Gamma_i})$.

In $X' = \mathbb{A}^{3n+4}$ with variables $x_1, \ldots, x_{3n+3}, u$, consider the ideals $\mathcal{I}'_1 = \langle u\rangle$ and $\mathcal{I}'_2 = \langle x_1, \ldots, x_{3n+3}\rangle$ and the open sets $U'_i = X' \setminus Z(\mathcal{I}'_i)$. Let $\mathbb{G}_m$ act with weight 1 on x_i and weight -3 on u. By Lemma 4.22, we have a stack isomorphism

$$\left[V_i \big/ \mathbb{G}_m^2 \times (\mathbb{G}_m)_{R_1}\right] = \left[U'_i \big/ \mathbb{G}_m \times (\mathbb{G}_m)_{R_1}\right] .A$$

Define the superpotential

$$\bar{w} = u\left(\sum_{i=1}^{n} x_i f_i(x_{n+1}, \ldots, x_{3n+3}) + f_0(x_{n+1}, \ldots, x_{3n+3})\right),$$

where f_0 is a cubic and $f_1, \ldots, f_n$ are quadrics. This is a specialization of w where x_{3n+4} is set to 1.

LEMMA 6.5. *The category* $\mathrm{D}^{\mathrm{abs}}(U_i, S_\nu \times (\mathbb{G}_m)_{R_1}, w)$ *is a crepant categorical resolution of the category* $\mathrm{D}^{\mathrm{abs}}(U'_i, \mathbb{G}_m \times (\mathbb{G}_m)_{R_1}, \bar{w})$ *for* $i = 1, 2$.

Proof. Consider the open immersion $V \hookrightarrow U$, where

$$V = X \setminus Z(x_{3n+4})\,, \quad U = X \setminus Z(ux_1, \ldots, ux_n, x_{3n+4})\,. \tag{6.8}$$

A direct computation shows that the ideal $\langle ux_1, \ldots, ux_n, x_{3n+4}\rangle$ is the irrelevant ideal associated to the cone in the GIT fan that is the common face between the chambers Γ_1 and Γ_2. The path γ_{12} in Figure 6.3 gives the following stratifications associated to its elementary wall crossing:

$$\begin{aligned} U &= U_1 \sqcup S_-\,, \quad V = V_1 \sqcup S_-\,, \quad S_- := Z(u) \cap U\,, \\ U &= U_2 \sqcup S_+\,, \quad V = V_2 \sqcup S_+\,, \quad S_+ := Z(x_1, \ldots, x_{3n+3}) \cap U\,. \end{aligned} \tag{6.9}$$

Note that

$$S_- \cap (U_1 \setminus V_1) = \varnothing \quad \text{and} \quad S_+ \cap (U_2 \setminus V_2) = \varnothing\,; \tag{6.10}$$

hence, the immersions are compatible with the elementary wall crossing. By Theorem 3.7, we have that $\mathrm{D}^{\mathrm{abs}}(U_i, S_\nu \times (\mathbb{G}_m), w)$ is a crepant categorical resolution of $\mathrm{D}^{\mathrm{abs}}(U_i', \mathbb{G}_m \times (\mathbb{G}_m)_{R_1}, \bar{w})$. ☐

Lemma 6.6. *We have the following derived equivalences:*

$$\mathrm{D}^{\mathrm{abs}}(U_1', \mathbb{G}_m \times (\mathbb{G}_m)_{R_1}, \bar{w}) \cong \mathcal{A}\,, \quad \mathrm{D}^{\mathrm{abs}}(U_2', \mathbb{G}_m \times (\mathbb{G}_m)_{R_1}, \bar{w}) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}})\,, \tag{6.11}$$

where $\mathcal{A}$ *and* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}})$ *are as defined in equation* (6.2).

Proof. The GIT problem for X' with the $\mathbb{G}_m$-action defined above is the same as that in Subsection 6.1:

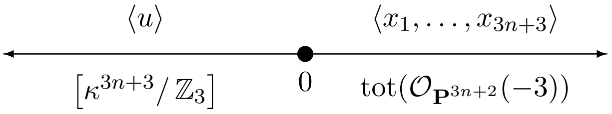

Figure 6.4. GIT fan for the $\mathbb{G}_m$-action

Recall from Subsection 6.1 that we have a fully faithful functor

$$\mathrm{D}^{\mathrm{abs}}(U_1', \mathbb{G}_m \times \mathbb{G}_m, \bar{w}) \longrightarrow \mathrm{D}^{\mathrm{abs}}(U_2', \mathbb{G}_m \times \mathbb{G}_m, \bar{w})\,,$$

and, by Theorem 3.5,

$$\mathrm{D}^{\mathrm{abs}}(U_2', \mathbb{G}_m \times \mathbb{G}_m, \bar{w}) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}})\,, \tag{6.12}$$

where

$$\mathcal{Z}_{\mathrm{sing}} := Z\left(\sum_{i=1}^{n} x_i f_i(x_{n+1}, \ldots, x_{3n+3}) + f_0(x_{n+1}, \ldots, x_{3n+3}) \right) \subset \mathbf{P}^{3n+2}$$

is the singular cubic $(3n+1)$-fold. The category $\mathcal{A}$ is $\mathrm{D}^{\mathrm{abs}}(U_1', \mathbb{G}_m \times \mathbb{G}_m, \bar{w})$. ☐

We finish with chamber Γ_3.

Proposition 6.7. *Let* $\mathrm{Bl}_Y(\mathbf{P}^{3n+2})$ *be the blow-up of* $\mathbf{P}^{3n+2}$ *along the hyperplane* Y *given by* $\{x_{n+1} = \cdots = x_{3n+3} = 0\}$. *Denote by* $\mathcal{Z}$ *the hypersurface stack*

$$\widetilde{\mathcal{Z}_{\mathrm{sing}}} = Z\left(\sum_{i=1}^{n} x_i f_i(x_{n+1}, \ldots, x_{3n+2}) + x_{3n+4} f_0(x_{n+1}, \ldots, x_{3n+3}) \right) \subseteq \mathrm{Bl}_Y\left(\mathbf{P}^{3n+2}\right).$$

Then we have the equivalence

$$\mathrm{D}^{\mathrm{abs}}(U_3, S_\nu \times (\mathbb{G}_m)_{R_1}, w) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \widetilde{\mathcal{Z}_{\mathrm{sing}}})\,.$$

Proof. In the chamber Γ_3, note that the fan Σ_3 corresponding to this chamber has the ray generated by a in all maximal cones. We can then consider the projection map $\pi\colon \mathbb{Z}^{3n+3} \to \mathbb{Z}^{3n+2} = \mathbb{Z}^{3n+3} / \langle e_{3n+3} \rangle$, which induces a fan Ψ_3 that is the image under π of all the faces in Σ_3. Then $X_{\Psi_3} = \mathrm{Bl}_Y(\mathbf{P}^{3n+2})$, where Y is the hyperplane given by $\{x_{n+1} = \cdots = x_{3n+3} = 0\}$. Call the exceptional divisor E. Then $X_{\Sigma_3} = \operatorname{tot}(\mathcal{O}_{\mathrm{Bl}_Y(\mathbf{P}^{3n+2})}(-3H - E))$. The equivalence

$$\mathrm{D}^{\mathrm{abs}}(U_3, S_\nu \times (\mathbb{G}_m)_{R_1}, w) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \widetilde{\mathcal{Z}_{\mathrm{sing}}})$$

is then immediately obtained by Theorem 3.5. ☐

6.4 Degree d $(2d-2)$-folds containing two planes

Fix $d \geqslant 3$. Consider the two planes $P_1 = \{x_{2d-3} = x_{2d-2} = x_{2d-1} = 0\}$ and $P_2 = \{x_1 = \cdots = x_{2d-4} = x_{2d} = 0\}$ in $\mathbf{P}^{2d-1}$. Let $\mathcal{Z}_{\mathrm{sing}}$ be a generic cubic that contains both P_1 and P_2. When $d = 3$, the cubic is smooth and this example's rationality was studied by Hassett [Has00]. When $d > 3$, the cubic is singular.

Recall that by Orlov's theorem, we have a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}}) = \langle \mathcal{A}, \mathcal{O}, \ldots, \mathcal{O}(d-1) \rangle \,. \tag{6.13}$$

In the case where $d > 3$, the cubic $\mathcal{Z}_{\mathrm{sing}}$ is not smooth, so $\mathcal{A}$ is not Calabi–Yau but has a crepant categorical resolution that is.

We will prove that a categorical resolution of $\mathcal{A}$ is geometric. As in the previous subsection, this is achieved by interpreting $\mathcal{A}$ as the absolute derived category of a Landau–Ginzburg model. We can also find a Landau–Ginzburg model interpretation of the crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}})$. The details of this will be provided in the exposition and proofs below. For now, we summarize our findings in the following way.

PROPOSITION 6.8. *There is a chain of fully faithful functors*

$$\begin{tikzcd}
 & & \mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_2}) \arrow[dr] & \\
\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{CY}}) \arrow[r] & \mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_{\mathrm{sing}}}) \arrow[ur] \arrow[dr] & & \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \widetilde{\mathcal{Z}_{\mathrm{sing}}})\,, \\
 & & \mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_3}) \arrow[ur] &
\end{tikzcd}$$

where

(i) *$\mathcal{Z}_{\mathrm{CY}}$ is a $(2d-4)$-dimensional Calabi–Yau complete intersection of two polynomials of bidegree $(d-2, 2)$ and $(d-1, 1)$ in $\mathbf{P}^{2d-4} \times \mathbf{P}^2$, and its derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{CY}})$ is a crepant categorical resolution of $\mathcal{A}$ in equation (6.13);*

(ii) *$\mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_{\mathrm{sing}}})$ is a crepant categorical resolution of the derived category of the degree d hypersurface $\mathcal{Z}_{\mathrm{sing}}$ in $\mathbf{P}^{2d-1}$ containing the two planes P_1 and P_2;*

(iii) *$\mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_2})$ is a crepant categorical resolution of the derived category of the degree d hypersurface $\mathcal{Z}_{\mathrm{sing}}$ blown up at the plane P_2;*

(iv) *$\mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_3})$ is a crepant categorical resolution of the derived category of the degree d hypersurface $\mathcal{Z}_{\mathrm{sing}}$ blown up at the plane P_1; and*

(v) *$\widetilde{\mathcal{Z}_{\mathrm{sing}}}$ is the degree d hypersurface $\mathcal{Z}_{\mathrm{sing}}$ blown up at both planes P_1 and P_2.*

Remark 6.9. As in the previous example, our proposition guarantees that $\mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_{\mathrm{sing}}})$, $\mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_2})$, and $\mathrm{D}^{\mathrm{b}}(\widetilde{\operatorname{coh} \mathcal{Z}_3})$ are crepant categorical resolutions in the sense of Kuznetsov [Kuz08] as well as our own.

As alluded to previously, the fully faithful functors in Proposition 6.8 are obtained using comparisons between toric Landau–Ginzburg models. The precise toric setup is as follows. Fix the lattice $N = \mathbb{Z}^{2d}$ with elementary basis vectors e_i and dual lattice M. Take the geometric point collection $\nu = \{v_1, \ldots, v_{2d+2}, a\}$, where

$$\begin{aligned} v_i &= e_i \text{ for } 1 \leqslant i \leqslant 2d-1\,, \\ v_{2d} &= -\sum_{i=1}^{2d-1} e_i + de_{2d}\,, \\ v_{2d+1} &= e_{2d-3} + e_{2d-2} + e_{2d-1} - e_{2d}\,, \\ v_{2d+2} &= -e_{2d-3} - e_{2d-2} - e_{2d-1} + 2e_{2d}\,, \\ a &= e_{2d}\,. \end{aligned} \tag{6.14}$$

The cone $\sigma := \operatorname{Cone}(\nu)$ is almost Gorenstein and $\mathfrak{m} = (1, \ldots, 1, 2)$. Note that the elements in the set $\nu_{=1} := \{v_i\}$ all pair to 1 with $\mathfrak{m}$, and $\langle \mathfrak{m}, a\rangle = 2$. Let $R_1 = \{v_{2d+1}, v_{2d+2}\}$ and $R_2 = \{a\}$.

Let $X := \mathbb{A}^{2d+3}$, and compute that $S_\nu = \mathbb{G}_m^3$. We denote the coordinates of $\mathbb{A}^{2d+3}$ by $x_1, \ldots, x_{2d}$ for the lattice points $v_1, \ldots, v_{2d}$ and u_1, u_2, u_3 for the points v_{2d+1}, v_{2d+2}, a. The weights for the action of S_ν are in the following table:

Coordinates	Weight of $\mathbb{G}_m^3$
$x_1, \ldots, x_{2d-4}, x_{2d}$	$(1, 0, 0)$
$x_{2d-3}, x_{2d-2}, x_{2d-1}$	$(1, 0, 1)$
u_1	$(0, 1, 0)$
u_2	$(0, 1, 1)$
u_3	$(-d, -1, -2)$

The GIT fan has eight chambers. We describe them explicitly. Let

$$\begin{aligned} &p_0 := (1/2, 1/2, 1/2)\,, && p_1 := (1, 0, 0)\,, && p_2 := (1, 0, 1)\,, \\ &p_3 := (0, 1, 0)\,, && p_4 := (0, 1, 1)\,, && p_5 := (-d, -1, -2) \end{aligned}$$

be points in $\mathbb{R}^3$ and

$$\mathcal{K}_1 := \langle x_1, \ldots, x_{2d-4}, x_{2d}\rangle\,, \quad \mathcal{K}_2 := \langle x_{2d-3}, x_{2d-2}, x_{2d-1}\rangle$$

ideals in $\kappa[x_1, \ldots, x_{2d}, u_1, u_2, u_3]$.

The following table describes the eight chambers of the GIT fan and the irrelevant ideals corresponding to the unstable locus for each chamber:

Chamber Γ_i	Cone in $\hat{G}_{\mathbb{R}}$	Irrelevant ideal $\mathcal{I}_i$
Γ_1	$\operatorname{Cone}(p_5, p_1, p_2)$	$\langle u_3\rangle\mathcal{K}_1\mathcal{K}_2$
Γ_2	$\operatorname{Cone}(p_0, p_3, p_4)$	$\langle u_1u_2\rangle\mathcal{K}_1 + \langle u_1u_2\rangle\mathcal{K}_2 + \langle u_2u_3\rangle\mathcal{K}_1 + \langle u_1u_3\rangle\mathcal{K}_2$
Γ_3	$\operatorname{Cone}(p_0, p_1, p_3)$	$\langle u_1u_2\rangle\mathcal{K}_1 + \langle u_1\rangle\mathcal{K}_1\mathcal{K}_2 + \langle u_2u_3\rangle\mathcal{K}_1$
Γ_4	$\operatorname{Cone}(p_0, p_2, p_4)$	$\langle u_1u_3\rangle\mathcal{K}_2 + \langle u_1u_2\rangle\mathcal{K}_2 + \langle u_2\rangle\mathcal{K}_1\mathcal{K}_2$
Γ_5	$\operatorname{Cone}(p_0, p_1, p_2)$	$\langle u_1, u_2\rangle\mathcal{K}_1\mathcal{K}_2$
Γ_6	$\operatorname{Cone}(p_5, p_1, p_3)$	$\langle u_1u_3, u_2u_3\rangle\mathcal{K}_2$
Γ_7	$\operatorname{Cone}(p_5, p_2, p_4)$	$\langle u_1u_3, u_2u_3\rangle\mathcal{K}_1$
Γ_8	$\operatorname{Cone}(p_5, p_3, p_4)$	$\langle u_1u_3\rangle\mathcal{K}_2 + \langle u_2u_3\rangle\mathcal{K}_1 + \langle u_1u_2u_3\rangle$

For $1 \leqslant i \leqslant 8$, let $U_i := \mathbb{A}^{2d+3} \backslash Z(\mathcal{I}_i)$ be the semistable locus corresponding to each chamber. Finally, consider a function $w = \sum_{m\in\Xi} c_m x^m$ for generic choices of constants $c_m \in \kappa$. We can rewrite w in the form

$$w = u_1u_3f_1(x_1,\dots,x_{2d}) + u_2u_3f_2(x_1,\dots,x_{2d}) \tag{6.15}$$

for some polynomials f_1, f_2 which are smooth on all of the U_i. Let $\mathcal{D}_i := \mathrm{D}^{\mathrm{abs}}(U_i, \mathbb{G}_m^3 \times \mathbb{G}_m, w)$ be the factorization category associated to the GIT chamber Γ_i.

Proof of Proposition 6.8. Using the fact that χ_{-K} corresponds to the point $(d,1,2)$ in $\hat{G}_{\mathbb{R}}$, we apply Theorem 3.12 to obtain a poset structure for the categories $\mathcal{D}_i$ given by the following diagram of fully faithful functors:

$$\begin{tikzcd}
 & & \mathcal{D}_3 \arrow[dr] & \\
\mathcal{D}_1 \cong \mathcal{D}_6 \cong \mathcal{D}_7 \cong \mathcal{D}_8 \arrow[r] & \mathcal{D}_2 \arrow[ur] \arrow[dr] & & \mathcal{D}_5\,. \\
 & & \mathcal{D}_4 \arrow[ur] &
\end{tikzcd} \tag{6.16}$$

The claim is now proven by giving geometric interpretations for the five distinct categories. This is done in Propositions 6.12, 6.10, and 6.11 below. □

The categories $\mathcal{D}_1$ and $\mathcal{D}_5$ are derived categories of algebraic varieties, while $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ are crepant categorical resolutions of derived categories of singular varieties. We will describe the categories in order.

PROPOSITION 6.10. *Let $\mathcal{Z}_{\mathrm{CY}}$ be the zero locus $Z(f_1,f_2) \subseteq \mathbf{P}^{2d-4} \times \mathbf{P}^2$, which is a $(2d-4)$-dimensional Calabi–Yau complete intersection. Then, there is an equivalence*

$$\mathcal{D}_1 := \mathrm{D}^{\mathrm{abs}}(U_1, \mathbb{G}_m^3 \times (\mathbb{G}_m)_{R_1}, w) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_{\mathrm{CY}})\,.$$

Proof. Consider the fan Σ_1 associated to the GIT chamber Γ_1. It is constructed by taking the cone generated by $v_1,\dots,v_{2d}$ and then star-subdividing along v_{2d+1} and v_{2d+2}. Consider the product of projective spaces $\mathbf{P}^{2d-4} \times \mathbf{P}^2$. Let H_1 and H_2 be the hyperplane divisors associated to $\mathbf{P}^{2d-4}$ and $\mathbf{P}^2$, respectively. One can compute that

$$X_{\Sigma_1} \cong \mathrm{tot}(\mathcal{O}_{\mathbf{P}^{2d-4}\times\mathbf{P}^2}(-(d-2)H_1 - 2H_2) \oplus \mathcal{O}_{\mathbf{P}^{2d-4}\times\mathbf{P}^2}(-(d-1)H_1 - H_2)\,.$$

Let $\mathcal{Z}_{\mathrm{CY}}$ be the zero locus $Z(f_1,f_2) \subseteq \mathbf{P}^{2d-4} \times \mathbf{P}^2$, which is a $(2d-4)$-dimensional Calabi–Yau complete intersection. By Theorem 3.5, we have the equivalence

$$\mathcal{D}_1 \cong \mathrm{D}^{\mathrm{abs}}\left(U_1, \mathbb{G}_m^3 \times (\mathbb{G}_m)_{R_1}, w\right) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_{\mathrm{CY}})\,.$$ □

We now move to the crepant categorical resolutions.

PROPOSITION 6.11. (i) *The absolute derived category $\mathrm{D}^{\mathrm{abs}}(U_1, S_\nu \times (\mathbb{G}_m), w)$ is a crepant categorical resolution of $\mathcal{A}$ given in equation (6.13).*

(ii) *The absolute derived category $\mathrm{D}^{\mathrm{abs}}(U_2, S_\nu \times (\mathbb{G}_m), w)$ is a crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_{\mathrm{sing}})$.*

(iii) *The absolute derived category $\mathrm{D}^{\mathrm{abs}}(U_3, S_\nu \times (\mathbb{G}_m), w)$ is a crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_2)$, where $\mathcal{Z}_2$ is the strict transform of $\mathcal{Z}_{\mathrm{sing}}$ in $\mathrm{Bl}_{P_2}(\mathbf{P}^{2d-1})$.*

(iv) *The absolute derived category $\mathrm{D}^{\mathrm{abs}}(U_4, S_\nu \times (\mathbb{G}_m), w)$ is a crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_1)$, where $\mathcal{Z}_1$ is the strict transform of $\mathcal{Z}_{\mathrm{sing}}$ in $\mathrm{Bl}_{P_1}(\mathbf{P}^{2d-1})$.*

Proof. Consider the open immersion $V \hookrightarrow U$, where

$$V = X \setminus Z(u_1u_2)\,, \quad U = X \setminus Z(u_1u_2, u_2u_3\mathcal{K}_1, u_1u_3\mathcal{K}_2)\,. \tag{6.17}$$

A direct computation shows that the ideal $\langle u_1u_2, u_2u_3\mathcal{K}_1, u_1u_3\mathcal{K}_2\rangle$ is the irrelevant ideal associated to the cone in the GIT fan that is the common face between the chambers Γ_2 and Γ_8. The path between these two chambers yields the following stratifications associated to its elementary wall crossing:

$$\begin{aligned} U &= U_8 \sqcup S_-\,, \quad V = V_8 \sqcup S_-\,, \quad S_- := Z(u_3) \cap U\,,\\ U &= U_2 \sqcup S_+\,, \quad V = V_2 \sqcup S_+\,, \quad S_+ := Z(x_1, \ldots, x_{2d}) \cap U\,. \end{aligned} \tag{6.18}$$

Note that

$$S_- \cap (U_8 \setminus V_8) = \varnothing \quad \text{and} \quad S_+ \cap (U_2 \setminus V_4) \ = \varnothing\,; \tag{6.19}$$

hence, the immersions are compatible with the elementary wall crossing. Consider the gauged Landau–Ginzburg model $(V_2, \mathbb{G}_m^3 \times (\mathbb{G}_m)_{R_2}, w)$. Consider the affine space $X_{u_1,u_2} = \mathbb{A}^{2d+1}$ found by taking $\operatorname{Spec}(\kappa[x_1, \ldots, x_{2n}, u_3])$. There is a stack isomorphism

$$\left[V_2 \middle/ \mathbb{G}_m^3 \times (\mathbb{G}_m)_{R_1}\right] = \left[(X_{u_1,u_2} \setminus Z(x_1, \ldots, x_{2d})) \middle/ \mathbb{G}_m \times (\mathbb{G}_m)_{R_1}\right] = \operatorname{tot}(\mathcal{O}_{\mathbf{P}^{2d-1}}(-dH))\,;$$

thus, U_2 is a partial compactification of $\operatorname{tot}(\mathcal{O}_{\mathbf{P}^{2d-1}}(-dH))$. The superpotential w is an extension of the section $f_1 + f_2$ on $\operatorname{tot}(\mathcal{O}_{\mathbf{P}^{2d-1}}(-dH))$. Note that by Hirano's theorem,

$$\mathrm{D}^{\mathrm{abs}}(X_{u_1,u_2} \setminus Z(x_1, \ldots, x_{2d}), \mathbb{G}_m \times (\mathbb{G}_m), u_3(f_1 + f_2)) = \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}})\,,$$

where $\mathcal{Z}_{\mathrm{sing}}$ is the zero locus of $Z(f_1 + f_2) \subset \mathbf{P}^{2d-1}$. Note that since the section $u_1f_1 + u_2f_2$ also defines a section of $-dH + E_1 + E_2$ in $\widetilde{\mathbf{P}}$, we know that $\mathcal{Z}_{\mathrm{sing}}$ contains the two planes $x_{2d-3} = x_{2d-2} = x_{2d-1} = 0$ and $x_1 = \cdots = x_{2d-4} = x_{2d} = 0$.

By Theorem 3.7, we have that $\mathcal{D}_2$ is a crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}})$. We have two semi-orthogonal decompositions:

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_{\mathrm{sing}}) = \langle \mathcal{A}, \mathcal{O}, \ldots, \mathcal{O}(d-1)\rangle\,,$$

where $\mathcal{A} = \mathrm{D}^{\mathrm{abs}}(V_8, \mathbb{G}_m^3 \times (\mathbb{G}_m)_{R_2}, w)$, and

$$\mathcal{D}_2 = \langle \mathcal{D}_8, \mathcal{O}, \ldots, \mathcal{O}(d-1)\rangle\,.$$

By Theorem 3.14,

$$\mathcal{D}_1 \cong \mathcal{D}_8 := \mathrm{D}^{\mathrm{abs}}(U_8, S_\nu \times (\mathbb{G}_m), w) \tag{6.20}$$

is a crepant categorical resolution of $\mathcal{A}$.

Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the resultant varieties from taking $\mathcal{Z}_{\mathrm{sing}}$ and blowing up the planes $x_{2d-3} = x_{2d-2} = x_{2d-1} = 0$ and $x_1 = \cdots = x_{2d-4} = x_{2d} = 0$, respectively. By doing the analogous comparisons between Γ_3 and Γ_7 and between Γ_4 and Γ_6, one can see that $\mathcal{D}_3$ is a crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_1)$ and $\mathcal{D}_4$ is a crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_2)$. ☐

We finish with the geometric interpretation of the category $\mathcal{D}_5$.

PROPOSITION 6.12. *Consider the two planes* $P_1 = \{x_{2d-3} = x_{2d-2} = x_{2d-1} = 0\}$ *and* $P_2 = \{x_1 = \cdots = x_{2d-4} = x_{2d} = 0\}$ *in* $\mathbf{P}^{2d-1}$. *Let* $\mathcal{Z}_{\mathrm{sing}}$ *be the cubic* $Z(f_1 + f_2)$ *in* $\mathbf{P}^{2d-1}$, *where* f_1 *and* f_2 *are the cubics defined in equation* (6.15). *Consider the blow-up* $\widetilde{\mathcal{Z}_{\mathrm{sing}}}$ *of* $\mathcal{Z}_{\mathrm{sing}}$ *along* P_1 *and* P_2. *Then,* $\mathcal{Z}_{\mathrm{sing}}$ *contains both* P_1 *and* P_2, *and we have the equivalence*

$$\mathcal{D}_5 := \mathrm{D}^{\mathrm{abs}}(U_5, \mathbb{G}_m^3 \times \mathbb{G}_m, w) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \widetilde{\mathcal{Z}_{\mathrm{sing}}})\,.$$

Proof. Start with the standard fan for $\mathbf{P}^{2d-1}$, then blow up the hyperplanes $x_{2d-3} = x_{2d-2} = x_{2d-1} = 0$ and $x_1 = \cdots = x_{2d-4} = x_{2d} = 0$ to obtain the variety $\widetilde{\mathbf{P}}$. Note that $\mathrm{Cl}(\widetilde{\mathbf{P}})$ equals $\mathbb{Z}^3$, and it is generated by the hyperplane section H and the exceptional divisors E_1 and E_2 given by the respective blow-ups described above. Consider the divisor $D = -dH + E_1 + E_2$. One can check that the fan Σ_5 is the total space of the line bundle $\mathcal{O}_{\widetilde{\mathbf{P}}}(-D)$. A generic global section of $\mathcal{O}_{\widetilde{\mathbf{P}}}(-D)$ is given by taking w and setting u_1 and u_2 to 1. Let $\widetilde{\mathcal{Z}_{\mathrm{sing}}}$ be the zero locus $Z(f_1 + f_2) \subseteq \widetilde{\mathbf{P}}$. By Theorem 3.5, there is an equivalence

$$\mathrm{D}^{\mathrm{abs}}(U_5, \mathbb{G}_m^3 \times (\mathbb{G}_m)_{R_2}, w) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \widetilde{\mathcal{Z}_{\mathrm{sing}}})\,.$$

The fact that $\mathcal{Z}_{\mathrm{sing}}$ contains P_1 and P_2 is clear from the definition of the divisor D. $\square$

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