

**Additivity, subadditivity and linearity:
automatic continuity and quantifier weakening
by
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To Charles Goldie.

Abstract. We study the interplay between additivity (as in the Cauchy functional equation), subadditivity and linearity. We obtain automatic continuity results in which additive or subadditive functions, under minimal regularity conditions, are continuous and so linear. We apply our results in the context of quantifier weakening in the theory of regular variation, completing our programme of reducing the number of hard proofs there to zero.

Key words. Subadditive, sublinear, shift-compact, analytic spanning set, additive subgroup, Hamel basis, Steinhaus Sum Theorem, Heiberg-Seneta conditions, thinning, regular variation.

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1 Introduction

Our main theme here is the interplay between additivity (as in the Cauchy functional equation), subadditivity and linearity. As is well known, in the presence of smoothness conditions (such as continuity), additive functions $A : \mathbb{R} \rightarrow \mathbb{R}$ are linear, so of the form $A(x) = cx$. There is much scope for *weakening* the smoothness requirement and also much scope for weakening the universal quantifier by *thinning* its range \mathbb{A} below from the classical context $\mathbb{A} = \mathbb{R}$:

$$A(u + v) = A(u) + A(v) \quad (\forall u, v \in \mathbb{A}). \quad (\text{Add}_{\mathbb{A}}(A))$$

We address the Cauchy functional equation in §2 below. The philosophy behind our quantifier weakening¹ theorems is to establish *linearity* of a function F on \mathbb{R} from its *additivity* on a thinner set \mathbb{A} and from additional (‘side’)

¹or ‘quantifier easing’

conditions, which include its *extendability* to a subadditive function; recall that $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is *subadditive* [HilP, Ch. 3] if for all $u, v \in \mathbb{R}$

$$S(u + v) \leq S(u) + S(v), \tag{Sub}$$

whenever meaningful on the right-hand side (cf. [Kuc, 16.1] for subbadditive and [Roc, p. 23 ffol] for convex functions); see also [MatS], [BinO4]. (This choice for the setting is more convenient than alternatively working on \mathbb{R}_+ , even though one-sided side-conditions are important here.) To motivate our main result we begin with an automatic continuity theorem, devoted entirely to subadditive functions; it implies a result about those linear functions that have subadditive extensions – see Proposition 7 below (on uniqueness of extension). This (Theorem 0 below) makes explicit an argument springing from a step in a proof by Goldie, of Th. 3.2.5 in [BinGT] (BGT below, for brevity), recently improved and generalized in [BinO14] (though still implicit even there).

We recall that for S subadditive and finite-valued, $S(0) \geq 0$, as $S(0) \leq S(0) + S(0)$, so that $S(0) = 0$ iff $S(-z) = -S(z)$ for some z , as will be the case below when S extends an additive function; cf. [Kuc, p. 457].

Theorem 0. *For subadditive $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ with $S(0+) = S(0) = 0$: S is continuous at 0 iff $S(z_n) \rightarrow 0$, for some sequence $z_n \uparrow 0$, and then S is continuous everywhere, if finite-valued.*

The last part above draws on [HilP, Th. 2.5.2] that, for a subadditive function, continuity at the origin implies continuity everywhere. Theorem 0 above, in the presence of right-sided continuity, asserts that the merest hint of left-sided continuity gives full continuity; contrast this with the behaviour of the subadditive function $\mathbf{1}_{[0, \infty)}$, which is continuous on the right but not on the left. This leads to the question of whether right-sided continuity can be thinned out. We are able to do so in the next two results below, but at the cost of imposing more structure, either on the left, or on the right. We need the following two definitions.

Definitions. 1. Say that Σ is *locally Steinhaus-Weil (SW)*, or has the *SW property locally*, if for $x, y \in \Sigma$ and, for all $\delta > 0$ sufficiently small, the sets

$$\Sigma_z^\delta := \Sigma \cap B_\delta(z),$$

for $z = x, y$, have the *interior-point property*, that $\Sigma_x^\delta \pm \Sigma_y^\delta$ has $x \pm y$ in its interior. (Here $B_\delta(x)$ is the open ball about x of radius δ .) See [BinO9] for conditions under which this property is implied by the interior-point property of the sets $\Sigma_x^\delta - \Sigma_x^\delta$ (cf. [BarFN]); for a rich list of examples, see §4.

2. Say that $\Sigma \subseteq \mathbb{R}$ is *shift-compact* if for each *null sequence* $\{z_n\}$ (i.e. with $z_n \rightarrow 0$) there are $t \in \Sigma$ and an infinite $\mathbb{M} \subseteq \mathbb{N}$ such that

$$\{t + z_m : m \in \mathbb{M}\} \subseteq \Sigma.$$

See [BinO11], and for the group-action aspects, [MilO].

For connections between these two, and related result, see §8.3. Thus armed, we begin with symmetric (two-sided) thinning. The second part of the result below is a variant of the Berz Theorem for *sublinear* functions ([Berz], §5, [BinO16]).

Theorem 0' [BinO14, Th. 3]. *If $S : \mathbb{R} \rightarrow \mathbb{R}$ is subadditive with $S(0) = 0$ and there is a symmetric set Σ containing 0 with:*

- (i) *S is continuous at 0 on Σ ;*
- (ii) *for all small enough $\delta > 0$, Σ_0^δ is locally Steinhaus-Weil*

– then S is continuous at 0 and so everywhere.

In particular, this conclusion holds if there is a symmetric set Σ , Baire/measurable and non-negligible in each $(0, \delta)$ for $\delta > 0$, on which

$$S(u) = c_\pm u \text{ for some } c_\pm \in \mathbb{R} \text{ and all } u \in \mathbb{R}_+ \cap \Sigma, \text{ or all } u \in \mathbb{R}_- \cap \Sigma \text{ resp.}$$

The alternative case to two-sided thinning is one-sided thinning accompanied by linear bounding. Although one tries to impose continuity conditions on a thin set below, these cannot be ‘too thin’ as the example of the indicator function of the irrationals shows: $\mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ is subadditive and additive on \mathbb{Q} (indeed \mathbb{Q} -homogeneous), but not continuous. More thinning is possible by involving more structure: the ability to ‘span’ (see below).

To motivate the accompanying side-condition, consider a subadditive S that is locally bounded, say by some $\varepsilon > 0$ on $(0, a]$. For any $x > 0$, choose $n \in \mathbb{N}$ with $(n - 1) < x/a \leq n$; then

$$S(x) \leq (n - 1)S(a) + S(x - (n - 1)a) \leq xS(a)/a + \varepsilon.$$

In particular, for $x \geq a$, $S(x) \leq c_a x$, for $c_a := [\varepsilon a + S(a)]/a$, i.e. S is linearly bounded away and to the right of the origin. Theorem 0⁺ below derives

global linear boundedness from similar behaviour on a thin set near 0 when c_a itself is bounded above on the thin set.

Theorem 0⁺. *Let $\Sigma \subseteq [0, \infty)$ be locally SW accumulating at 0. Suppose $S : \mathbb{R} \rightarrow \mathbb{R}$ is subadditive with $S(0) = 0$ and:
 $S|_{\Sigma}$ is linearly bounded above by $G(x) := cx$, i.e. $S(\sigma) \leq c\sigma$ for some c and all $\sigma \in \Sigma$, so that in particular,*

$$\limsup_{\sigma \downarrow 0, \sigma \in \Sigma} S(\sigma) \leq 0.$$

Then $S(x) \leq cx$ for all $x > 0$, so

$$\limsup_{x \downarrow 0} S(x) \leq 0,$$

and so $S(0+) = 0$.

In particular, if furthermore there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ with $z_n \uparrow 0$ and $S(z_n) \rightarrow 0$, then S is continuous at 0 and so everywhere.

Boundedness in the latter case could be provided on an open set $U \subseteq (0, 1)$ accumulating at 0; continuity in the former case on a set $\Sigma = (-U) \cup \{0\} \cup U$ with U as before. Indeed, in this context one may equivalently assume that the set Σ has precisely this form: see §4.

Theorem 1, our quantifier weakening theorem, was our original motivation, for reasons explained later in the paper. The Heiberg-Seneta side-condition of Theorem 1 is due to Heiberg ([Hei] in 1971) and Seneta ([Sen] in 1973) – see BGT Th. 1.4.3.

Theorem 1. *For $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ subadditive, and $\mathbb{A} \subseteq \mathbb{R}$ an additive subgroup, suppose that*

- (i) \mathbb{A} is dense;
- (ii) $A := S|_{\mathbb{A}}$ is finite and additive, i.e. $\text{Add}_{\mathbb{A}}(S)$ holds;
- (iii) S satisfies the one-sided (Heiberg-Seneta) boundedness condition

$$\limsup_{u \downarrow 0} S(u) \leq 0. \quad (HS(S))$$

Then S is linear: $S(u) = cu$ for some $c \in \mathbb{R}$, and all $u \in \mathbb{R}$.

Its proof relies on Theorem 0. Our next result, Theorem 1' based (in part) on Theorem 0', is formulated in the spirit of Theorem 1 so far as the

Heiberg-Seneta-style condition is concerned. However, the passage to the limit below is through a set Σ which may be (very) *thin* ('ghost-like'); the limit may be one-sided or two-sided, depending both on Σ and the ambient context.

Theorem 1' (cf. [BinO15, §6 Th. 5]). *Theorem 1 above holds with condition (iii) replaced by any one of the following:*

(iii-a) *S satisfies the Heiberg-Seneta boundedness condition thinned out to a symmetric set Σ that is locally SW, i.e.*

$$\limsup_{u \rightarrow 0, u \in \Sigma} S(u) \leq 0;$$

(iii-b) *S is linearly bounded above on a locally SW subset $\Sigma \subseteq \mathbb{R}_+ = (0, \infty)$ accumulating at 0, so that in particular*

$$\limsup_{u \downarrow 0, u \in \Sigma} S(u) \leq 0;$$

(iii-c) *S is bounded above on a locally SW subset $\Sigma \subseteq \mathbb{A}_+$ accumulating at 0, that is, the following \limsup is finite:*

$$\limsup_{u \downarrow 0, u \in \Sigma} S(u) < \infty; \quad (SW-HS(S))$$

(iii-d) *S is bounded on a subset $\Sigma \subseteq \mathbb{A}$ that is shift-compact (e.g. on a set that is locally SW, and so on an open set) and*

$$\mathbb{A} = \mathbb{A}_S := \{u : G(u) := \lim_{x \rightarrow \infty} [S(u+x) - S(x)] \text{ exists and is finite}\}.$$

Theorem 1'(c) above encompasses as an immediate corollary Ostrowski's Theorem [BinO11] and its classical generalizations. Below we use *negligible* to mean meagre or null, according as (Baire) category or (Lebesgue) measure is considered; we use *non-negligible* to mean a Baire/Lebesgue set that is not correspondingly negligible. Further discussion here involves set-theoretic assumptions (typically, alternatives to the alternatives to the Axiom of Choice, AC, see §8.5 for references).

Theorem O ([Darb], [Ostr], [Meh]; cf. [BinO11]). *If $A : \mathbb{R} \rightarrow \mathbb{R}$ is additive and bounded above on a non-negligible set, then A is linear.*

The measure-theoretic development here came earlier chronologically; it was then noticed that the Baire (or category) case was closely analogous.

The two cases are developed in parallel in BGT. It has emerged recently that the primary case is in fact the category case; see e.g. [BinO2,10], [Ost2].

The point of introducing side-conditions of Heiberg-Seneta type is that they enable one to avoid assuming that our functions are *either Baire or measurable*. This is in stark contrast to all the other major results in the theory of regular variation (for which see below), when such an assumption is necessary – e.g., to avoid the *Hamel pathology* (see e.g. [HilP, p. 238], BGT p. 5) of discontinuous additive functions (necessarily wildly non-measurable/non-Baire). See e.g. the property of local boundedness in Prop. 9 below (§4), and foundational results such as the Uniform Convergence Theorem [BinO3]. We strengthen results of this type by imposing the side-condition on *as thin a set as possible* (‘ghost-like Heiberg-Seneta conditions’), albeit with some regularity of structure.

Our results here concern, as well as quantifier weakening and automatic continuity, various results on additive, subadditive and sublinear functions. Our original motivation was (quantifier weakening in the context of) regular variation (§7: Karamata theory and its extensions; see e.g. BGT). This specific motivation turns out to be valuable here: our viewpoint on the area, informed by this, is complementary to (indeed, contrasts with) that of the standard work in the area, Kuczma ([Kuc, Ch. 16]).

The rest of the paper is structured as follows. The backdrop of the Cauchy equation is considered in §2; then we pass in §3 to developing the theme of linearity first from additivity then from subadditivity, proving Theorems 0 and 1 and their variants, and also Theorem 2. We stop to clarify the thinning aspect of the classical *Heiberg-Seneta condition* in §4. Using the theme of linearity from subadditivity we extend in §5 Berz’s Theorem on sublinearity [Berz] from the classical measurable case to the Baire category case, more natural for us here, since refining the Euclidean topology to the density topology (which converts measurable sets into *Baire* sets, i.e. sets with the Baire property – see e.g. [BinO10, 17]) yields a parallel new proof of Berz’s Theorem. In §6 we discuss thinning by *spanning*: we want to *weaken quantifiers* by *thinning their range* as much as possible. But limits are imposed on this: a set which is too thin will not be able to span. We are thinking here of the reals \mathbb{R} as a vector space over the rationals \mathbb{Q} , Hamel bases etc. All this is motivated by regular variation, for which see §7. This makes good on the claim we have already made elsewhere (see [BinO7,8,11,12]): this reduces the number of hard proofs in the theory of regular variation to zero.

2 Cauchy theory

Theorem 1 (to be proved in §3, where further results of this kind are established) is concerned with *weakening* the quantifier in the classical Cauchy functional equation, by thinning the range \mathbb{A} of the universal quantifier in $(\text{Add}_{\mathbb{A}})$ above from the classical context $\mathbb{A} = \mathbb{R}$. The philosophy behind the theorems is to establish *linearity* of a function F on \mathbb{R} from its *additivity* on a thinner set \mathbb{A} and from additional (‘side’) conditions, which include its extendability to a subadditive function. The principal motivation for such an approach arises in regular variation (§7) and rests on the following result, which identifies the *additive kernel* G of the function F^* below. The paper studies conditions under which F^* coincides with this kernel. The condition (i) below motivated further study in [BinO15, §5 especially Prop. 6] (within a more subtle group structure, and under the more demanding requirements of uniform convergence).

Proposition 1 (Additive Kernel). *For $F : \mathbb{R} \rightarrow \mathbb{R}$ put*

$$\mathbb{A}_F := \{u : G(u) := \lim_{x \rightarrow \infty} [F(u+x) - F(x)] \text{ exists and is finite}\},$$

and, for $u \in \mathbb{R}$ define

$$F^*(u) := \limsup_{x \rightarrow \infty} [F(u+x) - F(x)].$$

Then:

- (i) \mathbb{A}_F is an additive subgroup;
- (ii) G is an additive function on \mathbb{A}_F ;
- (iii) $F^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ is a subadditive extension of G ;
- (iv) F^* is finite-valued and additive iff $\mathbb{A}_F = \mathbb{R}$ and $F^*(u) = G(u)$ for all u .

Proof. (i) $0 \in \mathbb{A}_F$. Next, from

$$F(u+v+x) - F(x) = [F(u+v+x) - F(v+x)] + [F(v+x) - F(x)], \quad (*)$$

we see that \mathbb{A}_F is a subsemigroup of \mathbb{R} . In fact it is a subgroup: for $u \in \mathbb{A}_F$ one has $-u \in \mathbb{A}_F$, because on writing $y = u+x$ one has

$$F(-u+y) - F(y) = -[F(u+x) - F(x)]. \quad \square \text{ (i)}$$

(iii) The identity (*) also implies subadditivity of F^* and the partial additivity result that

$$F^*(u+v) = F^*(u) + F^*(v) \quad \forall u \in \mathbb{A}_F \quad \forall v \in \mathbb{R}. \quad (**)$$

We note that for $u \in \mathbb{A}_F$, i.e. when $G(u)$ exists, then $F^*(u) = G(u)$, so proving part (iii) for $u \in \mathbb{A}_F$. \square (iii)

(ii) With $v \in \mathbb{A}_F$, (**) above yields additivity of G . \square (ii)

(iv) If F^* is finite-valued and additive, then $F^*(-u) = -F^*(u)$ for all u . The substitution $y = u + x$ yields

$$\begin{aligned} \liminf_{x \rightarrow \infty} [F(u+x) - F(x)] &= -\limsup_{x \rightarrow \infty} [F(x) - F(u+x)] \\ &= -\limsup_{y \rightarrow \infty} [F(-u+y) - F(y)] \\ &= -F^*(-u) = F^*(u). \end{aligned}$$

That is, $\liminf_{x \rightarrow \infty} [F(u+x) - F(x)] = \limsup_{x \rightarrow \infty} [F(u+x) - F(x)]$, i.e. $F^*(u) = \lim_{x \rightarrow \infty} [F(u+x) - F(x)] = G(u)$ for all u ; so $\mathbb{A}_F = \mathbb{R}$. The converse follows from (i) and (ii). \square (iv). \square

Remark. With the Axiom of Choice AC replaced by an axiom (such as the Axiom of Determinacy, AD) under which all sets are Baire/measurable (a price that most mathematicians most of the time will not be willing to pay!), Prop. 1 (helped by Prop. 6 below) is all we need, and the remaining results below become unnecessary. See [BinO16, Appendix 1] and §8.5; cf. [BinO15, §7 Th. 6].

Proposition 2 below leads from Theorem 1 to Theorem 1' in §3 below.

Proposition 2. *For $A : \mathbb{R} \rightarrow \mathbb{R}$ additive, the following are equivalent:*

- (i) A is bounded above on a non-negligible Baire/measurable set;
- (ii) A is bounded above on an interval;
- (iii) for some $M \in \mathbb{R}$

$$\limsup_{u \downarrow 0} A(u) \leq M; \quad (\limsup_M)$$

- (iv) $(\limsup)_0$ holds, i.e. $(HS(A))$ holds;

Proof. (i)→(ii): If A is bounded above on a non-negligible Baire/measurable set L , then it is bounded above on $L + L$, which contains an interval by the *Steinhaus-Piccard-Pettis Sum-Theorem* ([BinO9, Th. E]; cf. [GroE], [BinO11] and the recent [BinO18]).

(ii)→(iii): By additivity, $(\limsup)_M$ holds for some $M \in \mathbb{R}$.

(iii)→(iv): Without loss of generality (w.l.o.g.) $M \geq 0$. For any $K > M$, if $\sup\{A(u) : 0 < u < \delta\} < K$, then by additivity $\sup\{A(u) : 0 < u < \delta/2\} <$

$K/2$, as $2A(u) = A(2u) < K$, so $(\limsup)_{M/2}$ holds, and so the least $M \geq 0$ for which the condition holds is $M = 0$.

(iv)→(i): Clear. \square

Remark. Once Theorem 1 is established, we may apply Theorem 1 to $S = A$ with $\mathbb{A} = \mathbb{R}$ to deduce a further equivalent condition:

(v) A is linear on \mathbb{R} : $A(u) = cu$ for some $c \in \mathbb{R}$, and all $u \in \mathbb{R}$.

The analogue of Prop. 2 for an additive subgroup is also relevant below.

Proposition 2' (Automatic continuity, after Darboux, [Darb]). For \mathbb{A} a dense additive subgroup and $A : \mathbb{A} \rightarrow \mathbb{R}$ additive, the following are equivalent:

- (i) A is continuous;
- (ii) A is right-continuous at 0;
- (iii) A is continuous at 0;
- (iv) A is locally bounded;
- (v) A is locally bounded above on some interval.

Proof. As this is routine, we refer to [BinO11] for details, save to say that (v)→(iv) is as in Prop. 5 below, and to show that (iv)→(i). If A is locally bounded at 0, there is $\delta > 0$ and M such that $|A(a)| \leq M$ for $a \in \mathbb{A}$ with $|a| < \delta$. Given $\varepsilon > 0$, choose an integer N with $N > M/\varepsilon$. For $a \in \mathbb{A}$ with $|a| < \delta/N$, $|NA(a)| = |A(Na)| \leq M$, so

$$|A(a)| \leq \varepsilon,$$

as $M/N < \varepsilon$. This gives continuity. \square

Remark. In the next section Prop. 6 establishes the further equivalent condition:

(vi) A is linear: $A(u) = cu$ for some $c \in \mathbb{R}$, and all $u \in \mathbb{A}$.

3 Linearity from subadditivity

Here we establish a paradigm for identifying circumstances *when linearity may be deduced from subadditivity* – encapsulated in Theorem 2 below – by showing that $S(y)/S(x) = y/x$ on a dense subspace and appealing to right-continuity.

We begin with a result linking the Heiberg-Seneta condition (HS) of Theorem 1 with automatic one-sided continuity. We need some preliminaries: we turn first to conditions implying finite-valued subadditivity. The first result requires a mixture of one-sided and two-sided information.

Proposition 3. *For $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ subadditive, write $\Sigma_+ := \{u \in \mathbb{R} : S(u) < \infty\}$. If $\Sigma_+ \cap [0, \infty)$ contains an interval and $\Sigma_+ \not\subseteq [0, \infty)$, then either S is finite everywhere or is identically $-\infty$. In particular, if*

- (i) *S is finite on a subset Σ unbounded below (e.g. a dense subset of \mathbb{R});*
- (ii) *S is bounded above on $(0, \delta)$ for some $\delta > 0$, e.g. S satisfies the condition $(HS(S))$,*

– then S is finite everywhere.

Proof. As Σ_+ is a subsemigroup of the additive group \mathbb{R} , and contains an interval, it contains a ray $[A, \infty)$ (see e.g. BGT Cor. 1.1.5). Choose $c \in (-\infty, 0) \cap \Sigma_+$. Then $nc + (A, \infty) \subseteq \Sigma_+$ for all $n \in \mathbb{N}$, so $\Sigma_+ = \mathbb{R}$, i.e. $S(u) < \infty$ for all $u \in \mathbb{R}$. If $S \not\equiv -\infty$, say $S(u_0) > -\infty$, then for any $u \in \mathbb{R}$

$$S(u) \geq S(u_0) - S(u_0 - u) > -\infty.$$

The particular case now follows, since by (i) $\Sigma \not\subseteq [0, \infty)$, and by (ii) S is bounded on an interval: there is $\delta > 0$ such that $S(x) < 1$ for all $x \in (0, \delta)$, so $\Sigma \cap [0, \infty)$ contains $(0, \delta)$. \square

We will soon prove in Proposition 3' a one-sided variant. The argument used can yield more; so, it is more convenient to prove first

Proposition 4. *If $S : \mathbb{R} \rightarrow \mathbb{R}$ with $S(0) = 0$ is subadditive and linearly bounded above by $G(x) = cx$ on an open set U accumulating to the right at 0, then S is linearly bounded above by G on \mathbb{R}_+ .*

Furthermore, if $S(x) = G(x)$ on a dense set D , then $S(x) = cx$ on \mathbb{R}_+ .

Proof. The set $\Sigma_+ := \{v : S(v) \leq cv\}$ is an (additive) semigroup containing U . By a theorem of Kingman [BinO18, Th. 3.5], Σ_+ is dense in \mathbb{R}_+ : for any interval $I \subseteq \mathbb{R}_+$ there is $\eta \in I$ such that $\eta/m \in U$ for infinitely many $m \in \mathbb{N}$; for such an η and any corresponding $m \in \mathbb{N}$,

$$S(\eta) \leq mS(\eta/m) \leq mc(\eta/m) = c\eta.$$

So $\eta \in \Sigma_+ \cap I$, proving density. Since Σ_+ is a semigroup $\eta + (a, b) \subseteq \Sigma_+$ for any interval $(a, b) \subseteq U$, with a chosen as small as desired, since U accumulates at

0. So the family of open intervals J contained in Σ_+ have dense union in Σ_+ : in sum, $\text{int}(\Sigma_+)$ accumulates to the right and left of any point of $(0, 1)$. Fix any $x \in (0, 1)$. There exists $h > 0$ as small as desired such that $x - h \in \Sigma_+$. For such an h ,

$$S(x) \leq S(x - h) + S(h) \leq c(x - h) + S(h).$$

Taking limits as $h \downarrow 0$ through such h , yields

$$S(x) \leq cx,$$

since $S(0+) = 0$, by Theorem 0⁺. This holds for any $x \in (0, 1)$ and by assumption for $x = 0$.

For the last part, a similar appeal to subadditivity and Theorem 0⁺ yields $S(x+) \leq S(x)$, for any $x > 0$. So, for any $x \geq 0$, if $x + h \in D$, then

$$S(x) \geq \limsup_{h>0} S(x+h) \geq \lim_{h>0} \sup_{x+h \in D} S(x+h) = \lim_{h>0} \sup_{x+h \in D} c(x+h) = cx.$$

Combining, $S(x) = cx$. \square

A similar but simpler argument yields:

Proposition 3'. *For $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ subadditive with $S(0) = 0$, write $\Sigma_+ := \{u \in \mathbb{R} : S(u) < \infty\}$. If $\Sigma_+ \cap [0, \infty)$ contains an open set accumulating on the right at 0, and S is finite on a dense subset of \mathbb{R}_+ , then $S|_{\mathbb{R}_+}$ is finite.*

If, further, S is finite on a subset unbounded below, then also $S|_{\mathbb{R}_-}$ is finite.

Proof. For the first assertion, argue just as before, replacing ' $\leq cx$ ' above by ' $< \infty$ ' to show that $\Sigma_+ \cap \mathbb{R}_+ = \mathbb{R}_+$. Then $S(x) < \infty$ for all $x > 0$. For any $x > 0$ choose $u_0 > x$ with $S(u_0)$ finite; then, as $u_0 - x > 0$,

$$S(x) \geq S(u_0) - S(u_0 - x) > -\infty.$$

For the last part, if S is finite at $u_0 < 0$, then for $u \in (u_0, 0)$

$$S(u) \leq S(u_0) + S(x - u_0) < \infty : \quad \mathbb{R}_- \subseteq \Sigma_+.$$

Finally, $S|_{\mathbb{R}_-}$ is finite, since $-\infty < -S(u) \leq S(-u)$, for $u > 0$. \square

Proposition 5. For $S : \mathbb{R} \rightarrow \mathbb{R}$ subadditive:

(i) if S is bounded above on some interval, say by K on $B_\delta(a)$, for instance if $(HS(S))$ holds, then for any $b \in \mathbb{R}$

$$S(b+a) - K \leq S(x) \leq S(b-a) + K \quad (x \in B_\delta(b)),$$

in particular it is locally bounded;

(ii) if S is locally bounded, then $\liminf_{t \rightarrow 0} S(t) \geq 0$, so $S(0+) = 0$ if $(HS(S))$ holds.

Proof. (i) If S is bounded above by K on $B_\delta(a) = a + (-\delta, +\delta)$, then for any $b \in \mathbb{R}$, S is bounded on $b + (-\delta, +\delta)$. Indeed, for any $x \in b + (-\delta, +\delta)$, since both $a + (x - b)$ and $a - (x - b)$ are in $B_\delta(a)$,

$$\left. \begin{aligned} S(x) &\leq S(b-a) + S(a+x-b) \leq S(b-a) + K, \\ S(x) &\geq S(b+a) - S(a+b-x) \geq S(b+a) - K. \end{aligned} \right\} \quad (\star)$$

If S satisfies (HS) , then $S(x) < 1$ for $x \in (0, \delta)$ for some $\delta > 0$.

(ii) Following [HilP, 7.4.3], select a sequence $\{z_n\}$ with $z_n \rightarrow 0$ and $S(z_n) \rightarrow \lambda_- := \inf_{u \rightarrow 0} S(u)$. By local boundedness, λ_- is finite, so for any $\varepsilon > 0$ and n large enough $\lambda_- - \varepsilon \leq S(2z_n) \leq 2S(z_n) < 2(\lambda_- + \varepsilon)$. So $\lambda_- \leq 2\lambda_-$, yielding $\lambda_- \geq 0$. \square

Having motivated right-sided continuity as in Theorem 0, we now prove it.

Proof of Theorem 0. The condition is evidently necessary. As for sufficiency, suppose given $z_n \uparrow 0$ as in the hypothesis, $x_n \uparrow 0$ with $S(x_n) \rightarrow \lambda$, and any $\varepsilon > 0$; we will show that $\lambda = 0$.

Choose $\delta > 0$ with $S(t) \leq \varepsilon$ for $0 \leq t \leq \delta$. W.l.o.g. we assume that $z_1 > -\delta$. Now choose $m(n)$ for $n \in \mathbb{N}$ with $z_n \leq x_{m(n)}$. Then $0 \leq x_{m(n)} - z_n \leq \delta$, as $x_{m(n)}, z_n \in (-\delta, 0)$, and so

$$S(x_{m(n)}) \leq S(x_{m(n)} - z_n) + S(z_n) \leq \varepsilon + S(z_n).$$

Passing to the limit gives

$$\lambda \leq \varepsilon + 0 = \varepsilon.$$

Taking limits as $\varepsilon \downarrow 0$ gives $\lambda \leq 0$. But, as $-x_{m(n)} \in (0, \delta)$,

$$0 = S(0) \leq S(x_{m(n)}) + S(-x_{m(n)}) \leq S(x_{m(n)}) + \varepsilon,$$

so taking limits gives

$$0 \leq \lambda + \varepsilon,$$

so $\lambda \geq 0$, as above. Combining, $\lambda = 0$, so S is continuous at 0. Finally, if S is finite-valued and continuous at 0, it is so at any x , since

$$-S(-h) \leq S(x+h) - S(x) \leq S(h). \quad \square$$

Proof of Theorem 0'. Since $S|\Sigma$ is continuous at 0 it is bounded above on $\Sigma_\delta := \Sigma \cap (-\delta, \delta)$ for some $\delta > 0$; but $\Sigma_\delta + \Sigma_\delta$ contains an interval, so S is bounded on an interval, and so locally bounded by Prop. 5(i). If S is not continuous at 0, then by Prop. 5(ii) $\lambda_+ := \limsup_{t \rightarrow 0} S(t) > \liminf_{t \rightarrow 0} S(t) \geq 0$. Choose a null sequence $\{z_n\}$ with $S(z_n) \rightarrow \lambda_+ > 0$. Let $\varepsilon := \lambda_+/4$. W.l.o.g. $S(z_n) > \lambda_+ - \varepsilon$ for all n . By continuity on Σ at 0 there is $\delta > 0$ with $|S(t)| < \varepsilon$ for $t \in \Sigma_\delta$. As before and using symmetry, $\Sigma_\delta + \Sigma_\delta = \Sigma_\delta - \Sigma_\delta$ contains an interval I around 0. For any n with $z_n \in I$, there are $u_n, v_n \in \Sigma_\delta$ with $z_n = u_n + v_n$; then

$$S(z_n) \leq S(u_n) + S(v_n) \leq 2\varepsilon < \lambda_+,$$

and so

$$3\lambda_+/4 = \lambda_+ - \varepsilon < S(z_n) \leq S(u_n) + S(v_n) \leq 2\varepsilon < \lambda_+/2,$$

a contradiction. So S is continuous at 0 and so continuous everywhere, as in Theorem 0. The last part follows since $\Sigma \cap (0, \delta)$, being Baire/measurable and non-negligible, has the SW property for each $\delta > 0$. \square

Proof of Theorem 0⁺. We may take $c = 0$, since $S'(t) := S(t) - ct$ is linearly bounded above by 0 on Σ , and S' is subadditive. (Also the thinned Heiberg-Seneta condition holds for S' .) Thus $S(t) \leq 0$ for $t \in \Sigma$.

Fix an arbitrary $x > 0$. We show that $S(x) \leq 0$. As Σ accumulates at 0, there is a point $\sigma_x \in \Sigma \cap (0, x/2)$. Then $\Sigma' := \Sigma \cap (\sigma_x, \frac{1}{2}(\sigma_x + x/2))$ has the SW property locally, and so $\Sigma' + \Sigma'$ contains a proper interval $[a, b]$ in $(2\sigma_x, \sigma_x + x/2)$.

With a, b fixed, choose $\sigma \in \Sigma \cap (0, b-a) \cap (0, a)$. By density of Σ , we may suppose that $a, x \notin \mathbb{N}\sigma$. Then there is $m \in \mathbb{N}$ with $a < m\sigma < m\sigma + \sigma < b$. Now, as $(m+1)\sigma < b < x$, we may choose $n > m+1$ in \mathbb{N} with $n\sigma < x < n\sigma + \sigma$. Then, as $0 < x - n\sigma < \sigma$, adding $m\sigma$ gives

$$a < m\sigma < x + (m-n)\sigma < (m+1)\sigma < b.$$

Now pick $u, v \in \Sigma'$ with $u + v \in (a, b)$ such that

$$u + v = x - (n - m)\sigma : \quad x = u + v + (n - m)\sigma.$$

By subadditivity, as $n - m \in \mathbb{N}$, and as $u, v, \sigma \in \Sigma$,

$$S(x) \leq (n - m)S(\sigma) + S(u) + S(v) \leq 0.$$

Thus $S(x) \leq cx$ for all $x > 0$. In particular

$$\limsup_{x \downarrow 0} S(x) \leq 0.$$

Being linearly bounded above on Σ , S is also relatively locally bounded above on Σ , hence also on $\Sigma + \Sigma$, and so on an interval; so by Prop. 5(ii) $\liminf_{x \downarrow 0} S(x) \geq 0$.

The final assertion follows from Theorem 0. \square

We may now turn to linearity from additivity rather than subadditivity, for which see later. A key step follows. By appealing to Kronecker's Theorem ([HarW, XXIII, Th. 438]), the proof rolls together the two ingredients of density and of routine use of continuity (as in Prop. 2' above); we thank the Referee for this elegant approach. See Proposition 6' in §5 for an alternative approach.

Proposition 6. *For \mathbb{A} a dense subgroup of \mathbb{R} , if $G : \mathbb{A} \rightarrow \mathbb{R}$ is additive and bounded above on $(0, \varepsilon) \cap \mathbb{A}$, for some $\varepsilon > 0$, then G is linear: $G(a) = ca$ for some $c \in \mathbb{R}$ and all $a \in \mathbb{A}$.*

Proof. Being subadditive, G may be assumed bounded on $I = (0, \varepsilon)$ for some $\varepsilon > 0$, by Prop. 5(i).

Fix any non-zero $u_0 \in \mathbb{A}$ and put $c = G(u_0)/u_0$. We prove that $G(a) = ca$ for all $a \in \mathbb{A}$ by showing that $H(a) := G(a) - ca$ ($a \in \mathbb{A}$) is identically zero.

Now H is bounded on $I \cap \mathbb{A}$, by M say. By additivity, $H(a) = 0$ for $a \in u_0\mathbb{Z}$, as $H(u_0) = 0$. Suppose that $H(u) \neq 0$ for some $u \in \mathbb{A}$; then for any $p \in \mathbb{N}$, $pu \notin u_0\mathbb{Z}$ (as otherwise $pH(u) = H(pu) = 0$). Fix $p \in \mathbb{N}$ arbitrarily. As pu and u_0 are incommensurable, the subgroup they generate, $u_0\mathbb{Z} + pu\mathbb{Z}$, is dense, by Kronecker's theorem. So $mpu + nu_0 \in I \cap (0, |u_0|) \cap \mathbb{A}$, for some $m, n \in \mathbb{Z}$. As $m \neq 0$ (since $nu_0 \in (0, |u_0|)$ is not possible),

$$M \geq |H(mpu + nu_0)| = |m|p |H(u)| \geq p|H(u)|.$$

So $M/p \geq |H(u)|$. But $p \in \mathbb{N}$ was arbitrary, so $H(u) = 0$, after all. \square

Proposition 7 (Unique extension). *For $\Sigma \subseteq \mathbb{R}$ dense and closed under integer scaling (e.g. a subgroup), let $G : \Sigma \rightarrow \mathbb{R}$ be linear: $G(\sigma) = c\sigma$ ($\sigma \in \Sigma$). If $S : \mathbb{R} \rightarrow \mathbb{R}$ with $S(0+) = S(0)$ is any subadditive extension of G , then S is also linear on \mathbb{R} and $S(u) = S(1)u = cu$ for all $u \in \mathbb{R}$.*

Proof. Here $S(0) = 0$, since $S(-\sigma) = G(-\sigma) = -G(\sigma) = -S(-\sigma)$ for any $\sigma \in \Sigma$. Take $z_n \in \Sigma \cap (-1, 0)$ converging to 0; then $S(z_n) = G(z_n) = cz_n \rightarrow 0$ for some c . By Theorem 0, S is continuous. Now $S(\sigma) = G(\sigma) = c\sigma$ on Σ ; so, as Σ is dense in \mathbb{R} , by continuity $S(t) = ct = tS(1)$ on \mathbb{R} . \square

Linearity from subadditivity is now a corollary of Prop. 5, 6 and 7:

Proof of Theorem 1. Here $S(0) = 0$ as $S|_{\mathbb{A}}$ is additive on \mathbb{A} . As $(HS(S))$ holds, for each $\varepsilon > 0$ there is $\delta > 0$ with $S(t) \leq \varepsilon$ for $0 < t < \delta$ and so $S(0+) = S(0)$. In particular $G = S|_{\mathbb{A}}$ is additive on \mathbb{A} and bounded above on $(0, \delta)$. By Prop. 6, $G(a) = ca$ for some $c \in \mathbb{R}$ and all $a \in \mathbb{A}$. By Prop. 7, $S(t) = ct$ for all $t \in \mathbb{R}$. \square

Proof of Theorem 1'. (a) This follows from Theorem 1 by Theorem 0'. \square_a

(b) This follows from Theorem 1 by Theorem 0⁺. \square_b

(c) Here $\mathbb{A} \supseteq \Sigma + \Sigma$, so contains an interval, and, being a dense additive subgroup, $\mathbb{A} = \mathbb{R}$. So S is additive and is bounded on $\Sigma + \Sigma$, and so on some interval; so S is linear, by Prop. 6. \square_c

(d) As before, S is subadditive, and by assumption is, in view of Prop. 1, finite on the dense (additive) subgroup $\mathbb{A} = \mathbb{A}_S$ of \mathbb{R} . As \mathbb{A} is shift-compact, $\mathbb{A} \supseteq \mathbb{A} + \mathbb{A}$ contains an interval, so by density again $\mathbb{A} = \mathbb{R}$, i.e. S is finite everywhere. So $S = S|_{\mathbb{A}}$ is *additive* on the subgroup $\mathbb{A} = \mathbb{R}$. In particular $[S(u+x) - S(x)] = S(u)$ and so $G(u) = S(u)$ for $u \in \mathbb{A}$. By Prop. 5, S is locally bounded, and so by Prop. 6 S is linear on $\mathbb{A} = \mathbb{R}$. $\square_d \square$

Theorem 2 (On linearity). *Let $S : \mathbb{R} \rightarrow \mathbb{R}$ be subadditive with $S(0+) = 0$. If, for some dense additive subgroup \mathbb{A} of \mathbb{R} , the restriction $S|_{\mathbb{A}}$ is additive, then S is linear on \mathbb{R} : $S(u) = cu$ for some $c \in \mathbb{R}$ and all $u \in \mathbb{R}$.*

Proof. S extends $G := S|_{\mathbb{A}}$, additive and continuous by Prop. 7. \square

Cautionary Example. Recall from §1 that $S := \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ is subadditive, and for $\mathbb{A} = \mathbb{Q}$, $S|_{\mathbb{A}} = 0$ is linear; but S is not linear. We return to the relation of this example to Theorem 2 later in §7.

Remark. In Theorem \mathcal{O}' above, it is not enough to assume only that the subadditive function S is locally bounded above and \mathbb{Q} -homogeneous on a set Σ that is dense (i.e. $S(q\sigma) = qS(\sigma)$ for all $\sigma \in \Sigma$ and rational q); indeed, the indicator function $\mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ just considered is subadditive and also \mathbb{Q} -homogeneous on $\Sigma = \mathbb{Q}$, but not continuous. Such a weaker assumption yields only that $\liminf_{t \rightarrow 0} S(t) = 0 = S(0)$. (Proof: As S is locally bounded, choose K and $\delta > 0$ with S bounded by K on $[-\delta, \delta]$. Fix $\sigma \in \Sigma \cap (0, \delta)$; then $|S(\sigma/n)| = |S(\sigma)|/n \leq K/n$ for all $n \in \mathbb{N}$.)

4 Thinnings of Heiberg-Seneta conditions and subadditive functions

We turn now from functional equations (whose prototype for us is the Cauchy functional equation (Add) from §1) to functional inequalities (the prototype of which for us is the corresponding inequality (Sub) from §1). The classical sources here are [HilP, Ch. 3] (for the measurable case only, but we need the category version also, for which see [BinO1]) and [Kuc, Ch. 16]; cf. [MatS]). Kuczma makes the contrast between the surprisingly great affinity between Cauchy's equation and Jensen's inequality, and the differences between (Add) and (Sub). Here matters are reversed: what is surprising in our context is the *similarity* between (Add) and (Sub).

The motivation in this section is the quantifier weakening of our title, in the context of regular variation (§7 below). The prototypical results here are *characterization theorems* (BGT, Th. 1.4.3, 3.2.4). The prototypical conditions for these are the *Heiberg-Seneta conditions*, of 'limsup limsup' type. There are links with *Tauberian conditions*, of 'lim limsup sup' type; see e.g. BGT Ch. 4. Here we start with the classical condition ($HS(S)$)

$$\limsup_{u \downarrow 0} S(u) \leq M$$

for $M \in \mathbb{R}$, re-phrased as follows:

$$\limsup_{n \rightarrow \infty} \sup \{S(x) : x \in (0, 1/n)\} \leq M,$$

which we generalize so as to 'thin out' the intervals $(0, 1/n)$ in various senses (including with the help of category and measure), appropriately expanding

the notation $(\limsup)_M$. We focus here on sets that have the local Steinhaus-Weil property of §1, and begin with a list of those relevant here. For an alternative mode of thinning (via spanning) see §6.

Examples of families of locally Steinhaus-Weil sets.

The sets listed below are typically, though not always, members of a topology on an underlying set.²

- (o) Σ a usual (Euclidean) open set in \mathbb{R} (and in \mathbb{R}^n) – this is the ‘trivial’ example;
- (i) Σ density-open subset of \mathbb{R} (similarly in \mathbb{R}^n) (by Steinhaus’s Theorem – see e.g. BGT Th. 1.1.1, [BinO18], [Oxt, Ch. 8]);
- (ii) Σ locally non-meagre at all points $x \in \Sigma$ (by the Piccard-Pettis Theorem – as in BGT Th. 1.1.2, [BinO18], [Oxt, Ch. 8] – such sets can be ‘thinned out’, i.e. extracted as subsets of a second-category set, using separability or by reference to the Banach Category Theorem [Oxt, Ch.16]);
- (iii) Σ the Cantor ‘middle-thirds excluded’ subset of $[0, 1]$ (since $\Sigma + \Sigma = [0, 2]$);
- (iv) Σ universally measurable and open in the *ideal* topology ([LukMZ], [BinO17]) generated by omitting Haar null sets (by the Christensen-Solecki Interior-points Theorem of [Sol]);
- (v) Σ a Borel subset of a Polish abelian group and open in the ideal topology generated by omitting *Haar meagre* sets in the sense of Darji [Darj] (by Jabłońska’s generalization of the Piccard Theorem, [Jab1, Th.2], cf. [Jab3], and since the Haar-meagre sets form a σ -ideal [Darj, Th. 2.9]); for details see [BinO18].

If Σ is *Baire* (has the Baire property) and is locally non-meagre, then it is co-meagre (since its quasi interior is everywhere dense).

Caveat. 1. Care is needed in identifying locally SW sets: Matoušková and Zelený [MatZ] show that in any non-locally compact abelian Polish group there are closed non-Haar null sets A, B such that $A + B$ has empty interior. Recently, Jabłońska [Jab4] has shown that likewise in any non-locally compact abelian Polish group there are closed non-Haar meager sets A, B such that $A + B$ has empty interior.

2. For an example on \mathbb{R} of a compact subset S such that $S - S$ contains an interval, but $S + S$ has measure zero and so does not, see [CrnGH].

3. Below we are concerned with subsets $\Sigma \subseteq \mathbb{R}$ where such ‘anomalies’ are assumed not to occur.

²Below we refer to ideal topologies in the sense of [LukMZ].

Definition. Say that $(SW)\text{-}\limsup_{u\downarrow 0} S(u) \leq M \in \mathbb{R}$ if there is $\Sigma \subseteq (0, 1)$ *accumulating* at 0 with the local Steinhaus-Weil property such that, for $\Sigma_n := \Sigma \cap (0, 1/n)$,

$$\limsup_{n \rightarrow \infty} \sup S(\Sigma_n) := \limsup_{n \rightarrow \infty} \sup \{S(x) : x \in \Sigma_n\} \leq M. \quad (SW\text{-}\limsup_M(S))$$

Evidently, $(SW)\text{-}\limsup_{u\downarrow 0} S(u) \leq M \in \mathbb{R}$ holds if $\limsup_{u\downarrow 0} S(u) \leq M \in \mathbb{R}$ holds (refer to $\Sigma = (0, 1)$).

For later use, say that $(SSW)\text{-}\limsup_{u\downarrow 0} S(u) \leq M \in \mathbb{R}$ if there is a *symmetric* set $\Sigma \subseteq (-1, 1)$, i.e. $\Sigma = -\Sigma$, with the local Steinhaus-Weil property and $\Sigma \cap (0, 1)$ accumulating at 0, such that $SW\text{-}\limsup_M(S)$ above holds for $\Sigma_n := \Sigma \cap (-1/n, 1/n)$.

It is thematic for us that, inasmuch as they affect subadditive functions, the quantifier weakening that thinness offers implies a level of informativeness equal to that of the trivial example (o). We thank the Referee for the following ‘bridging’ result, clarifying the relationship with $\limsup_{u\downarrow 0} S(u) \leq M \in \mathbb{R}$.

Proposition 8. *For $S : \mathbb{R} \rightarrow \mathbb{R}$ subadditive, $SW\text{-}\limsup_M(S)$ holds for some M iff: for some K ,*

$$S \text{ is bounded above by } K \text{ on some open } U \subseteq (0, \infty) \text{ with } 0 \in \bar{U}. \quad (\dagger)$$

Proof. Suppose that $SW\text{-}\limsup_M(S)$ holds for some M . Then, with the notation above, for some infinite set \mathbb{M}

$$S(u) < M + 1 \quad (u \in \Sigma_m, m \in \mathbb{M}).$$

So for $x = u + v$ with $u, v \in \Sigma_m$

$$S(x) < 2(M + 1).$$

By the Steinhaus-Weil property $\Sigma_m + \Sigma_m$ contains an interval I_m in $(0, 2/m)$; then $U := \bigcup_{m \in \mathbb{M}} I_m$ is open, $0 \in \bar{U}$ and S is bounded above by $K := 2(M + 1)$ on P .

The converse is clear: given U and K as in (\dagger) above, take $\Sigma := U$; then $SW\text{-}\limsup_M(S)$ holds for $M = K$. \square

The occurrence above of the infinite set \mathbb{M} justifies a combinatorial departure ‘beyond Lebesgue and Baire’. A wider combinatorial characterization

involving the embedding of a convergent subsequence (rather than only of a null subsequence) may be obtained by reference to the level sets of a function S , defined by

$$H^r, \text{ or } H^r(S) := \{x : |S(x)| < r\}.$$

Proposition 9. *If the subadditive function $S : \mathbb{R} \rightarrow \mathbb{R}$ is Baire/measurable, then for every convergent sequence $\{u_n\}$ with $u_n \rightarrow u$, there exist $k \in \mathbb{N}$, $t \in H^k$ and \mathbb{M} infinite with*

$$\{t + u_m : m \in \mathbb{M}\} \subseteq H^k.$$

In particular, S is locally bounded above, and so locally bounded.

Proof. We argue as in [BinO3]: since $\mathbb{R} = \bigcup_{k \in \mathbb{N}} H^k$, there is $K \in \mathbb{N}$ with H^K non-negligible Baire/measurable and $S|_{H^K}$ is bounded by K . Given a sequence $\{u_n\}$ with $u_n \rightarrow u$, put $z_n := u_n - u \rightarrow 0$; by the *bilateral* version of KBD [BinO5, §3], choose $t \in H^K$ and \mathbb{M} infinite with $\{t + z_m, t - z_m : m \in \mathbb{M}\} \subseteq H^K$. Then, with $a = t, b = t + u$ and $x = t + u + z_m$ for $m \in \mathbb{M}$, we may apply (\star) as in the proof of Prop. 5 but with H^K for $B_\delta(a)$, since $a + b - x = t - z_m \in H^K$ and $x + a - b = t + z_m \in H^K$. This yields, since $b + a = u + 2t$ and $b - a = u$,

$$S(2t + u) - K \leq S(t + u_m) \leq S(u) + K.$$

So for some k and infinite \mathbb{M} one has $\{t + u_m : m \in \mathbb{M}\} \subseteq H^k$.

Suppose that S is not locally bounded above; then for some u and sequence $\{u_n\}$ with $u_n \rightarrow u$, the sequence $\{S(u_n)\}$ is unbounded above. But for some $k \in \mathbb{N}$, $t \in H^k$ and \mathbb{M} infinite as above, since $t + u_m \in H^k$ for $m \in \mathbb{M}$,

$$S(u_n) \leq S(t + u_m) + S(-t) \leq k + S(-t),$$

a contradiction. Now apply Prop. 5(i). \square

Remark. For Σ shift-compact and $S : \mathbb{R} \rightarrow \mathbb{R}$ subadditive, if S is bounded above on Σ , then S is bounded above on \mathbb{R} [BinO1, Th. 2(ii)]. Proposition 3'' below extends this to the range $\mathbb{R} \cup \{-\infty, +\infty\}$.

Proposition 3'' (On finiteness). *Suppose that $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is subadditive and satisfies:*

(i) *S is finite on a subset Σ unbounded below (e.g. a dense subset of \mathbb{R}), and*

(ii) for some $M \in \mathbb{R}$, $SW\text{-lim sup}_M(S)$ holds.
 Then S is finite everywhere, and so locally bounded.

Proof. By (ii) S is bounded above, by K say, on some locally-SW set T (e.g. a Baire/measurable non-negligible); as above, by the Steinhaus property, $T + T$ contains an interval on which S is bounded above by $2K$. Apply Proposition 3 to deduce that S is finite everywhere.

As S is finite and subadditive and bounded above on an interval, by Prop. 4 (cf. [BinO1, Th. 2(ii)]) S is locally bounded. \square

We close with an alternative approach to Theorem 0'.

Proposition 4'. *If $S : \mathbb{R} \rightarrow \mathbb{R}$ is subadditive with $S(0) = 0$ and satisfies $SSW\text{-lim sup}(S)_0$, then S is continuous.*

Proof. Otherwise, S is not continuous at 0 (as at the end of Prop. 11), and so as above $\lambda_+ := \limsup_{t \rightarrow 0} S(t) > \liminf_{t \rightarrow 0} S(t) \geq 0$, the latter by Prop. 5(ii) (by the Remark above, combined with Prop. 5(i), S is locally bounded). Choose a null sequence $\{z_n\}$ with $S(z_n) \rightarrow \lambda_+ > 0$, and Σ as in $SSW\text{-lim sup}(S)_0$. For any $\varepsilon > 0$, take $N = N(\varepsilon)$ such that $\sup S(\Sigma_N) < \varepsilon/4$. As $\Sigma_N + \Sigma_N$ contains an open interval, $r + I$ with $0 \in I$ say, then $r + z_n \in r + I$ for all large n ; write $r = s + t$ and $r + z_n = u_n + v_n$ for $s, t, u_n, v_n \in \Sigma_N$. Then

$$S(z_n) \leq S(u_n) + S(v_n) + S(-s) + S(-t) < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 \leq \varepsilon,$$

as $-s, -t \in \Sigma_N$ (symmetry). Taking limits yields $\lambda_+ \leq \varepsilon$, for each $\varepsilon > 0$, so $\lambda_+ \leq 0$, a contradiction. So $\lambda_+ = 0$, and so $\lim_{t \rightarrow 0} S(t) = 0 = S(0)$, contradicting our initial assumption. \square

5 Sublinearity and Berz's Theorem

Theorem 2 above is reminiscent of the following classical result. Recall that for Σ closed under positive rational scaling S is *sublinear* on Σ in the sense of Berz [Berz] if S is subadditive and $S(nx) = nS(x)$ for $x \in \Sigma, n = 0, 1, 2, \dots$ (i.e. S is positively \mathbb{Q} -homogeneous and $S(0) = 0$).

Theorem B (Berz's Theorem, [Berz]; cf. [Kuc, Th. 16.4.3]). *If $S : \mathbb{R} \rightarrow \mathbb{R}$ is measurable and sublinear, then there are $c_{\pm} \in \mathbb{R}$ such that $S(u) = c_+u$ for $u \geq 0$ and $S(v) = c_-v$ for $v \leq 0$.*

Theorem 3 below is its category analogue; cf. [BinO16] and §8.4. We will need the following variant of Prop. 6, which in fact includes it (below take $\Sigma = \mathbb{A}$ a dense subgroup of \mathbb{R} , and $S = A$ an additive function: continuity below is then equivalent to local boundedness at 0, by Prop. 2').

Proposition 6' (Relative linearity). *Let $S : \Sigma \rightarrow \mathbb{R}$ be subadditive with $\Sigma \cap \mathbb{R}_+$ dense on \mathbb{R}_+ and closed under positive-integer scaling. If S is \mathbb{N} -homogeneous and right-continuous, then S is linear on $\Sigma \cap \mathbb{R}_+$: $S(\sigma) = c_+ \sigma$ for some $c_+ \in \mathbb{R}$ and all $\sigma \in \Sigma \cap \mathbb{R}_+$.*

Proof. The proof is adapted from BGT Th. 3.2.5 (see also [BinG, Proof of Th. 5.7]). Fix any *positive* $\sigma_0 \in \Sigma$ and put $c := S(\sigma_0)/\sigma_0$. We show that $S(\sigma) = c\sigma$ for all $\sigma \in \Sigma$. To this end, fix any $\sigma \in \Sigma$. Now define for $0 < \delta \in \Sigma$

$$i = i(\delta) := \min\{n \in \mathbb{N} : n\delta > \sigma\}, \quad i_0 = i_0(\delta) := \min\{n \in \mathbb{N} : n\delta > \sigma_0\},$$

so that $i(\delta)\delta \downarrow \sigma$ and $i_0(\delta)\delta \downarrow \sigma_0$ as $\delta \downarrow 0$; here w.l.o.g. $G(\delta) \neq 0$ (otherwise $G = 0$ on $(0, \varepsilon) \cap \Sigma$, for some $\varepsilon > 0$, implying below that $S(\sigma) = 0$ and so $S \equiv 0$ on $\Sigma \cap \mathbb{R}_+$).

Since $\delta, u_0 \in \Sigma \cap \mathbb{R}_+$ and $\Sigma \cap \mathbb{R}_+$ is closed under integer scaling, we have by \mathbb{N} -homogeneity of S that $S(i\delta) = iS(\delta)$; likewise $S(i_0\delta) = i_0S(\delta)$. Taking limits here and below through Σ as $\delta \downarrow 0$ and using right-continuity of S at σ_0 and σ ,

$$S(\sigma_0) = \lim_{\delta \downarrow 0} S(i_0(\delta)\delta), \quad S(\sigma) = \lim_{\delta \downarrow 0} S(i(\delta)\delta).$$

Dividing these two, as $\sigma_0 \neq 0$,

$$\sigma/\sigma_0 = \lim_{\delta \downarrow 0} i(\delta)\delta/i_0(\delta)\delta = \lim_{\delta \downarrow 0} i(\delta)/i_0(\delta) = \lim_{\delta \downarrow 0} i(\delta)S(\delta)/i_0(\delta)S(\delta) = S(\sigma)/S(\sigma_0).$$

Cross-multiplying, $S(\sigma) = c\sigma$ for any $\sigma \in \Sigma \cap \mathbb{R}_+$. \square

Theorem 3 (Baire-Berz Theorem). *If $S : \mathbb{R} \rightarrow \mathbb{R}$ is Baire and sublinear, there are $c_{\pm} \in \mathbb{R}$ such that $S(u) = c_+u$ for $u \geq 0$ and $S(v) = c_-v$ for $v \leq 0$.*

Proof. By a theorem of Baire, we may choose a meagre set M such that $S|(\mathbb{R} \setminus M)$ is continuous (see [Oxt, Th. 8.1]). Expand M to a union of closed nowhere dense sets, if necessary. Take $\Sigma := \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} qM$, which is closed under rational scaling and is dense on \mathbb{R} (by Baire's Theorem – as each $\mathbb{R} \setminus qM$ is a dense \mathcal{G}_δ). In particular $\Sigma = -\Sigma$. By Prop. 6', S is linear on $\Sigma \cap \mathbb{R}_+$. By Theorem 0+, S is continuous. So as S is linear on $\Sigma \cap \mathbb{R}_+$, a dense subset of \mathbb{R}_+ , it is linear on \mathbb{R}_+ ; likewise $S(-x)$ is linear on \mathbb{R}_+ . \square

6 Thinning via spanning

Weakening of quantifiers amounts to *thinning* of the relevant set. Theorem 1 may be viewed as a two-pronged test of the subadditive function S on thin sets: one prong a domain condition, density, the other a boundedness condition at 0. As \mathbb{A} is a subgroup, the first condition reduces to density of \mathbb{A} at 0. On \mathbb{R} it further reduces to a two-point condition: for Theorem 1 to hold, the set \mathbb{A} necessarily has at least two incommensurable members (recall the example $\mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$); by Kronecker's Theorem ([HarW] XXIII, Th. 438) \mathbb{A} is then dense. That is, \mathbb{A} needs to be at least a two-dimensional subspace of \mathbb{R} , regarded as a vector space over \mathbb{Q} .

Say that $T \subseteq \mathbb{R}$ is a *spanning* set if it spans \mathbb{R} regarded as a vector space over \mathbb{Q} (e.g. contains a Hamel basis). We work below with *analytic sets*; for background see e.g. [Rog], [BinO6]. The approach below is motivated by a theorem due to F. Burton Jones and its later strengthening by Z. Kominek – see below. Recall from §4 that T is *symmetric* if $-T := \{-t : t \in T\} = T$, and that T is *shift-symmetric* if for some τ the set $T + \tau := \{t + \tau : t \in T\}$ is symmetric ($T + \tau = -T - \tau$), and the latter is equivalent to *self-similarity*, in the sense of [BinO9]: that $T = a - T$ for some a (as then $T - a/2 = -(T - a/2)$).

The vectorial view brings further insights. Jones [Jon] proved that for additive $A : \mathbb{R} \rightarrow \mathbb{R}$, if $A|T$ is continuous on an analytic spanning set T , then it is continuous. Much later, Kominek [Kom] showed that, for such a set T , if $A|T$ is bounded above on T , then T is continuous – cf. [BinO6, Th. JK] and §8.6. Kominek's Theorem is stronger: it implies Jones's, for which see again [BinO6, §3]. The Jones-Kominek results show *how to test on thin sets* T *properties of interest on* \mathbb{R} .

In Theorem 1'_d above the oscillation condition is further thinned by using only shift-compact subsets in neighbourhoods of the origin³. This raises the question of establishing further quantifier weakening by testing only on analytic spanning sets, as these need not be shift-compact. This is indeed possible on both prongs, as follows.

As we have seen, Theorem 1 rests on the presence of enough of the fol-

³Say that T is *k-thinned* shift-compact if the k -fold sum: $k \cdot T := T + \dots T$ is shift-compact. These thinned subsets could in principle do the work of shift-compactness, as in [BinO4, §4]; the standard Cantor set C is 2-thinned shift-compact, as $C + C$ contains an interval (see Remarks, below): boundedness of a subadditive function on C yields local boundedness.

lowing three properties: finiteness, local boundedness and continuity. These are aided by density, and easy to achieve for additive functions by demanding them on a spanning set. Kominek's Theorem cannot be applied directly to S , as S is only subadditive, but a simple modification of its proof will work below. It is convenient to use a result of Erdős, for which we need the following.

Definition. For H a Hamel basis, and by extension for H a spanning set, we write

$$Z(H) := \{k_1 h_1 + \dots + k_n h_n : k_i \in \mathbb{Z}, h_i \in H\}$$

for the *Erdős set* of H (for which see e.g. [Kuc, §11.5]).

Theorem E (Erdős; see e.g. [Kuc, Lemma 11.5.3]). *For H a Hamel basis and, more generally, for H a spanning set, the Erdős set $Z(H)$ is a dense subgroup.*

We need a modified version of the Analytic Dichotomy Lemma in [BinO6, §2], which refers to the *k-fold sum* $k \cdot T := T + \dots + T$ (k times).

Proposition 10. *If $T \subseteq \mathbb{R}$ is a symmetric analytic spanning set, then for some $k \in \mathbb{N}$ the k -fold sum $k \cdot T$ contains the interval $(-1, 1)$. If T is a shift-symmetric analytic spanning set, then for some $k \in \mathbb{N}$ the k -fold sum $k \cdot T$ contains an interval.*

Proof. We indicate the necessary modification to the proof in [BinO6, §2]. As there, with T an analytic spanning set, for some m -tuple of rationals $r_i = p_i/N$ with $p_i \in \mathbb{Z}$, for $1 \leq i \leq m$, the set $(r_1 T + \dots + r_m T)$ is non-null. So too is $(p_1 T + \dots + p_m T)$, and so $(p_1 T + \dots + p_m T) - (p_1 T + \dots + p_m T)$ contains an interval around 0 (by the Steinhaus property, again). So, in the T symmetric case, as $T = -T$ we conclude that $T' := |p_1|T + \dots + |p_m|T + |p_1|T + \dots + |p_m|T$ contains an interval around 0, say $(-1/n, 1/n)$. Since $n \cdot T' \supseteq (-1, 1)$, one has $k \cdot T \supseteq (-1, 1)$ for $k = 2n(|p_1| + \dots + |p_m|)$.

If, however, $\tau + T$ rather than T is symmetric, then $kT = k(\tau + T) - k\tau$ and the shift-symmetric case follows from the symmetric case. \square

We may now give the analytic-spanning version of Theorem 1.

Theorem 4. *For $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ subadditive, and $A \subseteq \mathbb{R}$ an additive subgroup, suppose that*

- (i) $S|\mathbb{A}$ is finite and additive;
- (ii) there exists a shift-symmetric (self-similar), analytic, spanning set $T \subseteq \mathbb{A}$ such that $S|T$ is locally bounded above.

Then S is linear: $S(u) = cu$, for some $c \in \mathbb{R}$ and all $u \in \mathbb{R}$.

Proof. By Prop. 10, fix $k \in \mathbb{N}$ such that $k \cdot T$ contains $(-1, 1)$. By Theorem E above, the Erdős set $Z(T)$, and so also \mathbb{A} , is a dense subgroup. Now $k \cdot T \subseteq \mathbb{A}$, as \mathbb{A} is a subgroup. As $k \cdot T$ contains an interval, $\mathbb{A} = \mathbb{R}$ (cf. Th. S in §8.3). As $S|T$ is locally bounded above, by subadditivity S is locally bounded above on $k \cdot T$ (by continuity of addition), and so on an interval. But S is additive on \mathbb{R} and bounded above on an interval, so locally bounded, by Prop. 5(i). So S is linear by Prop. 7. \square

The condition that S be linearly bounded on a symmetric analytic spanning set, stronger than (ii), yields automatic linearity – without any need for (i) above. There are echos here of Theorem 0⁺.

Proposition 11. *If the subadditive function $S : \mathbb{R} \rightarrow \mathbb{R}$ is linear on a symmetric analytic spanning set, then S is linear.*

Proof. Suppose that $S(t) = ct$ for $t \in T$ with T a symmetric analytic spanning set. By Prop. 10 there exists $k \in \mathbb{N}$ with $k \cdot T$ containing $I := (-1, 1)$. So $\mathbb{R} = \bigcup_{n=k}^{\infty} n \cdot T$, since $n \cdot T$ contains the interval $(n/k)I$ for $n \in \mathbb{N}$ with $n > k$. Now consider any $k \in \mathbb{N}$ with $k \cdot T$ containing a symmetric interval J about 0. For $u \in J \subseteq kT$, choose $t_i \in T$ with $u = \sum_{i=1}^k t_i$; then

$$S(u) \leq \sum_{i=1}^k S(t_i) = \sum_{i=1}^k ct_i = cu,$$

by linearity of S on T . Likewise, as $-u \in J$,

$$S(-u) \leq c(-u) = -cu.$$

Note that $S(0) = 0$, by subadditivity, as $S(-t) = -ct = -S(t)$ for $t \in T$; so $-S(u) \leq S(-u)$, again by subadditivity. Combining,

$$-cu \leq -S(u) \leq S(-u) \leq -cu : \quad S(u) = cu \quad (u \in I).$$

As J is symmetric and can have arbitrary length, $S(u) = cu$ for all u . \square

Remarks. 1. We raise the question of whether symmetry can be omitted in Theorem 4.

2. With C the standard Cantor set, consider its translate $T := C - \frac{1}{2}$, which is symmetric. It is a spanning set as $C + C = [0, 2]$, $T + T = [-1, 1]$, as in Proposition 10; so, assuming the Axiom of Choice AC, a Hamel basis H may also be selected in T , and H is non-dense as C and T are. So a (nowhere dense) shift-symmetric Cantor set, such as T , would suffice to test for local boundedness. Though density is not explicitly mentioned, it remains present implicitly as T , being uncountable, contains incommensurables.

3. In view of the Jones-Kominek theorems above, the question arises as to whether a subadditive S with $S|T$ right-continuous on an analytic spanning set T is right-continuous on a dense set. Note that $S := \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ is continuous on the (analytic) spanning set $T := \mathbb{R} \setminus \mathbb{Q}$, yet S is not continuous.

A continuous function $f : T \rightarrow \mathbb{R}$ is termed (*extensibly continuous* or just) *precompact* in [BinO6, §5] if $\{f(t_n)\}$ is a Cauchy sequence whenever $\{t_n\}$ is a Cauchy sequence in T (cf. [Ful]). For $f : \mathbb{R} \rightarrow \mathbb{R}$ additive, precompactness of $f|T$ on an analytic spanning set T implies continuity, but this feature does not extend to subadditive functions, as $\mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}|(\mathbb{R} \setminus \mathbb{Q})$ is precompact.

Definition. Altering the property $SW\text{-lim sup}(S)_0$ so that the limsup is taken with reference to symmetric sets $P_n \subseteq (-1/n, 1/n)$ of the scaled form $P_n := T/n = \{t/n : t \in T\}$, with T a fixed *symmetric* analytic spanning set in $(-1, 1)$, yields $JK(T)_0$ – the $(JK)_0$ *property for T* .

This yields an analogue of Theorem 4 for general subadditive functions S rather than those that are linear on an additive subgroup \mathbb{A} containing an analytic spanning set T with $S|T$ bounded. As each P_n is a spanning set, the semigroup argument underpinning Prop. 3' remains valid, and so Prop. 3'' holds with $(JK)_0$ replacing $(SW\text{-lim sup})_M$.

Theorem 5 (Automatic right-continuity, after Jones-Kominek). *If $S : \mathbb{R} \rightarrow \mathbb{R}$ is subadditive, and $JK(T)_0$ holds for some symmetric analytic spanning set T with $S|T$ locally bounded above – then $S(0+) = 0$.*

Proof. By the Compact Spanning Approximation Theorem of [BinO6, §3], passing to a symmetric compact subset of T if necessary, we may assume that T is compact, and, scaling if necessary, that $T \subseteq (-1, 1)$. By Prop. 10, fix $k \in \mathbb{N}$ such that $k \cdot T$ contains $(-1, 1)$. If S is locally bounded above on

T by M , then S is locally bounded above on $(-1, 1)$ by kN , as $k \cdot T$ contains $(-1, 1)$; so S is locally bounded above, and so locally bounded by Prop. 5.

With this in mind, consider an arbitrary sequence $\{v_n\}$ with $v_n \rightarrow 0$ and $S(v_n) \rightarrow a$. For each n , as $k \cdot (T/n)$ contains $(-1/n, 1/n)$, we may assume, by passing to a subsequence of $\{v_n\}$ if necessary, that $|v_n| \leq 1/n$, and so $v_n = u_n^1 + \dots + u_n^k$ with $u_n^i \in T/n$ for each $i = 1, 2, \dots, k$. Again by passing to a subsequence if necessary, by $JK_0(T)$ we may assume that $\lim_n S(u_n^i) \leq 0$ for $i = 1, \dots, k$. By subadditivity,

$$S(v_n) \leq S(u_n^1) + \dots + S(u_n^k),$$

so $a = \lim_n S(v_n) \leq 0$. Suppose that $a < 0$. W. l. o. g., $w := \sum_i v_i < \infty$. Put $w_n := \sum_{i \leq n} v_i \rightarrow w$. By subadditivity,

$$S(w_n) \leq \sum_{i \leq n} S(v_i) \rightarrow -\infty.$$

This contradicts local boundedness at w . So $a = 0$; i.e. $S(0+) = 0$. \square

Remark. The very last step in the proof above is inspired by the Goldie argument in BGT, p. 142.

7 Quantifier weakening in regular variation

The standard work on the Karamata theory of regular variation is BGT. The present authors have returned to this area in a number of papers, together and separately, largely addressed to matters left open there. First, we address the *foundational question*: what is the appropriate generalization of the measurability and Baire-property settings of BGT? Secondly, we address the *contextual question*: what, beyond the real-line setting of BGT (and other settings briefly addressed in BGT Appendix 1 such as the complex plane, Euclidean space and topological groups), is the natural context for the theory? In Theorem 6 here, we complete our reduction of the number of hard proofs in the area to zero, thus making good on a claim we have already made elsewhere (see §1). It is striking that Th. 1.4.3 of BGT, in the context of Karamata Theory,

$$\lim_{x \rightarrow \infty} f(\lambda x)/f(x) = g(\lambda) \quad (\forall \lambda > 0), \quad (K)$$

is no harder than Th. 3.2.5 of BGT, in the context of *Bojanić-Karamata/de Haan* Theory ([BojK], [dHa]; cf. [BinO14])

$$\lim_{x \rightarrow \infty} \{f(\lambda x) - f(x)\}/g(x) = k(\lambda) \quad (\forall \lambda > 0). \quad (BKdH)$$

Here we weaken the quantifier \forall above as much as possible (cf. [BinG]). Results of this nature are false if the quantifier is weakened too much, for reasons connected with Hamel pathology (BGT § 1.1.4; cf. [Kuc, Ch. 11]). As is usual in this area, of course, one encounters a dichotomy: matters are either very nice or very nasty – even the merest hint of good behaviour being sufficient to guarantee the former; cf. [BinO7, 11]. Here, the condition needed, the *Heiberg-Seneta condition*, is a *one-sided* one of ‘liminf liminf’ type, as in the BGT results cited above, [BinG], [Hei], [Sen], or equivalently (below) of ‘limsup limsup’ type.

As usual for proofs, we work with Karamata theory written additively rather than multiplicatively. As in Prop. 1 above, we write

$$G(u) := \lim_{x \rightarrow \infty} F(u + x) - F(x)$$

for the limit on the right where this exists; \mathbb{A}_F the set on which the limit exists; and F^* for the limsup, again as in §1.

Theorem 6 (Quantifier-Weakening Theorem, cf. [BinO15, Th. 6]).

With F^ and \mathbb{A}_F as above, suppose that*

- (i) \mathbb{A}_F is dense in \mathbb{R} ;
- (ii) F^* satisfies the one-sided Heiberg-Seneta boundedness condition

$$\limsup_{u \downarrow 0} F^*(u) \leq 0 \quad (HS)$$

– then $\mathbb{A}_F = \mathbb{R}$ and F^* is linear: $F^*(u) = \lim_{x \rightarrow \infty} [F(u + x) - F(x)] = cu$ for some $c \in \mathbb{R}$, and all $u \in \mathbb{R}$.

Proof of Theorem 6. By Prop. 1, 3 and 6, F^* is a finite, subadditive, right-continuous extension of G . So G is continuous on \mathbb{A}_F , and so linear by Prop. 6: $G(\sigma) = c\sigma$ for some c and all $\sigma \in \mathbb{A}_F$. As \mathbb{A}_F is dense, by Prop. 7, $F^*(u) = cu$ for all u . By Prop. 1, $\mathbb{A}_F = \mathbb{R}$ and $F^*(u) = G(u)$. \square

Cautionary Example. For $F := \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ and $q \in \mathbb{Q}$, one has $F(q+x) - F(x) = 0$ for all x , and so \mathbb{A}_F is dense. Also $F^* = \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$. Indeed, fix $t \notin \mathbb{Q}$; then $F(t+q) - F(q) = 1$ for $q \in \mathbb{Q}$ and $F(t+x) - F(x) \in \{-1, 0\}$ for $x \notin \mathbb{Q}$, so

that $F^*(t) = 1$ and $\mathbb{A}_F = \mathbb{Q}$. As F^* does not satisfy $(HS(F^*))$, Theorem 6 does not apply, and indeed its conclusion fails. Also Theorem 2 (on linearity) – which is at the heart of Theorem 6 (via Theorem 1) – fails, as here the domain of G is \mathbb{Q} and $G = 0$ with a linear extension $S = 0$ to all of \mathbb{R} . This shows the full force of Prop. 1(iv).

Variants of Th. 6 are possible. In the preceding argument Theorem 1 may be replaced by Theorem 1'_a to yield:

Theorem 6' (Quantifier-Weakening Theorem). *Theorem 6 holds with (ii) replaced by:*

(ii_a)' *F^* satisfies the Heiberg-Seneta boundedness condition thinned to a symmetric set Σ that is locally SW, i.e.*

$$\limsup_{u \rightarrow 0, u \in \Sigma} S(u) \leq 0.$$

Likewise, using Theorem 1'_d yields:

Theorem 6'' (Strong Quantifier-Weakening Theorem). *Theorem 6 holds with (ii) replaced by:*

(ii_d)' *F^* is bounded on a subset of \mathbb{A}_F that is shift-compact (e.g. on a set that is locally SW, and so on an open set).*

Proof. Again \mathbb{A}_F is a subgroup and $G = F^*|_{\mathbb{A}_F}$ is additive (Prop. 1); the condition $(\limsup)_0$ follows from the assumption that $SW\text{-}\limsup_M(F^*)$ holds for some M . As F^* is bounded on a shift-compact subset of \mathbb{A}_F , Theorem 1' applies to $S = F^*$. \square

Cautionary Example Again. Recall that for $F := \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ one has $F^* = \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ and $\mathbb{A}_F = \mathbb{Q}$. So here \mathbb{A}_F is dense but not shift-compact, and F^* , though linear on \mathbb{A}_F , does not satisfy (ii).

The density assumption above may be weakened by using Theorem 4:

Theorem 7. *With \mathbb{A}_F as above, suppose that there exists a shift-symmetric, analytic, spanning set $T \subseteq \mathbb{A}_F$ such that $F^*|_T$ is locally bounded above. Then F^* is linear: $F^*(u) = \lim_{x \rightarrow \infty} [F(u+x) - F(x)] = cu$, for some $c \in \mathbb{R}$ and all $u \in \mathbb{R}$.*

Proof. As \mathbb{A}_F is a subgroup and $G = F^*|\mathbb{A}_F$ is additive (by Prop. 1), apply Theorem 4 to $S = F^*$. \square

Remark. The sharpenings of Theorem 6 make use of *relatives* of the Heiberg-Seneta condition. Their formulation draws on shift-compactness (and so sequential) properties of various ‘test sets’, T say. The classical development relies on the classic Steinhaus property, more properly: the Steinhaus-Weil⁴ (interior points) property of test sets (that $T - T$ has interior points): see BGT Th. 1.1.1; we study the links between the Steinhaus-Weil property and shift-compactness elsewhere [BinO17].

The classical *Quantifier Weakening Theorems* of regular variation (BGT §1.4.3 and §3.2.5) are re-stated below as Theorems K and BKdH. There, one needs as side-condition the Heiberg-Seneta condition HS restated multiplicatively here as (\limsup) (or a thinned version of it, as in Theorems 6', 6''). Recall from above the $*$ notation (as in g^*) signifying that \limsup replaces \lim .

Theorem K (cf. BGT: Th. 1.4.3). *Suppose that*

$$\limsup_{\lambda \downarrow 1} g^*(\lambda) \leq 1. \quad (\limsup)$$

Then the following are equivalent:

(i) *there exists $\rho \in \mathbb{R}$ such that*

$$f(\lambda x)/f(x) \rightarrow \lambda^\rho \quad (x \rightarrow \infty)(\forall \lambda > 0);$$

(ii) *$g(\lambda) = \lim_{x \rightarrow \infty} f(\lambda x)/f(x)$ exists, finite for all λ in a non-negligible set;*

(iii) *$g(\lambda)$ exists, finite, for all λ in a dense subset of $(0, \infty)$;*

(iv) *$g(\lambda)$ exists, finite for $\lambda = \lambda_1, \lambda_2$ with $(\log \lambda_1)/\log \lambda_2$ irrational.*

Theorem K is an immediate corollary of Theorem 6, as (\limsup) iff $(HS(F^*))$.

Theorem BKdH (cf. BGT: Th. 3.2.5). *For g with*

$$\lim_{x \rightarrow \infty} g(\lambda x)/g(x) = \lambda^\rho \quad (\lambda > 0),$$

and

$$\limsup_{\lambda \downarrow 1} f^*(\lambda) \leq 0, \quad (\limsup)$$

⁴Here, as with Steinhaus, the context is \mathbb{R} ; Weil’s context is (Haar) measurability in locally compact groups [Wei], cf. [GroE].

the following are equivalent:

- (i) $k(\lambda) := \lim_{x \rightarrow \infty} [f(\lambda x) - f(x)]/g(x)$ exists, finite for all $\lambda > 0$, and $k(\lambda) = c[\lambda^\rho - 1]/\rho$ for some c and all λ on a non-negligible set;
- (ii) $k(\lambda)$ exists, finite for all λ in a non-negligible set;
- (iii) $k(\lambda)$ exists, finite, for all λ in a dense subset of $(0, \infty)$;
- (iv) $k(\lambda)$ exists, finite for $\lambda = \lambda_1, \lambda_2$ with $(\log \lambda_1)/\log \lambda_2$ irrational.

Implicit in the proofs in BGT is the *Goldie functional equation* (GFE) and *Goldie functional inequality* (GFI). This is made explicit in [BinO14]. See (Add_A); cf. BGT, Equation (3.2.7)), and [Ost3] (on the relation between (GFE) and homomorphisms); we refer to these sources for background. In results of this type, the usual Baire/measurable assumptions are conspicuous by their absence. GFE in its simplest form below bears little relation to CFE:

$$\begin{aligned} K(u+v) &= K(u) + e^u K(v) && (u, v \in \mathbb{A}), && (GFE) \\ K(u+v) &\leq K(u) + e^u K(v). && && (GFI) \end{aligned}$$

It is immediate from (GFE) that either K is trivial: $K \equiv 0$, or for some $\rho \neq 0$

$$K(u) = (e^u - 1)/\rho \quad (u \in \mathbb{A}).$$

Only the latter case is of interest here. Define H by

$$H(x) := K(\log x) \quad (x \in \mathbb{E} := \exp \mathbb{A}).$$

Writing $u = \log x$ etc. gives

$$\begin{aligned} K(u+v) &= K(\log(xy)) = K(\log x) + xK(\log y), \\ H(xy) &= H(x) + xH(y). \end{aligned}$$

As $K(\log x) = (x - 1)/\rho$ for some $\rho > 0$, $H : \mathbb{E} \rightarrow \mathbb{R}$ is injective and order-preserving, so that $\mathbb{G} = K[\mathbb{A}]$ is dense, if \mathbb{A} is. Put

$$\eta(y) := 1 + \rho y,$$

so that $H^{-1}(y) = \eta(y)$ for $y \in \mathbb{G}$. Now consider on \mathbb{G} the Popa ‘circle’ operation (Popa [Pop] in 1965, and Javor [Jav] in 1968):

$$x \circ y = x \circ_\rho y := x + \eta(x)y.$$

This is indeed a group operation with neutral element $1_\rho = 0$ and inverse $x_\circ^{-1} = -x/\eta(x)$; for background see [BinO15] and [Ost3]. This group structure allows the Goldie equation to express homomorphy:

$$H(xy) = H(x) \circ H(y) = H(x) + \eta(H(x))H(y) = H(x) + xH(y) \quad (x, y \in \mathbb{E}).$$

Alternatively, again as H is invertible, with $X = H(x)$ etc, the equation may be re-configured to the celebrated *Gotab-Schinzel equation*:

$$\eta(X \circ Y) = \eta(X)\eta(Y) \quad (X, Y \in \mathbb{G}), \quad (GS)$$

introduced for the study of one-parameter subgroups of affine groups, for which see [AczD, Ch. 19] and the more recent [Brz].

Theorems 0, 0', and 0+ are transferable, on the basis of two simple facts stated in Prop. 12 below, to the context of functions $f : \mathbb{R}_+ \rightarrow \mathbb{G}$, with \mathbb{G} equipped with the Popa circle operation and with subadditivity replaced by:

$$f(xy) \leq f(x) \circ_\rho f(y)$$

so as to yield GFI. (This development with its associated side-conditions complements the alternative inequality

$$f(x + f(x)y) \leq f(x)f(y),$$

studied in [Jab2]; cf. the ‘suboperative’ functions of [HilP, 8.9].) Thus the BKdH-RV version comes at little cost, as do the corresponding Quantifier Weakening Theorems (Th. 6, 6', 6'') with linear ρx replaced by the affine $1 + \rho x$. We write

$$\mathbb{G}_+^\rho := \{x : 1 + \rho x > 0\}.$$

Proposition 12. *For $\rho \geq 0$, the set $[0, \infty)$ is a sub-semigroup of \mathbb{G}_+^ρ ; the induced order: $y \leq_\rho x$ iff $x \circ_\rho y^{-1} \in [0, \infty)$ coincides with $y \leq x$. Furthermore, if $c > 0$ and $a < b$, then*

$$a \circ_\rho c \leq b \circ_\rho c;$$

in particular,

$$(a, b) \circ_\rho c = (a \circ_\rho c, b \circ_\rho c),$$

i.e. the Euclidean topology on \mathbb{R}_+ is invariant under positive translation under \circ_ρ .

Likewise, for $\rho > 0$, if $0 < c < d$, and $a < b$ with $a, b \in \mathbb{G}_\rho^+$, then

$$a \circ_\rho c \leq b \circ_\rho d.$$

Proof. For the first assertion observe that

$$0 \leq x - (1 + \rho x)y / (1 + \rho y) \text{ iff } 0 \leq x(1 + \rho y) - (1 + \rho x)y = x - y, \text{ as } 1 + \rho y > 0.$$

The rest is immediate, since, η_ρ is positive on $\mathbb{G}_+^\rho \supseteq \mathbb{R}_+$ and order-preserving for $\rho > 0$. \square

We hope to give a more detailed account of this area elsewhere. Suffice it to say here that the two forms, Karamata and Bojanić-Karamata/de Haan, of regular variation above are related to – and indeed, subsumed within – a third form, *Beurling* slow and regular variation; see [BinO13, 10.3], [BinO15, §7]. The link here is the Popa circle operation above.

8 Complements

8.1 *Sources.* This paper is a sequel to three sources: [BinO11] (on the theorems of Steinhaus and Ostrowski, which underpin everything in this area), [BinO14] (on the *Goldie functional equation* (GFE) and inequality) and [Ost3] (on the relation between the more general *Goldie-Beurling equation* and homomorphisms between Popa groups) which here specializes to (GFE).

8.2 *Frullani integrals.* A classical situation where quantifier weakening is important (which, as it happened, was the original motivation for [BinG, I,II.6], and so for this paper and [BinO11] via BGT) is the theory of *Frullani integrals* ([BinG, II §6], BGT §1.6.4; Berndt [Bern]), important in many areas of analysis and probability. This in turn is a combination of two results from regular variation, the Aljančić-Karamata theorem (a result of *Mercerian* type) and the Characterisation Theorem (BGT §1.4), a central result in the area inseparable from quantifier weakening. One reason why regular variation is so ubiquitous and useful is its relevance to *scaling* [Bin].

8.3 *Shift-compactness and Theorem S.* Evidently any (non-degenerate) interval is shift-compact; more generally, so are non-negligible Baire/measurable sets – this is the *Kestelman-Borwein-Ditor Theorem*, KBD, for which see

[BinO11, Th. 4.2]. Any shift-compact set Σ has the *classic Steinhaus property* (terminology of [BarFN]): 0 is an interior point of $\Sigma - \Sigma$, see [BinO9, Th. 2]. The following combinatorial version of the Steinhaus Subgroup Theorem, Theorem S below, will be seen capable of bearing the burden of the proof of our version Theorem 1'_a above. See [BinO11] for further equivalences in Theorem S (e.g. that S has finite index in \mathbb{R} , and statements involving Ramsey theory).

Theorem S ([BinO11, Th. 6.2]) *For an additive subgroup \mathbb{A} of \mathbb{R} , the following are equivalent:*

- (i) $\mathbb{A} = \mathbb{R}$,
- (ii) \mathbb{A} contains a subset that is locally Steinhaus-Weil (e.g. a non-negligible Baire/measurable set),
- (iii) \mathbb{A} is shift-compact.

8.4 *Bitopological Berz Theorem.* The proof of Theorem 3 in §5 above can be dualized to yield a parallel alternative and new proof for Theorem B. Here, in place of Baire's continuity theorem, a careful use of Lusin's theorem ([Hal, §55]; cf. [BinO8, §2] for a 'near-analogue') demonstrates linearity on a subset $\Sigma \cap \mathbb{R}_+$ covering almost all of \mathbb{R}_+ , and likewise on a subset $\Sigma \cap \mathbb{R}_-$ covering almost all of \mathbb{R}_- ; then Props 10 and 7 above complete the proof. However, an argument proving simultaneously Theorem 3 and Theorem B can be given [BinO16], by appeal to *density-topology* arguments, for which see [BinO10,11], cf. [Wil] and [Ost1].

8.5 *Dependence on axioms of set theory.* For a summary of the background information needed to appreciate the various set-theoretic axioms which implicitly confront analysts we refer to Appendix 1 of the fuller arXiv version of [BinO16]; the earlier article [Wri] of 1977 had a similar motivation. This may be omitted by the expert (or uninterested) reader.

8.6 *Kominek's Theorem.* We include this (discussed in §6) here, as it is an immediate corollary of Prop. 6.

Kominek's Theorem ([Kom], cf. [Jon]). *For additive $A : \mathbb{R} \rightarrow \mathbb{R}$, if $A|T$ is bounded on an analytic spanning set T , then A is continuous.*

Proof. If T is analytic and spans \mathbb{R} then, as $A(T - T)$ is bounded, w.l.o.g. $T = -T$ (otherwise repace T by $T \cup (-T)$), and

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} \bigcup_{\mathbf{q} \in \mathbb{Q}^n} (q_1 T + \dots + q_n T).$$

So there are $n \in \mathbb{N}$ and $m_1, \dots, m_n \in \mathbb{N}$ with $S := m_1T + \dots + m_nT$ of positive measure. So A is bounded on $S + S$ and so on an interval. Now apply Prop. 6 (with $\mathbb{A} = \mathbb{R}$). \square

8.7 *Kingman's Subadditive Ergodic Theorem.* Detailed study of subadditivity is partially motivated by links with the Kingman subadditive ergodic theorem, for which see e.g. [Kin1, 2], Steele [Ste].

8.8 *Sublinearity and risk measures.* An important class of functions with the two properties of subadditivity and positive homogeneity but with a more general domain occurs in mathematical finance – the *coherent risk measures* introduced by Artzner et al. [ArtDEH]; for textbook treatments see [McNFE], [FolS, 4.1]. For the more general domains (and brief commentary on the context) see again [BinO16].

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Postscript. We take pleasure in dedicating this paper to Charles Goldie. The first author is happy to recall that the argument in [BinG] (and later in BGT) that gave rise to this paper was due to him, and was the beginning of their long and fruitful collaboration.

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