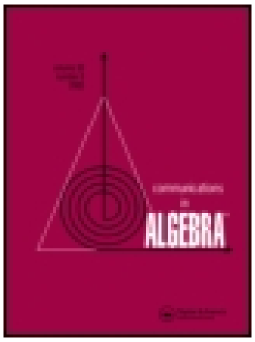

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Special subgroups of regular semigroups

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Abstract

Extending the notions of inverse transversal and associate subgroup, we consider a regular semigroup S with the property that there exists a subsemigroup T which contains, for each $x \in S$, a unique y such that both xy and yx are idempotent. Such a subsemigroup is necessarily a group which we call a *special subgroup*. Here we investigate regular semigroups with this property. In particular, we determine when the subset of perfect elements is a subsemigroup and describe its structure in naturally arising situations.

KEYWORDS: Quasi-ideal; regular semigroup; special subgroup

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1. SPECIAL SUBGROUPS

If S is a regular semigroup then for each $x \in S$ we denote the set of inverses of x by $V(x)$, and the set of associates of x by $A(x) = \{y \in S \mid xyx = x\}$. Well known is the concept of an *inverse transversal* of S, namely an inverse subsemigroup $S^\circ = \{x^\circ \mid x \in S\}$ with the property that $|S^\circ \cap V(x)| = 1$ for every $x \in S$ (1). Corresponding to this is the notion of an *associate subgroup* of S, namely a subgroup $S^\star = \{x^\star \mid x \in S\}$ with the property that $|S^\star \cap A(x)| = 1$ for every $x \in S$ (2). Here we shall be concerned with the subset

$$I(x) = \{y \in S \mid xy, yx \in E(S)\}$$

where $E(S)$ denotes the set of idempotents of S. Clearly, $V(x) \subseteq A(x) \subseteq I(x)$. Moreover, equality holds throughout if and only if S is completely simple (5)(6).

Our purpose here is to investigate the more general notion of a subsemigroup T of S with the property that

$$(\forall x \in S) \qquad |T \cap I(x)| = 1.$$

Given such a subsemigroup T, we define $x^\sim$ for every $x \in S$ by

$$T \cap I(x) = \{x^\sim\}.$$

Thus $xx^\sim, x^\sim x \in E(S)$ for every $x \in S$, and we observe first that every $x \in T$ is such that $x \in T \cap I(x^\sim) = \{x^{\sim\sim}\}$ whence $x = x^{\sim\sim}$ and consequently $T \subseteq S^\sim = \{x^\sim \mid x \in S\}$. The converse

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inclusion being clear from the definition of $x^{\sim}$, it follows that $T = S^{\sim}$. Then $x^{\sim} = x^{\sim\sim\sim}$ for every $x \in S$. Since T is a subsemigroup, for every $x \in T$ we have $x^{\sim}xx^{\sim} \in T \cap I(x)$ whence $x^{\sim}xx^{\sim} = x^{\sim}$ and so $xx^{\sim}x \in V(x^{\sim}) \subseteq I(x^{\sim})$. Since also $xx^{\sim}x \in T$ it follows that every $x \in T$ is such that $xx^{\sim}x = x^{\sim\sim} = x$ and so $S^{\sim} = T$ is regular. Now if y is any inverse of $x^{\sim}$ in $S^{\sim}$ then $y \in S^{\sim} \cap I(x^{\sim}) = \{x^{\sim\sim}\}$ whence $y = x^{\sim\sim}$ and consequently $S^{\sim}$ is an inverse semigroup in which $(x^{\sim})^{-1} = x^{\sim\sim}$. But if $e, f \in E(S^{\sim})$ then $ef = fe \in E(S^{\sim})$ gives $e, f \in S^{\sim} \cap I(ef) = \{(ef)^{\sim}\}$ whence $e = (ef)^{\sim} = f$. Thus we conclude that $S^{\sim}$ is a subgroup of S.

In what follows we shall call such a subgroup $S^{\sim}$ a *special subgroup* of S.

Theorem 1. *Let $S^{\sim}$ be a special subgroup of the regular semigroup S. If ξ is the identity element of $S^{\sim}$ then $S^{\sim} = H_{\xi}$.*

Proof. Since the maximal subgroups of S are precisely the $\mathcal{H}$-classes which contain idempotents, it follows that $S^{\sim} \subseteq H_{\xi}$. To obtain the reverse inclusion, let $x \in H_{\xi}$. Then $xx^{\sim}$, $x^{\sim}x \in E(H_{\xi})$ and so $xx^{\sim} = \xi = x^{\sim}x$, whence $x^{-1} = x^{\sim}$ and $x = (x^{\sim})^{-1} = x^{\sim\sim} \in S^{\sim}$. Thus $S^{\sim} = H_{\xi}$. □

Example 1. Let S be an orthodox completely simple semigroup. As is well-known, we can represent S as the cartesian product semigroup $S = G \times B$ of a group G and a rectangular band B. Choose and fix an element $\alpha \in B$. The $\mathcal{H}$-class of the idempotent $e = (1_G, \alpha)$, namely $H_e = G \times \{\alpha\}$, is a group transversal of S under the definition $(g, x)^{\circ} = (g^{-1}, \alpha)$. Let S^1 be the

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monoid obtained from S by the adjunction of a new identity element 1 and define, for each $t \in S^1$,

$$t^{\sim} = \begin{cases} t^{\circ} & \text{if } t \in S; \\ e & \text{if} t = 1. \end{cases}$$

Then $(S^1)^{\sim} = H_e$. For each $(g, x) \in S$ we have that

$$(h, y) \in H_e \cap I(g, x) \iff h = g^{-1}, \; y = \alpha$$

whence $H_e \cap I(g, x) = \{(g^{-1}, \alpha)\} = \{(g, x)^{\circ}\} = \{(g, x)^{\sim}\}$. Since also $H_e \cap I(1) = H_e \cap E(S^1) = \{e\} = \{1^{\sim}\}$, it follows that H_e is a special subgroup of S^1. Note that since $A(1) = \{1\}$ the semigroup S^1 has no associate subgroups. Indeed, any such subgroup would have to contain 1 and the only subgroup of S^1 which does so is $\{1\}$.

Example 2. Let $k > 1$ be a fixed integer and for each $n \in \mathbb{Z}$ let n_k be the biggest multiple of k that is less than or equal to n. Consider the cartesian product set $S = \mathbb{Z} \times \mathbb{Z}$ equipped with the multiplication

$$(m, n)(p, q) = (\max\{m, p\}, n + q_k).$$

Since $m_k + n_k = (m + n_k)_k = (m_k + n)_k$ it follows that S is a semigroup which is regular; for example, $(m, n)(m, -n_k)(m, n) = (m, n)$. Here $E(S) = \{(m, n) \mid n_k = 0\}$ and

$$I(m, n) = \{(p, q) \mid q_k + n_k = 0\} = \{(p, q) \mid -n_k \leqslant q \leqslant -n_k + k - 1\}.$$

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The subsemigroup $S^{\sim} = \{(0, t_k) \mid t \in \mathbb{Z}\}$ is such that $S^{\sim} \cap I(m,n) = \{(0, -n_k)\}$ and so is a special subgroup. Here also

$$V(m,n) = \{(p,q) \mid p = m,\ q_k + n_k = 0\}; \qquad A(m,n) = \{(p,q) \mid p \leqslant m,\ q_k + n_k = 0\}.$$

The following result, which involves formulae similar to those for inverse transversals, will be used throughout what follows.

Theorem 2. *Let S be a regular semigroup. If $S^{\sim}$ is a special subgroup of S then*

(1) $(\forall x, y \in S) \qquad (xy^{\sim})^{\sim} = y^{\sim\sim}x^{\sim}$ *and* $(x^{\sim}y)^{\sim} = y^{\sim}x^{\sim\sim}$;

(2) $(\forall x, y \in S) \qquad (xy)^{\sim} = (x^{\sim}xy)^{\sim}x^{\sim} = y^{\sim}(xyy^{\sim})^{\sim}$.

Proof.

(1) Let ξ be the identity element of $S^{\sim}$. Then on the one hand $xy^{\sim} \cdot y^{\sim\sim}x^{\sim} = x\xi x^{\sim} = xx^{\sim} \in E(S)$, and on the other,

$$y^{\sim\sim}x^{\sim}xy^{\sim} \cdot y^{\sim\sim}x^{\sim}xy^{\sim} = y^{\sim\sim}x^{\sim}x\xi x^{\sim}xy^{\sim} = y^{\sim\sim}x^{\sim}xy^{\sim}$$

whence also $y^{\sim\sim}x^{\sim} \cdot xy^{\sim} \in E(S)$. Consequently, $y^{\sim\sim}x^{\sim} \in S^{\sim} \cap I(xy^{\sim})$ and hence $(xy^{\sim})^{\sim} = y^{\sim\sim}x^{\sim}$. Similarly, $(x^{\sim}y)^{\sim} = y^{\sim}x^{\sim\sim}$.

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(2) Using (1), we have $(x^{\sim}xy)^{\sim}x^{\sim} = (xy)^{\sim}x^{\sim\sim}x^{\sim} = (xy)^{\sim}\xi = (xy)^{\sim}$, and similarly $y^{\sim}(xyy^{\sim})^{\sim} = y^{\sim}y^{\sim\sim}(xy)^{\sim} = \xi(xy)^{\sim} = (xy)^{\sim}$. □

2. PARTICULAR SUBSETS

In the presence of an inverse transversal S°, or of an associate subgroup $S^{\star}$, Green's relations are nicely describable. Indeed, in those situations they are as follows:

$$
\begin{aligned}
(x,y) \in \mathcal{R} &\iff xx^{\circ} = yy^{\circ} \qquad [\text{resp. } xx^{\star} = yy^{\star}];\\
(x,y) \in \mathcal{L} &\iff x^{\circ}x = y^{\circ}y \qquad [\text{resp. } x^{\star}x = y^{\star}y].
\end{aligned}
$$

This is not so in general for regular semigroups with special subgroups. For instance, in Example 1 we have $ee^{\sim} = e = 11^{\sim}$ but $(e,1) \notin \mathcal{R}$ since $1 \notin eS^1$. So a separate investigation of the sets $J = \{xx^{\sim} \mid x \in S\}$ and $K = \{x^{\sim}x \mid x \in S\}$ is warranted. For this, we note that J and K have the equivalent descriptions

$$J = \{x \in S \mid x = xx^{\sim}\}, \qquad K = \{x \in S \mid x = x^{\sim}x\}.$$

For example, if $x \in J$ then there exists $y \in S$ such that $x = yy^{\sim}$ whence, by Theorem 2(1), $xx^{\sim} = yy^{\sim}(yy^{\sim})^{\sim} = yy^{\sim}y^{\sim\sim}y^{\sim} = yy^{\sim} = x$.

Likewise, we shall consider the subsemigroups

$$S\xi = \{x\xi \mid x \in S\}, \qquad \xi S = \{\xi x \mid x \in S\}.$$

The subsets of idempotents of $S\xi$ and ξS are identified as follows.

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Theorem 3. *$E(S\xi) = J$ and $E(\xi S) = K$.*

Proof. If $j \in J$ then $j = jj^{\sim}$ whence $j = j\xi$ and therefore $J \subseteq E(S\xi)$. Conversely, if $e \in E(S\xi)$ then $e = e\xi$ gives $\xi e\xi e = \xi e$, so that $e\xi, \xi e \in E(S)$ and consequently $\xi \in S^{\sim} \cap I(e) = \{e^{\sim}\}$. Then $e^{\sim} = \xi$ and $e = e\xi = ee^{\sim} \in J$. Thus $E(S\xi) = J$ and dually $E(\xi S) = K$. □

A further subset that is of structural importance is

$$P = \{x \in S \mid x = xx^{\sim}x\}.$$

Concordant with the terminology of (5), this may be called the set of *perfect elements* of S.

By the formulae in Theorem 2, it is readily seen that, equivalently,

$$P = \{xx^{\sim}x \mid x \in S\},$$

with, moreover,

$$(\forall x \in S) \qquad (xx^{\sim}x)^{\sim} = x^{\sim}.$$

Since for every $j \in J$ we have $j = jj^{\sim}j$ we see that $J \subseteq P$, and similarly $K \subseteq P$. Also, for every $x \in S$, we have $x^{\sim}x^{\sim\sim}x^{\sim} = \xi x^{\sim} = x^{\sim}$ and therefore also $S^{\sim} \subseteq P$. Moreover, by the above, $S^{\sim} = P^{\sim}$.

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As seen in Example 1 above, special subgroups are in general distinct from associate subgroups. Precisely when they coincide is determined as follows.

Theorem 4. *A special subgroup $S^{\sim}$ of S is an associate subgroup of S if and only if $P = S$.*

Proof. If $P = S$ then every $x \in S$ is such that $x = xx^{\sim}x$ whence $S^{\sim} \cap A(x) \neq \emptyset$. But if $y \in S^{\sim} \cap A(x)$ then $y \in S^{\sim} \cap I(x) = \{x^{\sim}\}$. Hence $S^{\sim} \cap A(x) = \{x^{\sim}\}$ and so $S^{\sim}$ is an associate subgroup. Conversely, if the special subgroup $S^{\sim}$ is an associate subgroup then for every $x \in S$ we have $x = xx^{\star}x = xx^{\sim}x \in P$ whence $P = S$. □

Precisely when P is a subsemigroup of S is the substance of the following result.

Theorem 5. *The following statements are equivalent:*

(1) *P is a (regular) subsemigroup of S;*

(2) *$KJ \subseteq P$.*

Proof.

(1) ⇒ (2): If P is a subsemigroup of S then, since both J and K are contained in P, it follows that $KJ \subseteq P$.

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(2) $\Rightarrow$ (1): If $x, y \in P$ then since $x^{\sim}xyy^{\sim} \in KJ \subseteq P$ we have, by Theorem 2(2),

$$x^{\sim}xyy^{\sim} = x^{\sim}xyy^{\sim}(x^{\sim}xyy^{\sim})^{\sim}x^{\sim}xyy^{\sim} = x^{\sim}xy(xy)^{\sim}xyy^{\sim}.$$

Pre-multiplying by x and post-multiplying by y, we obtain $xy = xy(xy)^{\sim}xy$ so that $xy \in P$ and therefore P is a subsemigroup which is regular since $S^{\sim} \subseteq P$. □

In the case of an inverse transversal S° the sets which correspond to J and K are denoted by I and Λ. It is natural therefore to consider properties which are analogous to the principal properties listed in (4).

Recalling that $E(S^{\sim}) = \{\xi\}$, we shall say that the special subgroup $S^{\sim}$ is

prime if $KJ = \{\xi\}$	[cf. S° *multiplicative* if $\Lambda\mathrm{I} \subseteq E(S^{\circ})$];
weakly prime if $(KJ)^{\sim} = \{\xi\}$	[cf. S° *weakly multiplicative* if $(\Lambda\mathrm{I})^{\circ} \subseteq E(S^{\circ})$];
a quasi − ideal if $S^{\sim}SS^{\sim} \subseteq S^{\sim}$	[cf. $S^{\circ}SS^{\circ} \subseteq S^{\circ}$ or, equivalently, $\Lambda\mathrm{I} \subseteq S^{\circ}$].

In what follows we shall consider the characteristic properties of each of these types and how they are related. For this purpose, throughout what follows, *S will denote a regular semigroup with a special subgroup $S^{\sim}$ whose identity element is ξ, and the subsets J, K, P are as defined above*.

3. $S^{\sim}$ WEAKLY PRIME

The weakly prime special subgroups have the following characterisations.

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Theorem 6. *The following statements are equivalent:*

(1) $S^{\sim}$ *is weakly prime;*

(2) $(\forall x, y \in S)\ \ (xy)^{\sim} = y^{\sim}x^{\sim}$;

(3) *the mapping* $\zeta : S \to S$ *described by* $\zeta : x \mapsto x^{\sim\sim}$ *is a morphism;*

(4) $\big(\forall x \in \langle E(S)\rangle\big)\ \ x^{\sim} = \xi$;

(5) $KJ \subseteq E(P)$.

Proof.

(1) $\Rightarrow$ (2): If $S^{\sim}$ is weakly prime then, for all $x, y \in S$, we have $(x^{\sim}xyy^{\sim})^{\sim} \in (KJ)^{\sim} = \{\xi\}$. It follows by Theorem 2(2) that $(xy)^{\sim} = y^{\sim}(x^{\sim}xyy^{\sim})^{\sim}x^{\sim} = y^{\sim}\xi x^{\sim} = y^{\sim}x^{\sim}$.

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (2): By (3) and Theorem 2(1), $(xy)^{\sim} = (xy)^{\sim\sim\sim} = (x^{\sim\sim}y^{\sim\sim})^{\sim} = y^{\sim}x^{\sim}$.

(2) $\Rightarrow$ (4): If (2) holds and $e \in E(S)$ then $e^{\sim} = (ee)^{\sim} = e^{\sim}e^{\sim}$ gives $e^{\sim} \in E(S^{\sim})$ and therefore $e^{\sim} = \xi$. An inductive argument using (2) now gives $x^{\sim} = \xi$ for every $x \in \langle E(S)\rangle$.

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(4) $\Rightarrow$ (5): If (4) holds then for all $k \in K$ and all $j \in J$ we have $(kj)^{\sim} = \xi$. Consequently, by Theorem 3,

$$kj = kj\xi = kj(kj)^{\sim} \in J \subseteq E(P)$$

and (5) follows.

(5) $\Rightarrow$ (1): If (5) holds then $kj\xi = kj \in E(P)$ and likewise $\xi kj \in E(P)$ whence it follows that $\xi \in S^{\sim} \cap I(kj) = \{(kj)^{\sim}\}$. Thus $(KJ)^{\sim} = \{\xi\}$ and so $S^{\sim}$ is weakly prime. □

Corollary. *If $S^{\sim}$ is weakly prime then J, K are sub-bands and P is a regular subsemigroup. Also, $S^{\sim}$ is an associate subgroup of P and its identity element ξ is a medial idempotent of P.*

Proof. By Theorem 2, for every $x \in J$ we have $x^{\sim} = (xx^{\sim})^{\sim} = x^{\sim\sim}x^{\sim} = \xi$ and so $x = xx^{\sim} = x\xi$. Thus, if $x, y \in J$ then, using Theorem 6(2), $xy = xy\xi = xyy^{\sim}x^{\sim} = xy(xy)^{\sim} \in J$ whence J, and likewise K, is a sub-band of S. That P is a subsemigroup follows from Theorem 6(5) and Theorem 5. Now for every $x \in S$ we have that $x^{\sim} = (xx^{\sim}x)^{\sim}$. That $S^{\sim} = P^{\sim}$ is then an associate subgroup of P follows from Theorem 4 applied to P. Finally, by Theorem 6(4), for every $x \in \langle E(P)\rangle$ we have $x = xx^{\sim}x = x\xi x$ and so ξ is a medial idempotent of P (3). □

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A structure theorem for regular semigroups with an associate subgroup whose identity element is a medial idempotent (and therefore that of P when $S^{\sim}$ is weakly prime) was established in (2); see also (8).

Example 3. In Example 2, $P = \{(m,n) \mid m \geqslant 0\}$, $J = \{(m,n) \mid m \geqslant 0,\ n_k = 0\}$ and $K = \{(m,0) \mid m \geqslant 0\}$. Then $KJ = K$ and, since $K^{\sim} = \{(0,0)\}$, it follows that $S^{\sim}$ is a weakly prime special subgroup. Here $S^{\sim}$ is neither prime nor a quasi-ideal.

Example 4. If L is a semilattice and G is a group, consider the inverse semigroup $S = L \times G$. Here $E(S) = L \times \{1_G\}$ and, for every $(x,g) \in S$,

$$I(x,g) = L \times \{g^{-1}\}.$$

For every idempotent $\xi = (e, 1_G)$ of S, the group $H_\xi = \{e\} \times G$ is a special subgroup of S with $(x,g)^{\sim} = (e, g^{-1})$ for every $(x,g) \in S$. With respect to this,

$$P = \{(x,g) \mid x \leqslant e,\ g \in G\}, \qquad J = K = \{(x,1_G) \mid x \leqslant e\} = E(P).$$

It follows by Theorem 6 that H_ξ is a weakly prime special subgroup of S. It is prime or a quasi-ideal if and only if L has a bottom element 0 and $e = 0$.

Example 5. Consider the semidirect product semigroup $Q \times_\zeta G$ of a band Q with an identity element ξ and a group G. With respect to a given morphism $\zeta : G \to \operatorname{Aut} Q$ described by $\zeta : g \mapsto \zeta_g$, this consists of the set $Q \times G$ together with the multiplication defined by $(x,g)(y,h) =$

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$(x\zeta_g(y),\ gh)$. That $Q \times_\zeta G$ is regular follows from the observation that

$$(x,g)(\xi,g^{-1})(x,g) = \big(x\zeta_g(\xi)\zeta_{1_G}(x),\ gg^{-1}g\big) = (x\xi x,\ g) = (x,g).$$

Since $(x,g)(x,g) = (x\zeta_g(x), g^2)$ it follows that $E(Q \times_\zeta G) = \{(x, 1_G) \mid x \in Q\}$. Consider now the subset $H = \{\xi\} \times G$. This is clearly a subgroup of $Q \times_\zeta G$. Moreover,

$$(\xi,h) \in I(x,g) \iff \big(\xi\zeta_h(x), hg\big), \big(x\zeta_g(\xi), gh\big) \in E(Q \times_\zeta G) \iff h = g^{-1}$$

so that $H \cap I(x,g) = \{(\xi, g^{-1})\}$. Consequently, H is a special subgroup of $Q \times_\zeta G$ under $(x,g)^\sim = (\xi, g^{-1})$, and H is isomorphic to G. It follows by Theorem 6(2) that H is weakly prime. It is readily seen that if $Q \neq \{\xi\}$ then H is neither prime nor a quasi-ideal.

The situation described in Example 5 is highlighted in the following result.

Theorem 7. *If $S^\sim$ is weakly prime then the subsemigroup $\xi P\xi$ of P is isomorphic to a semidirect product of a band with an identity and a group. More precisely, if $\Delta = \{x \in P \mid x^\sim = \xi\}$ then $\xi\Delta\xi$ is a sub-band of P with identity ξ. For each $g \in S^\sim$ the mapping $\zeta_g : x \mapsto gxg^\sim$ is an automorphism of $\xi\Delta\xi$, and $\zeta : g \mapsto \zeta_g$ is a morphism. Finally,*

$$\xi P\xi \simeq \xi\Delta\xi \times_\zeta S^\sim$$

under the mapping $\vartheta : x \mapsto (xx^\sim, x^{\sim\sim})$.

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Proof. If $x, y \in \Delta$ then, by Theorem 6(2), $(xy)^{\sim} = y^{\sim}x^{\sim} = \xi$ and so Δ is a subsemigroup of P. Now, for every $x \in \Delta$, $x\xi = xx^{\sim} \in J$ whence it follows that $\xi x\xi \in J \subseteq E(P)$. Also, if $x, y \in \Delta$ then $x\xi y \in \Delta$ gives $\xi x\xi \cdot \xi y\xi \in \xi\Delta\xi$ and therefore $\xi\Delta\xi$ is a sub-band of P with identity element ξ. Moreover, for every $g \in S^{\sim}$ and every $x \in \xi\Delta\xi$, we have $gxg^{\sim} \in \xi\Delta\xi$.

For each $g \in S^{\sim}$ consider the mapping $\zeta_g : \xi\Delta\xi \to \xi\Delta\xi$ given by $\zeta_g(x) = gxg^{\sim}$. We have $\zeta_g(\xi) = g\xi g^{\sim} = gg^{\sim} = \xi$. Moreover, if $x, y \in \xi\Delta\xi$ then

$$\zeta_g(xy) = gxyg^{\sim} = gx\xi yg^{\sim} = gxg^{\sim}gyg^{\sim} = \zeta_g(x)\zeta_g(y)$$

and so $\zeta_g \in \operatorname{End}\xi\Delta\xi$. Since $g^{\sim}xg \in \xi\Delta\xi$ with $\zeta_g(g^{\sim}xg) = gg^{\sim}xgg^{\sim} = \xi x\xi = x$, it follows that ζ_g is surjective. Moreover, if $\zeta_g(x) = \zeta_g(y)$ then, applying $\zeta_{g^{-1}}$ to this, we obtain $x = \xi x\xi = \xi y\xi = y$. Consequently, each $\zeta_g \in \operatorname{Aut}\xi\Delta\xi$.

Since $\zeta_g[\zeta_h(x)] = ghxh^{\sim}g^{\sim} = ghx(gh)^{\sim} = \zeta_{gh}(x)$, the mapping $\zeta : S^{\sim} \to \operatorname{Aut}\xi\Delta\xi$ given by $\zeta(g) = \zeta_g$ is a morphism.

It follows from the above that we can construct the semidirect product $\xi\Delta\xi \times_\zeta S^{\sim}$.

Since $xx^{\sim} \in \Delta$ by Theorem 2, it follows that $xx^{\sim} \in \xi\Delta\xi$ for every $x \in \xi P\xi$. Consider therefore the mapping $\vartheta : \xi P\xi \to \xi\Delta\xi \times_\zeta S^{\sim}$ given by

$$\vartheta(x) = (xx^{\sim}, x^{\sim\sim}).$$

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That ϑ is a morphism follows from the observation that, for $x, y \in \xi P\xi$,

$$\begin{aligned}
\vartheta(x)\vartheta(y) &= (xx^{\sim}, x^{\sim\sim})(yy^{\sim}, y^{\sim\sim}) \\
&= (xx^{\sim}\zeta_{x^{\sim\sim}}(yy^{\sim}),\ x^{\sim\sim}y^{\sim\sim}) \\
&= (xx^{\sim} \cdot x^{\sim\sim}yy^{\sim}x^{\sim},\ x^{\sim\sim}y^{\sim\sim}) \\
&= (x\xi yy^{\sim}x^{\sim},\ (xy)^{\sim\sim}) \\
&= (xy(xy)^{\sim},\ (xy)^{\sim\sim}) \\
&= \vartheta(xy).
\end{aligned}$$

That ϑ is injective follows from the fact that if $\vartheta(x) = \vartheta(y)$ then $xx^{\sim} = yy^{\sim}$ and $x^{\sim\sim} = y^{\sim\sim}$ whence $x = \xi x\xi = \xi xx^{\sim}x^{\sim\sim} = \xi yy^{\sim}y^{\sim\sim} = \xi y\xi = y$. Finally, given $y \in \xi\Delta\xi$ and $g \in S^{\sim}$ we have $yg \in \xi P\xi$ with

$$\vartheta(yg) = (yg(yg)^{\sim}, (yg)^{\sim\sim}) = (ygg^{\sim}y^{\sim}, y^{\sim\sim}g^{\sim\sim}) = (y\xi, \xi g) = (y, g)$$

and so ϑ is also surjective. Consequently, $\xi P\xi \simeq \xi\Delta\xi \times_{\zeta} S^{\sim}$. □

4. $S^{\sim}$ A QUASI-IDEAL

The quasi-ideal special subgroups have the following characterisations.

Theorem 8. *The following statements are equivalent:*

(1) *$S^{\sim}$ is a quasi-ideal of S;*

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(2) $KJ \subseteq S^{\sim}$;

(3) $KJ = [E(S)]^{\sim}$;

(4) $\xi P \xi = S^{\sim}$;

(5) $(\forall x \in S)\ \ x^{\sim} = x^{\sim}xx^{\sim}$;

(6) $(\forall x \in S)\ \ x^{\sim\sim} = \xi x \xi$.

Proof. We establish (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): If $S^{\sim}$ is a quasi-ideal then for all $x, y \in S$ we have $x^{\sim}xyy^{\sim} \in S^{\sim}SS^{\sim} \subseteq S^{\sim}$ whence $KJ \subseteq S^{\sim}$.

(2) $\Rightarrow$ (3): If (2) holds then for all $j \in J$ and $k \in K$ we have that $kj = (kj)^{\sim\sim} \in S^{\sim} \subseteq P$, whence it follows that $(kj)^{\sim} \in V(kj)$.

Given $j \in J$ and $k \in K$, recall (7) that the *sandwich set* $S(k, j)$ is given by

$$S(k, j) = \{g \in E(S) \mid g = jg = gk,\ kgj = kj\} \ = j\,V(kj)\,k.$$

Consider now the element $g = j(kj)^{\sim}k$ which, by the above, belongs to $S(k, j)$. Since $kjg = kg \in E(S)$ and $gkj = gj \in E(S)$ we have that $kj \in S^{\sim} \cap I(g)$ whence $kj = g^{\sim}$ and consequently $KJ \subseteq [E(S)]^{\sim}$. Conversely, for every $e \in E(S)$ we have $e^{\sim}ee^{\sim} = e^{\sim}e \cdot ee^{\sim} \in KJ \subseteq S^{\sim}$ whence, using

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Theorem 2(1), $e^{\sim} = e^{\sim\sim\sim} = (e^{\sim\sim}e^{\sim}e^{\sim\sim})^{\sim} = (e^{\sim}ee^{\sim})^{\sim\sim} = e^{\sim}ee^{\sim} \in KJ$. Thus $[E(S)]^{\sim} \subseteq KJ$ and (3) follows.

(3) $\Rightarrow$ (1): If (3) holds then $x^{\sim}xyy^{\sim} \in S^{\sim}$ whence, taking $x = \xi$, we obtain $\xi yy^{\sim} \in S^{\sim}$. It follows that $\xi y\xi = \xi yy^{\sim}y^{\sim\sim} \in S^{\sim}$ whence $x^{\sim}yz^{\sim} = x^{\sim}\xi y\xi z^{\sim} \in S^{\sim}$ and consequently $S^{\sim}SS^{\sim} \subseteq S^{\sim}$.

(1) $\Rightarrow$ (4): If (1) holds then $S^{\sim} = \xi S^{\sim}\xi \subseteq \xi S\xi \subseteq S^{\sim}SS^{\sim} \subseteq S^{\sim}$. Thus $\xi S\xi = S^{\sim} \subseteq P$ whence $\xi P\xi = \xi S\xi$ and (4) follows..

(4) $\Rightarrow$ (5): If (4) holds then for every $p \in P$ we have, by Theorem 2, $\xi p\xi = (\xi p\xi)^{\sim\sim} = \xi p^{\sim\sim}\xi = p^{\sim\sim}$ whence $p^{\sim}pp^{\sim} = p^{\sim}p^{\sim\sim}p^{\sim} = p^{\sim}$. Consequently,

$$(\forall x \in S) \qquad (xx^{\sim}x)^{\sim}xx^{\sim}x\,(xx^{\sim}x)^{\sim} = (xx^{\sim}x)^{\sim}$$

which reduces to $x^{\sim}xx^{\sim} = x^{\sim}$ which is (5).

(5) $\Rightarrow$ (6): If (5) holds then on pre- and post-multiplying by $x^{\sim\sim}$ we obtain (6).

(6) $\Rightarrow$ (1): If (6) holds then for all $x, y, z \in S$, $x^{\sim}yz^{\sim} = x^{\sim}\xi y\xi z^{\sim} = x^{\sim}y^{\sim\sim}z^{\sim} \in S^{\sim}$ whence $S^{\sim}SS^{\sim} \subseteq S^{\sim}$ and we have (1). □

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Example 6. Consider the following sets of real 2×2 matrices:

$$A = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix}, \begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix}, \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x \neq 0 \right\}; \qquad B = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \mid x \neq 0 \right\}.$$

Let $S = A \cup B$. Then S is a regular monoid with

$$E(S) = \left\{ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

For each $X \in S$ define $X^{\sim} = \begin{bmatrix} x^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Then $S^{\sim} = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x \neq 0 \right\}$ is a special subgroup. Here

$$K = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}; \qquad J = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\},$$

and consequently

$$KJ = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} = [E(S)]^{\sim}.$$

Thus, by Theorem 8, the special subgroup $S^{\sim}$ is a quasi-ideal of S.

Clearly, $S^{\sim}$ is neither prime nor weakly prime.

When $S^{\sim}$ is a quasi-ideal, we have the following useful identification of the subsemigroups ξS and $S\xi$ as, respectively, the $\mathcal{R}$-class and the $\mathcal{L}$-class of ξ.

Theorem 9. *If $S^{\sim}$ is a quasi-ideal then*

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(1) $\xi S = R_\xi = \{x \in S \mid x\,\mathcal{R}\,\xi\}, \qquad S\xi = L_\xi = \{x \in S \mid x\,\mathcal{L}\,\xi\}$;

(2) $J = E(L_\xi)$ *and is a left zero semigroup;*

(3) $K = E(R_\xi)$ *and is a right zero semigroup;*

(4) $J \cap K = \{\xi\}$.

Proof.

(1) If $x\,\mathcal{L}\,\xi$ then $Sx = S\xi$ whence there exists $y \in S$ such that $x = y\xi$. Then $x = x\xi \in S\xi$. Conversely, if $x \in S\xi$ then $x = x\xi$ gives, on the one hand $Sx \subseteq S\xi$, and on the other by Theorem 8(5), $x^{\sim}x = x^{\sim}x\xi = x^{\sim}\xi x\xi = x^{\sim}x^{\sim\sim} = \xi$, whence $S\xi \subseteq Sx$. The resulting equality now gives $x\,\mathcal{L}\,\xi$. Thus $S\xi = L_\xi$ and similarly $\xi S = R_\xi$.

(2) and (3) follow immediately from (1) and Theorem 3.

As for (4), it follows from (2) and (3) that

$J \cap K = E(L_\xi) \cap E(R_\xi) = E(L_\xi \cap R_\xi) = E(H_\xi) = \{\xi\}$.

□

Theorem 10. *If $S^{\sim}$ is a quasi-ideal then P is an ideal of S, and $S^{\sim}$ is a group transversal of P.*

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Proof. If $S^\sim$ is a quasi-ideal then, from $S^\sim \subseteq P$ and Theorem 8(2), we obtain $KJ \subseteq P$. It follows by Theorem 5 that P is a regular subsemigroup of S. Moreover, if $p \in P$ and $x \in S$ then, by Theorem 9(2),

$$px(px)^\sim px = pp^\sim px(px)^\sim px = pp^\sim px = px$$

whence $PS \subseteq P$. Similarly, by Theorem 9(3), $SP \subseteq P$ and consequently P is an ideal of S.

Now since Theorem 8(5) holds, if $x \in P$ then $x^\sim \in V(x)$ so $S^\sim \cap V(x) \neq \emptyset$. But if $y \in S^\sim \cap V(x)$ then $y \in S^\sim \cap I(x) = \{x^\sim\}$. Hence $|S^\sim \cap V(x)| = 1$ and so $S^\sim$ is a group transversal of P with $x^\circ = x^\sim$. □

Corollary. *If $S^\sim$ is a quasi-ideal then P is completely simple.*

Proof. Since $S^\sim$ is a group transversal of P, this follows from a basic theorem of Saito (9); see also (1, Theorem 4.3). □

When $S^\sim$ is a quasi-ideal, the structure of P can be described in terms of $S\xi = L_\xi$ and $\xi S = R_\xi$. For this purpose, consider the *spined product set*

$$S\xi \,|\!\times\!|\, \xi S = \{(x, a) \in S\xi \times \xi S \mid x^\sim = a^\sim\}.$$

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For $x, y \in S\xi$ and $a, b \in \xi S$, it follows by Theorem 2 and Theorem 8(5) that

$$(xx^{\sim}ay)^{\sim} = (x^{\sim}xx^{\sim}ay)^{\sim}x^{\sim} = (x^{\sim}ay)^{\sim}x^{\sim} = (ay)^{\sim}x^{\sim\sim}x^{\sim} = (ay)^{\sim}.$$

Similarly, $(ayb^{\sim}b)^{\sim} = (ay)^{\sim}$. We can therefore define a law of composition on $S\xi \,|\!\times\!|\, \xi S$ by the prescription

$$(x, a)(y, b) = (xx^{\sim}ay,\ ayb^{\sim}b).$$

Then

$$\begin{aligned}
[(x, a)(y, b)](z, c) &= (xx^{\sim}ay, ayb^{\sim}b)(z, c) \\
&= (xx^{\sim}ay(xx^{\sim}ay)^{\sim}ayb^{\sim}bz,\ ayb^{\sim}bzc^{\sim}c) \\
&= (xx^{\sim}ay(x^{\sim}xx^{\sim}ay)^{\sim}x^{\sim}ayb^{\sim}bz,\ ayb^{\sim}bzc^{\sim}c) && \text{by Theorem 2} \\
&= (xx^{\sim}ay \cdot (x^{\sim}ay)^{\sim}x^{\sim}ay \cdot b^{\sim}b \cdot z,\ ayb^{\sim}bzc^{\sim}c) && \text{by Theorem 8(5)} \\
&= (xx^{\sim}ayb^{\sim}bz,\ ayb^{\sim}bzc^{\sim}c); && \text{by Theorem 9(3)} \\
(x, a)[(y, b)(z, c)] &= (x, a)(yy^{\sim}bz, bzc^{\sim}c) \\
&= (xx^{\sim}ayy^{\sim}bz,\ ayy^{\sim}bz(bzc^{\sim}c)^{\sim}bzc^{\sim}c) \\
&= (xx^{\sim}ayy^{\sim}bz,\ ayy^{\sim}bzc^{\sim}(bzc^{\sim}cc^{\sim})^{\sim}bzc^{\sim}c) && \text{by Theorem 2} \\
&= (xx^{\sim}ayy^{\sim}bz,\ a \cdot yy^{\sim} \cdot bzc^{\sim}(bzc^{\sim})^{\sim} \cdot bzc^{\sim}c) && \text{by Theorem 8(5)} \\
&= (xx^{\sim}ayy^{\sim}bz,\ ayy^{\sim}bzc^{\sim}c). && \text{by Theorem 9(2)}
\end{aligned}$$

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Since $y^{\sim} = b^{\sim}$, these products are equal and therefore $S\xi \,|{\times}|\, \xi S$ is a semigroup which we now show is isomorphic to P.

Theorem 11. *If $S^{\sim}$ is a quasi-ideal then $P = JS^{\sim}K$ and $P \simeq S\xi \,|{\times}|\, \xi S = L_{\xi} \,|{\times}|\, R_{\xi}$.*

Proof. For every $x \in P$ we have $x = xx^{\sim}x = xx^{\sim}x^{\sim\sim}x^{\sim}x \in JS^{\sim}K$ so that $P \subseteq JS^{\sim}K$. The reverse inclusion is immediate from the fact that $J, S, K \subseteq P$ and P is a subsemigroup.

Consider now the mapping $\vartheta : P \to S\xi \,|{\times}|\, \xi S$ given by $\vartheta(p) = (p\xi, \xi p)$.

(*a*) *ϑ is injective.*

If $\vartheta(p) = \vartheta(q)$ then, by Theorem 2(1),

$p = pp^{\sim}p = p\xi(p\xi)^{\sim}\xi p = q\xi(q\xi)^{\sim}\xi q = qq^{\sim}q = q.$

(*b*) *ϑ is surjective*. Let $(p, q) \in S\xi \,|{\times}|\, \xi S$. Then $p^{\sim} = q^{\sim}$ and, by Theorem 8(5),

$\vartheta(pp^{\sim}q) = (pp^{\sim}q\xi, \xi pp^{\sim}q) = (pp^{\sim}q^{\sim\sim}, p^{\sim\sim}p^{\sim}q) = (p\xi, \xi q) = (p, q).$

(*c*) *ϑ is a morphism.*

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In fact,

$$\begin{aligned}\vartheta(p)\vartheta(q) = (p\xi, \xi p)(q\xi, \xi q) &= \big(p\xi(p\xi)^{\sim}\xi pq\xi,\ \xi pq\xi(\xi q)^{\sim}\xi q\big)\\ &= (pp^{\sim}pq\xi, \xi pqq^{\sim}q)\\ &= (pq\xi, \xi pq)\\ &= \vartheta(pq).\end{aligned}$$

It follows by (a), (b), (c) that ϑ is a semigroup isomorphism. □

Example 7. In Example 6 we have $\xi = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and

$$S\xi = \left\{\begin{bmatrix}x & 0\\ x & 0\end{bmatrix}, \begin{bmatrix}x & 0\\ 0 & 0\end{bmatrix} \mid x \neq 0\right\}, \qquad \xi S = \left\{\begin{bmatrix}x & x\\ 0 & 0\end{bmatrix}, \begin{bmatrix}x & 0\\ 0 & 0\end{bmatrix} \mid x \neq 0\right\}.$$

In this example, as can readily be verified, P coincides with the subsemigroup A and Theorems 10 and 11 apply.

5. $S^{\sim}$ PRIME

Finally, we consider the special subgroups that are prime. In relation to the previous properties, these have the following characterisation.

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Theorem 12. *The following statements are equivalent:*

(1) $S^{\sim}$ *is prime;*

(2) $S^{\sim}$ *is weakly prime and a quasi-ideal of S;*

(3) $S^{\sim}$ *is a quasi-ideal of S and* ξ *is a middle unit of P.*

Proof.

(1) $\Rightarrow$ (2): If (1) holds then $KJ = \{\xi\}$ and consequently, by Theorem 6(5) and Theorem 8(2), $S^{\sim}$ is both weakly prime and a quasi-ideal.

(2) $\Rightarrow$ (3): If $S^{\sim}$ is weakly prime and a quasi-ideal then, by Theorem 8(5) and Theorem 6(4), for all $x, y \in P$,

$$xy = x\xi x^{\sim}xyy^{\sim}\xi y = x(x^{\sim}xyy^{\sim})^{\sim\sim}y = x\xi y$$

whence ξ is a middle unit of P.

(3) $\Rightarrow$ (1): If $S^{\sim}$ is a quasi-ideal and ξ is a middle unit of P then, by Theorem 9(2,3),

$$x^{\sim}xyy^{\sim} = x^{\sim}x\xi yy^{\sim} = x^{\sim}x\xi = \xi.$$

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Consequently $KJ = \{\xi\}$ and so $S^{\sim}$ is prime. □

Theorem 13. *If $S^{\sim}$ is prime then*

(1)$JK = E(P) = \{x \in P \mid x^{\sim} = \xi\}$;

(2)P is orthodox.

Proof.

(1) If $x = jk \in JK$ then, since $J, K \subseteq P$ and, by Theorem 12 and the Corollary to Theorem 6, P is a subsemigroup, we see that $x \in P$. Moreover, by Theorem 9, $x\xi = jk\xi = j\xi = j$ and $\xi x = \xi jk = \xi k = k$. Consequently, since ξ is a middle unit of P by Theorem 12, $x = jk = x\xi\xi x = x^2$. Thus $JK \subseteq E(P)$.

Conversely, if $e \in E(P)$ then, by Theorem 6(4) and Theorem 9,

$$e = ee^{\sim}e = e\xi e = e\xi\xi e \in JK$$

whence $E(P) \subseteq JK$ and we have equality.

If now $x \in P$ with $x^{\sim} = \xi$ then $x = xx^{\sim}x = x\xi x = x^2$, so $\{x \in P \mid x^{\sim} = \xi\} \subseteq E(P)$. The reverse inclusion follows by Theorem 5(4).

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(2) If $e, f \in E(P)$ then, by (1) and the fact that ξ is a middle unit of P,

$ef \in JKJK = J\xi K = JK = E(P)$

and so P is orthodox. □

If $S^{\sim}$ is prime then, by the Corollary to Theorem 10, P is a completely simple subsemigroup which, by Theorem 13, is orthodox. In this case, P can therefore be expressed as the cartesian product of a group and a rectangular band. The identification of these is the substance of the following.

Theorem 14. *If $S^{\sim}$ is prime then $J \times K$ is a rectangular band and $P \simeq J \times S^{\sim} \times K$.*

Proof. That $J \times K$ is a rectangular band is immediate from Theorem 9(2,3).

Consider the mapping $\vartheta : P \to J \times S^{\sim} \times K$ given by $\vartheta(x) = (xx^{\sim}, x^{\sim\sim}, x^{\sim}x)$.

Since every $x \in P$ is such that $x = xx^{\sim}x = xx^{\sim} \cdot x^{\sim\sim} \cdot x^{\sim}x$, it is clear that ϑ is injective.

To see that ϑ is also surjective, let $(j, g, k) \in J \times S^{\sim} \times K$ and consider the element jgk which, by Theorem 11, belongs to P. By Theorem 9(2), $jgk(jgk)^{\sim} = jj^{\sim} \cdot jgk(jgk)^{\sim} = jj^{\sim} = j$, and likewise, by Theorem 8(3), $(jgk)^{\sim}jgk = k$. Since, by Theorem 5, $(jgk)^{\sim\sim} = g^{\sim\sim} = g$ it follows that $\vartheta(jgk) = (j, g, k)$ and hence ϑ is surjective.

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Furthermore, for all $x, y \in P$, it follows by Theorem 9 and Theorem 6 that

$$\begin{aligned}
\vartheta(x)\vartheta(y) &= (xx^{\sim},\ x^{\sim\sim},\ x^{\sim}x)(yy^{\sim},\ y^{\sim\sim},\ y^{\sim}y) \\
&= (xx^{\sim}yy^{\sim},\ x^{\sim\sim}y^{\sim\sim},\ x^{\sim}xy^{\sim}y) \\
&= (xx^{\sim},\ (xy)^{\sim\sim},\ y^{\sim}y) \\
&= (xx^{\sim}\cdot xy(xy)^{\sim},\ (xy)^{\sim\sim},\ (xy)^{\sim}xy\cdot y^{\sim}y) \\
&= (xy(xy)^{\sim},\ (xy)^{\sim\sim},\ (xy)^{\sim}xy) \\
&= \vartheta(xy).
\end{aligned}$$

Consequently, ϑ is a semigroup isomorphism. □

Example 8. Consider the subsemigroup $T = -\mathbb{N} \times \mathbb{Z}$ of the semigroup S in Example 2. Here $T^{\sim} = \{(0, n_k) \mid n \in \mathbb{Z}\}$ with $J = \{(0, n) \mid n_k = 0\}$ and $K = \{(0, 0)\}$. Then $KJ = \{(0, 0)\}$ and so $T^{\sim}$ is prime. Here $P = \{(0, n) \mid n \in \mathbb{Z}\}$ and, since K is trivial, $P \simeq J \times T^{\sim}$.

Example 9. Similar to Example 4, let $S = L \times G$ where L is a left zero semigroup and G is a group. Here $E(S) = L \times \{1_G\}$ and $I(x, g) = L \times \{g^{-1}\}$. For every idempotent $\xi = (e, 1_G)$ the group $H_\xi = \{e\} \times G$ is a special subgroup with $(x, g)^{\sim} = (e, g^{-1})$. Here $(x, g)(x, g)^{\sim} = (x, 1_G)$ and $(x, g)^{\sim}(x, g) = (e, 1_G)$ so that $J = E(S)$ and $K = \{\xi\}$. Then $KJ = \{\xi\}$ and so every H_ξ is prime.

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