CORPORATE VALUATION AND OPTIMAL OPERATION UNDER LIQUIDITY CONSTRAINTS

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We investigate the impact of cash reserves upon the optimal behaviour of a modelled firm that has uncertain future revenues. To achieve this, we build up a corporate financing model of a firm from a Real Options foundation, with the option to close as a core business decision maintained throughout. We model the firm by employing an optimal stochastic control mathematical approach, which is based upon a partial differential equations perspective. In so doing, we are able to assess the incremental impacts upon the optimal operation of the cash constrained firm, by sequentially including: an optimal dividend distribution; optimal equity financing; and optimal debt financing (conducted in a novel equilibrium setting between firm and creditor). We present efficient numerical schemes to solve these models, which are generally built from the Projected Successive Over Relaxation (PSOR) method, and the Semi-Lagrangian approach. Using these numerical tools, and our gained economic insights, we then allow the firm the option to also expand the operation, so they may also take advantage of favourable economic conditions.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

Cash is the most liquid component in all types of corporate assets. One the one hand, it is very important for a company to hold a certain amount of cash both to maintain daily operation and to support long-term investments. On the other hand holding too much cash is inefficient from the economics perspective, as the return on the cash within the firm is low. If the firm's cash holdings are relatively small, and the firm is restricted from accessing external capital, we refer to this as a cash constrained firm (or financially constrained firm). Motivated by this, the overriding principle of this thesis is to determine the optimal amount of cash a firm should hold (or raise) in order to operate their firm in a value maximising fashion.

Many empirical studies have presented strong connections between cash holdings and various corporate decisions, but they have shown inconsistent qualitative conclusions on the effects of holding large amounts of cash: One branch concludes that holding large amount of cash improves the investment performance and reduces the credit risk (see, e.g., Denis and Sibilkov, 2009; Brown and Petersen, 2011; Harford et al., 2012; Acharya et al., 2012), whilst another branch provides evidence that plenty of cash assets would lead to problems of over-investment and agency conflicts (see, e.g., Ozkan and Ozkan, 2004; Dittmar and Mahrt-Smith, 2007; Oswald and Young, 2008; Nikolov and Whited, 2014). Since these empirical papers cannot precisely answer the question of what is the optimal amount of cash to hold, we believe that an efficient theoretical model is needed to help people quantitatively address this problem, and that this model will allow us to test the interactions between cash holdings and various corporate decisions, such as abandonment decisions, cash management decisions, financing decisions and investment decisions. We will concentrate our research on a cash constrained firm that has limited cash assets and wishes to fully exercise the operational flexibility it has in place.

It is acknowledged that cash holdings are directly or indirectly related to corporate profitability, dividend policy, financing flexibility and investment opportunities. Representative studies, such as Bond and Meghir (1994), and Opler et al. (1999), provide evidence that the ability to raise internal funds is strongly correlated with investment decisions. Denis and Sibilkov (2009) indicate that larger initial cash holdings are positively correlated to higher levels of investment, and it particularly helps those firms take value-increasing projects, thus releasing the financing constraints, and Fresard (2010) showed that cash unconstrained firms had a greater market-share than their counterparts of constrained firms, thus cash policy should be modelled with a strategic dimension. The literature suggests that a model to study the optimal cash holding problem has to be able to handle the complex interdependencies between different corporate decisions and cash constraints. However, the classical Real Options model, which has been widely applied to study corporate decisions, cannot directly help us answer the two main questions proposed at the start of this chapter. This is because the classical Real Options framework assumes that companies will have unlimited access to risk-free funds in the financial market, and thus holding cash or not is assumed to be indifferent to make corporate decisions for firm managers. In this thesis, we are going to reform Real Options framework by including cash holdings, so that we can answer these two questions with a realistic and tractable mathematical model.

The remaining contents of this chapter are organized as follows: We first review the background knowledge of Real Options. We then discuss what we feel is missing from recent studies in the corporate finance literature, before highlighting the research objectives of this thesis. At the end of this chapter, we present the full structure of this thesis.

1.1 Background of Real Options

Classical Real Options Analysis aims to apply financial option theory and techniques to capital budgeting decisions. It assumes that the value of a project comes not only from the discounted cash flows but also from the manager's flexibility. This idea was first exposed by Myers (1977), who pointed out that all corporate investment opportunities are the firm's right instead of an obligation, and he defined the asymmetric investment flexibility as Real Options. Sick (1995) extended this definition and presented a more general one: "*Real Options are all the flexibilities that a manager possesses for making decisions on real assets*". Trigeorgis (1993b) classified these flexibilities into seven categories: Option to Defer, Staged Investment option, Option to Alter Operating Scale, Option to Abandon, Option to Switch, Growth Option, and Interacting Option. The theoretical models of these options can be found in the books by Dixit (1994) and Vollert (2012). The study of Real Options has greatly enriched the analysis tools that support corporate decisions, however, it is not simple to extend those models to include cash holdings. There are many limitations to the Real Options model, many of which have been discussed extensively in the literature (see Lander and Pinches, 1998; Vollert, 2003; Wang, 2010). Here, we particularly highlight the parts that are relevant to the difficulties associated with modelling the cash holdings process.

As discussed in Lander and Pinches (1998) and Wang (2010), classical Real Options require strong assumptions to be made so that financial option valuation and analysis techniques can be implemented. To explain, we must assume that we trade in a complete market (see Modigliani and Miller, 1958) in which: there are no frictions of trading; people can always find a portfolio to replicate the underlying assets; dividend payments are known in size and date; and people can always raise capital at the risk-free rate. However, for a cash constrained corporation, most of these assumptions are violated. In addition, Trigeorgis (1993a,b, 1995, 1996) further illustrated the limitations of classical Real Options in handling interdependencies among multiple corporate decisions. Under the classical Real Options framework, a firm holding a single project and independent options can be simply evaluated as the net present value of the project plus the sum of option values. However, a firm having multiple correlated projects and interdependent decisions, particularly when these decisions are made in a cash-constrained environment, cannot be merely valued as the sum of net present value and option values. This non-linearity inherent in the problem is of particular interest and acts as motivation for us to investigate further.

The strict assumptions on classical Real Options plus the complex interdependencies between different operational flexibilities construct the main barrier to build a simple framework that is able to combine cash holdings with Real Options Analysis. The most recent literature has kept the idea of valuing management flexibility as an option, but tried to overcome these obstacles by going back to the original building blocks of the model. New models have looked at such as the Stochastic Optimal Control and Optimal Stopping Theory (we refer to the book Åström, Karl J, 2012 for more details), or reconsidered more realistic factors, for example, stochastic revenue, taxation, dynamic dividends, cash constraints, liquidity and insolvency risks (see, Denis and Sibilkov, 2009; Hennessy and Tserlukevich, 2008; Sotomayor and Cadenillas, 2013; Li et al., 2013; Gryglewicz, 2011). When a firm's dynamic uncertainty is modelled in a stochastic control framework and is then solved with the Dynamic Programming Principle (see, in particular, Pham, 2009), the continuous operational actions can be formulated as a series of operational stages, within which many complex factors can be involved, and any number of joint decisions are allowed to be considered. We will give a detailed introduction of the required mathematical theory in Chapter 2.

Although the involvement of cash holdings and other factors makes the mathematical model more realistic to cope with corporate decisions, it, at the same time, increases the complexity of the model and the difficulty in finding and understanding theoretical and numerical solutions. For example, since all information of past cash flows is needed to model cash holdings, when it is embedded into a stochastic control framework, including the cash holdings will inevitably increase the dimension of the original model. To find a solution of this new model, extra economic assumptions and more efficient numerical algorithms are required. Fortunately, recent developments in computational techniques and numerical algorithms, to some extent, alleviate the anxiety of solving high dimensional problems. We will give an independent review on the related numerical techniques in Chapter 3. In the next section, we focus on the latest and most relevant studies on the corporate decision under cash constraints.

1.2 Recent Studies

The recent literature can be classified based on various assumptions of corporate factors, research topics, and theoretical/empirical works. In this section, the literature is organized based on mathematical assumptions of corporate factors, such as asset value, cash flow, and dividends. Particularly, we discuss the links and difference between previous work and the issues we try to address in this thesis.

1.2.1 Corporate Decisions under Stochastic Firm Value

In the early framework of Real Options, the asset value of a firm was assumed to follow a stochastic process. Following this idea, different corporate uncertainties were integrated into the stochastic feature of the asset value, thus making it easier to derive an analytic solution for a corporate decision. Leland (1994) and Leland and Toft (1996) investigated the optimal capital structure problem and provided a Real Options approach to value corporate debts. Cossin and Hricko (2004) studied the financial benefits of holding cash when raising new capital is costly and time-consuming, and the available external capital is full of uncertainty. This paper assumed the firm value follows a Geometric Brownian Motion and the cost of raising capital follows a mean-reverting process. Based on these settings, the timing value of holding cash is measured by the difference of firm value that generated in two cases, i.e. a firm that can investment at any time without limitations, and a firm that has to wait for external capital to complete the investment. Asvanunt et al. (2009) assumed the entire firm asset follows a Geometric Brownian Motion and structured a framework to investigate the interaction of cash balance and investment opportunities for a firm that has some initial debts and holds a growth option, where the cost of exercising this option can be financed by either cash or costly equity insurance. They concluded that the increase of cash holdings does not significantly raise the firm value in the absence of a growth option. However, in our study, we find cash holdings significantly affect the firm value when the firm is in a non-profitable state (see Chapter 4). This is because we value the firm with an assessment of its liquidity risk, which can be significantly reduced by holding more cash.

The literature mentioned above discusses the issue, how to extend the classical Real Options framework to fit complex corporate decisions with cash consideration. However, the assumption on the asset value of the firm strongly limits our understanding of the internal capital structure, and its effects on the corporate decisions, such as cash flows, capital structure and dividend policy, and thus constrain the possibility of modelling liquidity risk.

1.2.2 Corporate Decisions under Stochastic Cash Assets

To study how the uncertainty of liquid assets affects a firm's operational strategy, many academics have assumed the cash asset to be a stochastic process. Under this assumption, the optimal dividend distribution problem must be considered, based upon the understanding that a firm operates solely to maximize the shareholders' benefits by optimal choosing the dividend payout. Significant work in this area include Décamps, Jean-Paul and Villeneuve, Stéphane (2007), Belhaj (2010), Sotomayor and Cadenillas (2011) and Jiang and Pistorius (2012), among others.

Décamps, Jean-Paul and Villeneuve, Stéphane (2007) considered a firm that has drifted Brownian Motion cash reserves, which runs to optimize the expected dividend payout by controlling the dividend amount, investment time, and corporate bankruptcy. With these settings, they derived a mixed stochastic control optimal stopping model and investigated the interaction between dividend distribution and investment decisions under liquidity constraints. Belhaj (2010) further considered effects of infrequent shocks in cash reserves on dividend payout policy, by assuming a Poisson-type uncertainty. They pointed out that this new type of uncertainty did not alter the main feature of optimal dividend payout boundary. In addition, they studied the optimal insurance problem when the firm has a low level of liquid assets, and concluded that it is optimal for the firm to buy insurance when the cash is below a certain level. Sotomayor and Cadenillas (2011), Jiang and Pistorius (2012), and Chevalier et al. (2013)

derived optimal control regime switching models to study the interaction of optimal dividend payment and investment under liquidity constraints. The difference between the first two studies is that the system regimes in Sotomayor and Cadenillas (2011) are defined by drift and volatility coefficients functions, while the regimes in Jiang and Pistorius (2012) are defined by the discounted rate. Both of these regime criteria vary with different economic situations. The regimes of the model presented in Chevalier et al. (2013) are given by the indebtedness, and the coupon rate is altered with the change of debt regimes.

The assumption that cash holdings can be modelled as a stochastic process helps us to investigate the optimal dividend and investment policy under one stochastic dimension, the process of which is not too complex to handle. However, the coefficients of the cash reserve process are not straightly observable. Many corporate decisions are based on revenue particularly when we involve considerations of the short-term liquidity and long-term insolvency risks. To explain, a cash constrained firm has to stop the business, when it has a negative cash flow and no cash assets; and a firm's long-term insolvency depends on its future profitability (revenue) and current debt burden. Therefore, both cash flow and cash assets are essential considerations to study optimal cash holdings problems.

1.2.3 Corporate Decisions under Stochastic Revenue and Cash Constraints

The most recent works tried to model corporate decisions with easily observed variables, for example, accounting balance items. They believed that the firm value depended on the revenue and the assets. The revenue properly reflects the uncertainty of the external market and allows people to consider more realistic factors, for instance, operational costs, taxes, continuous dividend and investment, while the assets dimension can be used to explain the internal structure of the capital, measure the liquidity risk and link the interaction of different corporate decisions. In addition, working on these assumptions, it is more convenient to distinguish the valuation and operational strategy of firms that have different running objectives (to maximise the equity and to maximise the firm value), and thus it facilitates other studies and tests, for example, we may consider the agency conflicts, Trade-off Theory or Pecking Order Theory. We now review a selected literature that is most relevant to the models in this thesis.

Apabhai et al. (1997), Epstein et al. (1998) and Li (2003) assumed that a firm has a stochastic income process (Geometric Brownian Motion or Mean Reverting Process) and a cash saving process that is defined by the accumulated earnings plus interest income, the sign of which can be simultaneously positive and negative. Based on this, the firm value was defined as the expected cash savings at a fixed expiry in the future. They provided a simple guidance on how to model a firm with both revenues and cash assets, however, they did not consider other important factors (liquidity and insolvency) and corporate operational flexibilities (abandonment, cash management, financing and investment).

Titman and Tsyplakov (2007) investigated the optimal capital structure and investment strategy based on a stochastic optional control framework. They assumed a concave and increasing function to model a firm's production capacity and the stochastic price of the product, thus the net cash flow is defined as the selling income minus the operating cost, dividends and depreciation. The assets of the firm were assumed to be a continuous time process which is a function of depreciation and investment costs. In addition, the firm holds the management flexibility to dynamically change the level of debts, if this happens the firm has to adjust their equity with the same amount instantaneously in order to keep the total assets unchanged. The firm operates to maximize its equity value by controlling the investment ratio. More precisely, a positive investment ratio decreases the speed of dividend payout but increases the growth rate of assets. This paper conducts a good study on optimal capital structure but relies on an unrealistic assumption on the financing flexibility that a firm cannot increase or decrease the total level of assets. The discussion of the firm's revenue and assets help us have a better understanding of how to structure a model of corporate decisions based these criteria.

Gryglewicz (2011) studied the impact of liquidity and solvency concerns on corporate finance by assuming a stochastic cash flow with an evolving drift and innovation driven uncertainty. The joint effects of liquidity and solvency are modelled in the evolving drift function, where the long-term expected drift value stands for solvency, while the instantaneous drift value stands for liquidity. Once an innovation happens, it directly alters the instantaneous drift value and indirectly affects the long-term expected drift. Based on this tricky modelling, the author investigated how the insolvency and liquidity influenced the optimal capital structure, cash holdings, dividends, and bankruptcy. This study presents a novel method to model liquidity and solvency risks, the relation of which strongly depends on the mathematical definition of the drift process, however, in reality the liquidity and solvency risks might not exist an explicit relation. Thus, although the discussion of liquidity and solvency enriches our understanding on the corporate risks, in our study, we do not follow their idea to model these risks.

Anderson and Carverhill (2012) studied the optimal cash holding policy by assuming a stochastic cash flow that had a Cox-Ingersoll-Ross type drift coefficient function, and a continuous asset process that can take positive value denoting cash holdings, and negative value denoting short-term debts. With these assumptions, a stochastic optimal control framework was built to involve important considerations, i.e. the operational costs, taxes, optimal dividends, automatic updating short-term debts, optimal equity issuing, and to model the liquidity and insolvency risks. This paper provided a nice framework for most corporate decisions. The idea of how to model the cash flow and asset is very close to our work. However, this paper did not fully model an investment decision, and what is more, it assumed that "There is no reason to hold cash and borrow short-term simultaneously", which substantially closed the door to investigate the trade-off theory and liquidity shortage. Based on this improper setting, they presented a misleading conclusion that "Growth opportunities do not greatly affect cash holding policy". In this thesis, we will show how a growth opportunity will significantly rely on the cash holdings under certain circumstances.

More recently, Kisser (2013) investigated the value of holding cash in order to capture an expansion opportunity. In this paper, the firm value was based on both the state variable (cash flow) and cash holdings, where the cash flow follows a Geometric Brownian Motion, and only a proportion of the cash flow, after taking out a quadratic agency cost, is retained within cash savings. The value of cash is derived by comparing total firm value under the optimal cash retention policy to the case when all earnings are paid out as dividends. As such it quantifies the maximum increase in firm value by optimally trading off costs of external finance against agency costs of free cash flow. The value of holding cash is defined by comparing the firm value under an optimal dividend policy to the case when all the net incomes are distributed as dividends. Based on their model, they found that high cash flow volatility decreased the value of cash, and that optimal cash policy delayed investment decision relative to the case of full outside financing. Our study gives supporting evidence by showing consistent results on corporate optimal expansion problem. This paper derived a tractable framework to study the Real Options value of cash, however, the study of general corporate decisions based on their stochastic framework is absent.

Hugonnier et al. (2014) developed a dynamic model for a firm having investment, financing, cash management flexibility. They assumed that the firm has a stochastic cash flow that follows a Brownian Motion, and it holds a growth opportunity, where the exercising cost can be financed via two types of capital: the accumulated cash or an uncertain external equity, which follows a Poisson-Jump process. The paper particularly studied the financing cost of management flexibility, and they concluded that this cost significantly changed the operational behaviour of the firm, i.e. the dividend boundary had both incremental and lumpy features. This paper presented a good framework to combine different types of corporate decisions within one cash dimension. However, due to the lack of analysis on cash flows, the short-term liquidity risk and long-term insolvency were not well discussed. In addition, debt issuing, in this paper, is not allowed in order to finance the growth opportunities. We will cover these points in our studies.

Other relevant and useful studies include several working papers: Décamps and Villeneuve (2013) studied how to do investment problems for a cash-constrained firm by only considering the stochastic cash flow (no cash holdings). They provided useful benchmark solutions (closed form) for the optimal dividend payment problem, the optimal investment with growth option, and optimal investment without growth option. However, this model assumes no financing flexibility. Bolton et al. (2014b) investigated the optimal investment strategy under financial constraints. They assumed the firm value being a function of the future uncertain revenue and the level of liquidity. The firm operated to optimize the expected revenue by optimally exercising an initial investment option. Based on this framework, they conclude that bringing in financial constraints significantly altered the Real Options Analysis results. However, in all their models, the firm is presumed to be a value-maximizing firm (no dynamic dividends), and equity and debt financing has no difference. We feel that this limits them somewhat when trying to conduct studies on the optimal investment problem that can choose from different financing resources. We will discuss the difference in the optimal investment strategy that can be financed with cash, equity or debt.

All the models in this thesis are based on continuous time, so we rightly concentrate most of our review of the literature on them, but we can also recommend several nice discrete time models: Both Gamba and Triantis (2008) and Li et al. (2013) described discrete time models in which the interactions of equity financing flexibility, investment flexibility, financial constraints and risk of defaults are studied. Since the discrete time models cannot provide very accurate operational strategy, particularly those based on the marginal value of cash, we do not expand much more on this.

To summarize, in this thesis, our models mainly follow from the ideas of Anderson and Carverhill (2012), Kisser (2013), Bolton et al. (2014b). We study a firm's two types of operational objectives: value-maximizing and equity-maximizing to make it capable of handling agency conflicts. We assume that the value of a firm depends on the revenues and assets, and the assets here have two types of division, cash assets plus fixed assets, and equity plus debts. This framework will help us to properly capture the cash-flow insolvency risk when a firm has negative net cash flow and zero cash assets; and balance-sheet insolvency, when the total equity or the market value of equity is less than the total debt obligation. In this thesis, we assume that total asset level will be a function of revenue instead of an independent stochastic process and it might also vary with non-continuous financing and investment activities. The reason we make these assumptions comes from the following considerations: Firstly, the volatility of assets is not a direct criterion to measure the market uncertainty compared with that of revenues', and also other coefficients of the assets process are not easily observable. As discussed in Gryglewicz (2011) and Anderson and Carverhill

(2012), the settings of independent cash flow, and structured assets make it more convenient to involve risks from both market and internal capital structure. And lastly, the revenue-based asset assumption leads to a degenerate type of PDE (no second order derivative item for some dimension), which compared with the PDE generated by assuming an independent stochastic asset, is easier to model and then to find an approximate numerical solutions (see Chapters 2, and 3).

1.3 Research Objectives

The objective of this thesis is to address the optimal cash holdings problem and contribute to the understanding of how cash constraints affect corporate decisions, such as abandonment, optimal dividend payments, optimal financing, and optimal investment. We are going to provide a mathematical framework based on stochastic control theory to investigate a firms multiple interdependent management flexibilities, where their joint behaviour is embedded into one system that is limited by one global cash holdings, and also, this framework should be able to capture important features that have been identified as essential considerations in the literature, i.e. financial constraints, liquidity risk, interdependencies of various operational flexibilities, and economic friction analysis (for example, financing and expansion cost analysis). More precisely, it sheds light on the following research questions.

1. How do cash holdings alter the classical Real Options Analysis?
2. How does the dividend distribution policy interact with cash constraints?
3. How does the financing flexibility relax the liquidity constraints, and what are the different features of debt financing and equity financing?
4. How the financial constraints can limit the investment activities, and what's the difference to support an investment decision with cash holdings, equity financing and debt financing.

1.4 Thesis Structure

The structure of this thesis is organized as follows: In Chapters 2 and 3, we illustrate the background of mathematical theories and fundamental numerical algorithms respectively, which will be used to conduct studies of the following sequential chapters. Chapter 4 provided a simple extension of Real Options to show how the liquidity constraints affect corporate abandonment decisions. In Chapter 5 we derive an equity-value-maximizing firm model and discuss the optimal dividend policy based on the a firm's balance sheet equation. This model is designed to be extendable to further corporate decisions. Chapters 6 and 7 provide a framework that involves optimal equity and optimal debt financing respectively. As contributions to the literature, we, in particular, investigate how the equity financing flexibility affects a firm's operational behaviour, and how to find the market coupon rate for a particular corporate debt. Chapter 8 investigate the optimal investment decisions via various financing sources. We discuss how the liquidity constraints affect the firm's investment and financing decisions. This work supplements the literature on investment decisions by investigating how deciding on the financing resource differs the investment decision itself. Chapter 9 presents the main conclusions of the thesis and discusses areas for future study.

Finally, in the Evatt et al. (2014), another important aspect of my Ph.D. study is given. This is a paper that I have been a contributor to, for the three years of its construction. The paper concerns banking regulation. Here we formulate a continuous-time model of a deposit-taking bank, which operates to maximise its shareholders' benefits by optimally paying out dividends, issuing new equity and loans, and endogenous closure. We also consider the role of the regulator, who seeks to minimise the probability of early closure, through either insolvency (due to loan defaults) or endogenous closure. The paper has rested upon much of the theory developed within this thesis, and can be viewed as a practical application of our more theoretical work.

Chapter 2

Stochastic Control and Dynamic Programming

2.1 Introduction

Stochastic control problems (SCPs) aim to answer the question of how to optimize a specified objective in a stochastic dynamic system by controlling some variables. They have been widely observed in business management, finance and economics. Depending on the control variables, they could be roughly classified into three basic classes: Continuous Stochastic Control Problems (CSCPs), Optimal Stopping Problems (OSPs) and Impulse Control Problems (ICPs). The first class refers to those situations when the decision makers are allowed to continuously react to the random changes of a dynamic system. One well-known example is Merton's optimal investment problem Merton (1971), in which the control variable is the weight of risky assets in a portfolio. The second class describes problems where the controls are all about the timing and in which the decisions are generally irreversible, for example, the problem when to abandon a project. As for the third class, ICPs deal with the situation involving irregular and infrequent control decisions, for instance, the decision to switch between two product lines (say gold and silver mining) depending on the market price of each product.

Modern control theory is based on the studies of Pontryagin and Bellman in the 1950s. They each proposed a distinct approach to solve an optimal control problem: the Maximum Principle (MP) approach and Dynamic Programming Principle (DPP) approach. Both of these approaches have been extended in many ways, and they are both applicable to solve control problems in stochastic systems. One area of exploration within the MP framework has opened up a new branch of research, namely, Backward Stochastic Differential Equations, in which it is hard to find both analytical and numerical solutions. It is therefore beyond the scope of this thesis. Here, we focus only on the DPP approach.

The DPP was first established by Bellman, an American mathematician, in the work by Bellman (1957). This principle solves the global optimal control problem by splitting it into a series of sub-problems. By deriving and using the relations between the sub-problems and the original one, one can find the optimal control strategy for the original problem according to the optimal solutions of the sub-problems. The reason that we prefer DPP methods to model and solve our problems comes from two aspects. First, the DPP, when used to solve a continuous stochastic control problem, usually generates a non-linear partial differential equation (nPDE), also called Hamilton Jacobi Bellman (HJB) equation. This HJB equation can then be solved and analysed using standard PDE techniques. Second, the process of dividing the whole decision time into smaller chunks and making decision backward at each chunk based on DPP is similar to that of a real life decision process.

Classically, in order to derive an HJB equation and show the existence and uniqueness of the solution, we would need to use Itô's lemma. A requirement for the lemma to be applicable is that the value function should be sufficiently smooth, which is usually the case in the most standard mathematical finance problems. However, when dealing with SCPs, the strict smoothness condition can only be verified in a few cases. Therefore, the rigorous application of the DPP has been limited when using classical PDE methods. Fortunately, recent literature on viscosity solutions and comparison principles for nPDEs show how we can relax the smoothness assumption. Numerical methods have also been developed, which have been shown to give the corresponding viscosity solutions, such as Semi-Lagrangian Methods (d'Halluin et al. (2005); Chen and Forsyth (2007); Debrabant and Jakobsen (2013)). These advanced methods make the DPP possible for more general SCPs meaning that we can apply the approach to the models in this thesis.

In this chapter, we review the necessary mathematical background to derive models in the rest of the thesis from a practical perspective. We first present a standard framework to model SCPs. Based on this framework, we then illustrate two main approaches to solve SCPs, the classic dynamic-programming-verification approach and the viscosity solution approach, by using two simple examples. In the following contents, we try to make the explanation as practical as possible. If detailed proofs are required, we aim to provide the relevant references for the interested reader. The notations of this chapter follow the book by Pham (2009).

2.2 Mathematical Framework of Stochastic Control Problems

SCPs can be formed in many ways, depending on the type of objective function to be maximised and the type of admissible controls. We choose in this section to build in as generic a way as possible the most standard SCP we can. A standard Stochastic Optimal Control Problem typically incorporates the following elements: the system state, the control variables (or stopping time), the performance criterion and the value function.

System states: The system states represent all of the possible future scenarios or conditions that a problem might face, which is defined within an uncertain system. Mathematically, we model the uncertain system as a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where, Ω is the set of all possible scenarios, $\{\mathcal{F}_s\}_{s\geq 0}$ represents all the historical information before s, with $\mathcal{F}_t$ $(0 \leq t \leq s)$ being the information available at time t, and $\mathbb{P}$ is the probability measure. Based on this filtered probability space, the possible system states can be defined as a controlled stochastic process, the type of which will vary according to the problem. In this thesis, we mainly focus on the following controlled diffusion process, which will be used to model a series of optimal corporate operation problems (see Chapters 4 to 8).

$$dX_s^\alpha = \mu(X_s, \alpha)ds + \sigma(X_s, \alpha)dW_s \tag{2.1}$$

where, W_s denotes one dimensional Brownian Motion, $\mu(\cdot)$ denotes the drift of the process, $\sigma(\cdot)$ defines the volatility level of the process, and α represents a control variable. More introductions on the controlled stochastic process can be found in book Gihman and Skorohod (2012).

Control variable: As we can see from the process (2.1), the system states path-wise depend on the control strategy α, which more precisely to say, is $\{\mathcal{F}_s\}_{s\geq 0}$ measurable process. We assume α takes value from an admissible set $\mathcal{A}$, which is defined according to the particular mathematical and problem requirements.

There are three types of widely used controls: adaptive control, feedback control, and the Markov control, which are distinguished by what information is required to make a control decision. For adaptive control problems, we need all the historical information ($\{\mathcal{F}_s\}_{s\geq 0}$) before time s (including that at time s). Feedback control decisions only depend on the historical records of the state process (the value of X_t, for all $t \in [0, s]$). Finally, Markov controls can only depend on the current information of the state process X_s at time s. In practice, all the control variables should satisfy certain admissible conditions, which can be either technical based conditions, for example, integrability or smoothness requirements, or physical based conditions, for example, constraints on the value of the state process or controls.

Stopping time: When the control criterion is all about timing, we called this type of optimal control problem an optimal stopping problem. We define the stopping time τ as the first time-point when the system satisfies some particular conditions, for example, $\tau = \min_s\{s : X_s < 0\}$ denotes the first time when the value of X_s becomes negative. The value range of τ depends on the problem horizon. We assume the stopping time τ can take a value up to and including T in a finite time horizon problem ($\tau \in [0, T]$ for $T \in (0, \infty)$), and τ can take value infinity ($+\infty$), in a perpetual horizon problem. More mathematical explanation of optimal stopping time can be found in book Peskir and Shiryaev (2006).

It should be noted that classifying this type of time horizon is very useful before modelling a stochastic optimal control problem. This is because, for the infinite time horizon problems, solutions and the control decisions (if they exist) only depend on the state process, and do not depend on time. However, in the finite horizon cases, we make control decisions based on the remaining time $T-t$, thus the control decision depends on both the starting time t and the initial information x. In numerical test, we generally take $T_{max} = 5/\rho := 10^3 \times \Delta t$ as an experiential approximation of $T \to \infty$, since in this case, $|e^{(-\rho(T_{max}+\Delta t))} - e^{(-\rho \times T_{max})}| < 10^{-6}$ for all $\rho < 1$. We believe this accuracy is enough for the problems in this thesis.

Performance criterion (Cost or Reward Function): To track the performance of the system, we define a performance criterion function $J(t, X_t, \alpha)$, which depends on time t, the system state X_t and the control strategy α. Suppose a system is driven by a controlled stochastic process X_s^α, and it will continuously generate rewards or punishments $g(s, X_s, \alpha)$, and a one-off payoff $h(\tau, X_\tau)$ when we decide to terminate the system at time τ. Mathematically, we have,

$$J(t, X_t, \alpha, \tau) = E\left[\int_t^\tau e^{-\rho(s-t)} g(s, X_s, \alpha) ds + e^{-\rho(\tau-t)} h(X_\tau) \big| X_t = x\right]. \tag{2.2}$$

Here, we assume τ can take three types of value: $\tau = T$ for finite horizons; τ could be a random value if we are modelling an optimal stopping problem; or $\tau \to \infty$ for perpetual horizon (in this case, we use $J(X_t, \alpha)$ instead of $J(t, X_t, \alpha, \tau)$). It should be noted, in many finance or economics SCPs, we need to include a discount factor, such as $e^{-\rho s}$. This is because people make decisions based on the time value of money in an economic sense. In the context of this study, they are also required to be able to define a bounded objective function in the mathematical sense. We will illustrate some sufficient conditions in the following contents.

Value function: We define the value function $V(t, X_t)$ as the minimum expected costs (or maximum rewards) of the system. It is obtained by optimizing the performance criterion over all admissible controls $\mathcal{A}$ and possible stopping time τ with a given initial state (t, X_t).

$$V(t, X_t) = \sup_{\tau;\alpha \in \mathcal{A}} J(t, X_t, \alpha, \tau). \tag{2.3}$$

When the time horizon is infinite, we use $V(X_t)$ instead of $V(t, X_t)$. In the modelling process, reader might replace sup operator by other operators depends on the requirement of the problem. For example, when the objective function is to minimum the cost, people might use inf operator or min operator[1].

The objective of the optimal stopping stochastic control problem is to find a set of optimal control strategy (τ, α^*) and obtain the unique optimal value $V(t, X_t; \tau, \alpha^*)$ by characterizing the above value function. We now illustrate how to go about this using the Dynamic Programming Principle (DPP) and PDE approaches.

2.3 PDE Approaches for Stochastic Control Problems

PDEs approaches were first mentioned in connection to SCPs in the 1950s by Bellman Bellman (1957), when he developed the dynamic programming principle. The basic idea of this approach is to associate an original SCP with the corresponding non-linear PDE (also named HJB equation). By solving the HJB equation, we can indirectly find both the optimal control strategy and the corresponding value function.

In order to implement a PDE approach and find a unique solution of the original problem, we need to consider questions from two opposite directions: how to prove that the original value function solves the corresponding PDE, and how to show that the solution of the non-linear PDE is also the solution of the original problem. Recent studies develop two different branches to answer these questions: the classical PDE approach (to solve a SCP via finding the classical solution of the corresponding PDE) and the viscosity solution approach (to solve a SCP via finding the week solution of the

[1] A maximum (minimum) is the largest (smallest) number within a set. A supremum (infimum) is a number that bounds a set. A supremum (infimum) may or may not be part of the set itself. If the supremum (infimum) is a part of the set, it is also the maximum.

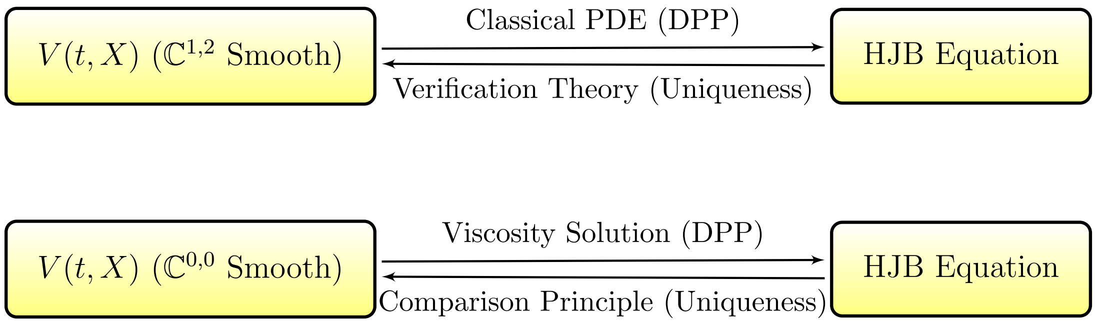


Figure 2.1: A sketch to show two PDE approaches and the corresponding theories that used to solve stochastic control problems.

corresponding PDE). We schematically show these two approaches and the relevant theories in Figure 2.1.

The classical PDE approach assumes the value function is sufficiently smooth. Thus, we can directly use Itô's lemma on a value function to derive an HJB equation. With the Verification Theory, we can further show that the smooth solution of the HJB is the value function. The viscosity solution approach relaxes the smoothness assumption. This approach directly implements DPP on the value function and shows that the value function is a weak (viscosity) solution of the corresponding HJB equation. According to the comparison theory, we can further prove that this solution is the unique viscosity solution of the HJB equation. Therefore, we can find a solution of the original problem by just studying the corresponding HJB equation.

2.3.1 Classical PDE Approach

The process to solve a stochastic optimal control problem with the classical PDE approach can be summarised as follows:

1. Derive the HJB equation formally with the dynamic programming principle;
2. Solve the HJB equation and obtain a smooth solution;
3. Verify that the smooth solution is the value function by using the Verification Theory. As a by-product, get the optimal control strategy.

We now show the corresponding mathematical theories that are used in these steps by choosing the following stochastic optimal control problem as an example. It should be noted that this example will not be used in the next series of chapters, here we only want to review the basic ideas of stochastic control theory.

Problem setting: Suppose there is a system that is driven by a controlled stochastic process X_s^α. The system will continuously generate instantaneous rewards $g(s, X_s, \alpha)$ during time $[t, T]$ and a termination rewards $h(X_T)$ at time T. We assume the stochastic process is given by the following Stochastic Differential Equation,

$$dX_s = \mu(X_s, \alpha)ds + \sigma(X_s, \alpha)dW_s, \tag{2.4}$$

based on which, the value function can be defined as,

$$V(t, X_t) = \sup_{\alpha \in \mathcal{A}} E\left[\int_t^T e^{-\rho(T-s)} g(s, X_s, \alpha)ds + e^{-\rho(T-t)} h(X_T)\big|X_t = x\right]. \tag{2.5}$$

Assumptions and conditions: To find out an optimal control strategy of the above problem with the classical PDE approach, we need the following sufficient assumptions and conditions (see Pham (2009) Chapter 3):

a. The controlled process (2.4) is a Markov Process (with Markov Control), and the coefficient functions $\mu(x, \alpha)$ and $\sigma(x, \alpha)$ should satisfy the following Lipschitz Conditions.

   There exists a constant $K \geq 0$, such that for any $y_1, y_2 \in \mathbb{R}$ and any control strategy $\alpha \in \mathcal{A}$, we have,

$$|\mu(y_1, \alpha) - \mu(y_2, \alpha)| + |\sigma(y_1, \alpha) - \sigma(y_2, \alpha)| \leq K|y_1 - y_2|.$$

   Particularly, for the initial state $x = 0$, we have,

$$E\Bigg[\int_t^T \big(|\mu(0, \alpha)|^2 + |\sigma(0, \alpha)|^2\big)dt\Bigg] < \infty.$$

b. The instantaneous rewards function g and termination rewards function h should satisfy the following conditions.

   i. The $h(x)$ is measurable and upper bounded function in $\mathbb{R}$, and it satisfies the following quadrature growth condition: For all initial state x ($x \in \mathbb{R}$), there are some constant K ($K \geq 0$), such that,

$$|h(x)| \leq K(1 + |x|^2). \tag{2.6}$$

ii. The continuous rewards function $g(t, x, \alpha)$ is a measurable function in $\mathbb{R}^3$, and it satisfies the following quadrature growth condition: For any $(t, x) \in [0, T] \times \mathbb{R}$ and $\alpha \in \mathcal{A}$, we have a positive constant C, such that,

$$|g(t, x, \alpha)| \leq C(1 + |x|^2) + \kappa(\alpha), \tag{2.7}$$

where $\kappa(\alpha)$ satisfies the following condition: For all $\alpha \in \mathcal{A}$, we have,

$$\kappa(\alpha) \leq C(1 + |\mu(0, \alpha)|^2 + |\sigma(0, \alpha)|^2). \tag{2.8}$$

c. The value function (2.5) is $\mathbb{C}^{1,2}$ in the space $[0, T] \times \mathbb{R}^2$.

The assumption and condition [a.] ensures that the controlled stochastic process is a Markov Process and it has a unique strong solution for all the possible control α and given initial condition (t, x). The condition [b.] guarantees a well-defined objective function which is bounded in the corresponding space. Assumption [c.] is a necessary condition to implement Itô's lemma to derive the corresponding HJB equation. For more details, we recommend the book by Pham (2009).

Dynamic Programming Principle: DPP is the key technique that is used to solve a stochastic control problem. Here, we first illustrate the DPP related to the classical PDE approach.

Theorem 2.3.1. *Let $(t, X_t) \in [0, T] \times \mathbb{R}$. For any time $\theta \in [t, T]$, we have,*

$$V(t, X_t) = \sup_{\alpha \in \mathcal{A}} E\left[\int_t^\theta e^{-\rho(\theta - s)} g(s, X_s, \alpha_s) ds + e^{-\rho(\theta - s)} V(\theta, X_\theta) \big| X_t = x\right]. \tag{2.9}$$

This equation shows that if we implement a strategy $\alpha_{[t,\theta]}$ from t to θ, and then implement the optimal strategy $\alpha^*_{[\theta,T]}$ from time θ onward, since $V(\theta, X_\theta^{t,x})$ represents the optimal value with optimal strategy at time θ, it is clear that the value generated by following this set of strategy $(\alpha_{[t,\theta)}, \alpha^*_{[\theta,T]})$ will never go over that generated by following the global optimal strategy during the entire time horizon. Thus, a strategy $\alpha^*_{[t,\theta]}$ that maximises the value function from t to θ with given $V(\theta, X_\theta^{t,x})$ must be a subset of the global optimization strategy $\alpha^*_{[t,T]}$ from t to T. So to determine the global optimization, we can partition the time interval into smaller chunks and optimize over each individually. When the chunk size is small enough, the calculation of the optimal control problem becomes a point-wise local minimization problem.

To keep notation simple, we use the following controlled linear operator $\mathcal{L}^\alpha V(t,x)$ in the PDE.

$$\mathcal{L}^\alpha V(t,x) = \mu(x,\alpha)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(x,\alpha)^2\frac{\partial^2 V}{\partial x^2}. \tag{2.10}$$

Theorem 2.3.2. ***HJB equation:*** *Based on dynamic programming principle, the original SCP defined by Equation* (2.5) *is associated with the following non-linear Partial Differential Equation (HJB equation),*

$$\frac{\partial V}{\partial t} + \sup_{\alpha\in\mathcal{A}}[\mathcal{L}^\alpha V(t,x) + g(t,x,\alpha) - \rho V] = 0. \tag{2.11}$$

It should be noted, to prove the above theory, we need to show two aspects: we can derive the HJB equation (2.11) from the original value function (2.5), and also, the original value function (2.5) is the solution of the HJB equation (2.11) at the same time. The detailed proof can be found in the book Pham (2009).

Theorem 2.3.3. ***Verification Theorem:*** *Let w be a $\mathbb{C}^{1,2}$ function in space $[0,T)\times\mathbb{R}$, and it satisfies the following quadratic growth condition: There exists a constant C such that*

$$|w(t,x)| \leq C(1+|x|^2), \quad \forall (t,x)\in[0,T]\times\mathbb{R}.$$

(i) Suppose that

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{\alpha\in\mathcal{A}}[\mathcal{L}^\alpha w(t,x) + g(t,x,a) - \rho V] \geq 0, \tag{2.12}$$

for all $x\in\mathbb{R}$,

$$w(t,x) \geq h(x),$$

then we have $w\geq V$ in space $[0,T]\times\mathbb{R}$.

(ii) Suppose further that $w(T, x) = h(x)$. and there exists a measurable function α^*, valued in $\mathcal{A}$, such that

$$\begin{aligned}
&-\frac{\partial w}{\partial t}(t,x) - \sup_{\alpha^* \in \mathcal{A}} [\mathcal{L}^{\alpha^*} w(t,x) + g(t,x,a) - \rho V] \\
&= -\frac{\partial w}{\partial t}(t,x) - \mathcal{L}^{\alpha^*} w(t,x) + g(t,x,\alpha^*) + \rho V \\
&= 0. \qquad (2.13)
\end{aligned}$$

(iii) According to our assumptions and conditions (see (a)*), the controlled stochastic process* (2.4) *admits a unique solution with the given initial conditions* (t, x) *and the optimal control strategy α^* ($\alpha^* \in \mathcal{A}$). We denote this solution as $\hat{X}_t^{\alpha^*}$.*

Based on (i), (ii) and (iii), we have that the following equation holds in space $[0, T] \times \mathbb{R}$,

$$w(t, x) = V(t, x),$$

and also, the control strategy α^* is an optimal Markovian control that maximised V.

More details about the verification theory can be found in the book Pham (2009) in Section 3.5. By combining the derivation process of the HJB equation and the Verification Theory, now we can say that the solution of the HJB equation is the solution of the optimal control problem (2.5).

2.3.2 Viscosity Solution Approach

The classical PDE approach requires the strong assumption that the value function is sufficiently smooth so that Itô's lemma can be used to derive the corresponding HJB equation. However, in most stochastic control problems, we cannot verify this smoothness condition and, in fact, it may not even exist. To work around this, the viscosity solution approach has been developed. Compared with the classical PDE approach, the process to solve an optimal stochastic control problem with a viscosity solution approach is quite different. We summarize the key steps of this approach as follows:

1. Derive the HJB equation of the original value function;

2. Prove that the value function is a viscosity solution of the associated HJB equation;
3. Show that the HJB equation has a unique solution in the viscosity sense; As a by-product, get the optimal control strategy of the original problem.

We now take the following optimal stopping stochastic control problem as an example to illustrate the viscosity solution approach.

Problem setting: Suppose there is an uncertain system that is driven by a controlled diffusion process with compound Poisson Jumps for $s \in [t, T]$,

$$dX_s^\alpha = \mu(s, X_s, \alpha)ds + \sigma(s, X_s, \alpha)dW_s. \tag{2.14}$$

The objective of operating this system is to maximize the instantaneous and termination rewards by controlling the variable α and stopping time τ. Therefore, we can define the value function as,

$$V(t, X_t) = \sup_{\tau;\alpha\in\mathcal{A}} E\left[\int_t^\tau e^{-\rho(\tau-s)}g(s, X_s, \alpha)ds + e^{-\rho\tau}h(X_\tau)\big|X_t = x\right]. \tag{2.15}$$

Here, the discount ρ denotes the opportunity cost, which was detailed discussed in the paper Evatt et al. (2014)

• ***Assumptions and conditions:*** According to Pham et al. (1998), to make sure there is a unique solution of value function (2.15), we need the following sufficient assumptions:

a. The controlled stochastic process (2.14) is a Markov Process (with Markov control). The coefficient functions μ, σ are continuous with respect to (t, x, α), and there exists a constant K $(K > 0)$, such that, for all $t, s \in [0, T]$, $x, y \in \mathbb{R}$ and $\alpha \in \mathcal{A}$, we have,

$$\begin{cases} |\mu(t, x, \alpha) - \mu(t, y, \alpha)| + |\sigma(t, x, \alpha) - \sigma(t, y, \alpha)| \leq K|x - y|, \\ |\mu(t, x, \alpha) + \sigma(t, x, \alpha)| \leq K(1 + |x|). \end{cases} \tag{2.16}$$

b. The value function is well defined, if the growth functions g and termination function h satisfies the following condition:

Suppose $g(t,x,\alpha)$ and $h(x)$ are two measurable functions, and there exists a constant $K \geq 0$, such that for all $t, s \in [0,T]$, $x, y \in \mathbb{R}$, and $\alpha \in \mathcal{A}$, we have the global Lipschitz condition,

$$|g(t,x,\alpha) - g(s,y,\alpha)| + |h(x,\alpha) - h(y,\alpha)| \leq K(|t-s| + |x-y|). \tag{2.17}$$

c. Assume the value function (2.15) is $\mathbb{C}^0$ continuous.

DPP for Optimal Stopping Problem: For a mixed stochastic control optimal stopping problem, we have that the following DPP holds.

Definition 2 .3.2. For all stopping time $\theta, \tau \in [t,T]$, and fixed values $x \in \mathbb{R}$, we have,

$$\begin{aligned} V(t,X_t) = \sup_{\tau;\alpha\in\mathcal{A}} E\Bigg[\int_t^{\tau\wedge\theta} & e^{-\rho(s-t)} g(s,X_s,\alpha)dt \\ & + \mathbb{1}_{\tau<\theta}\{e^{-\rho\tau}h(X_\tau)\} + \mathbb{1}_{\theta\leq\tau}\{e^{-\rho(\theta-t)}V(X_\theta)\}\Big|X_t = x\Bigg], \end{aligned} \tag{2.18}$$

where, $\mathbb{1}_{\tau<\theta}\{\cdot\}$ and $\mathbb{1}_{\theta\leq\tau}\{\cdot\}$ are indicator functions, which take value one when the condition holds, and take value zero in other cases.

We now illustrate how to link the original problem 2.15 with the following HJB Variational Inequality.

$$\min\left\{\frac{\partial\psi}{\partial t}(t,x) + \max_{\alpha\in\mathcal{A}}\left\{\mathcal{L}^\alpha\psi(t,x) + g(t,x,\alpha)\right\} - \rho\psi(t,x), \psi(t,x) - h(x)\right\} = 0, \tag{2.19}$$

where, ψ is a continuous and smooth enough function, which is generally called test function.

Definition 2 .3.3. ***Viscosity Solution:*** Any $V \in \mathbb{C}^0([0,T]\times\mathbb{R})$ is a viscosity super-solution or sub-solution of the Equation (2.19), if we have,

$$\begin{aligned} \min\Bigg\{-\rho V(t,x) + \frac{\partial\psi}{\partial t}(t,x) + \max_{\alpha\in\mathcal{A}}\left\{\mathcal{L}^\alpha\psi(t,x) + g(t,x,\alpha)\right\}&, \\ V(t,x) - h(x)\Bigg\} &\geq 0, \end{aligned} \tag{2.20}$$

or (≤ 0), respectively, whenever the test function ψ is $\mathbb{C}^{1,2}([0,T]\times\mathbb{R})$ continuous, and $V - \psi$ has a global minimum (or maximum). V is a viscosity solution of Equation (2.19) if it is both a super solution and a sub solution (see paper Pham et al. (1998)).

According to Pham et al. (1998), we can derive the 2.19 from 2.15 by using the above Dynamic Programming Principle. However, we cannot guarantee that the solution of HJBVI is the unique solution of the original optimal stopping stochastic optimal control problem. The simple logic here is that if the HJB has a unique viscosity solution, then this solution has to be V. In other words, the solution of the HJB is an approximated solution of the original problem. To achieve this, the Comparison Principle is developed to show the uniqueness of the solution.

Comparison Principle:

Theorem 2.3.4. *Let u be a $\mathbb{C}^0$ continuous viscosity sub-solution (or super-solution) of HJBVI* (2.20) *in space $[0,T] \times \mathbb{R}$. If $u(T,x) \leq V(T,x)$ (or $\geq V(T,x)$) for all $x \in \mathbb{R}$, then we have,*

$$u(t,x) \leq V(t,x) \quad \textbf{\textit{or}} \quad u(t,x) \geq V(t,x). \tag{2.21}$$

According to the Comparison Principle, if u is a viscosity solution of HJBVI (2.20) (both ≤ 0 and ≥ 0 hold), then, we can ensure that $u = V$. In other words, V is the unique viscosity solution. If this does not hold, say either $V \geq u$ or $V \leq u$, which violates our definition of the value function and viscosity solution. Therefore, by solving the HJBVI, we can find an optimal control strategy of the original problem. To keep focus, we do not show the proof of the Comparison Principle. People who are interested in the details are suggested to read Pham et al. (1998) and Fleming and Soner (2006).

The free boundary formula V.S. Variational Inequality formula: In an optimal stopping problem, generally, there are two ways to formulate a non-linear PDE: the free boundary approach (see Peskir and Shiryaev (2006)) and the Variational Inequality approach (Arkin and Slastnikov (2009)). The first approach has to assume the existence (and uniqueness) of a free boundary which defines the optimal stopping time. If one wishes to carry out some formal analysis of the free boundary, this approach is recommended, since we can use the closed form formula to study other problems that related this boundary. However, in most cases, it is not easy to prove that there exists a free boundary and derive an explicit formula for the boundary, in fact, Arkin and Slastnikov (2009) presents an optimal stopping problem where we cannot find a free boundary. Further, as shown in the HJBVI (2.19), the Variational Inequality provides more compact structure compared with the free boundary approach (see the Peskir and Shiryaev (2006)), which makes it is more convenient to model a problem and prove the existence and uniqueness of the viscosity solution. Therefore, in this thesis, we choose to represent our models and problems using the variational inequalities.

2.4 Conclusion

In this chapter, we reviewed the mathematical background for stochastic optimal control theory, and the different PDE approaches that can be used to solve stochastic control problems. By taking a pure stochastic control problem and a mixed optimal stopping and stochastic control problem as examples, we compared the assumptions and corresponding theories of the classical PDE approach versus the Viscosity Solution Approach. The background knowledge illustrated in this chapter, particularly the viscosity solution approach, will be used to derive models and design numerical algorithms in the following series of chapters.

Chapter 3

Semi-Lagrangian Methods

3.1 Introduction and Literature Review

Stochastic Control Problems (SCPs), when solved with Dynamic Programming Principle (DPP), usually associate with non-linear Partial Differential Equations (nPDEs). Most of these equations do not have analytical solutions meaning that we have to solve them numerically. In practice, there are two main obstacles to select an appropriate numerical method. First of all, the DPP approach suffers from the curse of dimensionality, particularly for multi-control problems: when one additional control criterion is introduced into the system, the nPDE generated by DPP has to be reformulated within a higher dimensional framework. Thus, an appropriate numerical method should be able to handle a highly dimensional equation with affordable calculation costs. Second, the value function generated from most stochastic control problems are not smooth enough, so that Itô's Lemma cannot be directly used to derive the corresponding PDE model. As a result, we normally have to find a weak (viscosity) solution of the corresponding HJB equation. This means that the numerical approach has to converge to a unique weak (viscosity) solution. In view of these obstacles, the numerical algorithms used in the stochastic optimal control area have been extensively developed. Here, we review these different numerical techniques by simply classifying them as either Classical Methods or Advanced Methods.

Classical Methods

Broadly speaking, we assume that the classical numerical algorithms consist of two branches, Simulating Methods and Approximating Methods. A representation of the first branch is Monte Carlo Simulations (MCs), which aim to simulate relative probabilities or stochastic characteristics by repeatedly generating random samples. Monte Carlo methods are very popular and widely used because they are flexible to program and are not subject to dimensional limitations (chiefly being used for more than four dimensions). However, given the fact they can only ever generate results from a random distribution they can not be expected to provide solutions with high accuracy. Moreover, when MCs are used to cope with a continuous non-bang-bang control problems, the original continuous control function has to be reduced to constant piecewise controls, thus resulting in a significant increase of computational cost along with a decrease in accuracy. Therefore, MCs are not suited for our model illustrated in Chapters 7 and 8.

The second branch is represented by Finite Difference Methods (FDMs), which obtain numerical results by using nearby grid points to approximate the Gradient and Hessian of a value function. They are typically used in engineering to solve low dimensional partial differential equations. In parabolic equations such as those often found in finance (all problem in this thesis are of this type), the resulting set of equations may be solved implicitly or explicitly depending on where approximations are taken in the time dimension. The explicit method requires strict limitations to keep the solution stable and convergent. It is easy to formulate and fast to program, but there is a cost in calculations due to the requirement of small enough time step size. Implicit methods, on the contrary, do not have such limitations to reach stabilities and convergence, but they need to calculate the inverse of a matrix at each time step which can be extremely computationally expensive. Both methods have their own limitations when solving high-dimensional nPDEs. This completes those methods which we term “Classical”.

Semi-Lagrangian Methods

Adaptations to the standard numerical methods are often required when working with HJB equations, for which we use the term Advanced Methods. The most widely used adaptations are Fast Marching, Fast Sweeping and Semi-Lagrangian methods. In this chapter, we mainly focus on SLMs.

Semi-Lagrangian Methods (SLMs), were firstly established by Wiin-Nielsen (1959) to overcome the drawbacks of Eulerian form derivations for a highly non-evenly distributed fluid. It then became very popular when solving weather forecast models in the 1980s due to its good performance in solving a PDE with large time steps (see Robert, 1981; Robert et al., 1985). With the recent emergence of stochastic control problems in modern finance and economics, Semi-Lagrangian algorithms are now widely used to solve the corresponding Hamilton-Jacobi-Bellman equations, which generally have characteristics of strong non-linearities, degeneracies and viscosity solutions. For example, in the paper d'Halluin et al. (2005), Semi-Lagrangian methods are used to solve American Asian Options under jump diffusions. In the later papers of Chen and Forsyth (2007) and Ware (2013), they are used to value natural gas storage with Real Option approaches. All of these problems have non-smooth value function and need to solve high-dimensional nPDEs.

Advantages and Drawbacks of SLMs

The main idea of a Semi-Lagrangian discretization is to integrate a PDE along the Lagrangian Trajectory, this makes it attractive when solving HJB equations for the following reasons. First of all, according to the authors of Ware (2013) and Warin (2015), the Semi-Lagrangian discretization is proven to be cost-effective with a high convergence rate and relatively low computational load (due to applicable large time steps and non-uniform grids). Secondly, Semi-Lagrangian discretization has a good capability to approximate non-smooth solutions. This is because Semi-Lagrangian methods can simulate nPDEs with large time steps, and still converge to the corresponding viscosity solutions (See Robert (1981) and Robert et al. (1985)). Third, when dealing with high dimensionality generated in Stochastic Control problems, Semi-Lagrangian methods are more effective than classic FDMs since they use interpolations on each Lagrangian Trajectory while the latter solves the equation dimension by dimension separately. Therefore, SLMs are applicable to high-dimensional (generally less than four dimensions) nPDEs with an affordable complexity. Last but not the least, according to recent work (Briani et al., 2004; Debrabant and Jakobsen, 2013; Ware, 2013; Tan, 2013; Warin, 2015), both the monotonic and non-monotonic convergence can be achieved with additional conditions on SLMs approximations. Monotonic numerical algorithms generally correspond to a low convergence rate, which requires more grids points, and they exhibit a highly viscous behaviour, which makes it hard to achieve the singularity of the solution. These papers extend the applications of SLMs further to more and more types of nPDEs, bringing great advantages to its use in practice.

Although Semi-Lagrangian Methods have lots of advantages, they are still not suitable for very high-dimensional nPDEs due to the fact that SLMs are essentially extensions of FDMs. SLMs require interpolation, so the accuracy and efficiency of the solution are strongly limited by the interpolation technique used. High dimensional interpolation can potentially be very computationally expensive. So, it is practical to say that SLMs are no longer competitive enough when solving nPDEs in more than four dimensions. Given all of the models in this thesis, all the problems can be modelled by nPDEs with no more than four dimensions. In addition to the mathematical improvements to the algorithms, such as non-regular grids and non-monotonic convergence, the continuous development of computer science makes it possible that parallel computing could reduce the total calculation time. In this thesis and outlined in this chapter, we are able to design a Semi-Lagrangian algorithm based on the OMP techniques[1] that greatly improves the efficiency of solving the high-dimensional nPDEs generated by our models. We now present how the SLMs can be used to solve an HJB step by step.

[1]OpenMP (Open Multi-Processing) is an API that supports multi-platform shared memory multiprocessing programming in C, C++, and Fortran, on most processor architectures and operating systems.

3.2 The HJB Variational Inequality (HJBVI)

In this thesis, most of our problems need to solve the following type of HJB Variational Inequality (HJBVI),

$$\min\left\{\underbrace{\partial_t u + \mathcal{L}(x, \partial x, \partial^2 x)u + \mathcal{H}(\partial y)u}_{HJB}, \underbrace{u - h}_{FBC}\right\} = 0. \tag{3.1}$$

The left-hand side of the equation includes two parts: the Hamilton Jacobin Bellman (HJB) part and the free boundary condition (FBC). Here, $\mathcal{L}$ is the elliptic linear differential operator, which is generally acting on the stochastic process; $\mathcal{H}$ is the non-linear first-order operator (or degenerate elliptic operator) derived from an optimal control process. We give more details of these operators in the following example. People can also reference Pham et al. (1998).

The basic idea to handle such type of nPDE is to discretize the HJB part in Equation (3.1) with Semi-Lagrangian Method, and then apply the PSOR method to find the relevant free boundary in the nPDE. To show this in more detail, in this chapter we consider the following generic three-dimensional model as an example,

$$\begin{aligned}
&\min\left\{\frac{1}{2}\sigma(t,x)^2\frac{\partial^2 u}{\partial x^2} + \mu(t,x)\frac{\partial u}{\partial x} - \rho u\right.\\
&+ \max_d\left\{\frac{\partial u}{\partial t} + e(t,x,y;d)\frac{\partial u}{\partial y} + g(t,x,y;d)\right\},\\
&\left. u - h\right\} = 0.
\end{aligned} \tag{3.2}$$

To focus on the numerical aspects we ignore the economic meaning of each variable, just bear in mind that u denotes the value function, t denotes the time, x denotes the stochastic dimension of the system, y is the continuous path that only have first-order partial derivative in the nPDE and the rest coefficients are all constant. The out-layer *min* operator is used to define the free boundary, and the internal maximum operator *max* is used to find the optimal control policy.

To simplify the notations, we firstly define the following operators:

a. The Linear Operator

$$\mathcal{L}(x, \partial x, \partial^2 x)u = \frac{1}{2}\sigma(t,x)^2 \partial_{xx} u + \mu(t,x)\partial_x u - \rho u, \tag{3.3}$$

where, $\mathcal{L}$ is elliptic since $\sigma(t, X)^2$ is always positive definite.

b. The Non-linear first-order Operator

$$\mathcal{H}(g, \partial t, \partial y)u = \max_d \big\{ \partial_t u + e(t,x,y;d)\partial_y + g(t,x,y;d) \big\}. \tag{3.4}$$

Based on these notations, we can simplify the notations of the nPDE (3.2) as,

$$\min\{\mathcal{L}(x, \partial x, \partial^2 x)u + \mathcal{H}(g, \partial t, \partial y)u, u - h\} = 0. \tag{3.5}$$

3.3 Semi-Lagrangian Methods

We have already mentioned how the basic idea of SLMs is to integrate the PDE along the Lagrangian Trajectory instead of solving each dimension independently, so to illustrate this by taking the Equation (3.5) as an example, we consider the following path,

$$dy = e(t, x, y; d)dt. \tag{3.6}$$

For any given value of control variable d, the variation of the value function on the (t, y) dimensions, if following the Lagrangian derivative formula can be defined as,

$$\frac{Du}{Dt}(d) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{3.7}$$

Following this trajectory, substituting Equation (3.7) into Equation (3.5), the original nPDE can be rewritten as,

$$\min\left\{ \mathcal{L}(x, \partial x, \partial^2 x)u + \max_d \left\{ \frac{Du}{Dt}(d) + g(t,x,y;d) \right\}, u - h \right\} = 0. \tag{3.8}$$

By comparing the original nPDE (3.5) with the new form of nPDE (3.8), we can see that the Semi-Lagrangian methods, to some extent, reformulate the three dimensional equation (t, x, y) as a new simplified three dimensional equation $(x, \{t, y\})$, with which we do not solve the nPDE on the dimension y independently, instead, we can solve point by point by approximating $\frac{Du}{D\tau}$ along the Lagrangian Trajectory using interpolations. Next we describe in detail how to discretize the nPDE following several crucial steps.

3.3.1 Discretization

The key steps required to discretize the Model (3.5) can be summarised as follows,

Step a: Define grids and relevant notations;

Step b: Approximate $\mathcal{L}$ operator with Finite Difference Methods;

Step c: Approximate $\mathcal{H}$ operator along a Lagrangian Trajectory;

Step d: Solve the tri-diagonal system and the free boundary by using PSOR methods.

Step a: Define grids and relevant notations

We use $\tau = T - t$ to denote the backward time, then the Equation (3.8) can be rewritten as,

$$\min\left\{\mathcal{L}(x, \partial x, \partial^2 x)u + \max_d\left\{-\frac{Du}{D\tau}(d) + g(\tau, x, y; d)\right\}, u - h\right\} = 0. \tag{3.9}$$

We consider a four dimensional space (τ, x, y), where $\tau \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$. To set up the grids on each dimension, we consider $\tau \in [0, T]$, $x \in [x_{min}, x_{max}]$, $y \in [y_{min}, y_{max}]$. If we are using unequally spaced grids, $[x_{min}, x_{max}]$ can be descretized by $[x_0, x_1, ..., x_i, ..., x_I]$ with step size $[\Delta x_0, \Delta x_1, ..., \Delta x_{I-1}]$; $[y_{min}, y_{max}]$ can be discretized by $[y_0, y_1, ..., y_j, ..., y_J]$ with step size $[\Delta y_0, \Delta y_1, ..., \Delta y_{J-1}]$. The discretization of τ could be written as $[\tau_0, \tau_1, ..., \tau_{N-1}, T]$ with step size $[\Delta\tau_0, \Delta\tau_1, ..., \Delta\tau_{N-1}]$. The mesh grids have to satisfy the following conditions in order to ensure the stability, consistency and monotonicity of the scheme in later sections.

Condition 3.3.1. *Define*

$$\begin{aligned}
\Delta x_{max} &= \max_i(\Delta x_i); \quad \Delta y_{max} = \max_j(\Delta y_j); \\
\Delta x_{min} &= \min_i(\Delta x_i); \quad \Delta y_{min} = \min_j(\Delta y_j); \\
\Delta\tau_{max} &= \max_n(\Delta\tau_n); \quad \Delta\tau_{min} = \min_n(\Delta\tau_n).
\end{aligned} \tag{3.10}$$

We assume there are mesh size/timestep parameters $\delta_{min} = C_0\delta_{max}$ *such that,*

$$\Delta x_{max} = C_1\delta_{max}; \quad \Delta y_{max} = C_2\delta_{max}; \tag{3.11}$$

$$\Delta x_{min} = C_1'\delta_{min}; \quad \Delta y_{min} = C_2'\delta_{max}; \tag{3.12}$$

where $C_0, C_1, C_2, C_1', C_2'$ *are independent of* δ_{min} *and* δ_{max}.

We denote u as the continuous function, v as the corresponding approximated numerical value, and particularly $*$ is used to mark the interpolated value when approximate $\frac{Du}{D\tau}$. For example, we define σ_i, μ_i, $e^n_{i,j}$, $f^n_{i,j}$, $g^n_{i,j}$ and $v^n_{i,j}$ as the approximated numerical value on the grid (τ_n, x_i, y_j), and $v^*_{i,j}$ as the interpolated objective function value that will arrive on grid (x_i, y_j) at time τ_n. We will give more explanation for each notation when we use them later on.

Step b: Approximate $\mathcal{L}$ operator with Finite Difference Methods (FDMs)
As a part of Semi-Lagrangian Method, we descretize the linear operator $\mathcal{L}u$ on the grid (x_i, y_j) at time τ_n using standard FDMs . We focus on the grids where $i = 1, 2, ..., I-1$, $j = 1, 2, ...J - 1$ and $n = 1, 2, ..., N$. As for the grids on the boundaries, we will show how to approximate them later. Also, we do not show the details of how to use the Taylor Expansion to derive the Formula (3.13), these can be found in Durran (2010).

To keep simplicity, in this part, we denote $\sigma_i, \mu_i, v_{i-1}, v_i, v_{i+1}$ as the numerical approximation of $\sigma(\tau_n, x_i)$, $\mu(\tau_n, x_i)$, $u(\tau_n, x_{i-1}, y_j)$, $u(\tau_n, x_i, y_j)$ and $u(\tau_n, x_{i+1}, y_j)$, respectively. With these notifications, we can define the approximation to the linear operator $\mathcal{L}$ as,

$$\mathcal{L}^n_{i,j} = \alpha_i v_{i-1} + \gamma_i v_i + \beta_i v_{i+1}, \tag{3.13}$$

where, α_i, β_i can be found according to the following formulas, and the γ is given by $\gamma_i = -(\alpha_i + \beta_i + \rho)$.

a. <u>Forward Approximation:</u>

$$\begin{cases} \alpha_i = \dfrac{\sigma_i^2}{\Delta x_{i-1}(\Delta x_i + \Delta x_{i-1})} \\ \beta_i = \dfrac{\sigma_i^2}{\Delta x_i(\Delta x_i + \Delta x_{i-1})} + \dfrac{\mu_i}{\Delta x_i} \end{cases}$$

b. <u>Central Approximation:</u>

$$\begin{cases} \alpha_i = \dfrac{\sigma_i^2}{\Delta x_{i-1}(\Delta x_i + \Delta x_{i-1})} - \dfrac{\mu_i}{\Delta x_i + \Delta x_{i-1}} \\ \beta_i = \dfrac{\sigma_i^2}{\Delta x_i(\Delta x_i + \Delta x_{i-1})} + \dfrac{\mu_i}{\Delta x_i + \Delta x_{i-1}} \end{cases}$$

c. Backward Approximation:

$$\begin{cases} \alpha_i = \dfrac{\sigma_i^2}{\Delta x_{i-1}(\Delta x_i + \Delta x_{i-1})} - \dfrac{\mu_i}{\Delta x_{i-1}} \\ \beta_i = \dfrac{\sigma_i^2}{\Delta x_i(\Delta x_i + \Delta x_{i-1})} \end{cases}$$

According to the studies of Chen and Forsyth (2007) and Ware (2013), when the Linear Operator (3.13) is used to structure an SLM, the coefficients α_i, β_i have to satisfy the following sufficient condition.

Condition 3.3.2.

$$\alpha_i \geq 0; \qquad \beta_i \geq 0 \ \ for \ \ i = 0, .., I; \quad j = 0, ..., J; \quad n = 1, ..., N. \tag{3.14}$$

We use the following forward-backward-selection algorithm to take the advantage of second order accuracy and also to meet the above condition.

Forward Backward Selecting Approach (3.15)

Starting with: $\alpha_i, \beta_i \Leftarrow$ **central**
if $\alpha_i \leq 0$ **or** $\beta_i \leq 0$ **then**
  if $\alpha_i \leq 0$ **then**
    $\alpha_i, \beta_i \Leftarrow$ **forward**
  else
    $\alpha_i, \beta_i \Leftarrow$ **backward**
  end if
end if

Step c: Approximate $\mathcal{H}$ operator along a Lagrangian Trajectory

To discretize the non-linear operator $\mathcal{H}$, we consider how the value changes between two discrete time points τ_n and τ_{n+1} by using the rectangular rules[2]. Our objective is to find the value $v_{i,j}^{n+1}$ at time τ_{n+1} by using all the formulas and pre-solved values $v_{i,j}^n$ at time τ_n.

[2]Using the rectangular rule provides a fully implicit time stepping scheme, while using the trapezoidal rule gives a Crank-Nicholson time stepping scheme. Although Crank-Nicholson scheme gives a higher order convergence rate, the fully implicit scheme present a dynamic which is closer to a real decision process Chen and Forsyth (2007)

We reformulate the y path that were illustrated at the beginning of this section going backwards in time,

$$dy = -e(\tau, x, y; d)d\tau. \tag{3.16}$$

Following the Lagrangian Trajectory defined by this path, we have,

$$\frac{Du}{D\tau}(d) = \frac{\partial u}{\partial \tau} - e\frac{\partial u}{\partial y}. \tag{3.17}$$

The task now becomes how can we discretize the Lagrangian derivative formula using all the given information at time τ_n. We do this with the following numerical approximation,

$$\frac{Du}{D\tau}(d) \approx \frac{v_{i,j}^{n+1} - v^*(d)}{\Delta\tau_n}, \tag{3.18}$$

where, $v_{i,j}^{n+1}$ is the discrete value of u at grid point (τ_{n+1}, x_i, y_j), which is our target value to solve. It should be mentioned that we assume that $v_{i,j}^{n+1}$ is independent with control parameter d, this is because we use the rectangle rule in the time dimension. We define v^* as the value on the exact point (x_i, y^*), from which position, if we depart on time τ_n, following the Lagrangian Trajectory, we can then arrive at (x_i, y_j) at time τ_{n+1}.

Since we do not have analytic formula of the path $y(\cdot)$, we need to find an estimated value (y_j^*) of (y^*) using their discrete growth formula at each time step τ_n based on the rectangle rule between τ_n and τ_{n+1},

$$y_j^{n+1} = y_j^* - e(\tau_{n+1}, x_i, y_j; d)\Delta\tau_n, \tag{3.19}$$

Here, we define y_j^* as the numerical approximation of y^* at time τ_n.

Although we find an approximation for (y^*), we still must address a way to find the value $u(x_i, y^*)$, because we do not have the analytical formula of u either. Fortunately, we know $v_{i,j}^n$, the numerical estimation of u, on all the predefined grids (x_i, y_j) at time τ_n. Therefore, we can approximate $u(x_i, y^*)$ by doing a numerical interpolation on the given surface $v_{i,j}^n$ to find $v_{i,j}^*$. Thus, we have,

$$\frac{Du}{D\tau}(d) \approx \frac{v_{i,j}^{n+1} - v_{i,j}^*(d)}{\Delta\tau_n} + \text{errors}. \tag{3.20}$$

It should be noted that the errors of this approximation will enter from two aspects: the numerical approximation of position (y^*) and the numerical interpolation on the discrete surface $v^n_{i,j}$. The accuracy ratio of the total algorithm is mainly limited by the lower accuracy of these two approximations. Therefore, theoretically at least, we need to increase the accuracy of both of these two approximations, so that we can get a more accurate numerical solution of the nPDE. More discussions can be found in the book Hamming (2012).

With the above discrete formulas, we can now discretize the $\mathcal{H}$ operator as,

$$\mathcal{H}^{n+1}_{i,j} \approx \max_d \left\{ -\frac{v^{n+1}_{i,j} - v^*(d)}{\Delta\tau_n} + g^n_{i,j}(d) \right\}. \tag{3.21}$$

Step e: Solve the tri-diagonal system and free boundary

Till now, we have illustrated how to discrete the linear operator $\mathcal{L}$ and the non-linear operator $\mathcal{H}$. We now show how to use a fully-implicit scheme on the whole Model (3.9) to get a tri-diagonal system.

Applying the operator formulas defined above to the HJB part of Equation (3.9), we have,

$$\mathcal{L}^{n+1}_{i,j} + \mathcal{H}^{n+1}_{i,j} = 0. \tag{3.22}$$

More precisely,

$$\alpha_i v^{n+1}_{i-1,j} + \gamma_i v^{n+1}_{i,j} + \beta_i v^{n+1}_{i+1,j} + \max_d \left\{ -\frac{v^{n+1}_{i,j} - v^*(d)}{\Delta\tau_n} + g^n_{i,j}(d) \right\} = 0. \tag{3.23}$$

To simplify the notations, we denote $v^{n+1}_{i-1,j}$ as v_{i-1} , $v^{n+1}_{i,j}$ as v_i, and $v^{n+1}_{i+1,j}$ as v_{i+1}, on fixed grids (y_j, τ_{n+1}). Thus, the above equation can be rewritten as,

$$a_i v_{i-1} + b_i v_i + c_i v_{i+1} = f_i, \quad \text{where} \ \ a_0 = 0, c_I = 0. \tag{3.24}$$

$$\begin{bmatrix} b_0 & c_0 & & & 0 \\ a_1 & b_1 & c_1 & & \\ & a_2 & \cdot & \cdot & \\ & & \cdot & \cdot & c_{n-1} \\ 0 & & & a_I & b_I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_I \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ f_I \end{bmatrix},$$

where, the coefficient of a_i, b_i, c_i, f_i $(i = 1, 2, ..., I - 1)$ are given by the following formulas,

$$
\begin{cases}
a_i = \alpha_i, \\
b_i = \gamma_i - \dfrac{1}{\Delta\tau_n}, \\
c_i = \beta_i, \\
f_i = -\max\limits_{d}\left\{\dfrac{v^*(d)}{\Delta\tau_n} + g^n_{i,j}(d)\right\}.
\end{cases}
$$

The coefficients of a_i, b_i, c_i, f_i on $i = 0$ and $i = N$ require the discretization of the appropriate boundary conditions. We leave the exact formulation until the particular case is defined in the following chapters.

Now we show some algorithms that can be used to solve the tri-diagonal system and find the corresponding free boundary. We use Thomas methods [3] to solve a tri-diagonal system generated by a pure HJB equation, and PSOR to solve a system with free boundaries.

Thomas:

```
Require: a, b, c, f
  for i=1 → n do
    m = a_i / b_{i-1}
    b_i = b_i − m c_{i-1}
    f_i = f_i − m f_{i-1}
  end for
  v_n = f_n / b_n
  for i=n-1 → 0 do
    v_k = (f_k − c_k v_{k+1}) / b_k
  end for
```

PSOR:

```
Require: a, b, c, f, ω
  set v_0, v_n
  repeat
    set error = 0
    for i = 1 → n − 1 do
      ς = (f_i − a_i v_{i-1} − c_i v_{i+1}) / b_i
      v_i = max(v_i + ω(ς − v_i), g_i)
      error+ = (ς − v_i)^2
    end for
  until (error < tol)
```

The whole scheme

Based on the above discretization, we now define the entire scheme for one type of Model (3.1),

[3] http://www.cfd-online.com/Wiki/Tridiagonal_matrix_algorithm_-_TDMA_%28Thomas_algorithm%29

```
Set termination payoff v at τ = 0
//backward time loop
for n = 0 → N do
   for i = 0 → I, j = 0 → J do
      Approximate v*_{i,j} by using v^n
   end for
   // By sharing memory of v*, we implement parallel computing with OpenMP tech-
   niques to solve each tri-diagonal system independently on x dimension.
   for j = 0 → J do
      Set tri-diagonal system on x dimension
      Solving tri-diagonal system with PSOR or Thomas to get v^{n+1}
   end for
end for
```

3.3.2 Convergence, Stability, Consistency and Monotonicity

We know that for a highly non-linear HJB, there might be no unique solution to the problem. Therefore, it is important to ensure that the numerical solution that generated by following the above scheme converges to a unique viscosity solution in the weak sense. To achieve this, we need to analyse the algorithm from two different aspects: the approximation in the smooth region and the calculation of the boundary.

There are many examples in the literature that prove an SLM converges to the unique viscosity solution of the corresponding nPDE. Since the SLM is not our original contribution, we just simply show the sufficient conditions to guarantee the convergence, stability and consistency. As for the proofs, we refer the reader to Pham et al. (1998); Briani et al. (2004); Chen and Forsyth (2007) and the Chapter seven of Durran (2010).

Condition 3.3.3. ***Sufficient conditions on boundaries:***

Statement 1: *If the coefficient of the diffusion term vanishes at a region on a boundary with an outgoing characteristic, independent of the value for the control variable, then the viscosity solution on this boundary region is the limit of the viscosity solution from interior points.*

Statement 2: *If the characteristic at a region on the boundary, associated with the* *first-order term in the PDE, is incoming to the domain independent of the choice of the control value, then the viscosity solution at the region corresponds to the specified boundary data in the classical sense.*

We now link these two statements with the models that we need to solve in this thesis. First of all, we summarize the boundary conditions associated with the HJB equation defined in the HJBVI (3.2):

$$\frac{1}{2}\sigma(t,x)^2\frac{\partial^2 u}{\partial x^2} + \mu(t,x)\frac{\partial u}{\partial x} - \rho u + \max_d\left\{\frac{\partial u}{\partial t} + e(t,x,y;d)\frac{\partial u}{\partial y} + g(t,x,y;d)\right\} = 0, \tag{3.25}$$

when

$$(x,y,t) \in (0,x_{max}) \times [0,y_{max}] \times (0,T);$$

$$\mu(t,x)\frac{\partial u}{\partial x} - \rho u + \max_d\left\{\frac{\partial u}{\partial t} + e(t,x,y;d)\frac{\partial u}{\partial y} + g(t,x,y;d)\right\} = 0, \tag{3.26}$$

when

$$(x,y,t) \in \{0,x_{max}\} \times [0,y_{max}] \times (0,T);$$

$$u(T,x,y) = payoff(T,x,y) \tag{3.27}$$

when

$$(x,y,t) \in [0,x_{max}] \times [0,y_{max}] \times \{T\}.$$

To link these equations with *Statement* 1, we can treat the HJB (3.25) as a three dimensional degenerate equation (the second order derivatives of dimension t and y are vanished). In addition, our numerical test are all based on GBM, the $\mu(t,x)$ is always positive. By combing these two points, the information flow along t and y dimensions are outgoing characters, which means that the viscosity solution on this boundary region is the limit of the viscosity solution from interior points. As for *Statement* 2, we will show in the later chapters that the optimal control parameters are independent from the boundary conditions (i.e. $x = 0$ and $x = x_{max}$). There are basically two approaches can help us to achieve this: One way is to address the control parameter on these boundaries based on the economic conditions and the requirements of the real problem; Another approach is to limit the internal calculation region by free boundaries and solve the system within a format of Variational Inequality

Equation. With these approaches, our models used in this thesis can satisfy the second statement. We then define the following sufficient conditions to guarantee the stability and consistency for the whole scheme.

Condition 3.3.4. ***Sufficient conditions for scheme applied in the interior region:*** *Suppose the discretization satisfies the mesh grid size setting* (3.3.1)*, the linear operator satisfies the positive coefficient condition* (3.14)*, the non-linear operator using bilinear interpolation, while also the boundary conditions satisfy the statements* (3.3.3)*, then the scheme converges to the unique viscosity solution of the model* (3.2)*. Further, the scheme satisfies the Stability, Consistency and Monotonicity defined in the work Chen and Forsyth (2007). (The proof can be found in the suggested paper above.)*

3.3.3 Useful Suggestions in Practice

1. Non-Uniform Grids

   The Semi-Lagrangian algorithm allows for a discretization with highly non-uniform grids. This gives us an additional choice to balance the accuracy and calculation cost. For example, if our aim is to try and find the position of a free boundary with high accuracy but don't want to pay too much attention to the value, we can just set more grid points in the region where the boundary is located and fewer grids elsewhere. It should be mentioned that to use non-uniform grids, the parameter setting has to strictly meet the grid size condition (3.14).

2. Use parallel computing techniques

   Parallel Computing is an important technique to help speed up calculations. The fundamental idea of this technique is to split the whole job into different parallel sub-jobs. By running these sub-jobs independently on different computer resource at the same time, we can reduce the total calculation time. Theoretically we can say that this technique can at most increase the efficiency by $\frac{1}{N}$ where N is the number of sub-jobs. If the Semi-Lagrangian methods are programmed with parallel computing, we can solve the high-dimensional model more efficiently. Some calculations in this thesis are based on the OpenMP techniques.

3.4 Conclusion

In this chapter, we reviewed the numerical approaches we shall employ to solve our stochastic control problems, and in particularly discussed the motivations and limitations of using Semi-Lagrangian methods. By taking a four-dimensional partial integral differential equation (3.1) as an example, we illustrated all the techniques that are required to discretize an nPDE. Based on these techniques, we further discussed the necessary conditions that guarantee the consistency, stability and monotonicity of the Semi-Lagrangian algorithm. At the end of this chapter, we also listed several approaches to speed up the calculation, which is very useful in practice. This chapter presents enough information to reproduce the results for the subsequent chapters. In each of the following chapters, we will show how to implement these algorithms to solve a particular problem case by case.

Chapter 4

Cash-constrained Abandonment Option

4.1 Introduction

Classical Real Option Analysis aims at using financial option valuation theory and techniques to help companies make capital budgeting decisions. These methods are proposed based on people's understanding that corporate decision making is a firm's right rather than obligation. Thus, when evaluating a firm, we need to consider not only the discounted cash flows, but also the potential value of optimising corporate flexibility.

With the growing applications of Real Options, people noted that classical Real Option models cannot properly handle the interdependent decisions that share the same cash constraints. As explained by Trigeorgis (1993a, 1996) and Wang (2010), Real Options do not consider the effects of cash limitations. This is because they assume firms have unlimited access to external financial markets for risk-free money, thus holding or not holding cash has no difference. However, recent studies from both theoretical and empirical fields show that cash holdings are the essential consideration in corporate decision-making. For example, Kisser (2013) presented the theoretical value of holding cash to support a growth option, and Bond and Meghir (1994), Opler et al. (1999), Denis and Sibilkov (2009) and Fresard (2010) provided empirical evidences of strong connection between cash holdings and various corporate decisions. In addition, as discussed in Anderson and Carverhill (2012), a firm might suffer liquidity risk if there are no enough cash assets to afford the running cost or debt obligation, even it is a valuable project in the long run. Thus, cash holdings can be used as a natural measure of liquidity risk.

Although the importance of involving cash holdings in analysing corporate decisions is realized, to model cash holdings within a Real Option framework has a lot of barriers. A firm's current cash level depends on its past decisions and cash flows, thus a path dependent model is required. This extra dimension will eliminate in one stroke the possibility of an analytic solution. To study this problem, we need to handle a three-dimensional partial differential equation based on its approximated numerical solution, and that requires an efficient numerical algorithm.

In this chapter, we show how to overcome these obstacles and study the impacts of liquidity restrictions on making a corporate decision. By taking a firm's abandonment decision as an example, we first show how to model this option with cash holdings, and then solve it numerically with Semi-Lagrangian methods. Based on the results, we pay particular attention to how our model is different with the NPV and the classical Real Option methods. The advantages and disadvantages of these models in valuing a firm's decision are also discussed in the empirical analysis.

4.2 Mathematical Model

4.2.1 Problem Setting

Before presenting any mathematical formula, it is useful to overview the main settings of the problem. We consider a firm that has a project that will generate uncertain revenues. To keep the project running, the firm has to pay a constant running cost. Since the future revenue is full of uncertainty, the firm might suffer short-term non-profitable periods when the revenue is less than the operational cost. In order to hedge this short-term risk and keep paying operational costs, we assume the firm holds a certain amount of cash as reserves. This precautionary cash, along with the instantaneous profits, is to be deposited into a saving account with a risk-free interest rate. When the business is profitable, the profits instantaneously get paid into the savings account, whereas, in the opposite case, the firm has to continuously consume its retained cash in order to keep normal operating. We define internal insolvency as the scenario when a firm has no cash left to pay the operational costs. In this case, it has no option but to close. To mitigate losses and the probability of internal insolvency, we assume the firm has an option to close the project when it believes that the expected accumulated cash flows in the future cannot even surpass the abandonment cost.

To study this type of firm, we combine classical Real Option theory with the firm's cash holdings. We now go step by step to specify the mathematical formulas of the firm model and present relevant numerical solutions.

4.2.2 Stochastic Framework

The Normalized Annual Revenue: We assume the revenue generated by the project follows a stochastic process, which is defined on the time interval $s \in [t, T]$ with a constant initial value $R_t = R$.

$$dR_s = \mu(R_s, s)ds + \sigma(R_s, s)dW_s^P, \tag{4.1}$$

where $\mu(R_s, s)$ is the drift function which measures the average growth rate of the revenues along with time, $\sigma(R_s, s)$ is the variance function that measures the level of the revenues uncertainty, and W_s^P is the Brownian Motion under the market probability measure P.

It should be noted that, in this thesis, we mainly consider Geometric Brownian Motion (GBM) due to the following considerations: First of all, GBM provides a positive value process, which is useful to model a firm's revenue; Second, there are plenty of closed form solutions for GBM based Real Option models (see, particularly Dixit and Pindyck (1994)), thus we can compare our work with the corresponding classical Real Option methods. As such, in the following chapters, we define $\mu(R_s, s) = \mu R_s$ and

$\sigma(R_s, s) = \sigma R_s$.

Cash Holdings: We assume the firm holds some initial cash assets C_t at time t, and this amount of money combined with the firm's instantaneous net profits $(R_s - \varepsilon + rC_s)ds$, can be invested in a savings account immediately with interest rate r, where ε is the operational cost. Since R_s is stochastic, the net profits $(R_s - \varepsilon + rC_s)ds$ can be either negative or positive. When the net profit is positive, the cash holdings build up, otherwise, the cash level reduces until zero. With this, we can define the cash holdings process on the time interval $s \in [t, T]$ as,

$$dC_s = (R_s - \varepsilon + rC_s)ds \quad \text{where} \quad C_t = C \quad \& \quad C_s \geq 0. \tag{4.2}$$

Here, we assume C_s can only take no-negative value. This is because, in our problem setting, the firm cannot have negative cash holdings. Whenever, $C_s \leq 0$, we need to check the following abandonment or insolvency conditions.

Abandonment and Insolvency: According to the cash holdings formula, when the income is lower than the operational cost, the firm has to consume the retained cash until the revenue goes over ε again. If this negative profit situation lasts for a long time, there are two possible consequences: the manager actively abandons the business, or the firm declares bankruptcy due to the lack of cash to keep normal operating.

In the first scenario, we assume the firm has an option to abandon its business before time T. If the business stays in a negative profit situation for a long time, the future value of the project can become very low, or even negative. This abandonment option gives the firm the flexibility to limit the loss, or hedge the downside risk. Suppose an abandonment cost $\kappa(C)$ $(\kappa(C) \geq 0)$[1] is incurred at the abandonment time. If we denote $V(s, R_s, C_s)$ as the firm value at time s, then time τ at which the firm decides to abandon can be defined by the following stopping time[2],

$$\tau = \min_{s \in [t,T]} \{s : V(s, R_s, C_s) \leq -\kappa(C)\}. \tag{4.3}$$

[1]Essentially, the insolvency cost can be any types of functions, whereas which type of the function to use does not significantly affect the main structure of our cash-constrained Real Option model. Here we define it the same as the abandonment cost to help us focus on the main concern of cash holdings analysis. An independent insolvency cost function might be useful for people who particularly want to extend the model for corporate bankruptcy probability analysis, for example, to implement the idea of the paper Evatt et al. (2011).

[2]The theoretical explanation of stopping time can be found in the Section 2.2 or book by Peskir and Shiryaev (2006).

The second scenario happens when the firm has consumed all its cash holdings $C_s = 0$, and its revenue still cannot cover its operational cost, i.e. $R_s + rC_s < \varepsilon$. In this case, the firm has no option but to declare bankruptcy due to a liquidity shortage. We assume the cost of a forced abandonment due to insolvency is the same as the cost $-\kappa(C)$ when the firm exercises the option to abandon. Under this setting, the terminal payoff functions of Abandonment and Bankruptcy can be simplified by one formula,

$$h(\tau, R_\tau, C_\tau) = -\kappa(C). \tag{4.4}$$

It should be mentioned that the stopping time τ can only take a value from the interval $[t, T]$. When τ takes a value that is greater than or equal to T, it means that the project does not abandon or go insolvent before the end of its natural life. This happens only when the firm holds sufficient cash or the project is always profitable enough.

Objective Function: The objective of exercising this abandonment option is to maximize the project's discounted future cash flows. We define the project's instantaneous cash flow g as follows,

$$g(s, R_s, C_s) = R_s - \varepsilon \quad \text{for all} \quad s \in [t, T]. \tag{4.5}$$

We assume a general *market discount rate* ρ, which is different from risk-free rate r ($\rho > r$). This setting is very useful since a real investment or decision generally cannot be completely hedged with other portfolios of assets, which is quite different compared with financing options. The value of ρ can be understood as the external investment opportunity costs (or defined as the external risky opportunities returns). In practical applications, we can estimate it from many ways. The widely used approaches are Weighted Average Cost of Capital (WACC) (see book Brealey et al. (2010)) and the Capital Asset Pricing Model (CAPM) series of models (see more discussions in our paper Evatt et al. (2014).

By using the predefined revenue (4.1), cash holdings (4.2), and instantaneous income (4.5), the expected cash flow or project value at time t with a given abandonment time τ can be defined as,

$$J(t, R_t, C_t, \tau) = E\left[\int_t^\tau e^{-\rho(s-t)} g(s, R_s, C_s) ds + e^{-\rho(\tau-t)} h(\tau, R_\tau, C_\tau)\right]. \tag{4.6}$$

Thus the project value with an optimal abandonment time is given by $v(t, R_t, C_t)$,

$$v(t, R_t, C_t) = \sup_{\tau \in [t,T]} J(t, R_t, C_t, \tau). \tag{4.7}$$

4.2.3 HJB Variational Inequality (HJBVI)

According to the stochastic optimal control theory described in the mathematical background chapter, if the stochastic process (4.1) and the objective function (4.7) are well defined, we can derive a corresponding HJB Variational Inequality (HJBVI) to solve this optimal abandonment problem (see Chapter 2 or the book Pham (2009) for more details). To simplify the notation, we recall the following linear operator for a given function $v(x)$,

$$\mathcal{L}v = \mu \frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial v}{\partial x^2} - \rho v, \tag{4.8}$$

where x denotes some stochastic processes defined by the Equation (4.1), μ and σ are the corresponding drift and variance functions. With this linear operator, we can define the HJBVI of the perpetual project model (when the project lifetime is infinite) and the non-perpetual project model (when the project lifetime is finite) respectively as follows.

<u>For a perpetual project</u>, the Variational Inequality equation is given by,

$$\min\left\{ \mathcal{L}v + (R - \varepsilon + rC)\frac{\partial v}{\partial C} + R - \varepsilon, v + \kappa(C) \right\} = 0. \tag{4.9}$$

<u>For a non-perpetual project</u>, we define the HJB Variational Inequality as,

$$\min\left\{ \frac{\partial v}{\partial t} + \mathcal{L}v + (R - \varepsilon + rC)\frac{\partial v}{\partial C} + R - \varepsilon, v + \kappa(C) \right\} = 0. \tag{4.10}$$

The HJBVIs subject to boundary conditions,

$$\begin{cases} v = 0 & \text{when } t = T \textbf{ only for the non-perpetual model} & (4.11) \\ v = -\kappa(C) & \text{on} \quad C = 0 \;\&\; R - \varepsilon < 0, & (4.12) \\ \dfrac{\partial v}{\partial C} = 0 & \text{as} \quad C \to \infty \;\&\; 0 < R + rC - \varepsilon < \infty, & (4.13) \\ v = \hat{\alpha} R + \hat{\beta}\varepsilon + \hat{\gamma} & \text{as} \quad R \to \infty, & (4.14) \end{cases}$$

We now explain the economic meaning for each boundary condition.

1. The optimal abandonment boundary condition (see Equations (4.10) and (4.9))
   To deal with the firm's optimal abandonment, we can specify a free boundary condition[3]
$$v = -\kappa(C) \;\;\&\;\; \frac{\partial v}{\partial R} = 0 \;\; \text{on} \;\; (R, C) = (R^*, C^*), \tag{4.15}$$
   where the free boundary (R^*, C^*) is the set of points at which it is optimal to exercise the abandonment option. The first equation ($v = -\kappa(C)$) defines the exercise condition of the firm's abandonment option, that is, if the manager believes that the future profits cannot even surpass project value after payment the abandonment cost ($-\kappa(C)$), the firm would like to close its business. The second equation ($\frac{\partial v}{\partial R} = 0$) defines the smooth pasting condition, which ensures no arbitrage opportunity.

2. Termination boundary for a non-perpetual model (see Equation (4.11))
   The value of the project equals zero at the expiration date if the firm had not exercised the abandonment option.

3. The firm's bankruptcy boundary condition (see Equation (4.12))
   When the firm does not have any cash holdings, and its revenues are unable to pay the operational costs, the manager has to stop the project and pay the bankruptcy cost, thus $V = -\kappa(C)$.

4. Unlimited cash boundary condition (see Equation (4.13))
   To deal with the case in which there are unlimited cash holdings we arrive at (4.13). If there is enough cash to hedge any future risks, the marginal value of one extra dollar to the project is zero. So since the cash holdings no longer affect the project value, we can treat this situation as a classical Real Option framework that does not have any financial constraints.

5. The firm's no-risk boundary condition (see Equation (4.14))
   When the firm is very profitable ($R - \varepsilon \to \infty$), the firm does not need to hold extra cash to hedge future risks. Therefore, the expected value of the project is a linear function of its revenue level. In this case, we can use either a Neumann type boundary condition (or zero gamma condition from finance

[3]It should be noted that this boundary condition is already packaged into the Variational Inequality Equations (4.10) and (4.9). So, we do not list it independently with other boundary conditions.

perspective), $\frac{\partial^2 v}{\partial R^2} = 0$ as $R \to \infty$, or a Dirichlet-type boundary condition $v = \hat{\alpha}R + \hat{\beta}\varepsilon + \hat{\gamma}$ *as* $R \to \infty$ ($\hat{\gamma}$ denotes a constant). Here, we use the second type, since this analytic formula is more convenient to compare with the value of classic abandonment option. We now give two examples of how to define coefficient $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$.

$$v(t, R, C) = \frac{R}{\rho - \mu}(1 - e^{-(\rho-\mu)(T-t)}) - \frac{\varepsilon}{\rho}(1 - e^{-\rho(T-t)}). \tag{4.16}$$

If the revenue process is defined as a Mean Reverting process, this boundary can be defined by,

$$v(t, R, C) = \frac{R - \bar{R}}{\rho + \theta}(1 - e^{-(\rho+\theta)(T-t)}) + \frac{\bar{R} - \varepsilon}{\rho}(1 - e^{-\rho(T-t)}). \tag{4.17}$$

b. For a perpetual project (when $T \to \infty$), the coefficients functions $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are independent of time. Therefore we have the corresponding equations,

$$v(t, R, C) = \frac{R}{\rho - \mu} - \frac{\varepsilon}{\rho}, \tag{4.18}$$

and,

$$v(t, R, C) = \frac{R - \bar{R}}{\rho + \theta} + \frac{\bar{R} - \varepsilon}{\rho}. \tag{4.19}$$

It should be noted that the value R and C are the initial values. The value v is the project value with given initial t, R and C. In the numerical section, for simplicity, we assume $t = 0$. The derivation of the above equations can be found in many literatures, we suggested Dixit (1994), Myers and Majd (2001) and Bolton et al. (2014b) for more details.

4.3 Numerical Methods

The PDE model we derived in the previous section is a degenerate elliptic partial differential equation (without second order derivative $\frac{\partial^2 v}{\partial C^2}$ in the cash dimension). For this type of PDE, it is very difficult to find an analytical solution or even to approximate the solution with a finite different algorithm. This is because the PDE exhibits characteristic lines that have two opposite directions within the Revenue-Cash space, and the resulting boundary conditions on the Cash dimension are therefore piecewise distributed (see Figure 4.1). To overcome this, we use a Semi-Lagrangian discretization on the PDE. More discussion about the algorithm itself can be found in the numerical Chapter 3.

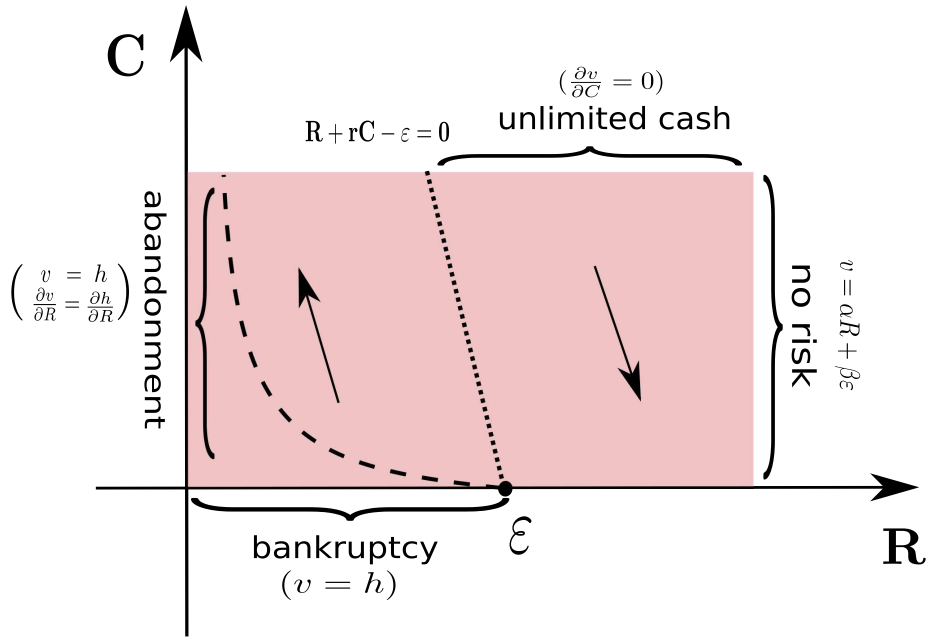


Figure 4.1: A schematic sketch to show the calculation domain and boundary condition layouts, where the arrows denote the characteristic line directions, the middle straight dashed line represents zero income barrier, and the curly dashed line denotes the abandonment boundary.

To explain the reason for this we must look more closely at Figure 4.1, and note that the term in Equation (4.10) that is important is the one multiplying $\frac{\partial v}{\partial C}$. When the revenue ($R$) plus the return on cash (rC) is less that the operational cost (ε), the instantaneous cash flow ($(R+rC-\varepsilon)dt$) is negative. In this case, the firm's cash levels are decreasing as we move forwards in time meaning that information flows from the bottom up when solving backwards in time (see arrow on the left). However, when the firm's Revenue (R) plus the return on cash (rC) is higher than operational cost (ε), the firm's cash level increases moving forwards in time meaning that information flows from the top down (see arrow on the right). In the negative cash flow region, since information is travelling upwards we require two boundary conditions: the free boundary of abandonment and the solid boundary of bankruptcy. Similarly, in the positive cash flow region, we have two boundary conditions: the no-risk boundary and the unlimited cash boundary. Given this problem setting, an outline of the algorithm required to solve this problem is given by the following equation,

$$\min\left\{\alpha_i v_{i-1} + \gamma_i v_i + \beta_i v_{i+1} - \frac{v_{i,j,k}^{n+1} - v^*}{\Delta\tau_n} + g_{i,j,k}^n, v_{i,j,k}^n + h_{i,j,k}^n\right\} = 0. \tag{4.20}$$

with boundaries,

$$\begin{cases} v_J = v_{J-1} \;\; \textbf{unlimited cash}, \\ v_i = \max(v_i, -\kappa(C)) \;\; \textbf{abandonment}, \\ v_{j=0} = 0 \;\; \textbf{bankruptcy}, \\ V_{I,j} = \alpha_{I,j} R_I + \beta_{I,j}\varepsilon \;\; \textbf{no risk}. \end{cases} \tag{4.21}$$

4.4 Solution and Analysis

Before doing any calculation, we first specify the abandonment and insolvency cost function as,

$$\kappa(C) = K. \tag{4.22}$$

Here we assume the cost K is a constant, and unless otherwise informed, it should be assumed that for all of the following calculations any unreferenced parameters should take the values given in Table 4.1. These parameters form the base set of our tests, and they represent a set of scenarios that best illustrate the features of our model.

Parameters	$\mu(y^{-1})$	σ	ρ	$\varepsilon(My^{-1})$	$r(y^{-1})$	$K(M)$	$T(y)$
Value	0.02	0.15	0.05	1	0.01	0	100

Table 4.1: Input parameter values used for numerical test and solutions, where μ and σ are the growth rate and volatility of the annualised revenue (given by the Equation (4.1)); ρ is the market discounted rate (used in the Equation (4.7)); ε is the annualised cost (given by the Equation (4.2)); r is the short term interest rate (used in the Equation (4.2)); K is the abandonment cost (defined in the Equation (4.22)); T is the expected life-time of the project (defined in Equations (4.16) and (4.17), and the discussion how to choose a T can be found in Chapter 3).

To simplify the explanation of the results, we normalize the parameters based on the operational cost $\varepsilon = 1$ (My^{-1}), so that we do not need to explain the unit of each variable.

4.4.1 Cash-Constrained Abandonment Option

Before solving a cash limited abandonment option model, we review the Net Present Value Model (see Brealey et al. (2010)) and classic abandonment option model (see Dixit and Pindyck (1994) or Bolton et al. (2014b)), and use them as benchmark models to highlight our contribution. In order to find an analytical solution as a benchmark, in this section, we only consider the GBM type of revenue. The closed-form solution will be used to compare these three kinds of models.

- **Continuous NPV:**

We assume the project will continuously generate deterministic net profits $R - \varepsilon$ in a perpetual time. Then the NPV of this project can be formulated by

$$\mathbf{NPV} = \max\left\{\frac{R-\varepsilon}{\rho-\mu}, 0\right\}. \tag{4.23}$$

It should be noted that the above definition is not a standard Net Present Value of a project. Here, we define it in this form to make our comparison more convenient.

- ***Classic Real Options Model without Cash Limitations:***

According to Dixit and Pindyck (1994), a firm's abandonment decision can be valued as its expected discounted cash flow before abandonment. They assume that a firm has no cash constraints, so the value of abandonment decision is independent of the firm's cash holdings i.e. $\frac{\partial v}{\partial C} = 0$. In contrast, in our model, we consider a cash constrained firm, and thus the marginal value of holding cash for the abandonment decision varies with the firm's cash holdings and it always satisfies that $\frac{\partial v}{\partial C} \geq 0$. When a firm has unlimited cash holdings, the firm can be treated as a financial unconstrained firm. From this perspective, the model given by Dixit and Pindyck (1994) can be treated as a special case of our cash constrained option model (4.9), when $C \to \infty$.

Following the idea of Dixit and Pindyck (1994), we now rebuild the classic Abandonment Option model. We assume the project will continuously generate net cash flow $g(R) = R - \varepsilon$ in a perpetual time, thus, the option value is only based on its uncertain revenue and we denote it as $v(R)$. This function can be solved by the following ordinary differential equation (ODE),

$$\mathcal{L}v + g = 0, \tag{4.24}$$

with the value-matching and smooth-pasting conditions,

$$\begin{cases} v = -K \quad \text{on} \quad R = R^*, \\ \dfrac{\partial v}{\partial R} = 0 \quad \text{on} \quad R = R^*, \\ v = \hat{\alpha} R + \hat{\beta}\varepsilon \quad \text{as} \quad R \to \infty. \end{cases}$$

In the above conditions, R^* denote the free boundary to exercise the option, and $\hat{\alpha}$ and $\hat{\beta}$ are the coefficients that defined in (4.18). For simplicity, in our comparison case, we assume $K = 0$. With these assumptions, we can get the unique closed-form solution when $R \geq R^*$,

$$v^*(R) = \begin{cases} \dfrac{1}{\gamma - 1}\left(\dfrac{R}{R^*}\right)^{\gamma}\left(K - \dfrac{\varepsilon}{\rho}\right) + \left(\dfrac{R}{\rho - \mu} - \dfrac{\varepsilon}{\rho}\right), \\ -K \quad \textbf{when} \quad R < R^*. \end{cases}$$

where,

$$\begin{cases} \gamma = \dfrac{1}{\sigma^2}\left[-\left(\mu - \dfrac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \dfrac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}\right], \\ R^* = \dfrac{\gamma}{1-\gamma}(K - \dfrac{\varepsilon}{\rho})(\rho - \mu). \end{cases}$$

and γ is the negative root of the character function.

We now present a set of results to compare three models: the Net Present Value model, the classical Real Option model and the cash-constrained Real Option model. The Net Present Value is directly solved by Equation 4.23.

The classical Real Option model is solved numerically by using standard finite difference methods based on PSOR algorithms (see Wilmott (1995)), whilst the cash-constrained Real Option model derived in this chapter can be solved by using Semi-Lagrangian methods illustrated in the above section. It should be noted, to simply the comparison and explanation, in the following contents, we use option value to denote the project value that solved by the cash-constrained Real Option model.

Figure 4.2 shows the difference between the Net Present Value, the classical Real Option and the cash-constrained Real Option with different cash holdings when valuing an abandonment option. The top solid line shows the option value generated by the classical Real Option method, which only depends on the revenue without any cash limitations. The middle three dashed lines present how the option value changes against the revenue for a company that currently holds half year, one month and zero precautionary cash holdings to pay the operational cost. The relevant solid circle at the bottom on each line marks the free boundary to abandon the business in each particular case. The lowest straight line shows the discounted Net Present Value defined by Equation (4.23). It is bounded by zero and grows linearly with the increasing positive net revenue. According to this figure, we can see that the classical Real Option gives the highest evaluation, the NPV gives the lowest estimation, whereas, the cash-constrained Real Option approach sits somewhere in the middle. This is because the more optionality a project has, the more value it has. Under this setting, the classical Real Option model overvalues projects and provides misleading decisions.

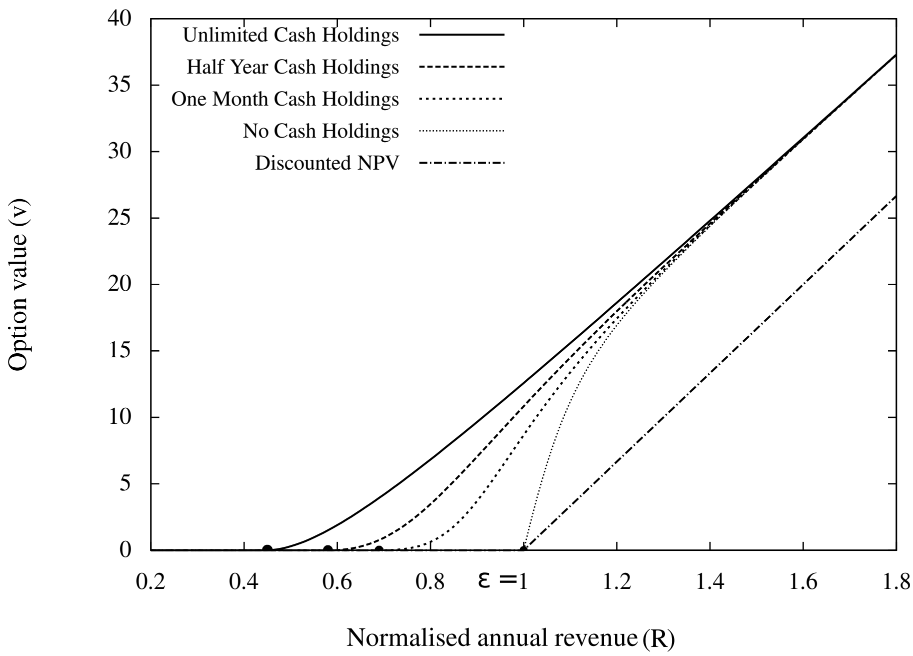


Figure 4.2: A comparison of three different approaches in project valuation: the Net Present Value (the bottom straight dash dotted line), the classical Real Option (the top solid line) and the Real Option with cash constraints (dotted lines in the middle). Here, the operational cost ε is 1; the abandonment cost K takes 0; and the cash holdings C take $0, \frac{1}{12}, \frac{1}{2}$ and 10 times of operational cost (equivalently to unlimited cash holdings).

Figure 4.3 presents how the option value varies with different cash levels by taking fixed R such that the net revenues are $-0.05, 0$ and 0.05. We use the vertical line to mark the position at which the marginal value of cash holdings first-time hit zero for that particular value of R. The left-hand side circle on each line curve defines the abandonment option value in each particular case when the cash holdings are zero.

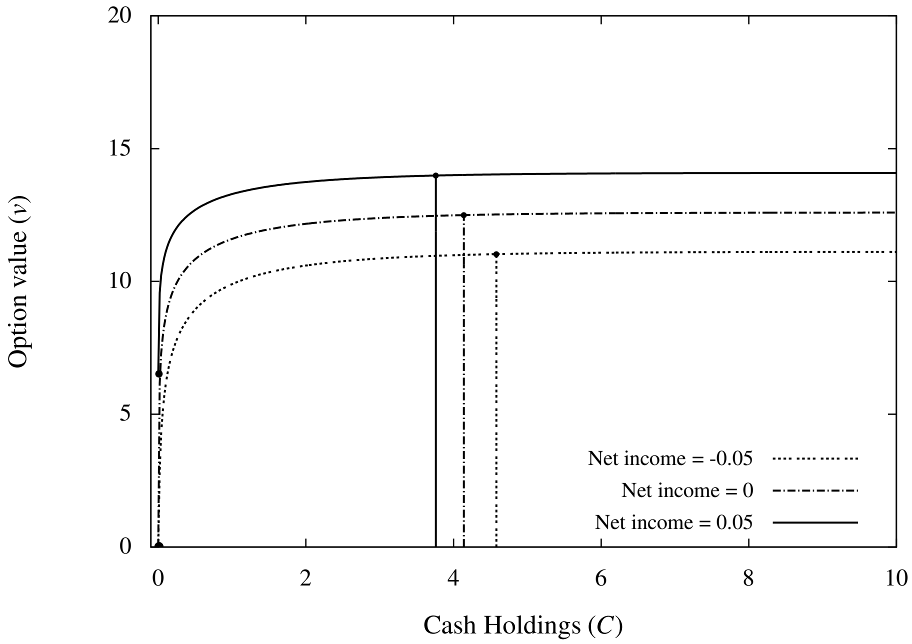


Figure 4.3: The value of the cash-constrained abandonment option against the cash holdings, where the Net income takes value $-0.05, 0$ and 0.05.

According to this figure, we can see that the marginal value of cash decreases with higher cash holdings. Particularly, the first small amount of cash held in reserve can play a significant role to increase the option value. Taking the $R - \varepsilon = 0.05$ line as an example, 0.25 years cash holdings of running costs can increase the project's value from 6.5 to 13. In addition, by comparing the optimal cash holding levels of three cases $R - \varepsilon = -0.05, 0$ and 0.05, we can see the higher profitability the firm has, the fewer cash holdings that a firm needs to hold to fully exercise its flexibility.

In Figure 4.4 we look at how the abandonment boundaries that are generated by a cash-constrained Real Option model and a classical Real Option differ. According to this figure, we can see that the whole operational space (Cash-Revenue) is divided into two regions by the abandonment boundary (the curved-dashed line): the abandonment region and the continuation region. According to the solid line, we can see that a small increase in the amount of cash (here, when $C \leq 5$) can significantly affect the abandonment decisions. This observation suggests that classical Real Option approach might provide misleading decisions since it assumes unlimited cash holdings. By further comparing the solid line and the dashed vertical line, we can see that the abandonment boundary generated by the cash-constrained Real Option model convergences to that generated by the classical Real Option model when the cash level is high enough. This is because when a project has enough cash it can be treated as one that has no financial constraints. This is what naturally assumed in the classical Real Option model.

We are also interested in how the effects of cash holdings vary with project lifetime since long-life projects generally face more uncertainty in the future. Figure 4.5 presents various abandonment boundaries for projects that have different life spans, i.e. ten years, twenty years, fifty years and a hundred years in turn. By comparing these four lines, we can see that cash holdings are more important to projects with a longer expiration date. For example, a project with 100 years to run needs only 6 units of cash to reach the abandonment boundary that solved by classical Real Option model, whereas a project of 10 years needs less than 1 unit of cash.

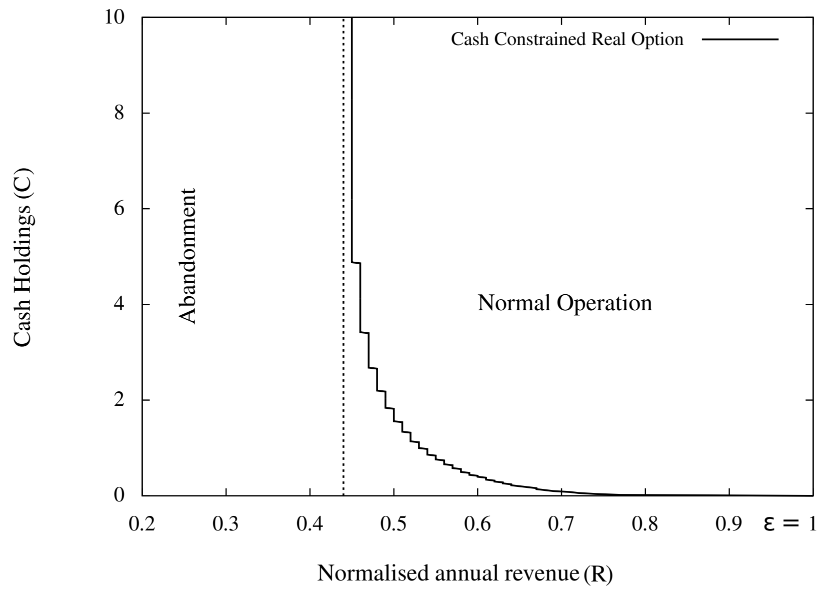


Figure 4.4: A comparison of the abandonment boundaries between a classical Real Option model and a cash-constrained Real Option model (at the initial time $t = 0$), where the solid line is the abandonment boundary of a cash-constrained Real Option model, while the dashed vertical line is the abandonment boundary of a classical Real Option model. We assume the operational cost is $\varepsilon = 1$.

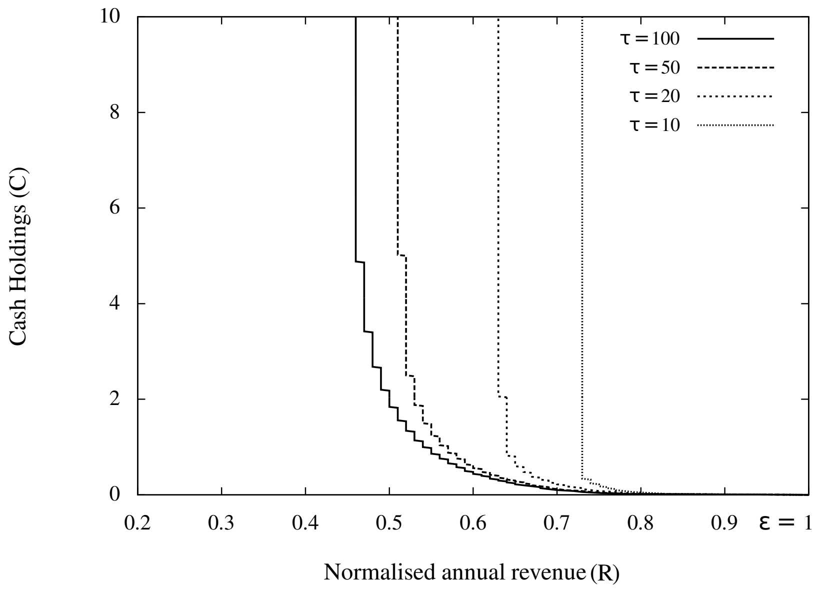
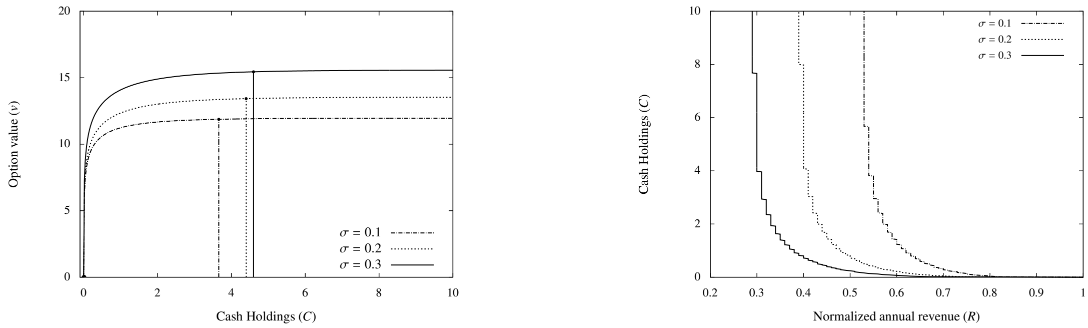


Figure 4.5: A plot to show the effects of expiration time ($\tau = T - t$) on the abandonment boundaries, where τ takes value $100, 50, 20$ and 10.

4.4.2 Parameter Tests

We have presented a set of important numerical solutions based on parameters defined in the Table 4.1. We now conduct parameter tests to show how the cash-constrained Real Option valuation and suggested operation strategy varies when taking different parameter values.

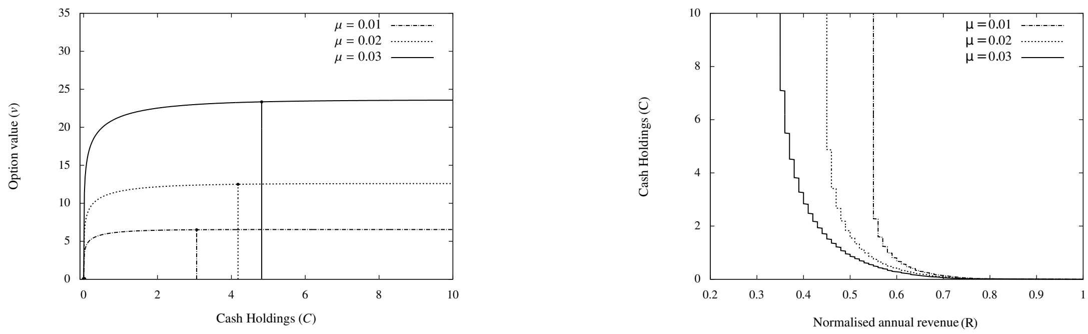
The volatility σ defines a measure of the fluctuations of a revenue process around its expected growth rate, which we can think of as a benchmark of external risk. Figure 4.6(a) shows the option value against cash holdings for a project that has revenues with volatilities $\sigma = 0.1, 0.2$ and 0.3. Figure 4.6(b) presents the corresponding abandonment boundaries with the same volatilities. According to these figures, we can see that the option value increases with the increase of the revenue volatility, and that the firm is more conservative in exercising the abandonment option, especially when it holds enough cash holdings. The option value increases with the increase of volatility because the flexibility to stop the business is more significant in avoiding further losses when a firm faces more uncertainty. The exercise boundary drifts left with increasing σ because when the project is in a poor financial state and the cash holdings can keep operating for an enough long time, the firm have a greater probability to go back to a profitable region due to the larger fluctuations in revenue.



(a) Option value (b) Exercise boundary

Figure 4.6: The volatility effects on the option value and exercise boundaries, where we take $\sigma = 0.1, 0.2$, and 0.3 as examples.

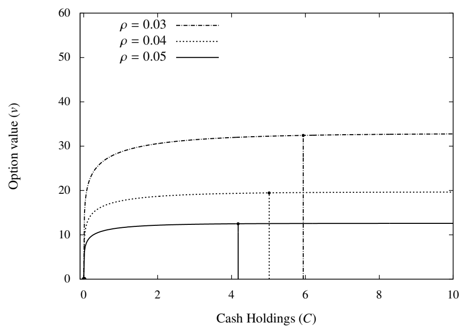
By assuming a GBM process, the drift μ defines the revenue's long-term expected growth rate. Figure 4.7(a) shows how the project's value varies with revenue drift by taking $\mu = 0.01, 0.02$ and 0.03 as examples. Figure 4.7(b) presents the corresponding abandonment boundaries for each particular case. According to these figures, we can see that, for a project that has a higher expected return revenue, its value is relatively higher, and also, the firm is more reluctant to abandon the project even when a project is not that profitable. Thus holding cash is more significant in order to fully take the potential value of a project that has a higher μ.



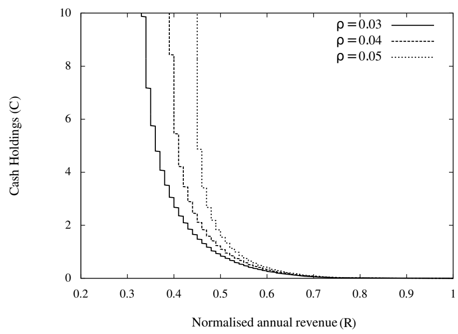
(a) Option value variations (b) Exercise boundary variations

Figure 4.7: The revenue drift effects on the option value and exercise boundaries, where we take $\mu = 0.01, 0.02$, and 0.03 as examples.

Figures 4.8(a) and 4.8(b) show the significant effects of investors' discount rate on the option value and abandonment boundary. According to these two figures, we can see that the option value decreases with increasing of discount rate ρ. What's more, the discount rate affects the change of the marginal value of cash. We can explain this by noting that when the discount rate is higher, the marginal value of cash decreases at a slower rate, thus the abandonment position for a higher discount rate, compared with a lower one, converges faster to a stable value with increasing cash levels. This is because the discount rate ρ defined in Equation (4.7) measures investors expected return (or opportunity cost). So, investors that have a higher ρ will value a project less that those having lower ρ. From the opportunity cost point of view, when the ρ is low, investors have low opportunity costs, therefore they are reluctant to abandon the business.



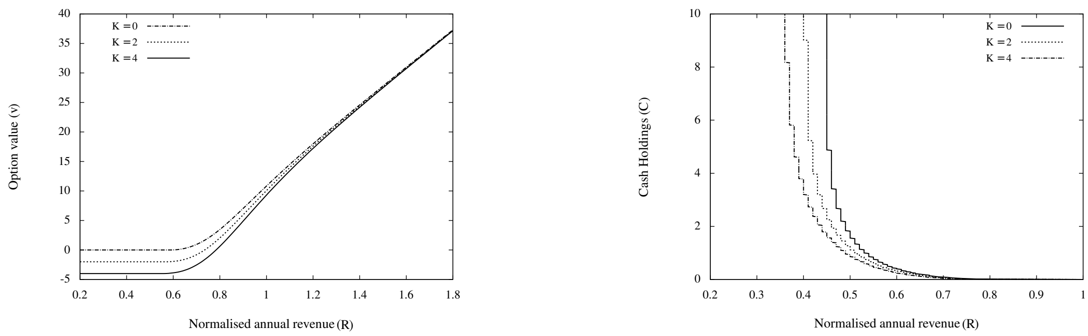
(a) Option value variations



(b) Exercise boundary variations

Figure 4.8: The discount rate effects on the option value and exercise boundaries, where we take $\rho = 0.03, 0.04$, and 0.05 as examples.

We assume a firm has to pay a fixed level of cost K when it exercises the abandonment option. Figures 4.9(a) and 4.9(b) summary how the abandonment cost affect a project's valuation and operation by choosing $K = 0, 2$ and 4 as examples. As one might expect, if we examine these two figures we can see that the option value decreases and the abandonment boundary drifts left, when the abandonment cost decreases. This is because the abandonment cost defines economic frictions of exercising this flexibility, thus the firm is more reluctant to abandon the business when the cost is low.



(a) The effects on option valuation

(b) The effects on exercise boundary

Figure 4.9: The abandonment cost effects on option value and exercise boundaries, where we take fixed cost $K = 0, 2$, and 4 as examples.

4.5 Conclusion

Classically, investment under uncertainty problems can be valued via Real Option approaches with the assumption that there are no cash constraints (see Dixit (1994) and Myers and Majd (2001)). In this chapter, we investigated how to relax this assumption by involving cash holdings. We derived a formal framework to combine cash constraints with classical Real Option methods, and solved the model numerically using Semi-Lagrangian methods. By comparing our solutions with those obtained from the classical Abandonment Real Options Analysis, we presented how the cash holdings quantitatively affect the option value and restrict the abandonment flexibility. Our study and analysis demonstrated that the classical Real Option Analysis overvalued a project and thus provided misleading decisions. In addition, we found that the value of a cash-constrained Real Option converged to the value of classical Real Option as the cash holdings growth large. The classical Real Options generally overvalued a firm's decision due to the ignorance of the cash constraints. We showed that a firm should abandonment its business at a higher threshold (revenue level) compared with the classical Real Option solution when the firm has limited cash holdings. The effects of financial constraints will be expanded when the volatility increase and discount rate decrease. Our research and results are consistent with other work (for examples, Décamps and Villeneuve (2013) and Hugonnier et al. (2014)). As the cash level grows high enough (over a barrier), the marginal value of cash decreases to one, making it unnecessary for a project to hold large amounts of cash. This suggests that a firm should redistribute the excess cash as dividends (or as new investments), when the marginal value of cash is lower than the cost. This study of dividend payback will be carried out in the next chapter.

Chapter 5

Optimal Dividend Policy

5.1 Introduction

Implementing an optimal dividend policy is essential for a firm's success in making other operational decisions, particularly when there are cash constraints. On the one hand, a high level of dividend payment might reduce cash holdings, and thus increase liquidity risk and limits a firm's investment flexibility. On the other hand, a critically low level of dividend payment might reduce shareholders' investment flexibility and leave the investors' wealth at risk, in that affects the firm's ability to raise external funds. This is because the market value of a firm is believed to be the future expected dividend payouts. How to model the optimal dividend payment and reveal the interactions of the payout policy and other corporate decisions thus is an interesting problem.

Early studies tried to address this problem by assuming a stochastic cash reserves (see Sotomayor and Cadenillas, 2011; Chevalier et al., 2013). Based on this assumption, they can model a firm's dynamic dividend payments and cash dependent decisions within one cash reserve process to study the dynamic interactions. However, as discussed in the introduction chapter, the assumption of cash holdings being a stochastic process has many limitations in studying corporate decisions, because it stops one from being able to explore cash flow dependent decisions arising from liquidity risk. In addition, the coefficients of the cash reserve are not easily observable. Therefore, in this chapter, we mainly follow the idea of Anderson and Carverhill (2012), Hugonnier et al. (2014), and Bolton et al. (2014b) to study the optimal dividend problem by assuming the revenue process of the firm following a stochastic process. Our model provides an innovative approach to combine the liquidity risk (via cash holdings), dynamic dividend payment (via optimal dividend payment structure) and corporate optionality (via management flexibility), thus people can use it to study the optimal capital structure by including debts and conduct other studies, for example agency conflicts (see Chapters 6 and 7, or literature Mauer and Sarkar (2005), John and Knyazeva (2006) and Hennessy and Tserlukevich (2008)).

To keep simplicity and focus, we begin with this chapter by assuming the firm has no debts, financing flexibility and investment opportunities before gradually including these extensions in the later chapters of the thesis. We keep the settings and assumptions of Chapter 4 and further add balance sheet items (i.e. fixed assets, cash assets and equity). This will enable us to study the optimal dividend payment policy in this chapter, and also more complex investment problems in the following chapters.

5.2 The Optimal Dividend-Payment Firm Model

5.2.1 Problem Setting

We consider a firm that operates to maximise its shareholders' benefits by optimally paying out dividends. Suppose the firm has a project to generate uncertain revenues in future several years. To keep normal operation, the firm has to continuously pay a running cost, thus the net revenue (after taking out the operational cost) can be positive or negative since the revenue is full of uncertainty. In order to hedge a possible short-term risk and keep paying operational expenses, the firm has to prepare a certain amount of cash as reserves at the initial time. When the firm is profitable, managers have options to either invest the instantaneous profits as cash savings with interest or redistribute the excess cash as dividends. When the firm is losing money, the firm has to continuously consume its retained cash in order to maintain normal operation. Therefore, it is possible for the firm to stop operating in one of three cases: the active abandonment case, this is when the expected value of shareholders less than the current book value of equity (after taking off the abandonment costs); the passive bankruptcy case, which is when the cash holdings are not sufficient to keep normal operation and the firm is forced to close by creditors; and finally there might be a natural predetermined end to operation, named termination time. In the next section, we take this economic interpretation and build it into the mathematical framework for this optimal dividend paying firm.

5.2.2 Stochastic Framework

Initial State: We assume a firm starts a business at time t with initial assets A_t, all of which come from shareholders' investment E_t, thus $A_t = E_t$. The firm uses these funds setting up a project, where A_t^f amounts are used to buy the solid assets (including plants, building and facilities, etc.) and the remaining capitals C_t ($C_t = C$) is used as cash holdings for operation. Therefore, we have $A_t = A_t^f + C_t = E_t$ at the initial time t. We assume the project will generate uncertain revenues R_s ($s \in [t, T]$), which are defined by the following Geometric Brownian Motion with an initial value $R_t = R$,

$$dR_s = \mu R_s ds + \sigma R_s dW_s^P. \tag{5.1}$$

Here μ is the drift function which measures revenue's instantaneous average increasing rate with time, σ is the variance function that measures the level of revenue's uncertainty, and W_s^P is a Brownian Motion under the market probability measure P.

During Operation: The firm obtains uncertain revenues R_s and pays operational cost ε. So the instantaneous profits that generated by the project per unit time are given by $R_s - \varepsilon$. By further considering the instantaneous dividend payout flow d and the interest income rC_s, we can get an instantaneous cash flow function as

$\Phi(s, R_s, C_s, d)$[1],

$$\Phi(s, R_s, C_s, d) = R_s + rC_s - \varepsilon - d. \tag{5.2}$$

With this cash flow function, we can define the firm's cash holdings process C_s ($s \in [t, T]$) as,

$$dC_s = \Phi(s, R_s, C_s, d)ds \quad \text{where} \quad C_t = C. \tag{5.3}$$

We assume constant fixed assets A^f, therefore, we can immediately get the total assets by using the above cash holdings process A_s ($s \in [t, T]$),

$$A_s = A^f + C_s. \tag{5.4}$$

The active abandonment case: In this case the firm chooses to terminate operation when they believe that holding this business cannot benefit the shareholders' benefits. Thus, we can define an optimal abandonment time τ_a by,

$$\tau_a = \min_{s \in [t,T]} \{s : V(s, R_s, A_s) \leq A_s - \kappa^a(A_s)\}, \tag{5.5}$$

where, V is the expected equity-value of share investors at abandonment time, and $\kappa^a(A_s)$ is the abandonment cost function for a firm that has A_s assets.

The passive bankruptcy case: This happens when the firm runs out of all their retained cash and is no longer profitable. Since the firm does not have enough cash to pay the operational costs, we assume that they are unable to gain credit and therefore must cease operation immediately. We denote the stopping time in this case as τ_b, which is associated with a liquidity shortage,

$$\tau_b = \inf_{s \in [t,T]} \{s : C_s \leq 0 \ \ \& \ \ R_s + rC_s - \varepsilon < 0\}. \tag{5.6}$$

It should be noted that we use inf operator to define the passive bankruptcy time because this time, from mathematical perspective, is a limitation value of a series of τ_a.

[1] It should be noted, the instantaneous cash flow function can be extended with more complex considerations, for example, depreciation, taxation, and infrequent losses. To help us focus on the optimal dividend policy, we take the simplest formula. People can easily include the necessary factors in further studies since what form of cash flow function to use here do not significantly change our main model structure. For extensions, we particularly recommend paper Anderson and Carverhill (2012).

The natural end of the project: In this case, the remaining equity in the firm is redistributed to the shareholders, thus, the firm generates an instantaneous cash outflow,

$$\max\big(A_T - \kappa^l(A_T), 0\big), \tag{5.7}$$

where, $\kappa^l(A_T)$ is the liquidation cost.

By summarising these three stopping cases, we define the τ as the closure time, where $\tau = \tau_a \wedge \tau_b \wedge T$. The stopping payoff function is given by,

$$h(\tau, R_\tau, A_\tau) = \begin{cases} \max\big(A_{\tau_a} - \kappa^a(A_{\tau_a}), 0\big) & \text{when} \quad \tau = \tau_a, \\ \max\big(A_{\tau_b} - \kappa^l(A_{\tau_b}), 0\big) & \text{when} \quad \tau = \tau_b, \\ \max\big(A_T - \kappa^T(A_T), 0\big) & \text{when} \quad \tau = T. \end{cases} \tag{5.8}$$

For simplicity, we assume that the firm's liquidation cost $\kappa^l(\cdot)$, abandonment cost $\kappa^a(\cdot)$ and termination cost $\kappa^T(A_T)$ can be expressed as the same function $\kappa(A_\tau)$. Then we can simplify Equation (5.8) as,

$$h(\tau, R_\tau, A_\tau) = \max\big(A_\tau - \kappa(A_\tau), 0\big). \tag{5.9}$$

The Objective Function: The objective of the firm is to optimise the shareholders' benefits, which can be valued by the accumulated dividends plus the remaining equity at the life end of the project. Using the above notations, we can write the value function of the firm as,

$$v(t, R_t, A_t) = \sup_{\tau, d} E\left[\int_t^\tau e^{-\rho(s-t)} d\ ds + e^{-\rho(\tau-t)} h(\tau, R_\tau, A_\tau)\right]. \tag{5.10}$$

The discount rate ρ is different from risk-free rate r ($\rho > r$). It has the same economic explanation as illustrated in the cash-constrained abandonment model. ρ can be understood as the external investment opportunity costs (or be defined as the external risky opportunity return). The value of ρ can be defined from many ways. The widely used approaches are Weighted Average Cost of Capital (WACC) and the Capital Asset Pricing Model (CAPM) series of models. Since, estimating the discount rate is not our focus in this thesis, we suggest the textbook Brealey et al. (2010) for more details of WACC, and our paper Evatt et al. (2014) for an implementation of CAPM.

The feasible control set of τ is given by $\tau \in [t, T]$, while the feasible control set of the dividend payment ratio d can be defined from real world consideration, i.e. it should be a non-negative real number $d \in [0, \infty)$. Although the control set is not a compact set (see the theoretical definition of compact set in the book by Kelley (1975)), we now illustrate how to find an optimal control strategy within this feasible control set via the following HJB Variational Inequality.

5.2.3 HJB Variational Inequality (HJBVI)

The above equation (5.10) is a mixed optimal stochastic control and stopping problem. By using the stochastic control theory illustrated in the mathematical background chapter, we can derive the associated HJB Variational Inequality[2] as,

$$\min\left\{\frac{\partial v}{\partial t} + \mathcal{L}v - \rho v + \sup_{d\in[0,\infty)}\left\{\varPhi(t,R,C,d)\frac{\partial v}{\partial A} + d\right\}, v - h\right\} = 0. \tag{5.11}$$

By further analysing the HJBVI Model (5.11), the first part of HJB is a linear function of the control parameter d. As illustrated in the literature Balakrishnan (1980), when the HJB is a linear function of the dividend ratio d, the solution to this control problem should be a Bang-Bang type of control, which means that the optimal control value d^* can only take the minimum value ($d = 0$) or the maximum value (denoted as $\tilde{d}$). The switching boundary between $d = 0$ and $d = \tilde{d}$ can be found by taking the derivative on the left-hand side of HJB Equation (5.11). In this way, we can get $\frac{\partial V}{\partial A} = \frac{\partial V}{\partial C} = 1$. The economic explanation of this boundary is that the firm should distribute excess cash as dividends when the marginal value of cash no more than one[3]. Based on this understanding, the original HJBVI can be reduced as the following type of HJBVI,

$$\min\left\{\frac{\partial v}{\partial t} + \mathcal{L}v - \rho v + \varPhi(t,R,C,0)\frac{\partial v}{\partial A}, \frac{\partial v}{\partial A} - 1, v - h\right\} = 0. \tag{5.12}$$

[2] We use sup operator because the control set of d is not compact.

[3] Similar discussions of the dividend payment boundary can be found in Belhaj (2010), Chevalier et al. (2013) and Li et al. (2013).

which is subject to boundary conditions,

$$\begin{cases} v = E & \text{as} \quad t = T, & (5.13) \\ v = h & \text{as} \quad C = 0 \ \& \ R + rC - \varepsilon \leq 0, & (5.14) \\ \dfrac{\partial v}{\partial A} = 1 & \text{as} \quad C \to \infty \ \& \ R + rC - \varepsilon > 0, & (5.15) \\ v = \hat{\alpha} R + \hat{\beta} \varepsilon + h & \text{as} \quad R \to \infty, & (5.16) \end{cases}$$

where, $\mathcal{L}$ is the second order linear operator (see Definition (4.8)), h is the termination payoff when the firm stops operating (see Equation (5.9)), $\hat{\alpha}$ and $\hat{\beta}$ are coefficient functions that depend on the revenue process (see Subsection 4.2.3).

Boundary Conditions: We now explain each boundary condition.

1. The firm's natural end boundary condition (see Equation (5.13))
   If the firm operates its business successfully until the natural end of its project lifetime, we set the firm value equals its equity value.
2. The firm's insolvency boundary condition (see Equation (5.14))
   When the firm does not have any cash holdings, and its revenues are unable to pay the operational costs, the manager has to stop the business, and then shareholders can get instantaneous payoff h.
3. Unlimited cash boundary condition (see Equation (5.15))
   When there is enough cash to hedge future risk and the firm is profitable, holding one more unit of cash cannot increase the extra benefits for shareholders in addition to the cash itself. Thus, the marginal value of this new dollar is 1.
4. The firm's no-risk boundary (see Equation (5.16))
   When the firm is very profitable ($R - \varepsilon \to \infty$), the firm does not need to hold extra cash to hedge future risks. Therefore, the shareholders prefer to redistribute all residual cash at the start of the business, and continuous pay out the net profits during the operation instantaneously ($d = R_t - \varepsilon$). So the boundary condition at $R \to \infty$ can be given by $v = \hat{\alpha} R + \hat{\beta} \varepsilon + h$. The definition of coefficients function $\hat{\alpha}$ and $\hat{\beta}$ can be found in Subsection 4.2.3.

According to the new HJBVI (5.12), the optimal control variable d vanishes. This is very useful since we do not need to worry about the non-compact property of the feasible control set of $d \in [0, \infty)$, since the optimal dividend payout boundary can be addressed by the free boundary condition $\frac{\partial v}{\partial C} = 1$. We now analyse the dividend policy from the economic perspective, and illustrate how to use this analysis to improve our numerical algorithm.

5.2.4 Dividend Policy Analysis

From financial viewpoint, we assume the managers have the flexibility to redistribute the retained cash as dividends at an instantaneous payment rate d. The payout decision can be made based on the following principles: First, the objective of dividend payments is to maximize shareholders' benefits; Second, dividends are non-negative, and its maximum available payment rate is limited by the total amount of extra cash plus the instantaneous net income. These principles are useful to help us in designing numerical schemes.

From the numerical scheme viewpoint, according to Chen and Forsyth (2007) and our practice, taking multiple discrete value of d in set $[0, \tilde{d}]$ can prove higher accuracy solutions than purely solve the HJBVI 5.12 via the free boundary. However, for a non-compact control set $d \in [0, \infty)$, it is very difficult to address the up-limited value $\tilde{d}$ used in numerical test. Thus, we need to consider an approximated up-bounded value of $(d = \tilde{d})$ within an infinitesimal time interval ds. We assume that $\hat{C}_s$ is the threshold in Revenue-Asset space at time $s \in [t, T]$, above which it is optimal to payout all excess cash as dividends. The position of $\hat{C}_s$ must be found implicitly based on the following equation,

$$\hat{C}_s = \left\{ C_s : \frac{\partial v}{\partial C}\bigg|_{C_s} = 1 \right\}, \quad \text{where} \quad \frac{\partial v}{\partial A} = \frac{\partial v}{\partial C}. \tag{5.17}$$

We further assume that the optimal policy will be for the firm to pay out a dividend sufficiently large to ensure that $C_s = \hat{C}_s$ after the dividend has been paid. Given our assumption of the form of the solution, we can define the admissible control set of dividends d as $\mathcal{D} = \{d | d \in [0, \tilde{d}]\}$, where $\tilde{d}$ is the upper-bound value defined by the following formula,

$$\tilde{d} = \max\left(\frac{C_s - \hat{C}_s}{ds} + \varPhi(s, R_s, C_s, 0), 0 \right). \tag{5.18}$$

Following this policy, a firm's cash assets level will never go above the threshold $\hat{C}_s$, and when a firm is on the dividend payment boundary and has a positive cash flow, it will moves along the dividend payment boundary since the value $\tilde{d}$ is large enough to force it back to the dividend payment boundary where $\hat{C}_s$. This formula (5.18) will be used as part of our numerical method.

5.3 Numerical Methods

The optimal dividend-payment firm model we have described in the last section is, from the mathematical viewpoint, a mixed stochastic control and optimal stopping problem. As discussed in Chapter 2, it is very difficult to find an analytical solution in such a framework, therefore, we need to find an approximate solution of this model by solving its corresponding HJB Variational Inequality (5.11) numerically. In this section, we illustrate how to use the Semi-Lagrangian approach to discrete the HJB equation and then combine it with a PSOR method to address the free boundary in the Variational Inequality. Since we have a complete discussion of the Semi-Lagrangian approach and PSOR methods in numerical algorithm Chapter 3, we only focus on how to implement them in this section.

We use Figure 5.1 to illustrate the calculation domain and boundary condition layouts in the $R-A$ space. According to the figure, we can see the whole $R-A$ space is divided into three separate regions: the continuation region, the abandonment region, and dividend payment region. The free boundary between the continuation region and dividend payment region can be solved by the following controlled HJB Variational Inequality,

$$\frac{\partial v}{\partial t} + \mathcal{L}v - \rho v + \max_{d\in[0,\tilde{d}]}\left\{(R + rC - \varepsilon - d)\frac{\partial v}{\partial A} + d\right\} = 0. \tag{5.19}$$

Since the controlled parameter d is linear in the HJBVI equation, we know that the optimal solution to the control problem is a Bang-Bang type control, which means that theoretically, the optimal dividend policy d^* can only take the minimum value $d = 0$ or maximum value $d = \tilde{d}$, which is defined by Equation (5.18). However, in the practical numerical calculation, we use $d_m \in [0, d_1, ..., d_M = \tilde{d}]$ since many researchers

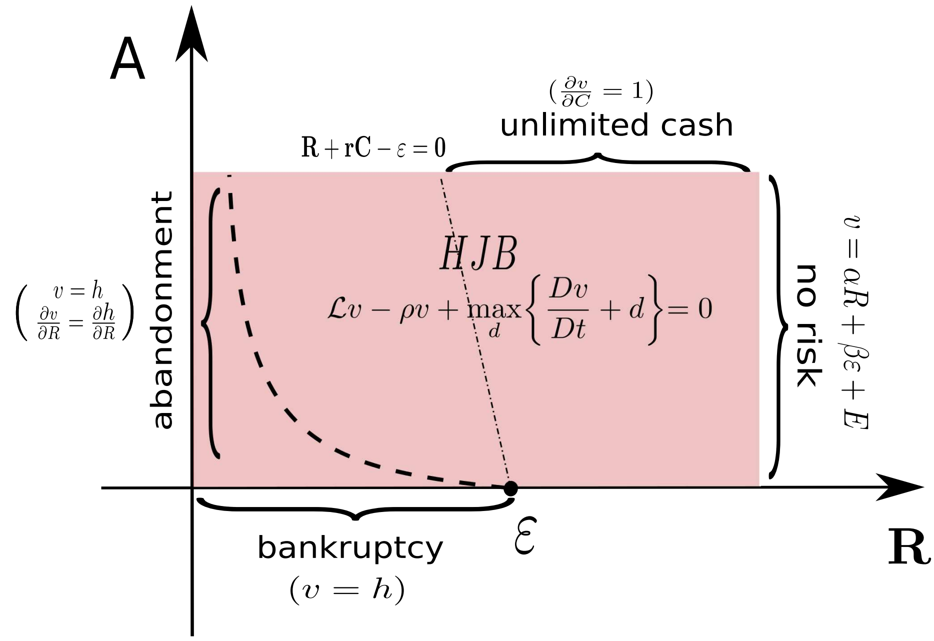


Figure 5.1: A schematic diagram to show the calculation domain and boundary layouts for the HJB Variational Inequality (5.11) within the Revenue-Asset space. Here the straight dot-dashed line represents the zero net income boundary, and the long dashed line represents the abandonment free boundary. d takes discrete value in the set $[0, \tilde{d}]$.

have shown that using discrete Bang-Bang control criteria can generate more accurate boundaries, for example, see Chen and Forsyth (2007).

Although there are restrictions on the size of the dividend payment, we cannot give an explicit formula to instantaneously find the exact up-limit of the dividends. Numerically, we use the value solved in backward time τ_{k+1} to approximate $\frac{\partial v}{\partial C} = 1$ (or $\frac{\partial v}{\partial A} = 1$ when $A^f = 0$) at τ_k with a fully explicit approach in time dimension, and then we can define $\tilde{d}$ with this approximated gradient $\frac{\partial v}{\partial C} = 1$. Therefore, we can discretize the controlled HJBVI (5.11) by using the fully implicit time step,

$$\alpha_i v_{i-1,j} + \gamma_i v_{i,j} + \beta_i v_{i+1,j} = -\min_{d_m} \left\{ \frac{v_{i,j} - v^*(d_m)}{\Delta \tau_n} + d \right\}, \tag{5.20}$$

where, d_m denotes the discrete value of control d, and α_i, γ_i, and β_i are coefficients that defined by the standard finite difference methods on R_i, A_j (see more details in Chapter 3). The value $v^*(d_m)$ is the interpolated value with a dividend payment ratio d_m found by following a Semi-Lagrangian trajectory at time point τ_{n+1}. The boundary conditions could be technically discretized as,

$$\begin{cases} v_J = v_{J-1} + \Delta C_{J-1} \quad \textbf{unlimited cash}, \\ v^n_{i,j} = \max(v^n_{i,j}, h_{i,j}) \quad \textbf{abandonment}, \\ v_{i,0} = h_{i,0} \quad \textbf{when} \quad R - \varepsilon \leq 0, \\ v_{I,j} = \hat{\alpha} R_I + \hat{\beta}\varepsilon + E_j, \end{cases} \tag{5.21}$$

where, I, J denote the index of the maximum value of dimensions R and A respectively.

5.4 Solution and Analysis

So far we have discussed the model itself and illustrated the basic numerical methods to solve such a model. In this section, we present a set of numerical solutions designed to illustrate how the dividend payout policy and internal capital structure affect both a firm valuation and operational strategy (i.e. the decision to abandon).

In table 5.1 we list a set of base case parameters which will be used in all calculations unless stated otherwise.

Parameters	$\mu(y^{-1})$	σ	ρ	$A_f(M)$	$\varepsilon(My^{-1})$	$r(y^{-1})$	$\kappa^a(A)(M)$	$T(y)$
Value	0.02	0.15	0.05	0	1	0.01	0	100

Table 5.1: A basic set of parameter values for numerical solutions. Here μ and σ are the growth rate and volatility of the annualised revenue (given by Equation (5.1)); ρ is the market discounted rate (used in Equation (5.10)); ε is the annualised running cost (defined in Equation (5.3)); r is the short-term interest rate (used in Equation (5.3)); $\kappa^a(A)(M)$ is the stopping cost (abandon and insolvency); $A_f(M)$ is the fixed assets; T is the expected lifetime of the project.

To simply the explanation of the results, we normalize the parameters based on the operational cost $\varepsilon = 1(My^{-1})$, so that we do not need to explain the unit of each variable.

Now, the value function defined in Equation (5.10) includes the book value of equity E ($E = A$) which we assume is equal to the initial investment of the shareholders. In order to show the shareholders' investment return, we define the following metric **Return on Equity** (ROE) by using the total future profits minus the money that they invest at the beginning.

$$\mathbf{ROE} = v - E, \quad \text{where} \quad A = E \quad \text{when} \quad D = 0. \tag{5.22}$$

Using this metric allows us to better compare the difference between the cash-constrained Real Option model and the optimal dividend-payment firm model.

5.4.1 Firm Valuation and Operational Strategy

Based on the parameter values defined in Table 5.1, we first present a set of numerical solutions of the valuation and operational strategy of an optimal dividend paying firm.

Figure 5.2 summarises how the Return on Equity varies against the revenue for a firm that starts operation with a different initial level of cash assets to pay the operational costs based on parameters defined in table 5.1. The top four lines show the equity return when the firm has zero cash holdings, one-month cash holdings, five-year cash holdings, and ten-year cash holdings for operational costs respectively. The bottom straight line shows the project's discounted net present value. The solid circles at the bottom of each line represent the corresponding abandonment boundaries for each level of cash assets.

By comparing these different lines, we can see that the equity return monotonically increases with an increase in cash assets, particularly, when the firm has no cash assets and its revenue is lower than the operational cost, the firm's equity return is zero. According to this figure, we can see that cash holdings play a significant role in valuing a firm's equity return. Thus, we cannot ignore the financial constraints when valuing a firm, and must consider what level of cash assets the firm should keep in order to overcome the financial difficulties.

In order to see more details on how this marginal value of liquid assets varies in different business cases, we choose a firm's three different types of profitable levels: positive profits ($R - \varepsilon = 0.05$), zero profits ($R - \varepsilon = 0$) and negative profits ($R - \varepsilon = -0.05$). Figure 5.3 shows how equity return varies along with liquid assets in these three cases.

According to this figure, we can see that the marginal value of cash decreases with

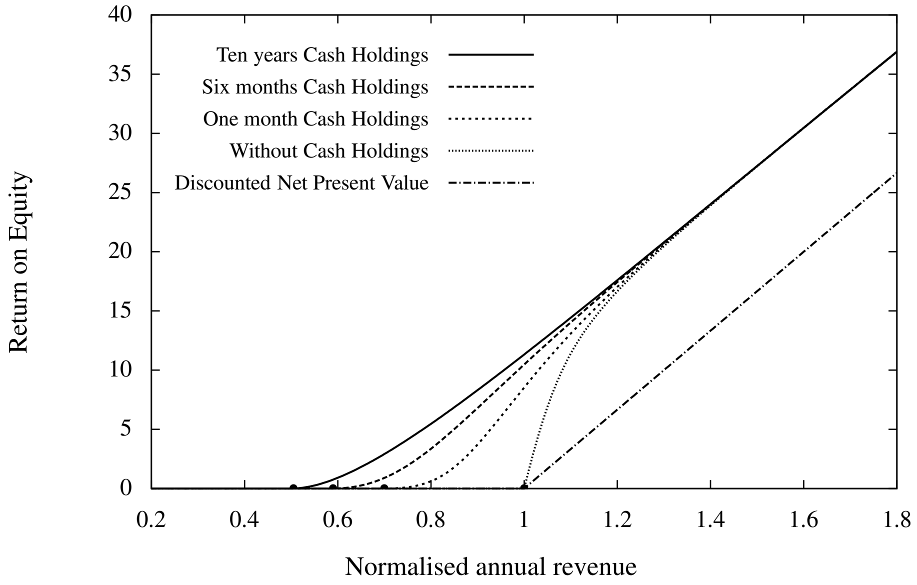


Figure 5.2: The Return on Equity against normalised annual revenue (taking operational cost as a benchmark) for a firm that has different cash holdings to pay the operational cost. Here, we assume the fixed operational cost $\varepsilon = 1$.

the increase of the cash holdings. Particularly, the first small amount of cash plays a significant role to increase the Return on Equity of the firm. This is because it reduces the bankruptcy probability due to liquidity shortage. Taking the $R - \varepsilon = 0.05$ line as an example, 0.25 years cash holdings can increase the project's value from 6.5 to 13. In addition, by comparing the optimal cash holding levels of these three cases $R - \varepsilon = -0.05, 0, 0.05$, we can see the higher profitability the firm has, the fewer cash holdings that the firm needs to hold to fully exercise the firm's flexibility.

Figure 5.4 shows the abandonment and dividend payment thresholds in the $R - A$ space at the start time. Those two boundaries split the whole space into three different operating regions: the Abandonment Region, the Normal Operation Region and the Dividend Payment Region. The abandonment boundary has similar character compared with the cash-constrained Real Option model. It starts from zero net cash flow point at the bottom, rapidly moves left with the increasing of a few liquid assets, and turns to be vertical when the cash level is relatively high. The dividend boundary shows that when the firm is profitable or has large amounts of cash holdings, the firm prefers redistributing the excess cash as dividends.

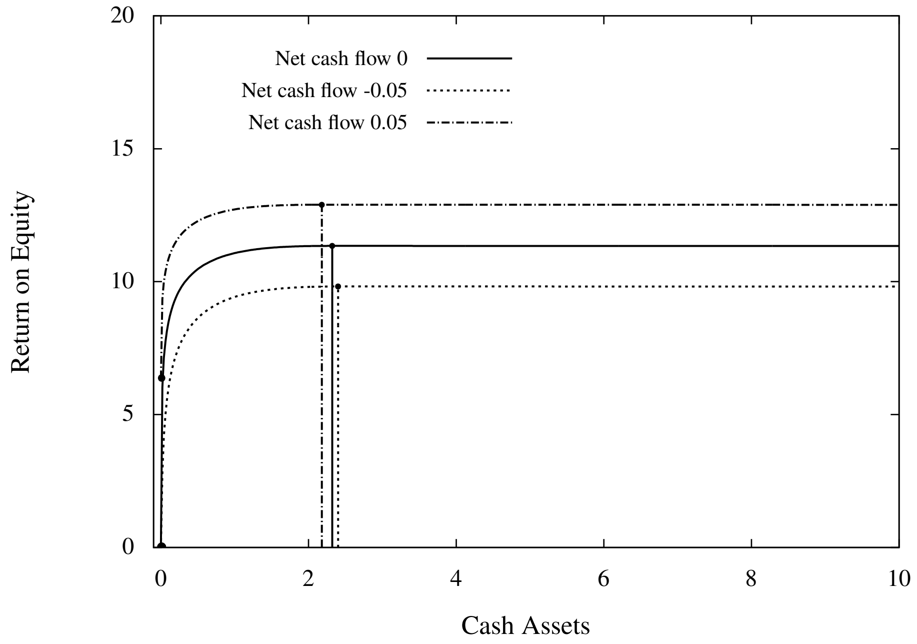


Figure 5.3: The Return on Equity against Cash Assets for a firm that has different net cash flows, where the vertical lines represents the corresponding dividend payment boundary for each net cash flow.

It is interesting to see that the dividend boundary is not monotonically varying against the firm's revenue when the net income is negative. This is because, in this situation, the characteristic line of the HJB part goes from the top to the bottom when moving forwards in time. Therefore, there does not exist a real dividend payment boundary that has an economic meaning. Here we keep both the negative and positive parts of the dividend boundary for visualization simplification.

5.4.2 The Optimal Dividend-Payment Firm Model V.S. the Cash-Constrained Real Option Model

The main difference between the optimal dividend-payment firm model (here we called it the firm model for simplicity) and the cash-constrained Abandonment Real Option model is whether there is an optimal control of the cash flow. To show how this flexibility affects a firm's valuation and operating strategy, we compare these two models by the following figures (see 5.5 and 5.6).

Figures 5.5 compares the operating strategy of these two models by taking savings

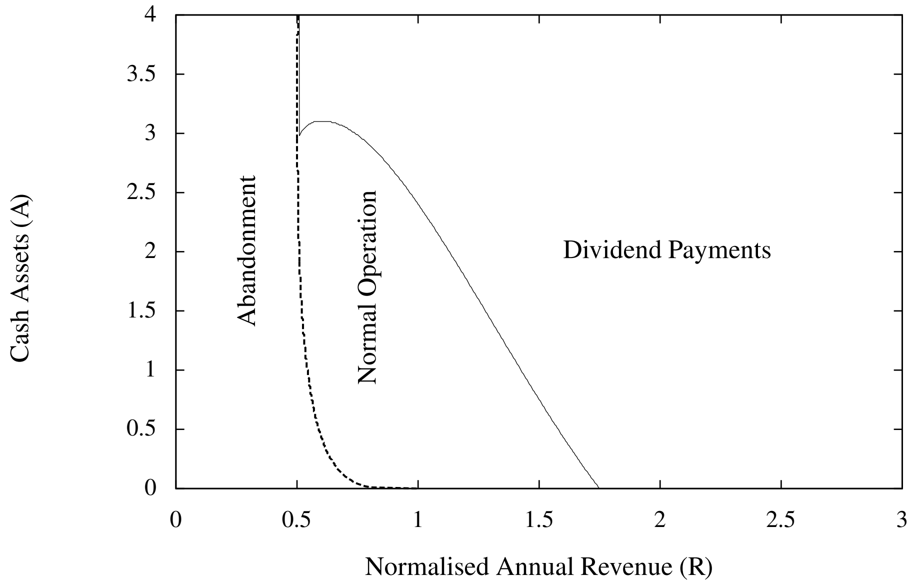
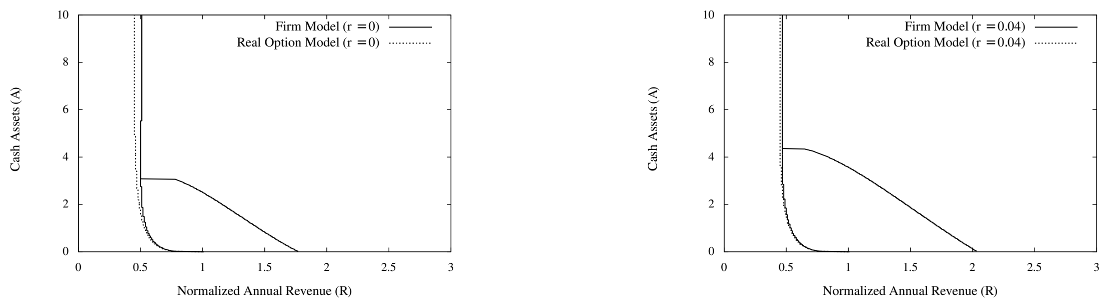


Figure 5.4: A plot to show the corporate operational regions within the Revenue-Asset space, where the whole space was divided into three regions: the abandonment region, the normal operational region and the optimal dividend payment region. The parameter values are based on Table 5.1.

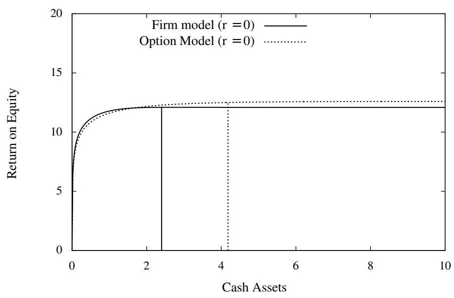


(a) Operational Regimes for $r = 0$

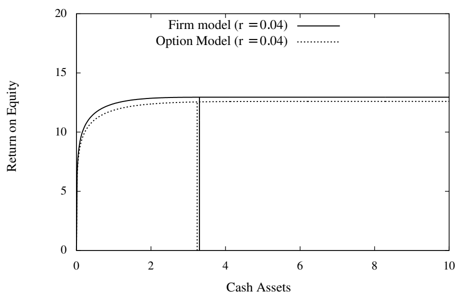
(b) Operational Regimes for $r = 0.04$

Figure 5.5: A comparison of the abandonment boundaries between the optimal dividend-payment firm model and the cash-constrained Real Option model, where the left part of the solid line and the right part of the solid line represents the abandonment and dividend payment boundaries of the firm model respectively, and the dashed line represents the abandonment boundary of the cash constrained Real Option model. Here, we assume the risk-free rate r takes value 0 and 0.04.

interest rate $r = 0$ and 0.04 respectively. According to these two figures, we can see that the abandonment boundary of the firm model locates further right compared with that of the cash-constrained Real Option model. The distance between these two boundaries gets smaller along with the increasing savings interest rate. This is



(a) Return on Equity for $r = 0$



(b) Return on Equity for $r = 0.04$

Figure 5.6: A comparison of the value between the optimal dividend-payment firm model and the cash-constrained Real Option model, where the solid line represents firm's expected Return on Equity, and the dashed line represents the option value of the cash constrained Real Option model. Here, we assume the risk-free rate r takes value 0 and 0.04.

because the cash-constrained Real Option model assumes that all the income goes into the savings account automatically, whereas, the dividend-payment firm model assumes the excess cash should be returned to shareholders. Therefore, managers are more reluctant to abandon the business by following the solution of the cash-constrained Real Option model than following the optimal dividend-payment firm model.

Figures 5.6 compares the value generated by these two models when the project has zero net cash flow. It is interesting to see that there is a cross point between the two value lines. When the cash level is relatively low, the firm model gives higher value than the Real Option model, while, when the cash level is relatively high, it is opposite. This is because the firm model has extra flexibilities to manage the cash flow, i.e. holding cash in a liquidity shortage phase and redistribute cash as dividends when the marginal value of cash equals one. We further show how the savings' interest rate affects the solution via Figure 5.6(b). When $r = 0.04$, we can see clearly that the firm model generates higher value since the growing savings income significantly increases the shareholders' equity, even the cash level is relatively high. However, the increasing savings income does not change the value of the cash-constrained Real Option model.

5.4.3 Parameter Tests

We have presented the main results of the firm model above based on the parameters defined in the Table 5.1. In this subsection, we show how these results vary with different parameters by individually testing A^f, σ, μ, ρ and r.

Figures 5.7 and 5.8 present how a firm's fixed assets level affect the operational strategy. By comparing these two figures, we can see how the inclusion of fixed assets will affect the firm's bankruptcy region. When the firm has no cash assets ($A = A^f$), and its profits are negative, we assume that the firm is forced to declare a bankruptcy even though it might be profitable in the future. We can also see that the dividend payment boundary drifts upward due to the inclusion of fixed assets. This is because the ratio of the fixed assets to the total assets defines the retained cash level, and it affects the whole business's bankruptcy probability. However, the dividend payment boundary in Figure 5.7 is not simply the boundary that we see in Figure 5.8 plus one unit of distance, which means that the effects of the inclusion of fixed assets on the operational strategy are not exactly linear.

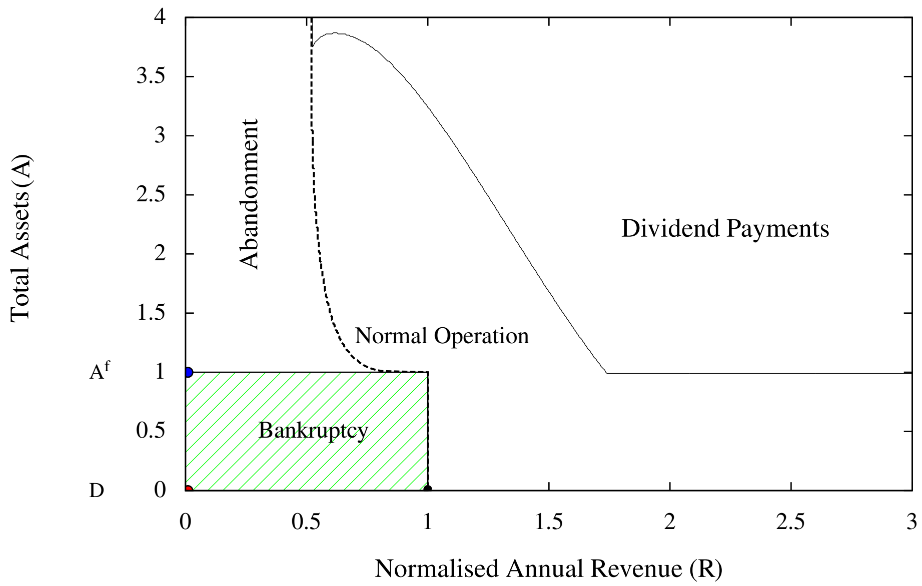


Figure 5.7: A plot to show how the fixed assets affects a firm's operational strategy within the Revenue-Asset space. Here, we assume the fixed operational cost $\varepsilon = 1$ and the fixed assets ($A^f = 1$).

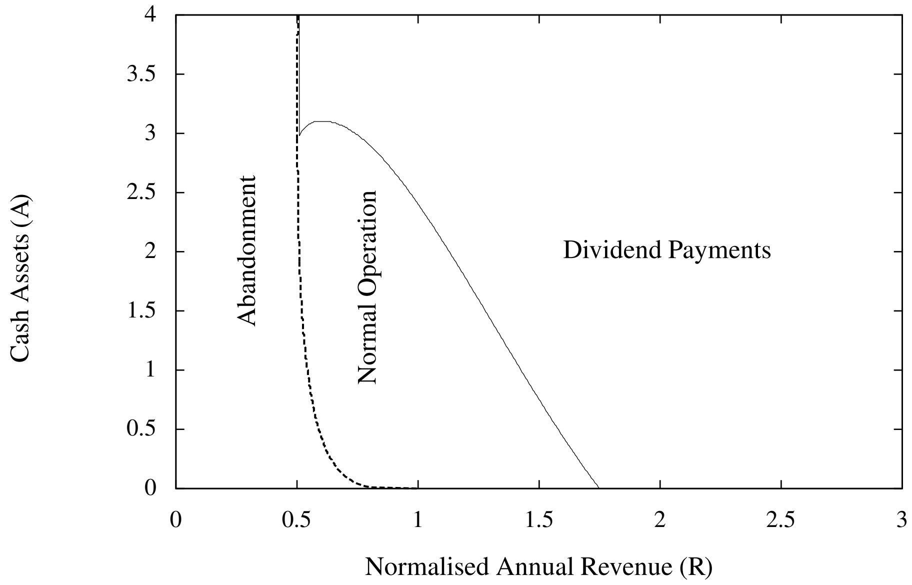


Figure 5.8: A plot to show the corporate operational regions within the Revenue-Asset space, where the whole space was divided into three regions: the abandonment region, the normal operational region and the optimal dividend payment region. The parameter values are based on Table 5.1.

Figure 5.9 and 5.10 summary how revenue volatility affects firm's valuation and operational strategy. In Figure 5.9, we show how equity return varies along with assets by fixing revenue $R = 0.95$ and testing $\sigma = 0.1, 0.2$ and 0.3 individually. It is obvious that when the Return on Equity increases, its optimal liquid asset level drifts left with the increase of revenue's volatility. We know that volatility defines the uncertainty level of the future revenue, and the cash holdings are used as a financial buffer to overcome any short-term risk. Therefore, when σ is high, the revenue becomes ever more volatile and therefore, it is more likely to move to either a lower or higher position, and simultaneously, the firm requires more cash holdings to hedge such volatile changes.

Figure 5.10 presents the operational boundaries by taking the same parameters. According to this figure, we can see that the abandonment boundary drifts left with the increasing σ. Besides, the firm is more reluctant to abandon the business and pay out dividends with growing uncertainty of the revenue. It is interesting that the change of volatility has greater effects on the value and boundaries when liquid assets are low, but the effects of varying σ are not that significant when the firm has a higher level of assets. The reasons are the same.

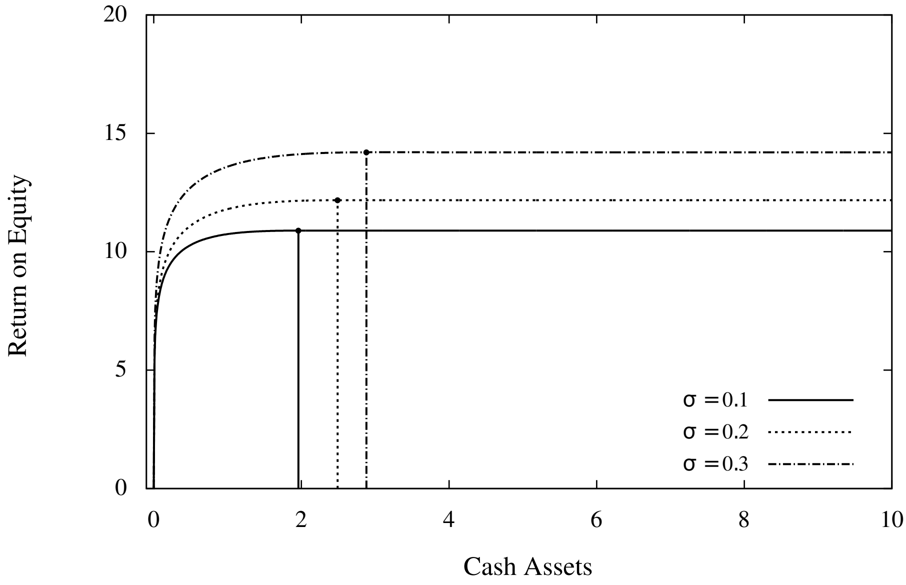


Figure 5.9: The Return on Equity of the business against the cash assets for firms that have different revenue volatility σ, where the other parameters take values from the Table 5.1.

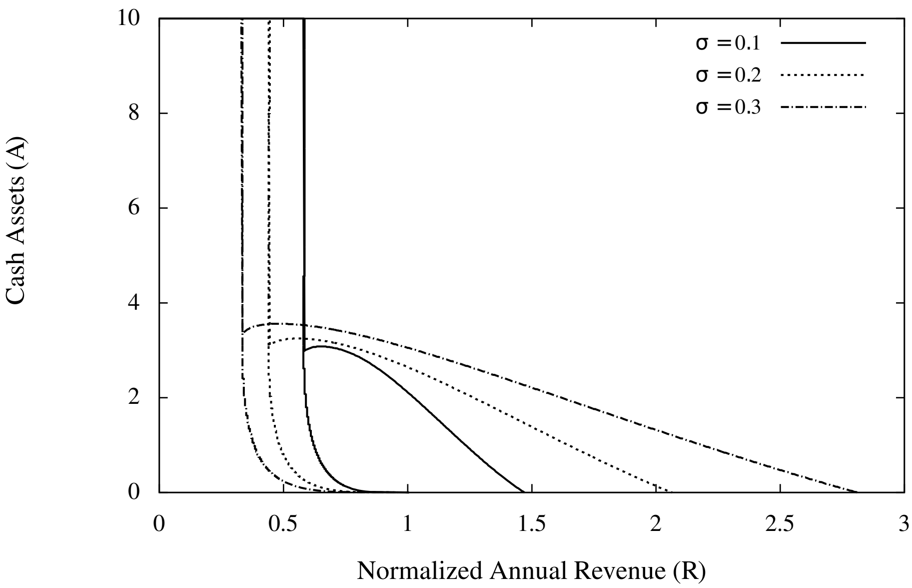


Figure 5.10: Corporate abandonment and dividend payment boundaries for firms that have different revenue volatility σ, where the rest parameter values are based on Table 5.1. Operational decisions vary with different volatilities (i.e. $\sigma = 0.1$, $\sigma = 0.2$, $\sigma = 0.3$)

The drift μ defines the expected growth rate of the revenue. Figures 5.11 and 5.12 show how this parameter affects the equity return and operational policies by fixing revenue $R = 0.95$ and testing $\mu = 0.1, 0.2, 0.3$ respectively. More precisely, according to Figure 5.11, we can see that the increase of drift has significant effects on equity return. The value growth generated by increasing drift is more sensitive than that generated by an increase in volatility. According to Figure 5.12, we can see that increasing drift has tiny effects on the boundaries when the liquid assets are relatively low (less than one), while its effects on the value and boundary are more significant when liquid assets are relatively high (more than one).

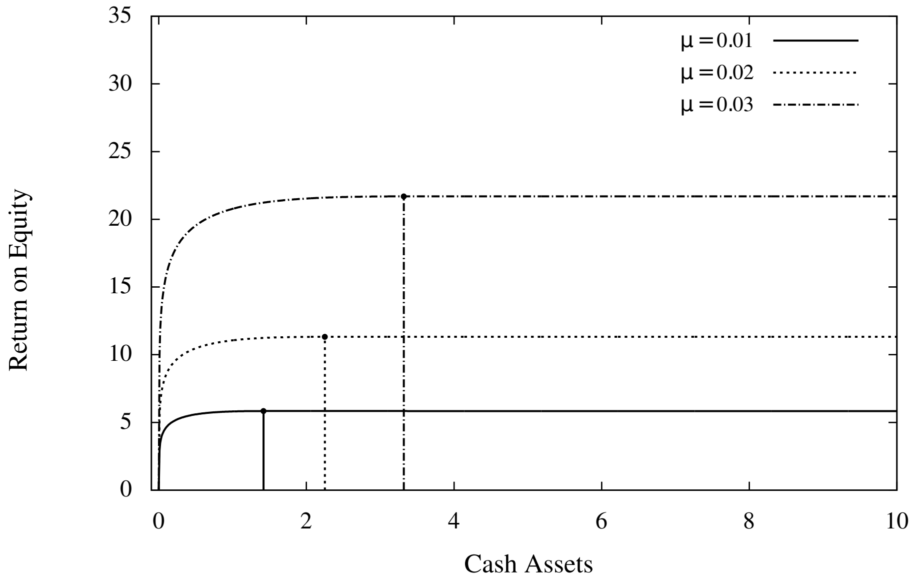


Figure 5.11: A plot to show the Return on Equity for a firm that has different revenue drift rate μ, where the rest parameters take values from the Table 5.1. Here, we take $\mu = 0.01, 0.02$ and 0.03 as examples.

The discounted rate ρ defines the external rate of returns. The numerical solutions of the corporate value and the abandonment boundary are very sensitive to the discounted rate, which can be observed in Figures 5.13 and 5.14. If the revenue has no volatility $\sigma = 0$, the difference of discount rate and revenue drift $(\rho - \mu)$ measures the margin of investing in a public market compared with investing the firm project. The more difference there is, the more incentives that a firm would redistribute the excess cash as dividends. This conclusion matches the observation in Figure 5.14 by comparing dividend payout boundaries with different discount rates. The abandonment

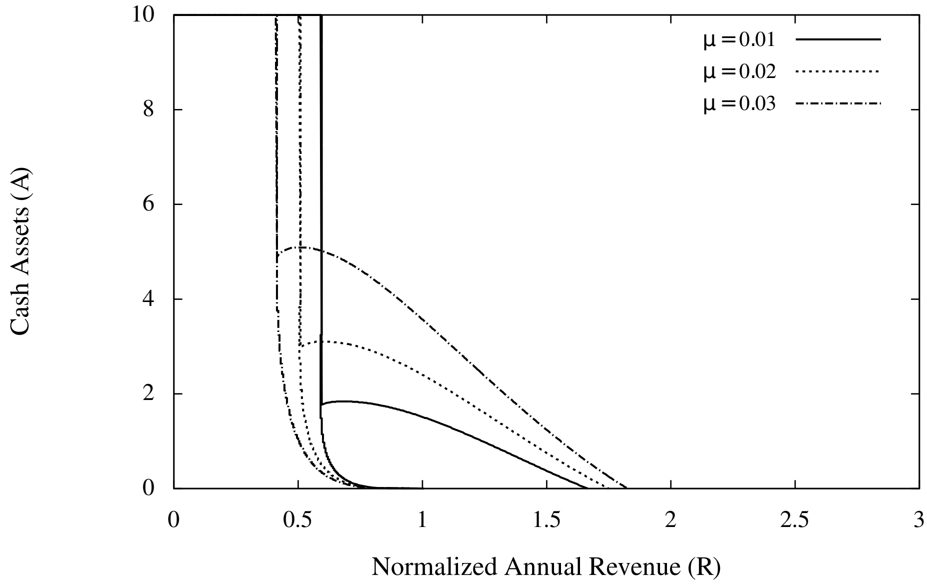


Figure 5.12: Corporate abandonment and dividend payment boundaries within Revenue-Asset space for a firm that implements different revenue drift rate μ, where the rest parameters take values from the Table 5.1.

boundaries also show the potential losses that hold excess cash. Compared with the external return rate ρ, the more difference the external return and internal return, the more motivations investors would terminate the business in order to reduce losses, as shown in Figure 5.14.

We are also interested in how the operating strategy varies against the time to expiration. Figure 5.15 presents a firm's operation when there are 80 years, 40 years, 20 years and 10 years time to expiration. According to this figure, we can see that the firm is more reluctant to abandon the business or redistribute dividend when there is still a long time to expiration. This is because, for a self-financing firm, it is better to hold more cash for a longer life project in order to hedge the future operational uncertainty and take the potential profits. Thus, when moving from short life project to long life project, the dividend payment boundary expands and the abandonment boundary shifts towards the left. We can observe expanding dividend payment boundary and left shifting abandonment boundaries.

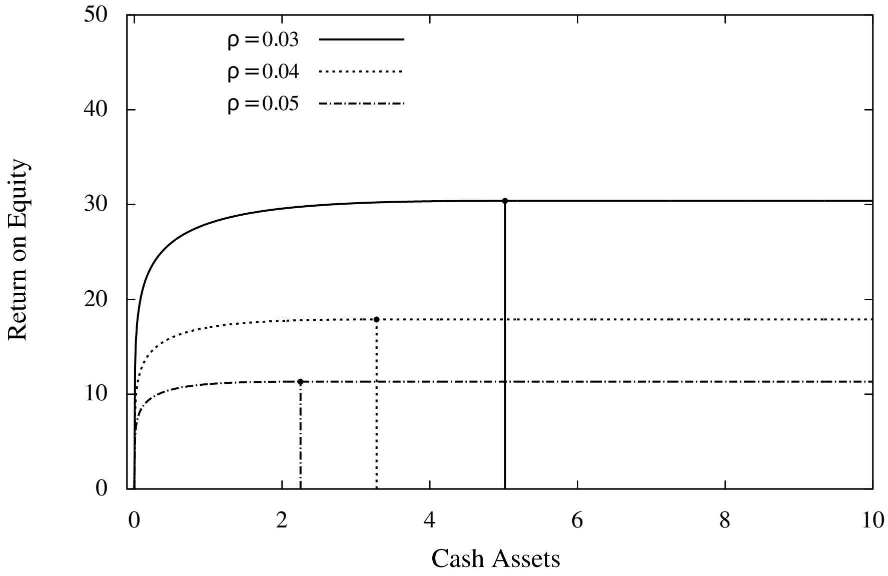


Figure 5.13: A plot to show the Return on Equity against the cash assets for a firm that employs different discount rates ρ, where the rest parameters take values from the Table 5.1. Here, we take $\mu = 0.01, 0.02$ and 0.03 as examples.

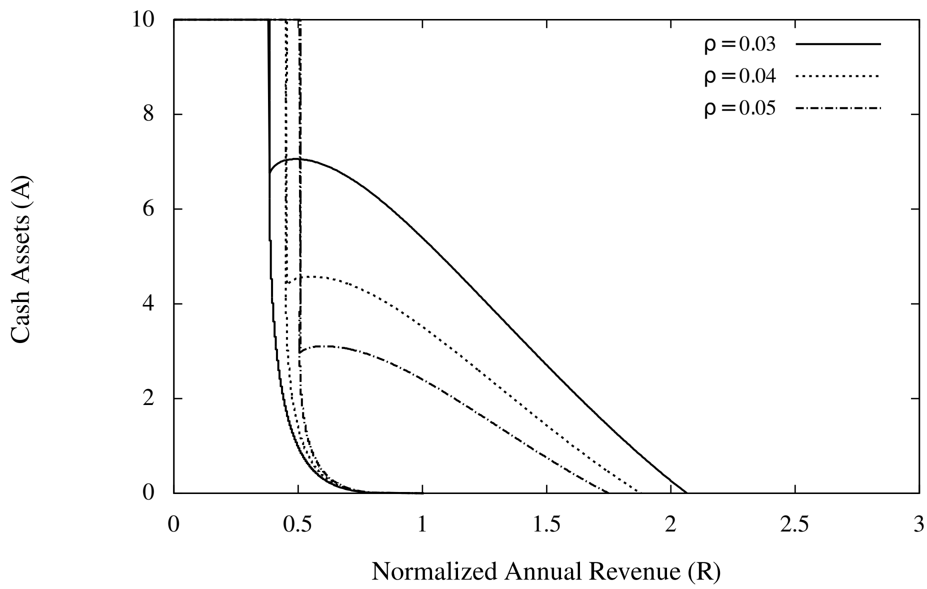


Figure 5.14: Corporate abandonment and dividend payment boundaries within Revenue-Asset space for a firm that implements different discount rate ρ, where the rest parameters take values from the Table 5.1.

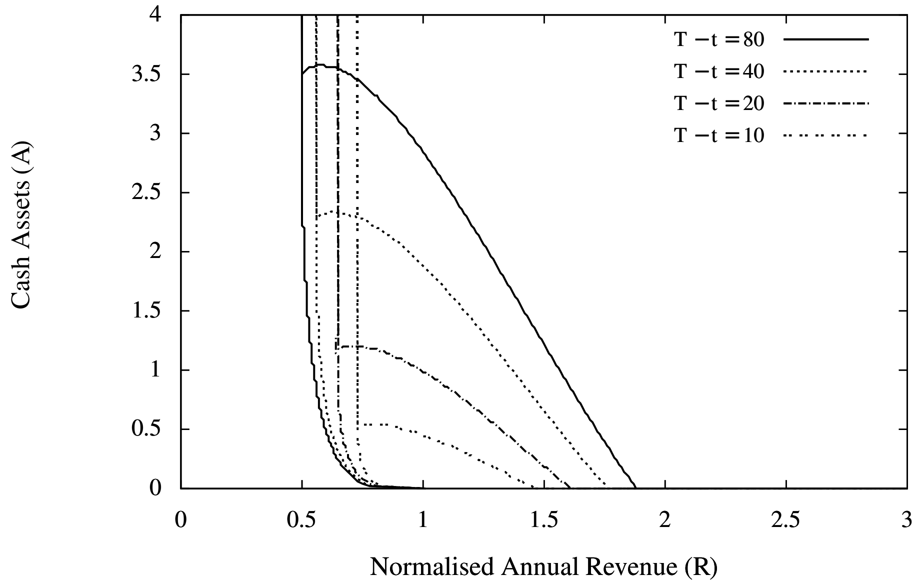


Figure 5.15: Corporate abandonment and dividend payment boundaries for firms that have different expiration time ($\tau = T - t$), where the rest parameter value are based on Table 5.1.

5.5 Conclusion

In corporate finance studies, the shareholders benefits are believed to be the expected dividend payment, thus the dividend payout policy significantly affects the shareholders' long term return. Recent studies on optimal dividend payment model generally assumes that the firm itself to be a stochastic process (see Décamps, Jean-Paul and Villeneuve, Stéphane (2007), Belhaj (2010), Jiang and Pistorius (2012) and Chevalier et al. (2013)). This assumption fails to involve the interaction between the dividend policy with the firm's liquidity risk (see the discussion in Gryglewicz (2011)). In this chapter, we presented how to model a cash constrained firm that had the flexibility to optimally pay out dividends and abandon the business. We found that the optimal dividend payout method was a more efficient way to study the interest of shareholders, and the marginal value of cash was the key criterion to define the optimal dividend payout boundary. In addition, we found a firm's optimal dividend payment boundary does not monotonically vary along with the firm's revenues. When the firm is in a non-profitable region, the optimal dividend payment threshold decreases as revenue decreases until meets abandonment boundary. This result shows that shareholders have incentives to distribute dividends in a bad business situation, which reveals a possible existence of the agency conflicts between when a firm has shareholders and debt investors. We leave this study for Chapter 7. The model presented in this chapter allows people to study how a firm optimally redistributes the excess cash as dividends, and how this optionality interacts with a firm's cash constraints and abandonment decision. In this chapter, we assumed the firm had no option to raise external funds. We investigate the impact of optimal financing problems in the subsequent chapters.

Chapter 6

Optimal Financing Problems - Equity Financing

6.1 Introduction

Apart from using internal cash reserves, external financing is another important approach to help a firm overcome its liquidity shortage and capture attractive investment opportunities. There are generally two possible avenues for a firm to raise external funds, equity financing and debt issuance, and they involve different types of participants, i.e. equity and debt investors. As we know, these two types of investors have very different investment objectives and risk preferences, therefore, it is difficult to build the equity and debt financing problems under one unified framework. Given the complexities involved, we initially investigate and model them separately. Here, in this chapter we only focus on the optimal equity financing problems for a cash constrained firm that has no debts. We will study the optimal debt financing problem in the next chapter.

A firm that raises new capital through equity financing will incur a stock dilution due to the issuing of additional common shares of the company, for example, a primary market offering. The understanding of the share dilution would help us build a practical model. In a dilution process, two core participants shall be involved: the original shareholders and the new share investors. The initial shareholders build up the firm, so they have the right to obtain all the capital returns as they also bear the obligation to all the risks. When a firm does not have enough cash assets to support normal operation, the original shareholders have an incentive to raise external capital by issuing new equity, otherwise they have to give up the business as the firm goes insolvent. If new equity is issued we say that the financing option is exercised, and the original investors (shareholders) will have to give up part of their ownership. In return, the firm can obtain a certain amount of cash (equity) from the new share investors. As for the new investors, they invest cash into the firm in exchange for a part of ownership and a proportion of the firm's future dividend payments and the remaining equity when the firm closes its business.

In the previous studies of modelling optimal equity financing problems, there are only a few papers that address similar issues to ours. Here, we particularly refer to the work by Anderson and Carverhill (2012) and Bolton et al. (2014b). They assumed that a firm operated to maximize the shareholders' benefits by optimally paying out dividends, and they modelled the optimal equity financing decision as an optimal stopping problem based on the firm value and cash holdings process. This idea is quite helpful, however, this approach assumes that a firm has only one opportunity to raise new equity in the external market. In addition, they do not present enough discussions on how the financing cost (often termed the economic friction) can affect the exercising of the financing option. In view of this, to complement these deficiencies, in this chapter, we investigate how the financing limitations affect the financing strategy, by taking three types of financing options as examples: financing at the initial time, financing at an optimal time, and financing unlimited times. In particular, we test how the financing cost affects the exercising of each financing flexibility. To achieve this, we extend our optimal dividend-payment firm model as illustrated in Chapter 5 by further assuming that a firm has an option to optimally issue new equity with the above three types of financing flexibility.

6.2 Mathematical Model

6.2.1 Problem Setting

To keep consistency with Chapter 5, we still consider a cash constrained firm that aims at maximising its shareholders' future benefits. This firm has the same revenue, cash holdings, dividend payment option and abandonment option, as illustrated in that chapter. We are now going to further assume that the firm has the flexibility to sell a proportion of the shareholders' ownership in exchange for an injection of cash capital into the firm. The firm achieves its aim (maximising its shareholders' benefits) by choosing both the optimal financing time and controlling the amount of capital that is raised. In order to verify the effect of equity financing flexibility on the firm's valuation and operational strategy, we consider three types of the financing options. Each of the three options is differentiated by a gradual increase in the financing flexibility: The option to finance the firm once at a fixed time-point for any amount of capital; the option to finance the firm once at an optimal time for any amount; and the option to finance the firm unlimited times at any time points for any amount (refinancing is available). First, we summarize the entire operational timeline of the equity financing firm. Then, we can derive the corresponding stochastic framework for a firm that has each of the above financing options independently.

6.2.2 Operating Timeline for an Equity Financing Firm

We inherit notations illustrated in Chapter 5 for models in this chapter, and assume that a firm starts with initial revenue R_t ($R_t = R$) and cash assets A_t ($A_t = C_t = A = C$) at time t. To investigate the whole structure of an equity financing firm, we now study the firm's two operating phases of a financing decision: the before-financing phase and the after-financing phase (and the inherent economic connection between them).

Before-financing Phase: We define the before-financing phase as the time-period when a firm operates normally and has not yet exercised any financing options. In this phase, the firm operates with initial capital A_t, which acts as a cash buffer to help maximise its initial shareholders' benefits by controlling the dividend payments and seeking an optimal exercise of its financing option or abandonment option. When the firm operates non-profitably for a long time then its cash assets A_s ($s \in [t, T]$) continuously decreases to a critically low level, the firm needs to weigh up which strategy should it implement: to abandon the business immediately or to seek external financing, according to which strategy can utmost benefit the original shareholders' interest.

Exercising a Financing Option: Suppose the firm raises $E^F + \Psi(E^F)$ total amount of cash equity at an exercise time τ_f. Here, $\Psi(E^F)$ is the financing cost, which satisfies $\Psi(0) = 0$; E^F is the net capital left after financing, which satisfies $E^F \in [0, +\infty)$; and τ_f can take either a fixed time or an optimal time depending on the type of financing flexibility subject to $\tau_f \in [t, T]$. When the firm exercises this decision at τ_f, its cash assets increase immediately from A_{τ_f} to $A^+_{\tau_f}$, where,

$$A^+_{\tau_f} = A_{\tau_f} + E^F. \tag{6.1}$$

Also, the exercise of this option will immediately result in a share dilution, which means a reduction of the existing shareholders' ownership in exchange for the new cash equity. We now illustrate the inherent economic connections between the market value of the firm, the value of original shareholders and the value of new shareholders.

Suppose the original shareholders' proportional ownership changes from 1 to λ ($0 \leq \lambda \leq 1$) in a share dilution. We denote the firm value just before the financing time τ_f as $V(\tau_f^-, R_{\tau_f}, A_{\tau_f})$, and just after the financing time as $V(\tau_f^+, R_{\tau_f}, A^+_{\tau_f})$. For the original and new shareholders, in a no arbitrage market, their market value during the issue should satisfy the following equations, respectively.

$$\begin{array}{ll} & \text{before financing} \ : \ \text{after financing} \\ \text{Original Shareholders:} & V(\tau_f^-, R_{\tau_f}, A_{\tau_f}) = \lambda V(\tau_f^+, R_{\tau_f}, A^+_{\tau_f}) \\ \text{New Shareholders:} & E^F + \Psi(E^F) = (1-\lambda) V(\tau_f^+, R_{\tau_f}, A^+_{\tau_f}). \end{array}$$

Combing these two equations together and removing λ, we can get the value conservation equation,

$$V(\tau_f^-, R_{\tau_f}, A_{\tau_f}) = V(\tau_f^+, R_{\tau_f}, A_{\tau_f}^+) - E^F - \Psi(E^F). \tag{6.2}$$

Since the original shareholders have the flexibility to choose a financing amount E^F, they shall find an optimal level $\check{E}^F$ to maximise their own benefits, here it is the firm value just before dilution, i.e. $V(\tau_f^-, R_{\tau_f}, A_{\tau_f})$. Thus, if there is one optimal financing amount, it should satisfy the following equation,

$$\check{E}^F := \underset{E^F}{\operatorname{argmax}} \big\{ V(\tau_f^+, R_{\tau_f}, A_{\tau_f} + E^F) - E^F - \Psi(E^F) \big\}. \tag{6.3}$$

Mathematically, we can find $\check{E}^F$ from its extreme value by differentiating Equation (6.3) against E^F, and setting the right hand side to zero. So we end up with

$$\frac{\partial V(\tau_f^+, R, A^+)}{\partial A^+}\frac{dA^+}{dE^F} - 1 - \frac{d\Psi}{dE^F} = \frac{\partial V(\tau_f^+, R, A^+)}{\partial A^+} - 1 - \frac{d\Psi}{dE^F} = 0, \quad \text{at} \quad E^F = \check{E}^F. \tag{6.4}$$

Here, we assume $\Psi(E^F)$ is a non-concave function ($\frac{\partial \Psi}{\partial E^F}$ is decreasing). In addition, we know that $\frac{\partial V}{\partial A}$ is a serious decreasing function of A (see (5.3)). By combing these two conditions, the function $\frac{\partial V}{\partial A} - 1 - \frac{\partial \Psi}{\partial E^F}$ is serious decreasing function, so that the extreme value $\check{E}^F$ is the optimal value to maximise the financing payoff function.

The economic interpretation of this equation is that the marginal value of the last dollar raised must equal one plus the marginal cost of external financing. So, for us to be able to find $\check{E}^F$, we first need to address the market value of the firm $V(\tau_f^+, R, A)$ in the after-financing phase.

After-financing Phase: The firm value in after-financing phase depends on what type of financing flexibility the firm has, since different flexibilities generate different instant payoffs that affect the firm's current value. Also, we are strongly interested in how changing the degree of financing flexibility can avoid a liquidity shortage. By combining the firm's overall operational analysis illustrated above, we now derive the stochastic framework of an equity-financing firm by independently taking three financing options as examples and gradually increasing the financing flexibility.

6.2.3 Option to Finance Once at a Fixed Time (Type A)

In the first example, we assume a firm has an option to raise any amount of capital at the fixed time, i.e. $\tau_f = T_f$ ($T_f \in [t, T]$), but will never be able to do so before or after this time. This simplest option provides a useful framework for us to study how the firm acts to maximise its shareholders' benefits by optimally choosing the financing amount.

Since the firm can only finance once at time T_f, for any time $s \geq T_f$ the firm can be treated as a cash constrained firm that has initial assets $A^+_{T_f}(A^+_{T_f} \geq A_s)$ but without any financing options. Based on this, the after financing value $V(T_f^+, R_{T_f}, A^+_{T_f})$ in Equation (6.2) can be equivalently considered as a non-financing firm $v(s, R_s, A^+_s)$ ($s \in [T_f, T]$) with new level of cash assets, and we know the solution of v can be found by solving Equation (5.11). Therefore, we can define the firm value at the financing exercise time T_f as,

$$V(T_f, R_{T_f}, A_{T_f}) = \max_{E^F}\big\{v(T_f, R_{T_f}, A_{T_f} + E^F) - E^F - \Psi(E^F)\big\}. \tag{6.5}$$

According to this equation, we can see that the firm would only exercise this financing option if there exists an optimal financing amount $\check{E}^F > 0$ such that the optimal value of the firm in the new state $(v(T_f, R_{T_f}, A_{T_f} + \check{E}^F) - \check{E}^F - \Psi(\check{E}^F))$ exceeds the value of doing nothing ($v(T_f, R_{T_f}, A_{T_f})$ when $\check{E}^F = 0$).

Optimal Financing Amount Analysis: Based on our study of firm value v in Chapter 5, the marginal value of cash decreases towards to one as cash increases in assets dimension while the marginal cost of equity financing is $1 + \frac{d\Psi}{dE^F}$ which is always great than one. This observation suggests that there should be a threshold on assets dimension, over which the firm has no endogenous incentive to raise new equity. Therefore, by assuming $A_s = C_s$, we can find this threshold $\check{A}_s$ using the following equation,

$$\check{A}_s = \left\{A_s : \frac{\partial v}{\partial A}(s, R_s, A_s) = 1 + \frac{d\Psi}{dE^F}\bigg|_{E^F=0}\right\}. \tag{6.6}$$

We assume the firm is currently in one operational state (s, R_s, A_s), based on the threshold $\check{A}_s$, we know the optimal financing amount $\check{E}^F$ for a firm in this state should satisfy the following necessary condition,

$$\check{E}^F \leq \max(\check{A}_s - A_s, 0), \tag{6.7}$$

We say this is a necessary condition instead of sufficient-necessary condition to define $\check{E}^F$ because we do not how different financing cost function $\Psi(\cdot)$ will affect the firm's behaviour on the R dimension. That's also why we have to apply numerical techniques to solve this problem. Although, this condition can still at least help us to set and check the upper limits of the financing amount E^F in the numerical optimization.

6.2.4 Option to Finance Once at an Optimal Time (Type B)

Suppose the firm has an option to finance once at an optimal financing time ($\tau_f \in [t, T]$ is random). After exercising this flexibility, for any time $s \in [\tau_f, T]$, the firm can once again be treated as an optimal dividend payment firm that has no more financing options but with new initial assets $A^+_{\tau_f}$. Therefore, the payoff function for the current shareholders when they make the decision as to whether they should raise an amount E^F of capital at time τ_f can be defined as,

$$V(\tau_f, R_{\tau_f}, A_{\tau_f}) = v(\tau_f, R_{\tau_f}, A_{\tau_f} + E^F) - E^F - \Psi(E^F). \tag{6.8}$$

Based on this payoff equation, the original shareholders' benefits at time t (in the before-financing phase) can be valued from three possible parts: the optimal dividend payments, the payoff when the firm abandons the business or the payoff when the firm exercises its financing option, where, the latter two parts are mutually exclusive. We use the indicator function to express the optimal choice between the last two options. Then, we can now define the value function of the firm at time t as,

$$V(t, R_t, A_t) = \sup_{d,\tau,\tau_f,E^F} J(t, R_t, A_t, d, \tau, \tau_f, E^F), \tag{6.9}$$

where,

$$\begin{aligned} J(t, R_t, A_t, d, \tau, \tau_f, E^F) =& E\bigg[\int_t^{\tau_f \wedge \tau} \big(de^{-\rho(s-t)}\big)dt \\ &+ e^{-\tau_f \rho} \mathbb{1}_{\{\tau > \tau_f\}} V(\tau, R_\tau, A_\tau) \\ &+ e^{-\tau\rho} \mathbb{1}_{\{\tau_f \geq \tau\}} h(\tau, R_\tau, A_\tau)\bigg], \end{aligned} \tag{6.10}$$

and, τ is the time when the firm closes the business, which is defined by Equation (5.8); τ_f is the optimal financing time; $\tau \wedge \tau_f$ denotes the smaller one between τ and τ_f; $h(\tau, R_\tau, A_\tau)$ is the optimal abandonment payoff, which is defined by Equation (5.9).

According to stochastic control theory, the value Function (6.9) is a viscosity solution of the following HJB Variational Inequality,

$$\begin{aligned}\min\bigg\{&\frac{\partial V}{\partial t} + \mathcal{L}V - \rho V + \sup_{d\in[0,\infty)}\bigg((R + rC - \varepsilon - d)\frac{\partial V}{\partial A} + d\bigg),\\ &V - \max_{E^F}\bigg(v(t, R, A + E^F) - E^F - \Psi(E^F)\bigg),\\ &V - h\bigg\} = 0,\end{aligned} \tag{6.11}$$

subjecting to the boundary conditions,

$$\begin{cases} V = E & \text{as} \quad t = T & (6.12)\\ \dfrac{\partial V}{\partial A} = 1 & \text{as} \quad C \to \infty \;\&\; R + rC - \varepsilon > 0, & (6.13)\\ V = \hat{\alpha}R - \hat{\beta}\varepsilon + h & \text{as} \quad R \to \infty, & (6.14)\end{cases}$$

where, the linear operator is defined by Equation 4.8. The economic meanings of each boundary condition can be found in Subsections 4.2.3 and 5.2.3.

Analysis on Optimal Financing Amounts and Optimal Financing Time:

The Type B financing option offers the extra flexibility to choose an optimal time to exercise the financing option. Comparing this with the Type A option, we need to further consider whether it is better to wait than to exercise the option immediately, i.e. the optimal financing time. However, without specify the financing cost function, it is very difficult to address the optimal financing time and optimal financing amounts, since the benefits of financing depends on the gradient of v, the type of cost function. Therefore, we leave this in the numerical results test.

6.2.5 Option to Finance Unlimited Times (Type C)

We assume a firm that has a Type C financing option can refinance an unlimited number of times. Compared with the previous two options, this option offers significant flexibility for the firm to avoid liquidity shortage. To illustrate, the firm still has unlimited opportunities to access to the external capital markets regardless of whether some financing decisions have been made in the past or not. Therefore, after any particular exercise of the option, we still assume the firm has an unlimited number of financing option available in the future, but with refreshed cash assets and expiration. Thus, the payoff function for each exercise date can be derived from the firm value itself.

To model a firm with this financing option, we consider a series of ordered financing times $\{\tau_f^0, ..., \tau_f^n, ..., \tau_f^N\}$, where $\{t = \tau_f^0 \leq ...\tau_f^{n-1} \leq \tau_f^n ... \leq \tau_f^N = T\}$, and the corresponding time intervals: $[t, \tau_f^1), [\tau_f^1, \tau_f^2), ...[\tau_f^{n-1}, \tau_f^n), ...[\tau_f^{N-1}, T]$. With this series of stopping times, the value function $V(s, R_s, A_s)$ for all $s \in [t, T]$ can be defined by a Càdlàg type of process (see definition in book Applebaum (2009)), which is right continuous and has left limits.

To simplify the analysis, we consider the firm's dynamic behaviour in each time period $[\tau_f^n, \tau_f^{n+1})$. We use subscript '$-$' denotes the time just before-financing, and subscript '$+$' denotes the time just after-financing. The payoff function when the firm raises the E^F amount of new funds at time τ_f^{n+1} can be expressed as,

$$V(\tau_{f-}^{n+1}, R, A) = V(\tau_{f+}^{n+1}, R, A + E^F) - E^F - \Psi(E^F). \tag{6.15}$$

This equation gives a useful relationship of the firm value between two near time intervals, based on which, we can define the firm value at time τ_f^n as,

$$\begin{aligned} V(\tau_{f+}^n, R_{\tau_{f+}^n}, A_{\tau_{f+}^n}) = \sup_{d,\tau,\tau_{f+}^{n+1},F} E\Bigg[& \int_{\tau_{f+}^n}^{\tau_{f-}^{n+1}\wedge\tau} \left(de^{-\rho(s-t)}\right) dt \\ & + e^{-\tau_f^{n+1}\rho} \mathbb{1}_{\{\tau > \tau_f^{n+1}\}} \left(V(\tau_{f+}^{n+1}, R, A + E^F) - E^F - \Psi(E^F)\right) \\ & + e^{-\tau\rho} \mathbb{1}_{\{\tau_f^{n+1} \geq \tau\}} h(\tau, R, A) \Bigg]. \end{aligned} \tag{6.16}$$

Here, τ is the optimal abandonment time and $\tau \in [t, T]$. By implying the Stochastic Control Theory and Dynamic Programming Principle on each equation (with Formula (6.16)) through all the time intervals, we can derive a uniform Variational Inequality as follows,

$$\min\left\{\frac{\partial V}{\partial t}+\mathcal{L}V-\rho V+\max_{d}\left[(R+rC-\varepsilon-d)\frac{\partial V}{\partial A}+d\right],\right.$$
$$V-\max_{E^F}\left[V(R,A+E^F)-E^F-\Psi(E^F)\right],$$
$$\left.V-h\right\}=0, \tag{6.17}$$

which is subject to the same boundary conditions that of the Model (6.11).

Analysis on Optimal Financing Amounts and Optimal Financing Time: Since the Type C option offers unlimited flexibility to choose a financing time, following the same economic incentives, the firm will exercise each option only when the cash runs out. In Bolton's work (see Bolton et al., 2011), they discussed the optimal financing policy from an economic viewpoint. They showed that, for a firm that has continuous financing opportunities, it has no incentives to issue equity unless it has depleted all of its internal cash holdings. This is because 'Cash within the firm earns a below-market interest rate while there is also the time value for the external financing costs' (see Bolton et al., 2011). In the results section, we will show consistent solutions to support this conclusion.

Therefore, we can define the optimal financing time as,

$$\mathcal{T}:=\left\{\tau_f^n : A_{\tau_f^n}=0\right\} \quad \text{where} \quad n=1,2,3,...,N, \tag{6.18}$$

and the corresponding optimal financing amount $\check{E}^F_{\tau_f^n}$ should satisfy the following necessary condition,

$$\check{E}^F_{\tau_f^n}\leq \max\{\check{A}_{\tau_f^n},0\}, \tag{6.19}$$

where, following the same logic on firm's incentive to raise external equity, $\check{A}_{\tau_f^n}$ is given by,

$$\check{A}_{\tau_f^n}:=\left\{A_{\tau_f^n}:\frac{\partial V}{\partial A}(\tau_f^n,R_{\tau_f^n},A_{\tau_f^n})=1+\left.\frac{d\Psi}{dE^F}\right|_{E^F=0}\right\}. \tag{6.20}$$

6.3 Numerical Methods

In this section, we illustrate the corresponding numerical algorithms to solve the mathematical models.

6.3.1 The Type A Financing Option

For the fixed time financing option model (6.5), we need to solve the no-financing firm value (5.10) first in order to get the payoff of financing. We inherit the approach used in Chapter 5 to find v, and then use the following algorithm to find the numerical approximation of firm value V and the optimal financing amount $\check{E}^F$.

Type A Financing Option (6.21)

Get v for all (R, A) at time t
for $j = 0 \to J$ and $i = 0 \to I$ (loop A and R) **do**
  for $l = 0 \to L$ (loop E^F) **do**
    solve $\check{E}^F = \underset{E_l^F}{\text{argmax}}\{v(R_i, A_j + E_l^F) - E_l^F - \Psi(E_l^F)\}$
  end for
  solve $V(R_i, A_j) = v(R_i, A_j + \check{E}^F) - \check{E}^F - \Psi(\check{E}^F)$
end for

6.3.2 The Type B Financing Option

For the Type B financing option model (6.11), we need to instantaneously compare the continuation value of holding onto the option and the payoff of exercising a financing option in order to find the optimal financing time. To achieve this, we calculate in parallel the continuation value (6.11) and the no financing v (see model (5.10)) for each time point t by following the algorithm,

Type B Financing Option (6.22)

for $n = 1 \to N$ (backward time loops) **do**
  solve v^n for all (R, A)
  solve V^n for all (R, A) with Semi-Lagrangian Methods
  find optimal financing time and amount by using
  $V^n(R, A) = \max_{E^F}(v(R, E^F + A) - E^F - \Psi(E^F), V^n(R, A))$
end for

Figure 6.1 presents a schematic analysis of the calculation domain and boundary condition layouts for the Type B financing option. It shows two calculation regimes:

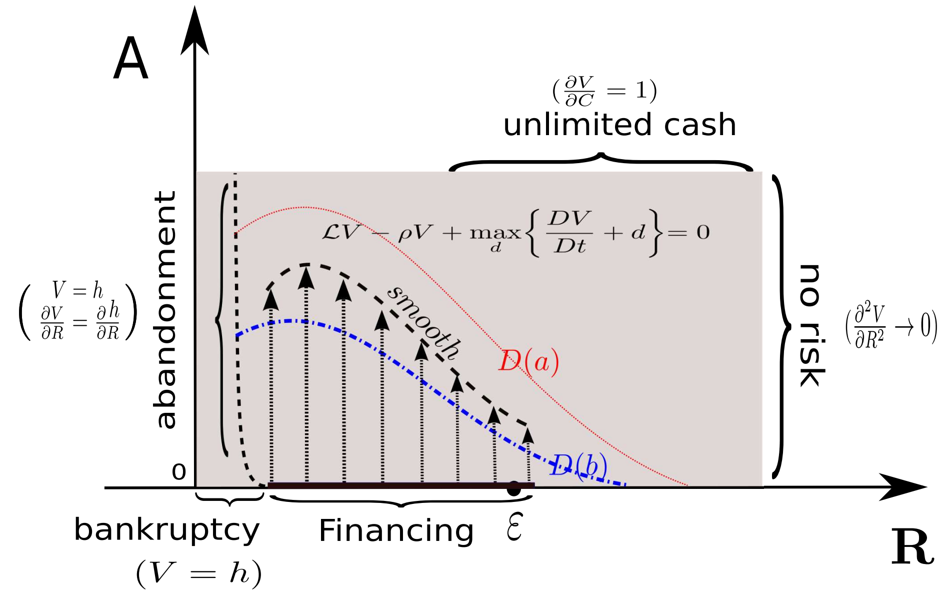


Figure 6.1: A sketch showing the calculation domain and boundary condition layouts for the Type B equity financing option. Here, R is the normalised annual revenue, A is the assets, ε is the operating costs, $D(a)$ denotes the dividend payment boundary after-financing, $D(b)$ denotes the dividend payment boundary before-financing. The length of the arrows denotes the financing amount. The start points of the arrows are where firm exercises this financing option. The end points of the arrows are where the marginal value of cash assets equals the marginal cost of financing.

the before-financing regime and the after-financing regime. The abandonment boundary, insolvency boundary, no risk boundary and unlimited financing boundary in each regime have same settings as illustrated in Figure 5.1. Here, we particularly visualize our understanding of the properties of the optimal financing time, optimal financing amount and the dividend payment boundaries in both regimes. The theoretical optimal financing time is when the firm first runs out the cash assets (at the bottom of the (R, A) space). When a financing option is exercised, the firm's operational state immediately jumps from the bottom line up to an optimal position with higher cash assets. We use arrows to denote these jumps of the firm's operational regimes, where the start positions of the arrow denote the operational status in the before-financing phase, the end positions of the arrows denote the state that in after-financing phase and the length of the arrows denote the financing amount. By further analysing the marginal value of cash, we can non-opinionatedly conclude that the operational status after financing should be located between the dividend payment boundaries of the before-financing phase and the after-financing phase. This is because a firm has the motivation to raise new equity only when the marginal value of cash is very high, whereas a firm has the motivation to pay dividends only when the marginal value is no less than one. We will give more discussion on this in the results section.

6.3.3 The Type C Financing Option

According to Equation (6.17), the financing payoff depends on V itself. Therefore, we can solve this model by using the following algorithm,

Type C Financing Option (6.23)

for $n = 1 \to N$ (backward time loops) **do**

  solve V^n for all (R, A) with Semi-Lagrangian Methods

  solve $V^n(R, A) = \max_{E^F}(V(R, E^F + A) - E^F - \Psi(E^F), V^n(R, A))$

end for

Figure 6.2 shows a schematic analysis of the calculation domain and boundary condition layouts for the Type C financing option. We can more efficiently understand the information showed in this figure by combing it with Figure 6.1. The Type C financing option offers an extra flexibility of choosing how many times to raise equity. This flexibility significantly affects the financing amount, since the firm doesn't need to raise enough external cash assets once, it just needs to meet the demands of the current situation. However, when there is a frictional financing cost, the firm has to balance the financing frequency and financing amount. Thus, the optimal financing amount has to be solved numerically. We show more analysis in the numerical solution chapter.

6.4 Solution and Analysis

We have derived the mathematical framework of three basic optimal financing options and presented the relevant algorithm to solve each of them. In this section, we present numerical results on how a firm's financial flexibility affects its valuation and operational strategy. Particularly, we conduct analysis on how the marginal value of cash

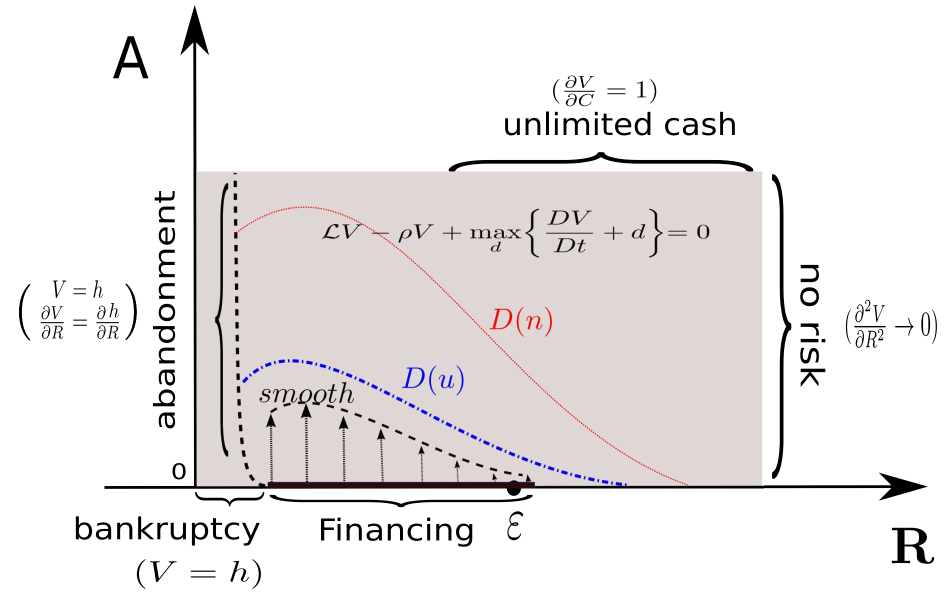


Figure 6.2: A schematic sketch to show the calculation domain and boundary layouts for a firm who has an unlimited financing option. Here, R is the normalised annual revenue, A is the assets, ε is the operating costs, $D(n)$ denotes the dividend payment boundary for a firm has no financing option, and $D(u)$ denotes the dividend payment boundary for a firm has an unlimited financing option. The length of the arrow denotes the financing amount. The start points of the arrows are where firm exercises this financing option. The end points of the arrows are where the marginal value of cash assets equals the marginal cost of financing.

and financing costs impacts a firm's financing decision.

In the following contents, we assume a linear financing cost function Ψ,

$$\Psi(E^F) = \psi_0 + \psi E^F, \tag{6.24}$$

and we use the parameter values in Table 6.1.

Parameters	$\mu(y^{-1})$	σ	ρ	$\varepsilon(My^{-1})$	$r(y^{-1})$	$\psi_0(M)$	ψ
Value	0.03	0.15	0.05	1	0.01	0.5	0.05

Table 6.1: A basic set of parameter values for numerical solutions. Here, μ and σ are the growth rate and volatility of the annualised revenue (see Equation (5.1)); ρ is the market discounted rate (used in Model (5.10)); ε is the annualised running cost (defined in Equation (5.3)); r is the short term interest rate (used in Equation (5.3)); A_f is the fixed assets. $\psi_0(M)$ and ψ are the coefficients required by Function (6.24).

To simply the explanation of the results, we normalize the parameters based on the operational cost $\varepsilon = 1(My^{-1})$, so that we do not need to explain the unit of each variable.

6.4.1 Corporate Valuation and Operational Strategy

In this section, the analysis will focus mainly on how the equity financing option affects the firm's valuation and operational strategy.

a. The Type A option:

A firm with Type A financing option has the flexibility to choose an optimal financing level at a fixed time t or never. Thus, the firm makes the optimal financing decision by comparing the Return on Equity (or Return on Equity $V - E$) with exercising this option and that without exercising the option (see the definition in Chapter 5).

Figure 6.3 shows three pairs of comparisons on the Return on Equity (see Definition 5.22) of a firm that has a financing option (solid lines) and a firm that has no financing option (dashed lines) by taking the net cash flow (NCF, which includes the interest income) $NCF = 0, -0.1$ and 0.1 individually. According to this figure, we can see that the fixed time financing option significantly increases the Return on Equity where the firm's cash holdings are relevantly low, so that the firm's abandonment boundaries are changed. To illustrate, we take the comparison of a firm's return on equity when the net cash flow is -0.1 as an example. We use '(n)' denoting the no-financing firm and '(y)' denoting the firm that has financing options. According to the figure, we can see that for a no-financing firm ($NCF = -0.1(n)$), it abandons the business when the cash holdings equal the abandonment cost, i.e. here it is zero ($V = 0$). However, for a firm that has the financing option ($NCF = -0.1(n)$), it never abandons the business since its value is always greater than the abandonment cost (zero). We show more on the operation policy analysis in the following two figures.

Figure 6.4 shows the firm's operational strategy when it has a Type A financing option. According to this figure, the whole space of the firm states is divided into four parts by three boundary lines: the dividend payment boundary (blue line), the financing boundary (green line) and the new abandonment boundary (red line). Comparing this figure with Figure 5.4, we can see that the firm exercises its financing option to raise funds externally instead of going to abandonment or bankruptcy when it has relatively low cash assets and negative cash flow. What is more, the exercising region is one part

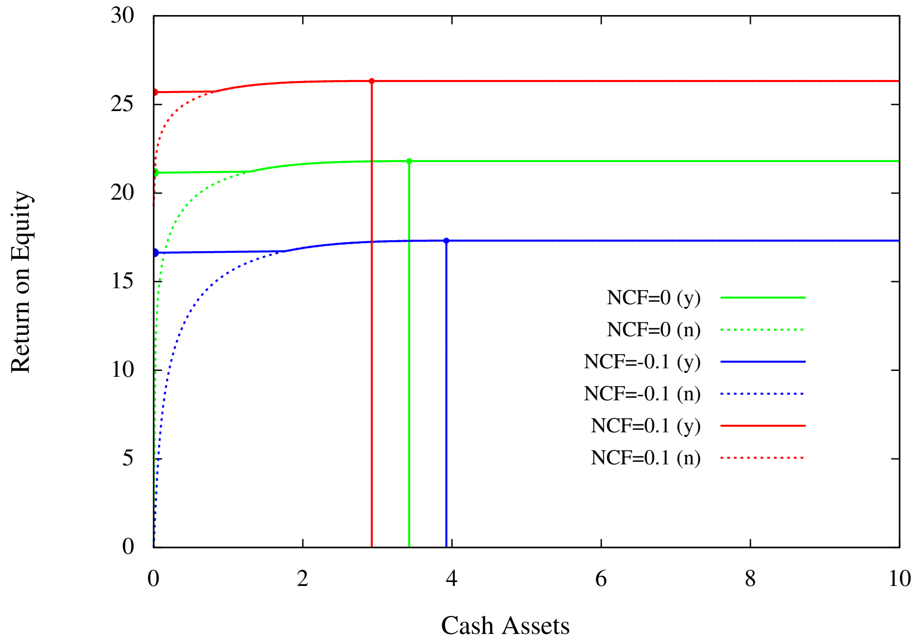


Figure 6.3: A comparison of the Return on Equity between a financing firm and a non-financing firm, where the net cash flow (NCF) takes value, $0, -0.1$ and 0.1. The vertical lines define the corresponding dividend payment boundary.

of the normal operational region for a firm that has no financing option. It has no overlap with the dividend payment region since these two boundaries are defined by two different levels of the marginal value of cash assets.

Figures 6.5 show how much new equity should a firm raise in order to maximise the original shareholders' benefits. According to these figures, we can see that the optimal financing amount of the firm decreases with the increase of the firm's self-sustaining funding. The main part of the financing region is located in the negative profit region, where $R \leq \varepsilon$. However, it is interesting to see that when the firm has a few cash holdings and low level of positive net cash flow, the firm still prefers to raise new funds by exercising the fixed time financing option. This is because the firm cannot choose an optimal financing time, and also, it is not allowed to refinance. We will show how this affects a firm's operational strategy with these further financing flexibility in the following results. By comparing the sub-figure (a) and (b), we can see that the financing cost has a significant effect on the optimal financing amount. We will study the financing cost effects in more depth in later sections.

To further illustrate the contribution of the financing flexibility to the value of the

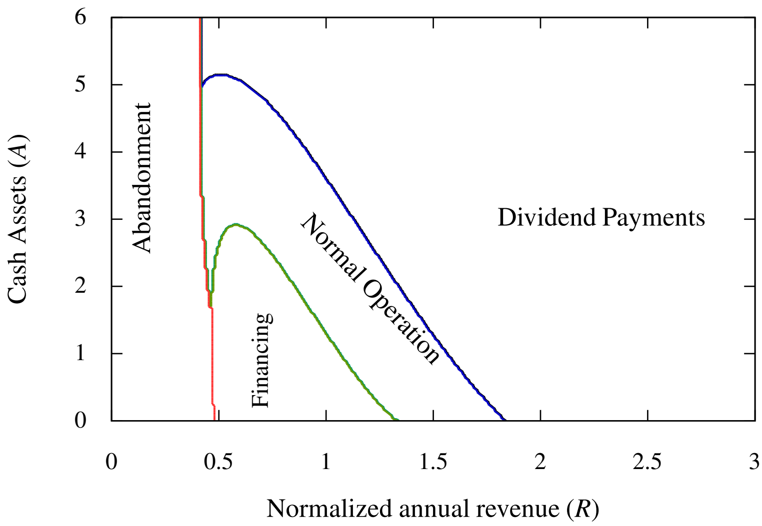
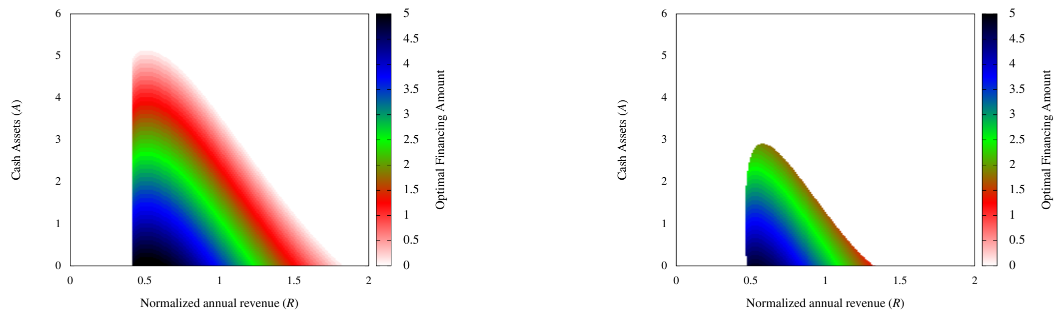


Figure 6.4: A plot to show the optimal operational strategy of a firm that has the Type A financing option within Revenue-Asset (R_t, A_t) space, where the parameter values are selected from the Table 6.1.



(a) Optimal financing amount without cost (b) Optimal financing amount with cost

Figure 6.5: A figure to show the optimal financing amount for a firm that has the Type A financing option, where the colour white represents the lowest financing amount and colour black represents the highest level. Figure (a) shows the optimal financing amount with no costs, whereas, Figure (b) shows the optimal financing amount with fixed cost $\psi_0 = 0.5$ and ratio cost $\psi = 0.05$ (The colourful region shown in this figure is the same as the 'Financing regime' region shown in Figure 6.4)

firm, we define the Financing Option Value (FOV) as,

$$\mathbf{FOV} = V - v, \tag{6.25}$$

where, V is the value of a firm that holds a financing option, which is defined by Equation (6.5), and v is the value of firm without any options to finance, which is defined by Equation (5.11).

Figures 6.6 and 6.7 present how the Financing Option Value (FOV) varies with revenue and cash assets. According to Figure 6.6, we can see that the FOV is always positive and it increases from zero to the highest level and then decreases with an increase in revenue. The option is always more valuable when the firm has fewer cash holdings. In Figure 6.7, the financing option value decreases with the increase of cash assets, and the rate of decrease also depends on the revenue level. This makes intuitive sense, since the more cash assets the firm has, the less impact a financing option can have.

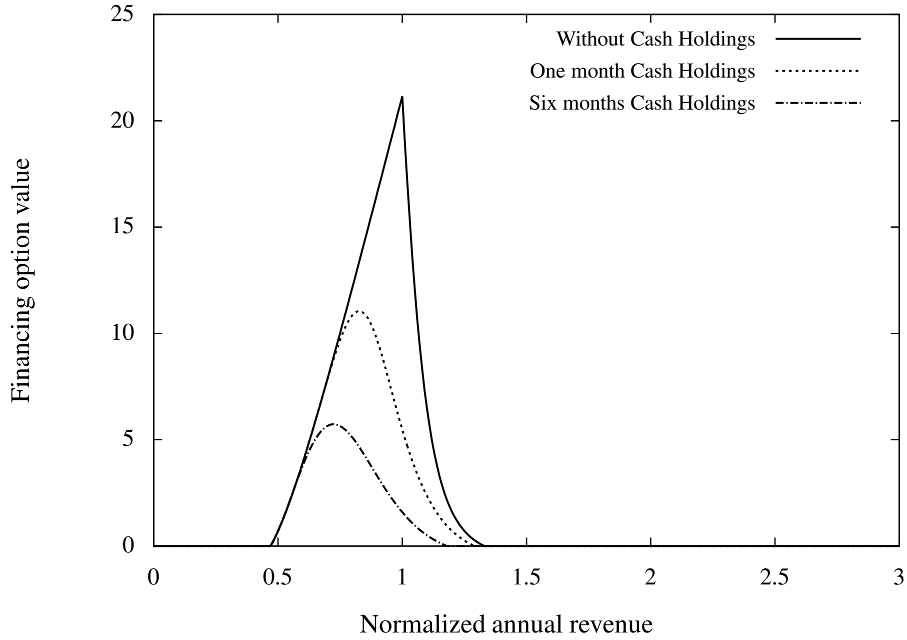


Figure 6.6: The Type A financing option value against revenues for a firm that has no cash holdings, one month cash holdings and six month cash holdings, where the fixed financing cost $\psi_0 = 0.5$ and the percentage financing cost $\psi = 0.05$.

b. The Type B financing option:

This option gives the firm extra flexibility to optimally choose a financing time. Figure 6.8 presents the Return on Equity of the firm against different initial cash assets by selecting net cash flow equals $-0.1, 0$ and 0.1 as examples. By comparing this figure with Figure 6.3, we can see that the optimal financing time flexibilities further reduces the risk of insolvency. Also, with this option, the firm tends to redistribute its excess cash at a relatively low level of cash assets, which means that the firm does not keep that much of the cash as a buffer for the future operation.

Figures 6.9 and 6.10 present how the FOV varies with different revenues and initial

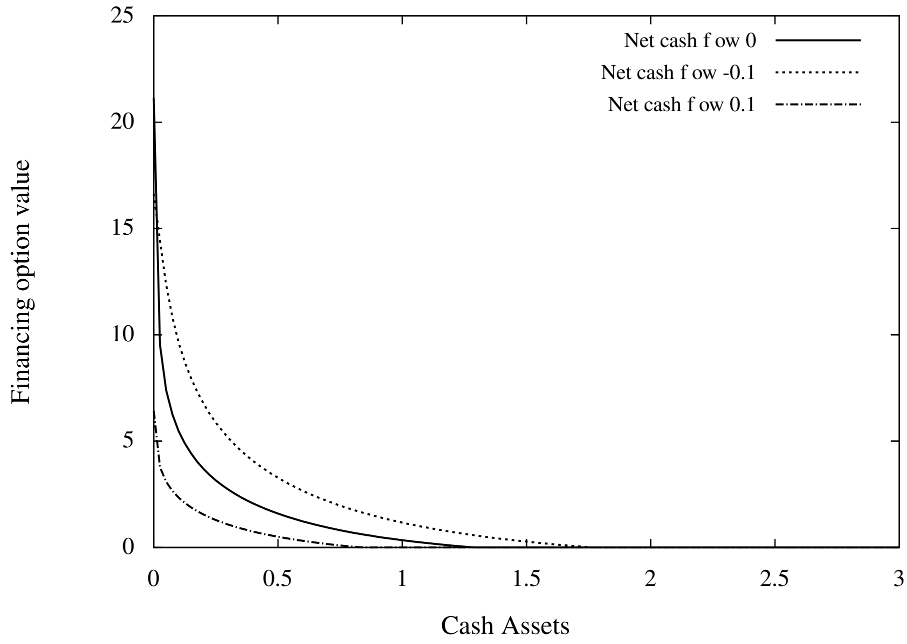


Figure 6.7: The Type A financing option value against cash assets for a firm that has different net cash flows, where the fixed agency cost $\psi_0 = 0.5$ and the percentage financing cost $\psi = 0.05$.

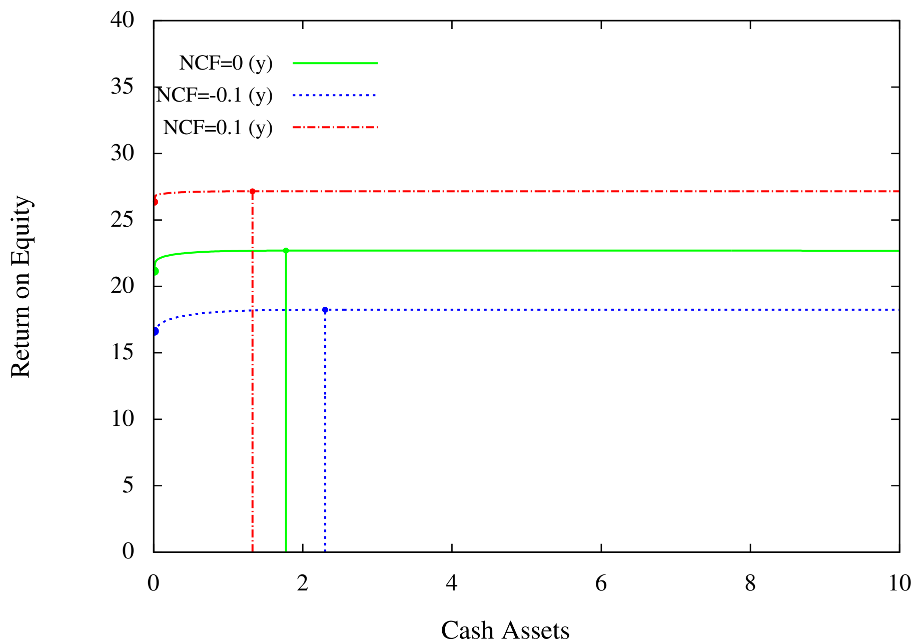


Figure 6.8: The Return on Equity against initial cash assets for a firm that has the Type B financing option, where we take net cash flow $-0.1, 0$ and 0.1 as examples. The vertical lines define the corresponding dividend payment boundary.

cash assets, respectively. By comparing these two figures with Figures 6.6 and 6.7, we can see this further flexibility of choosing an optimal financing time contributes extra value to the financing option, particularly when the firm has relevant high revenue and initial cash holdings. This is because the option allows the firm to wait and finance until it really has to raise external funds, therefore, the firm has the opportunity to take advantage of the time value of the money.

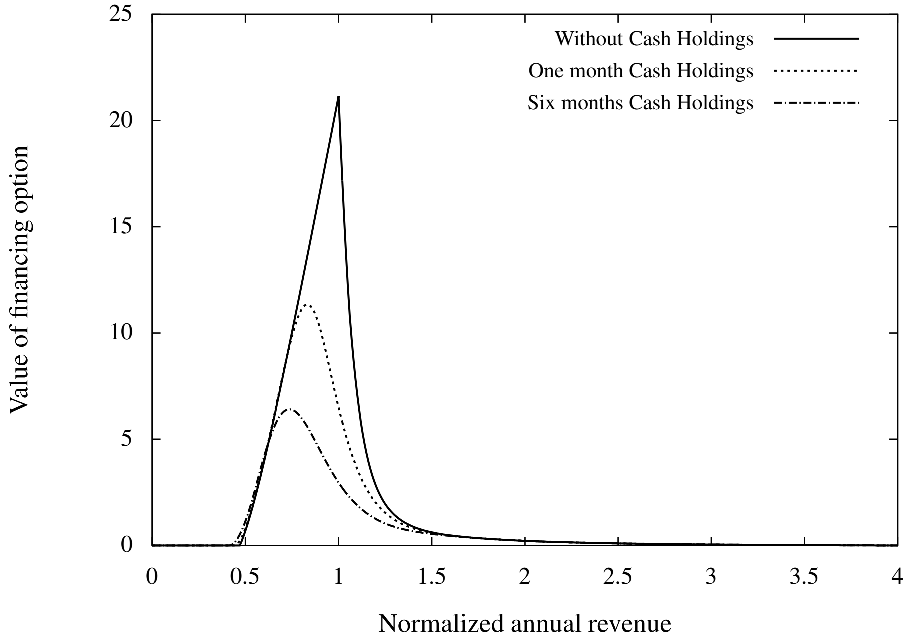


Figure 6.9: The Type B financing option value against revenues for a firm that has no cash holdings, one month cash holdings and six month cash holdings, where the fixed agency cost is $\psi_0 = 0.5$ and the percentage agency cost is $\psi = 0.05$.

Figure 6.11 illustrates the operating strategy for a firm that has an optimal time financing option. In this figure, the whole operating space is divided into three regions: the abandonment region, the normal operational region and the dividend payment regime. The blue arrows denote the optimal financing time and the optimal financing amount. More precisely, the start point of this arrow marks where the firm exercises the financing option and the length of arrows denotes how much new equity the firm raises when exercising this option. We can see that the firm exercises its financing option when the cash assets almost run out and the net cash flow is close to zero (this defines the optimal financing time), which is quite different compared with the results for the Type A financing option.

We further combine a firm's operating strategy in the before-financing phase with that in the after-financing phase, (see Figure 6.12). It is interesting to see the firm's

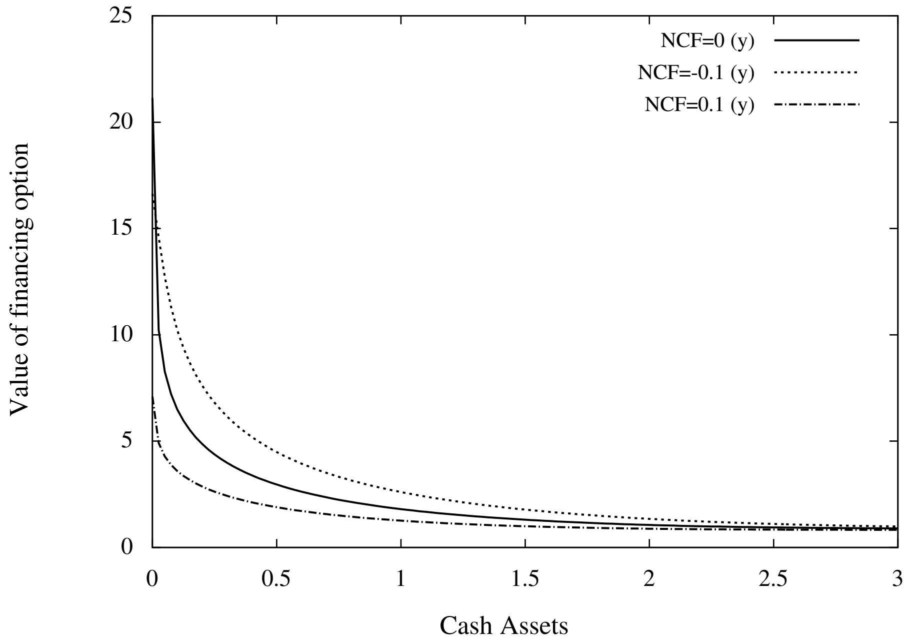


Figure 6.10: The Type B financing option value against cash assets when a firm has net cash flow $-0.1, 0$ and 0.1, where the fixed agency cost is $\psi_0 = 0.5$ and the percentage agency cost is $\psi = 0.05$.

abandonment boundary drifts left when the firm has relatively low cash assets, and it's dividend payment boundary moves downwards due to the effects of financing option. This is because the financing activity reduces the firm's insolvency risk. Therefore, in this less risky situation, a firm can redistribute dividends even if it has a lower level of cash assets.

c. The Type C financing option:

Figure 6.13 presents how the Return on Equity varies against the assets for a firm that has an unlimited financing option. According to this figure, we can see that the limitations of cash constraints are further reduced, and the dividend payment boundary moves to a lower level compared with Figure 6.8. This is because the firm has further flexibility to refinance during the rest operation time.

To further look at the effects of the financing flexibility, we solve the Financing Option Value (FOV) based on the Definition (6.25), where V is solved by Model (6.16), and v is the value of the firm that has no financing flexibility (see Equation (5.11)). With this definition, Figures 6.14 and 6.15 presents how the FOV varies with different revenues and initial cash assets. By comparing these two figures with the corresponding FOV

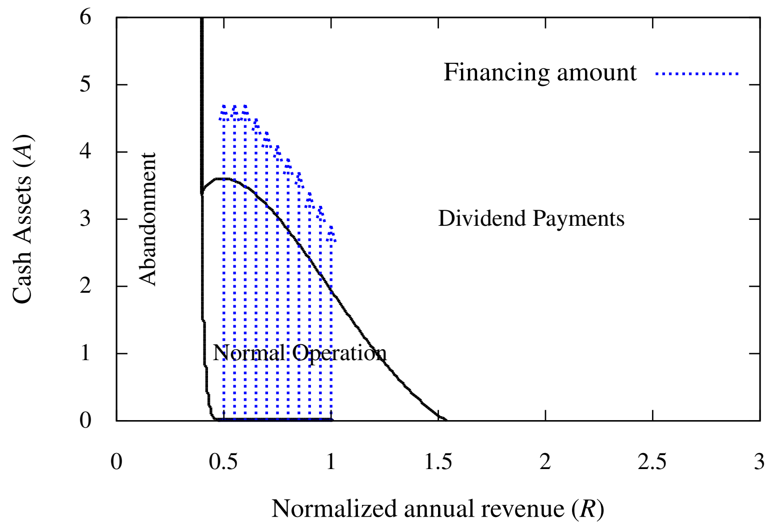


Figure 6.11: Operational strategy for a firm that has extra flexibility to choose an optimal financing time (in the before-financing phase). The start points of these arrows mark the debt financing exercise boundary. The length of the arrows define the financing amount. Here, we assume a fixed agency cost $\psi_0 = 0.5$ and a percentage agency cost $\psi = 0.05$.

figures presented in the Type A and B model sections, we can see the flexibility of unlimited access to the financial market further raises the value of financing option, particularly when the firm has relatively low revenue and initial cash holdings. This is because a firm with such an option has further flexibility to refinance. Therefore, it can take the benefits of this flexibility via the tradeoff process of deciding the financing frequency and amounts, where the firm cannot finance infinite times when the fixed financing cost is not zero. For example, the firm can raise the least required amount cash to overcome a short-term liquidity shortage via multiple financing times, instead of raising a large amount of capital only once. This strategy helps the firm fully take the time value of money via waiting for the exhaustion of internal cash, whereas, it might generate extra costs due to frequently using financing flexibility.

Figure 6.16 summarises the operating regimes for a firm having unlimited access to the financing markets, where the black solid line presents the corresponding abandonment and dividend payment boundaries and the arrows give information when and how much the firm decides to finance from external markets. We further summarise

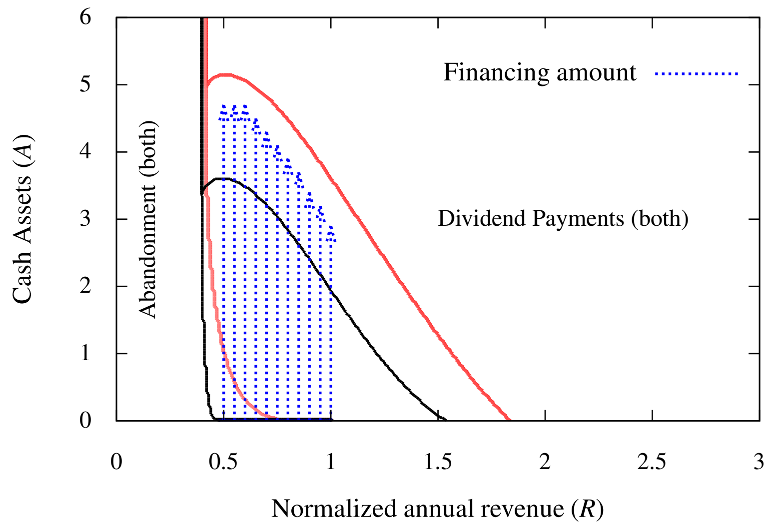


Figure 6.12: A comparison of the firm's operation between the before-financing phase and the after-financing phase. The black lines denote the abandonment and dividend payment boundaries in the before-financing phase while the red lines denote the abandonment and dividend payment boundaries in the after-financing phase. The start positions of arrows locate on the debt financing exercise boundary. The length of the arrows defines the financing amount. Here, we assume a fixed agency cost $\psi_0 = 0.5$ and a percentage agency cost $\psi = 0.05$.

the operating strategy into Figure 6.17 for both the before-financing and the after-financing. According to these two figures, we can see that the optimal financing time is when the firm runs out all its cash assets (there are only arrows at the bottom). The optimal financing amount is less than that shown in Figure 6.12, due to setting that this option allows a firm to finance any times.

6.4.2 Financing Cost Analysis

In this subsection, we test how the financing cost can affect a firm's financing decisions based on a linear agency cost function (see Equation (6.24)).

a. Option to issue equity once at a fixed time

Figure 6.18 present how the financing cost affects the optimal financing amount for a firm that has the Type A financing option. According to the different cases shown in

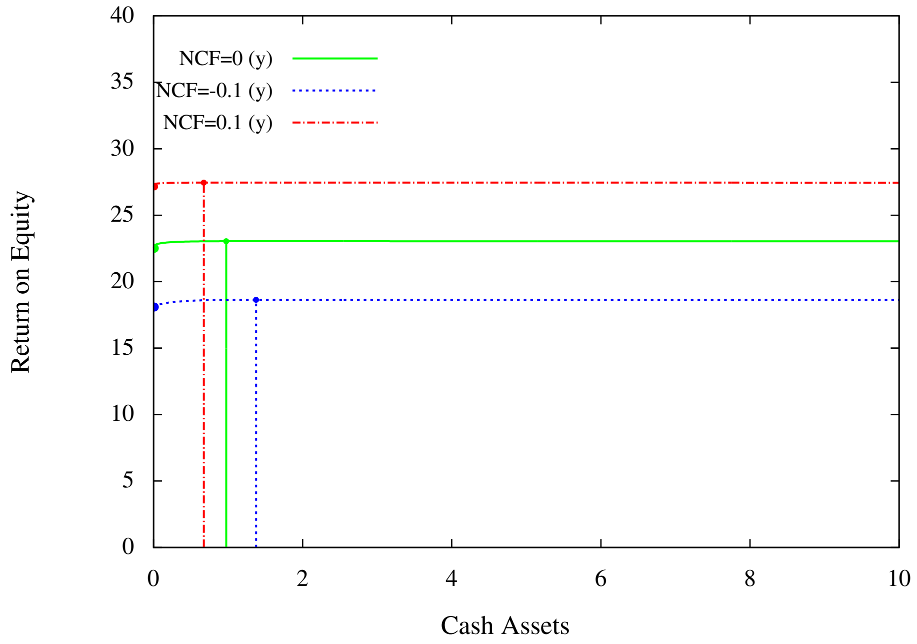


Figure 6.13: The Return on Equity against initial cash assets for a firm having the Type C financing option, where net cash flow equals $-0.1, 0$ and 0.1, and the financing cost function has coefficients $(\psi_0, \psi_1) = (0.5, 0.05)$. The vertical lines define the corresponding dividend payment boundary.

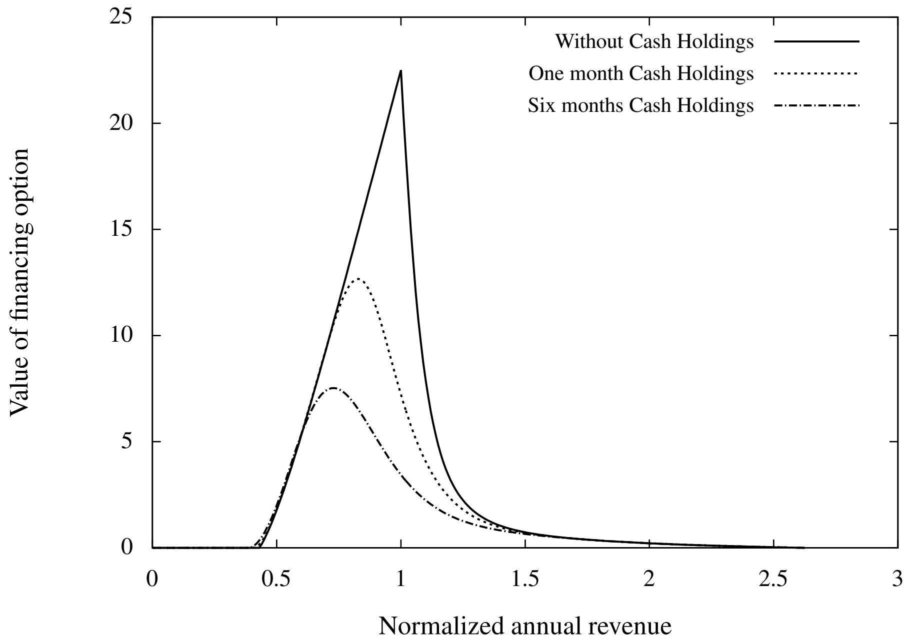


Figure 6.14: The Type C financing option value against revenues for a firm that has no cash holdings, one month cash holdings and six month cash holdings. Here, we assume a fixed agency cost $\psi_0 = 0.5$ and a percentage agency cost $\psi = 0.05$.

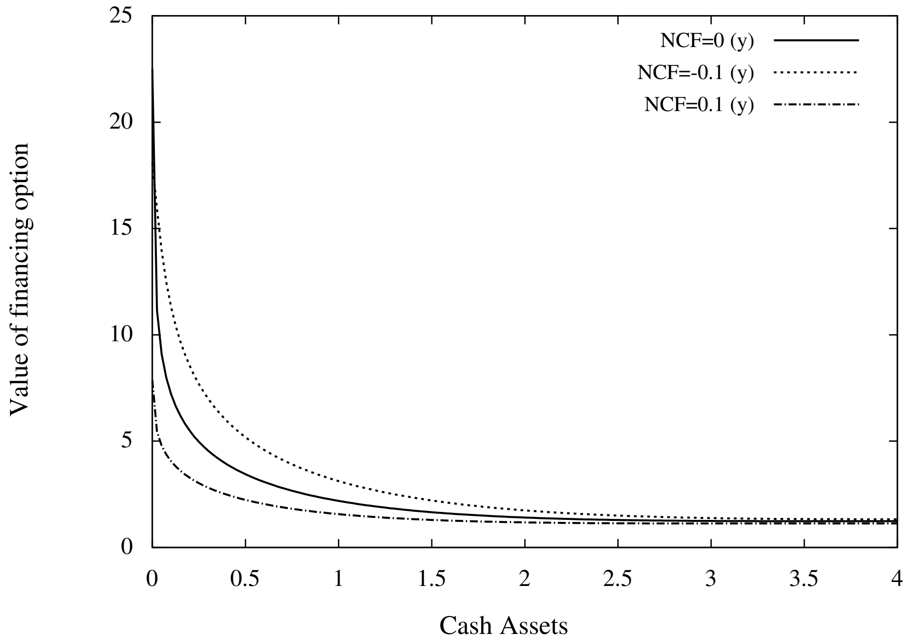


Figure 6.15: The Type C financing option value against cash assets for a firm that has different net cash flows (NCFs) $R - \varepsilon$, where the operational cost $\varepsilon = 1$.

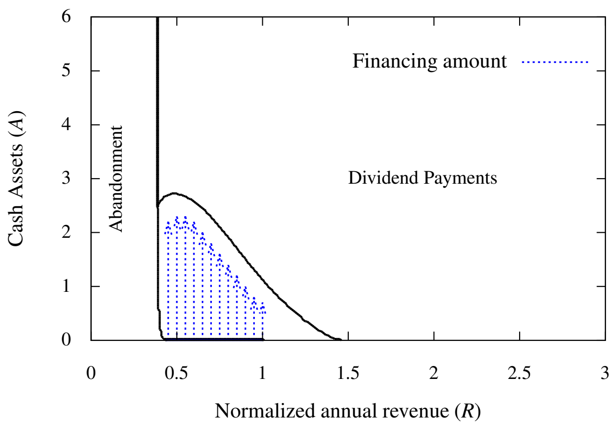


Figure 6.16: The operation regimes for a firm that has the Type C financing option (the before-financing phase). The start positions of arrows locate on the debt financing exercise boundary. The length of the arrows defines the financing amount. Here, we assume a fixed agency cost $\psi_0 = 0.5$ and a percentage agency cost $\psi = 0.05$.

this figure, we can see that the optimal financing amount significantly decreases with an increase in financing cost (either fixed cost or proportional cost), and the optimal financing amount is more sensitive to the fixed financing cost than to the proportional financing cost, which can be observed by comparing figures in the second row with

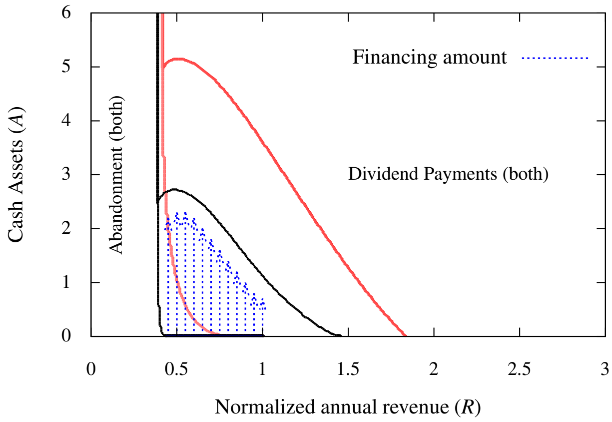
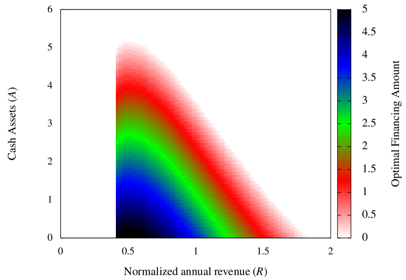
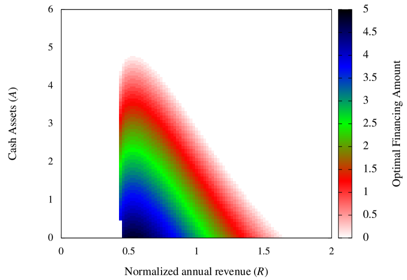


Figure 6.17: A comparison of the firm's operation between the before-financing phase and the after-financing phase. The black lines denote the abandonment and dividend payment boundaries in the before-financing phase while the red lines denote the abandonment and dividend payment boundaries in the after-financing phase. The start positions of arrows locate on the debt financing exercise boundary. The length of the arrows defines the financing amount. Here, we assume a fixed agency cost $\psi_0 = 0.5$ and a percentage agency cost $\psi = 0.05$.

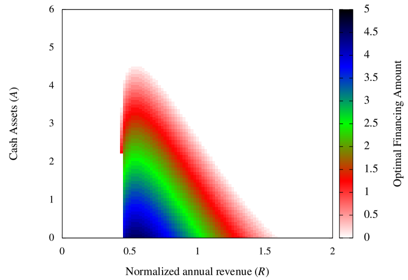
that in the first row, respectively.



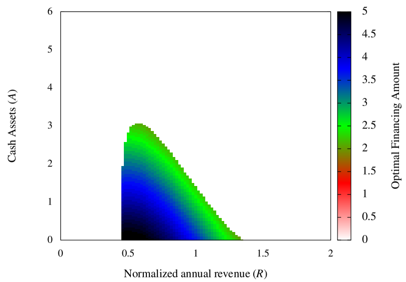
(a) Set $\psi = 0, \psi_0 = 0$



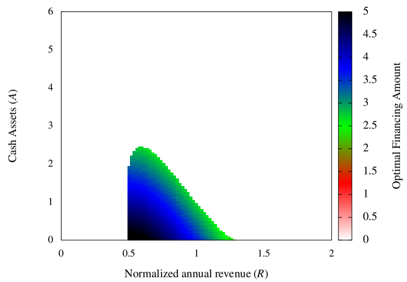
(b) Set $\psi = 0.05, \psi_0 = 0$



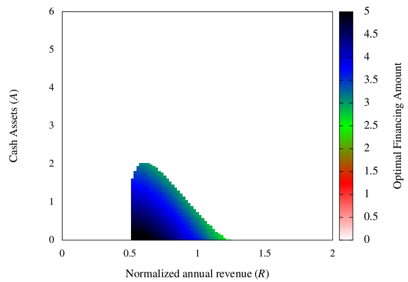
(c) Set $\psi = 0.1, \psi_0 = 0$



(d) Set $\psi = 0, \psi_0 = 0.5$



(e) Set $\psi = 0, \psi_0 = 1$



(f) Set $\psi = 0, \psi_0 = 1.5$

Figure 6.18: The cost effects on the optimal financing amount for a firm that has the Type A financing option. The white colour represents the lowest financing amount and black colour represents the highest level. We assume equity financing have a linear agency cost function defined by Equation (6.24) and choose different sets of fixed and proportional costs to give a simple analysis, where the corresponding parameter value are presented under each sub-figure.

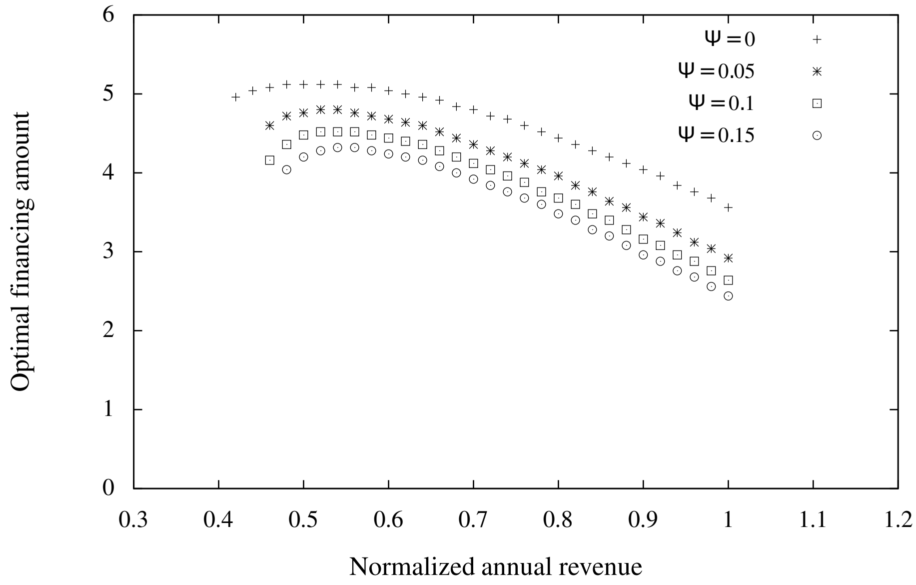


Figure 6.19: The optimal financing amount against firm revenues by varying proportion cost ψ and fixing the constant cost as $\psi_0 = 0$. Here the cash assets $A = C = 0$.

b. Option to finance the corporation once at an optimal time

Figure 6.19 shows how the firm's optimal financing policy varies with different proportion costs when exercising this option. According to this figure, we can see a firm that has relatively high financing cost, prefer to raise less capital at a relatively high level of revenue. The optimal financing amount goes up first and then goes down with the increase of revenue level. This is because, the firm needs to balance the benefits of raising new funds with the benefit of abandoning the business. When the firm's revenue level is close to the abandonment boundary R^*, it is optimal for the firm to abandon its business instead of raising external funds.

Figure 6.20 shows how the optimal financing policy varies with the fixed financing cost ψ_0. It is interesting that the fixed cost is not strongly associated with the optimal financing amount, whereas, it affects the optimal exercise boundary. A firm that has a higher fixed financing cost exercises its financing decision at a higher level of revenue. This is not hard to understand since the firm only has one chance to raise external funds in this model. We then compare these results with that generated by the unlimited financing model to show more evidence.

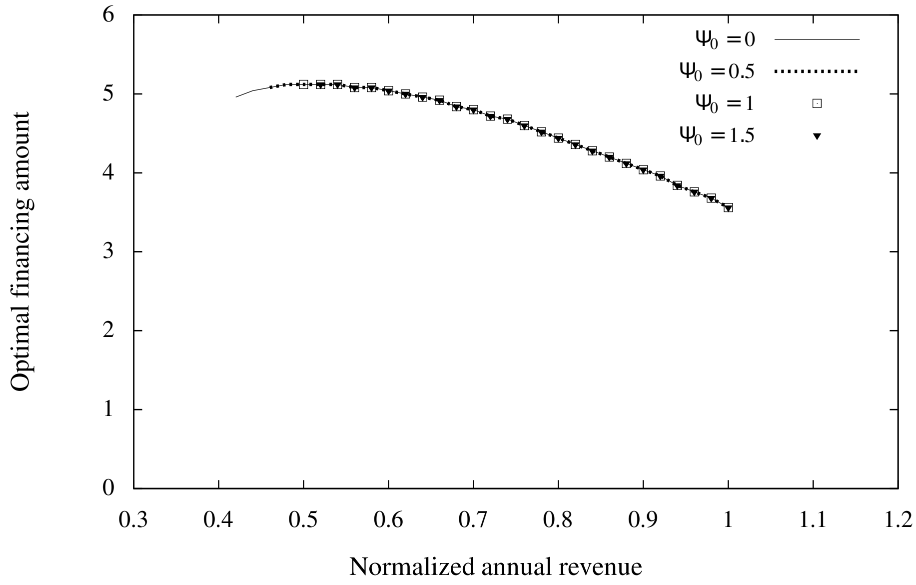


Figure 6.20: The optimal financing amount against firm revenues by varying constant cost ψ_0 and fixing the proportion cost as $\psi = 0$. Here the cash assets $A = C = 0$.

c. Option to finance unlimited times

The Type C financing option allows a firm to raise external cash unlimited times. Without financing cost, a firm can be treated as a financial unconstrained business. Therefore, here we particularly study how the financing cost affects the optimal equity financing decision, and thus distinguishes the financially constrained firm and financial unconstrained firm.

Figures 6.21 present how the firm's optimal financing amount varies along the firm's revenue by taking fixed financing cost $\psi_0 = 0, 0.5, 1$ and 1.5 as examples. By comparing the optimal financing amount between these four lines, we can see that firm finance itself for more cash when the financing cost is relevantly high. This is because, with a higher level of fixed costs, the firm chooses to finance more cash but less frequently. By comparing the revenue level of the first point of each line, we can see that the revenue level increases as the financing cost increases. This means that the firm would exercise a high frictional cost financing option only when it has a high probability to go back to a profitable region (revenue is relevantly high).

Figure 6.22 how important to involve the financing cost when we study a financially constrained firm. According to this figure, we can see that the optimal abandonment boundary for a financially constrained firm (financing cost great than zero) convergent to a straight vertical line with the decreasing of fixed cost. This observation shows that when the financing cost turns to zero, a financially constrained firm value convergence to the unconstrained firm value, where there are no financing constraints.

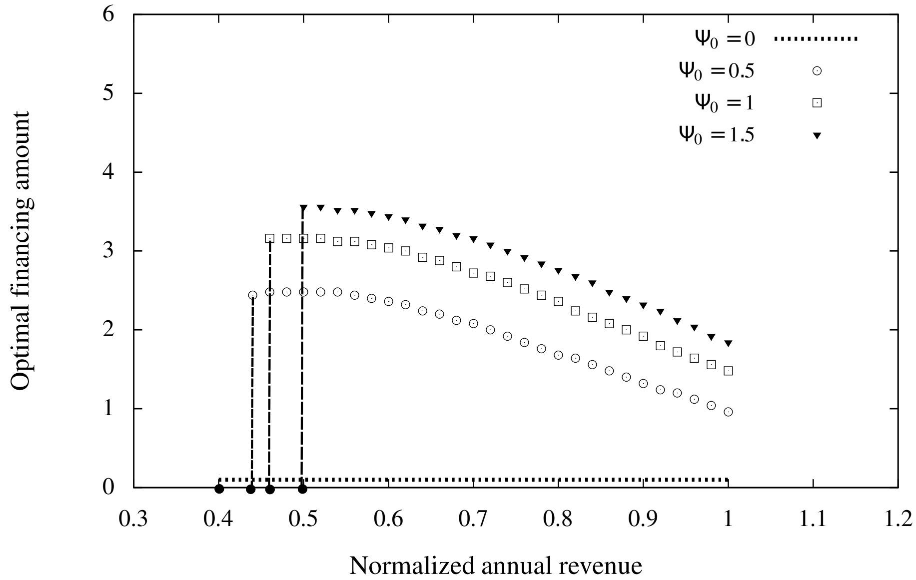


Figure 6.21: The optimal financing amount against revenues by varying constant cost ψ_0 and fixing proportion cost $\psi = 0$. There the cash assets $A = C = 0$.

The above options Type A to C have widely varying implications. The existence of any finite set of decision points (Type A and B) ensures that financing will be "Bang-Bang" in nature. However Type C includes that case of continuous opportunities for continuous changes in financing. In that case the optimum financing amount will be a Brownian function of the Brownian Type A variable, as the price of a Call option on a stock price S varies stochastically with S.

The financing cost function has significant effects on the financing policy. To explain, we take the linear cost function, non-concave cost function and non-convex functions as examples. Suppose a firm has a linear financing cost function. The percentage cost mainly defines the optimal financing amount, while the constant cost mainly defines the optimal financing time and frequency. For example, when the constant cost equals zero, the type C option will take unlimited financing times with infinitesimally small

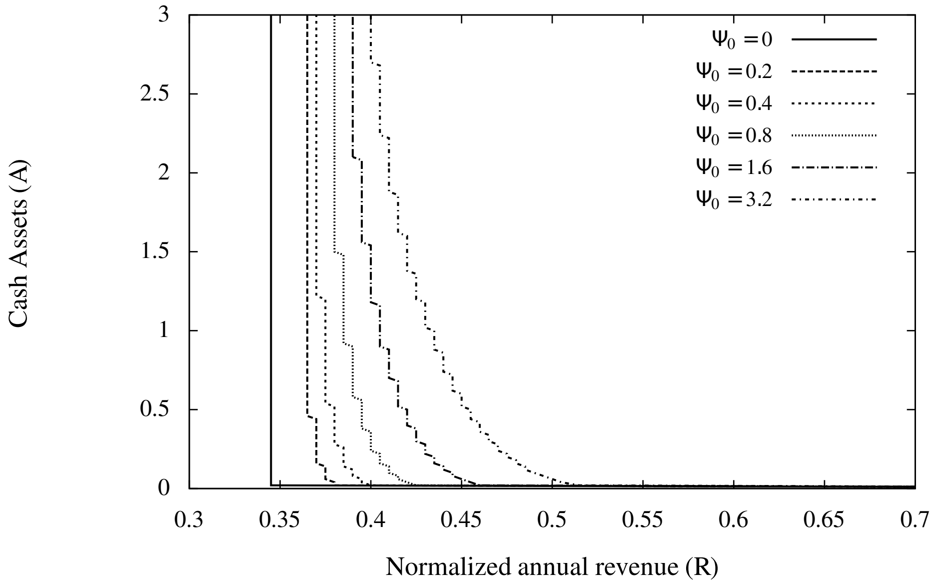


Figure 6.22: The optimal abandonment boundary by varying fixed cost ψ_0 and fixing the proportional cost $\psi = 0$.

amounts. In real business, generally there is constant financing cost. When the cost function is non-concave, people can always find a unique maximum value of Equation (6.4). However, the optimal financing amount might not be unique for higher order cost functions. When the financing function is not non-concave, the effects of financing cost on the financing policy are uncertain. According to the research given by Hugonnier et al. (2014), a large enough constant cost is necessary to guarantee a barrier type of exercising strategy and if the constant cost is small, the barrier strategy might be suboptimal. We will leave this for further study.

6.5 Conclusion

In this chapter, we investigated how the equity financing flexibility affected the valuation and operational strategy of a cash constrained firm. We derived three types of equity financing options by gradually increasing the financing flexibility: a fixed-time financing option, a flexible-time option and an unlimited financing option. The corresponding numerical technique were designed and presented for each model.

By analysing the numerical solutions, we found that a firm with equity financing flexibility had fewer cash constraints and higher value when compared to a firm with no financing flexibility. We also found that the financing cost was a significant consideration in exercising this flexibility. We showed that when the financing cost tends to zero, the liquidity constrained firm model degrades to a Real Option Model (as shown in Chapter 4). What is more, when a firm has the flexibility to choose the optimal financing time, it is always the best to defer financing until the cash assets run out. We also analysed the financing cost effects on firms' optimal financing decisions.

The mathematical models illustrated in this chapter provide useful frameworks to study equity financing. However, it is still not capable of handling debt financing problems, since debt investors have different objectives and risk preference compared with shareholders, which increases the complexity. In the next chapter, we are going to study the debt financing problems, which comes with its own set of unique problems as we must decide on the cost of the debt as well as the amount.

Chapter 7

Optimal Financing Problems - Debt Financing

7.1 Introduction

Alongside equity financing, debt financing is another crucial tool to help overcome a liquidity shortage. The chief challenge of modelling an optimal debt financing problem lies in how to estimate the financing costs. Compared with an equity financing problem (considered in Chapter 6), the cost of debt financing mainly includes two parts: the explicit agency fees and the implicit coupon payments (sometimes, we also called it coupon interest payments or interest payments). For different firms or the same firm in different financing circumstance, they are generally charged with different coupon rates (also called coupon interest rates or coupons, which are the fixed annual interest paid by the issuer to the bondholder). This is because the setting of a corporate debt coupon will depend on how the market assesses the future operational risk of the firm taking the loan out, and the required coupon level can then influence the amount of debts the firm decides to issue. In turn, the amount of cash the firm decides to borrow and the commitment to pay down the debts will influence the future credit risks. As we can see, it will be very difficult to estimate the coupon payments independently without considering the possible reaction of the debt investors.

Reviewing the previous studies of the optimal debt financing problems, few works appear to involve all the various corporate risks, and they generally tend not to contemplate how the debt investors' assessment and investing demands affect the firm's debt issuance. For example, in Leland's series of work (Leland, 1994; Leland and Toft, 1996; Leland, 1998), it is assumed that a firm has unlimited cash holdings, so there is no need for them to consider liquidity risk. Mauer and Sarkar (2005) examined the impact of a stockholder-bondholder conflict over the timing of the exercise of an investment option on firm value and corporate financial policy, whereas they did not involve liquidity risk and optimal dividend payment policies. Titman and Tsyplakov (2007) and Bolton et al. (2014a) do investigate both the liquidity and credit risks, but they use a constant coupon rate, so they fail to consider the effects of debt investors' assessments on each debt offering, thus misestimate the financing costs.

In this chapter, we are going to derive a mathematical model for an optimal debt financing problem with a risk-adjusted coupon rate, which is defined based on the a firm's future operational risk. This model allows people to study the behaviour of both debt issuers and debt investors within one framework. We assume the debt issuer is going to maximise the shareholders' benefits via an optimal financing strategy (using the amount of cash and the time at which they finance as control variables), whereas the market debt investors decide a coupon rate based on the required financing level and the potential risks they see in the firm.

The contents are organised as follows: At the beginning of this chapter, we discuss the optimal financing problem for a firm that has unlimited operating discretion, and we illustrate why it is not practical to model and solve such a problem in real time without an appropriate simplification. Based on this discussion, we simplify the original optimal financing problem by assuming that the corporation only has the option to finance with debt at a fixed time. We derive the mathematical model of this problem, and particularly illustrate how to find the market coupon rate by devising a bargaining game between debt issuer and investors, and why this coupon rate properly reflects the firm's future operational risks. We further present the numerical approach to solve this model, based on which, a set of numerical solutions are generated to show the interactions between the debt financing flexibility, the cash level, the abandonment decision and the dividend policy.

7.2 Stochastic Framework

Problem Setting: We consider a cash constrained firm that has the same settings in revenues, cash holdings, dividend payment option and abandonment option as illustrated in the optimal dividend-payment firm model (5.10) in Chapter 5. We now further assume the firm has the flexibility to issue debt, with a risk-adjusted coupon rate, to increase their liquid assets. As such, the firm's objective is to maximise its initial shareholders' investment return by managing three optimal operational decisions: dividend distribution, debt issuing, and closure.

Unlike the previous optimal debt financing literature, we will allow the debt coupon rate to dynamically depend on the firm operational risks, i.e. the external risk (from the uncertain revenue), the liquidity risk (from the cash level) and the credit risk (from the debt level). To model all three parts simultaneously, we assume the debt investors can obtain enough information of the underlying corporation (the financial states, i.e. the A_s , C_s, D_s and E_s, and the operational strategy, i.e. R_s, d and abandonment time (R^*, A^*)) for each $s \in [t, T]$, with which, they can estimate the operational risks of the firm. Based on the analysis of corporate risks, debt investors would provide a 'Financing Amount- Coupon Rate' $(D - r_b)$ table for the firm to make financing decisions. We assume that the firm will choose an optimal financing level from the table to maximise its shareholders' benefits.

7.2.1 Optimal Debt Financing with Unlimited Operating Discretion

When we study a company's optimal debt financing strategy, it is natural to assume that its financing discretion is unlimited, which means that the firm can issue any amount of debt for an unlimited number of times at any time points given an affordable price. However, to model and solve such an unlimited debt financing problem is extremely challenging. The main difficulties come from two aspects: the complexity of the dynamic interaction between the debt financing decisions and the interest costs; the limitations of the numerical algorithm and computing resources since there are generally no analytic solutions for such a complex problem (see more discussions in Glover and Hambusch, 2012).

For a firm that has unlimited access to the debt markets, its current coupon payments are the accumulated results of the previous financing activities (the financing amounts and the corresponding coupon rate at each financing time). Also, the level of coupon rate required by the market for the new debt issuance depends on both its past financing activities (i.e. the current capital structure) and future operational risks. Furthermore, the firm's optimal financing choices, in the whole operational period, are based on a trade-off between the benefits and the overall financing costs. The benefits can be measured by the marginal value of cash, while, the total costs depend on the financing frequency (which can affect the agency cost), financing time (affects the debt structure) and financing amounts (interest costs). However, these aspects cannot be analysed independently without considering the combined effects of the others. To illustrate these complexities, we denote the firm value at the operational state (s, R_s, A_s) $(s \in [t, T])$ as,

$$v\big(s, R_s, A_s, \{D\}_s, \{r_b\}_s, \varepsilon(\{D\}_s, \{r_b\}_s)\big), \tag{7.1}$$

and the value of corporate debt as,

$$u\big(s, R_s, A_s, \{D\}_s, \{r_b\}_s, \varepsilon(\{D\}_s, \{r_b\}_s)\big), \tag{7.2}$$

where, R_s denotes the revenue at time s; A_s denotes the total assets at time s; $\{D\}_s$ denotes a set containing the records of historical debt financing in the time interval $[t, s]$; $\{r_b\}_s$ denotes a set containing the corresponding records of historical coupon rate in the time interval $[t, s]$; ε represents the operational costs, which depends on the historical financing activities and coupon payment each time. According to these two value functions, we can see that for the optimal financing decision to be determined, the firm needs to contemplate its current operational state (s, R_s, A_s), historical financing records $(\{D\}_s, \{r_b\}_s)$ and the efficient accumulative operational costs $\varepsilon(\{D\}_s, \{r_b\}_s)$; Further, the coupon rate of each debt offering will be externally determined by the debt investors' assessment of debt value, which depends on the firm's operational strategy and risk. To solve such a problem, we have to frame a mixed feedback-control-dynamic-game model in a high dimensional space constructed with time, the revenue, the asset, the debt, the coupon rate and the operational cost. Even if we can find a proper mathematical framework for the unlimited debt financing problem, since there is generally no analytic solution, we need to find an efficient numerical technique to estimate the solution. However, the calculation costs greatly increase with the number of controls and dimensions. To give some intuition but a less stringent interpretation, we take the optimal dividend-payment firm model as an example (see Chapter 5). To find an accurate solution of this model, it takes around one hour. If we further include the controls of financing debt level D and coupon rate r_b on say, a 200×200 grid, and consider the interaction between debt investors and issuer at each of the 1000 possible time points, it would take *at least* $200 \times 200 \times 1000$ hours. Therefore, it is not practical to find real-time numerical solutions for this complex system and we will instead try to solve a simplified version of the more complex problem.

In the next sections, we will outline the framework for a simplified debt financing option, the option to issue debt at a fixed time with any amounts, and the corresponding techniques to find a numerical solution.

7.2.2 Optimal Debt Financing at Initial Time

Based on the description in the problem review, we consider a cash constrained firm that has the same settings of revenues R_s ($s \in [t, T]$), cash holdings C_s ($s \in [t, T]$), dividend payment option and abandonment option as illustrated in Chapter 5. For simplicity, we assume all the initial assets are raised from shareholders ($A_t = E_t$). We further assume the firm holds a debt financing option to issue D amount of debts at the initial time t with a risk-adjusted coupon rate r_b paid in perpetuity, where r_b is defined externally by the debt investors. When exercising this option, we assume the firm has to pay $\Psi(D)$ amount of agency fees, where the agency cost function is subject to $\Psi(0) = 0$. After that, the firm's assets immediately increase from initial amount A_t to $A_t^+ = A_t + D - \Psi(D)$ and the operational cost also goes up from ε to $\varepsilon + r_b D$

accounting for the annualised coupon payment $r_b D$.

Since the firm can only finance at time t or never, after financing, such a firm will become a leveraged firm that has no further financing options with fixed debt level, fixed coupon payment and new total assets. Therefore, we can derive the firm value just before financing by studying the corresponding leveraged corporation after financing. We now illustrate how to value a leveraged firm and its debts, and then, how to use these models to study the optimal fixed-time debt financing problem.

The Models for a Leveraged Firm and Its Debts

The model of a leveraged firm that has a fixed level of debt: A leveraged firm can be treated as an extension of a non-leveraged firm that has new operational cost, dividend policy and capital structure. We now derive the leveraged business model by extending the fully equity financed firm framework (5.11).

New Capital Structure: After the debt financing time t, suppose the firm holds a constant level of debts D and initial equity E_s ($s \in [t, T]$), among which, A^f are fixed assets and C_s ($s \in [t, T]$) are cash assets. Since the cash levels affect the liquidity risk, and the debt level affects the insolvency risks, when we model the leveraged firm, we need to consider the following asset divisions on time interval ($s \in [t, T]$),

$$A_s = A^f + C_s = D + E_s. \tag{7.3}$$

New Cash Flow: Along with the operational cost ε, a leveraged firm has to pay $r_b D$ debt repayments every year until it ceases operation. With this new cost, we redefine the cash flow function (5.2) on the time interval $s \in [t, T]$ as,

$$\Phi(s, R_s, C_s; d, D, r_b) = R_s + rC_s - \varepsilon - r_b D - d. \tag{7.4}$$

New Dividend Policy: The approach to distributing dividends in a leveraged firm is different from a pure equity financed firm, since there are potential agency conflicts between debt investors and share investors. To make the model extendable for agency conflicts studies, we present two types of dividend distributing policies here: the Radical Dividend Policy and the Conservative Dividend Policy.

- Radical Dividend Policy

  In this case, we assume the firm can distribute excess cash as dividends even the corporation has a negative cash flow. Thus we can reframe the estimated up-limit dividend ratio formula $\tilde{d}$ within the infinitesimally time step ds (see original definition (5.18)) as,

$$\tilde{d} = \max\left(\frac{\min(C_s, A_s - \tilde{A}_s)}{ds} + \Phi(s, R_s, C_s; 0, D, r_b), 0\right). \tag{7.5}$$

- Conservative Dividend Policy

  In this case, we assume the firm cannot distribute dividends when it has a negative cash flow or negative equity in order to protect debt investors' interests. Then, the up-limit dividend ratio formula $\tilde{d}$ (see original definition (5.18)) can be re-framed as,

$$\tilde{d} = \begin{cases} 0, \ \textbf{if } \Phi(s, R_s, C_s; 0, D, r_b) \leq 0, \\ \max\left(\dfrac{\min(C_s, E_s, A_s - \tilde{A})}{ds} + \Phi(s, R_s, C_s; 0, D, r_b), 0\right), \ \textbf{if } \Phi(s, R_s, C_s; 0, D, r_b) > 0. \end{cases} \tag{7.6}$$

These models imply that dividend payments can vary continuously with S, so long as cash flow is positive, even if dividends are paied over fixed and finite regions of the (S, C) space. These formulas are defined based on an infinitesimal time dS, thus theoretically the maximum value of d turns to infinity. Therefore, as illustrated in Chapter 5, they are only used to help us implement our algorithm.

New Stopping Conditions: When the firm stops its business (either Abandonment or Insolvency) at time τ, according to the liquidating process, the firm has to pay a closure cost $\kappa(A)$[1] first, then debts D, and lastly all the remaining equity to shareholders. Therefore, the payoff for shareholders in a stopping case is given by the following function,

$$h(\tau, R_\tau, A_\tau; D, r_b) = \max\{A_\tau - \kappa(A_\tau) - D, 0\}. \tag{7.7}$$

The Leveraged Firm Value: We assume the firm holds a constant debt D with a coupon rate r_b, so the value function of this firm can be defined as,

$$v(t, R_t, A_t; D, r_b) = \sup_{\tau, d} E\left[\int_t^\tau e^{-\rho(s-t)} d \ ds + e^{-\rho(\tau-t)} h(\tau, R_\tau, A_\tau; D, r_b)\right]. \tag{7.8}$$

[1]See more explanations of closure function in Chapters 4, 5 and 6.

This model can be solved via the corresponding HJB Variation Inequality,

$$\min\left\{\frac{\partial v}{\partial t}+\mathcal{L}v-\rho v+\sup_{d\in[0,\infty)}\left[\Phi(t,R,C;d,D,r_b)\frac{\partial v}{\partial A}+d\right],v-h\right\}=0, \tag{7.9}$$

or,

$$\min\left\{\frac{\partial v}{\partial t}+\mathcal{L}v-\rho v+\Phi(t,R,C;d,D,r_b)\frac{\partial v}{\partial A},\frac{\partial v}{\partial A}-1,v-h\right\}=0, \tag{7.10}$$

which are subject to the following boundary conditions,

$$\begin{cases} v=E & \text{when} \quad t=T, & (7.11)\\ \dfrac{\partial v}{\partial A}=1 & \text{as} \quad C\to\infty \ \&\ R+rC-\varepsilon-r_bD>0, & (7.12)\\ v=\hat{\alpha}R-\hat{\beta}(\varepsilon+r_bD)+E & \text{as} \quad R\to\infty, & (7.13)\\ v=h & \text{if} \quad R+rC-\varepsilon-r_bD<0 \ \&\ A<D, & (7.14)\end{cases}$$

The explanations of these boundary conditions can be found in Subsections 4.2.3 and 5.2.3 and the definition of h is given by Equation (7.7).

The market value of the firm debts: To keep consistent with the assumptions of a leveraged firm, we assume debt investors hold the D amount of debts with a promised coupon rate r_b. The market value of this corporate debt can be defined as the expected value of discounted debt repayments and its principle. What is different here to the previous models of debt valuation that can be found in the literature is that we assume the debt value depends on the firm's operational state (i.e. (s,R_s,A_s)), capital structure (i.e. the asset division) and the operational strategy (i.e. the stopping boundary (R^*,A^*) and the dividend payment boundary $(\tilde{R},\tilde{A})$).

When the firm stops its business (either abandonment or insolvency), debt investors might only get a part of the promised return depending on the remaining assets at the stopping time. Thus, the debt investors' payoff can be defined as,

$$h_P\big(\tau,R_\tau,A_\tau;D\big)=\max\{\min\big(A_\tau-\kappa(A_\tau),D\big),0\}, \tag{7.15}$$

where, τ is the firm's corresponding early closure time defined by equation (7.7), $\kappa(A_\tau)$ is the stopping costs.

We assume the debt investors have an independent discount factor ϱ, which measures their expected return (or the opportunity cost) when investing in the public market. Denoting $u(t,R_t,A_t,\tau,\tilde{d};D,r_b)$ as the value function of the debt investors. We have,

$$u\big(t,R_t,A_t,\tau,\tilde{d};D,r_b\big)=E\left[\int_t^\tau r_bDe^{-\varrho s}ds+e^{-\varrho(\tau-t)}h_P\big(\tau,R_\tau,A_\tau;D\big)\right], \tag{7.16}$$

where, τ is the optimal stopping time and $\tilde{d}$ is the optimal dividend payment amount, both of which are calculated in the firm model; ϱ is the discount factor for debt investors. We can solve this model with the following Partial Differential Equation,

$$\frac{\partial u}{\partial t} + \mathcal{L}u + \Phi(t, R, C; d^*, D, r_b)\frac{\partial u}{\partial A} - \varrho u + c = 0, \tag{7.17}$$

with boundary conditions,

$$\begin{cases} u = D & \text{when} \quad t = T, & (7.18) \\ u = \dfrac{r_b D}{\varrho}(1 - e^{-\varrho(T-t)}) + De^{-\varrho(T-t)} & \text{as} \quad R \to \infty, & (7.19) \\ \dfrac{\partial u}{\partial A} = 0 & \text{on} \quad (\tilde{R}, \tilde{A}), & (7.20) \\ u = h_P & \text{on} \quad (R^*, A^*), & (7.21) \end{cases}$$

where, $(\tilde{R}, \tilde{A})$ denotes the optimal dividend payment boundary and (R^*, A^*) denotes the optimal stopping boundary. They will be found as part of the solution to (7.8), $\mathcal{L}$ is defined by Equation 4.8, here works on the value function u. We now explain the economic explanation for each boundary condition.

1. Terminal boundary condition (see Equation (7.18))
   When the firm successfully operates the project until the termination time, the debt investors can immediately get the principle, therefore the market value of the debt is D.

2. No-risk boundary conditions (see Equation (7.19))
   When the firm is very profitable, the default probability of the debt is zero. Therefore, the debt investors can get all the promised coupon payment and principle as promised, therefore, the market value of the debt can be defined by,
$$u = \int_t^T e^{-\varrho(T-s)} r_b D ds + De^{-\varrho(T-t)} = \frac{r_b D}{\varrho}(1 - e^{-\varrho(T-t)}) + De^{-\varrho(T-t)}. \tag{7.22}$$

3. Liquidity risk boundary condition (see Equation (7.20))
   According to our discussion in Chapter 5, the firm decides to pay out dividends only when there is no liquidity shortage. Therefore, increase or decrease one more unit of cash assets does not change the debt value.

4. Default boundary condition (see Equation (7.20))
   When the firm stops the business before T, the debt investors can only get the

remaining assets after the firm paying out the closure costs, which is defined by Equation (7.15). It should be noted, this situation happens in a firm's two different operational cases: the abandonment and the insolvency.

Optimal Debt Financing with Risk-adjusted Coupon Rate

The biggest challenge to model a fixed-time financing decision with a risk-adjusted coupon rate is to address the coupon rate required by the debt investors. As discussed in the introduction, the coupon rate will depend on the firm's future operational risk, which is also affected by the current debt level. However, the coupon rate defines the financing cost, which in turn affects the financing amount. Therefore, to study such an optimal debt financing problem, we need to put the debt investors and debt issuer into one uniform framework so that we can capture the complex interplay of these two participants. We now show how to model this problem within a bargaining game framework.

To frame this problem and without losing generality, we consider the case a firm raises an optimal amount of new capital $D^*(r_b^*)$ with a risk-adjusted coupon rate $(r_b^*(D^*))$ at initial time t, where $t < T$. More precisely, the risk-adjusted interested rate $r_b^*(D)$ of a particular corporate debt depends on the financing level D and the firm's future operational risks. While, the firm's optimal financing decision (the financing amount $D^*(r_b^*)$) depends on the trade-off of financing costs (measured by agency cost $\Psi(D^*)$ and coupon payment rate $(D^* r_b^*)$) and financing benefits (the extra benefits generated in this debt financing for the firm's shareholders).

We have presented how to model a leveraged firm that has a fixed level of debts, as well as the market value of its debts. These studies can help us to define the objective functions of these two participants. Based on these studies, we now show how to model the optimal debt financing problem within a bargaining game step by step.

Step one: Define the objectives of the debt issuer and investors
We consider two participants in a debt issuing process: the debt issuer (a firm) and the debt investor (for example a bank). The firm wants to maximise the extra benefits

generated for its shareholders in this financing activity, and it has the flexibility to decide the financing amount D, where $D \in \mathcal{D} := [0, D^m]$ and D^m is the maximum value that D can take; The bank wants to obtain a return that at least cover their investment risk (say the internal rate of return of the debt investment), and they have the flexibility to set a coupon rate r_b, where $r_b \in \mathcal{R} := [r, r_b^m]$. Here, the up-limit value D^m and r_b^m are defined based on the governance and regulation policy[2].

According to the no-arbitrage theory, the shareholders' benefits just before financing and after financing should be consistent. With a fixed time debt financing option, if the financing amount D and interested rate r_b are given, after financing the firm becomes a leveraged corporation which can be valued by the Model (7.8). Therefore, we can derive the firm's value before financing with the following equation,

$$V\big(t, R_t, A_t; D, r_b\big) = v\big(t, R_t, A_t + D - \Psi(D); D, r_b\big). \tag{7.23}$$

Following the same logic, the corresponding value of the debt issued can be found by solving the equation,

$$U\big(t, R_t, A_t; D, r_b\big) = u\big(t, R_t, A_t + D - \Psi(D), \tau, \tilde{d}; D, r_b\big). \tag{7.24}$$

Based on these two value functions, it is straightforward to define the payoff of both the debt-issuer and -investor separately as a matrix of values depending on the coupon rate and the amount of the debt. By denoting $\mathbf{V^F}\big(t, R_t, A_t; D, r_b\big)$ as the corporate financing return at operational state $\big(t, R_t, A_t\big)$ and financing state $\big(D, r_b\big)$, then we have,

$$\mathbf{V^F}\big(t, R_t, A_t; D, r_b\big) = \max\big\{V\big(t, R_t, A_t; D, r_b\big) - V\big(t, R_t, A_t; 0, 0\big), 0\big\}. \tag{7.25}$$

By denoting $\mathbf{V_D}\big(t, R_t, A_t, \tau, \tilde{d}; D, r_b\big)$ as the investment return of debt investors at operational state $\big(t, R_t, A_t, \tau, \tilde{d}\big)$ and financing state $\big(D, r_b\big)$, then we have,

$$\mathbf{V^D}\big(t, R_t, A_t, \tau, \tilde{d}; D, r_b\big) = \max\big\{U\big(t, R_t, A_t, \tau, \tilde{d}; D, r_b\big) - D, 0\big\}. \tag{7.26}$$

Step two: Find the optimal market coupon rate for debt investors

Suppose the capital market is a buyer's market, which means there are many debt

[2] Discussions about the Equity-Debt ratio can be found in some researches about the corporate governance and regulation, see Mande et al. (2012), Berk et al. (2013), Acharya et al. (2013) and Graham et al. (2014) for more details.

investors, but only a limited number of the debt issuers. In this case, the market coupon rate is the level which just covers the firm's future operational risks and has no other premiums. Therefore, for a given financing level D, we can find the market coupon rate via this investment's internal rate of return ($r_b^* = IRR$),

$$r_b^*(t, R_t, A_t; D) = \arg_{r_b}\{\mathbf{V^D}(t, R_t, A_t; D, r_b) = 0\}. \tag{7.27}$$

It should be noted that mathematically, there might be more than one r_b satisfies the above equation. In the optimal debt financing problem, we define the break even coupon rate r_b^* as the first r_b that satisfies the above equation when increasing r_b from r to the regulation up-limit value (this is similar to how to find the IRR with discounted cash flow methods). In addition, as shown in Figure 7.9 in the numerical section, if the debt market is competitive, lenders may enter at the lowest coupon rate which breaks even for them. However if the market is very uncompetitive lenders may voluntarily limit their lending to ensure higher coupon rates. An exception to this is visible in the uppermost curve of Figure where maximum lender profit occurs at less than the maximum coupon rate that a monopoly lender could charge. In this chapter, we mainly focus on a competitive debt financing market, where the debt investors would only consider buying at a coupon rate which is no less than the corresponding $r_b^*(t, R, A; D)$ (see more discussions in the paper Glover and Hambusch, 2012).

From an economic viewpoint, the break even coupon rate considers different kinds of risks: the market risk, which comes from the uncertainty of revenue; the liquidity risk, which is measured by the firm's liquid assets; the firm's credit risks, which depends on when and how the firm goes into insolvency. Thus, the value of r_b^* could be used to show the risk premium of a particular corporate debt, which reflects both the external market risks and the internal risks of the firm.

Step three: Find the optimal debt financing strategy for debt issuer

The debt issuer has the flexibility to decide how much debt they wish to issue, and to achieve this it can use the market defined 'Financing Amount - Coupon Rate' table (D, r_b). Thus, we can solve the optimal financing decision (D^*, r_b^*) by maximizing the firm's objective function,

$$(D^*, r_b^*) = \operatorname*{argmax}_{D}\big\{\mathbf{V^F}(t, R_t, C_t; D, r_b^*) \mid r_b^*\big\}. \tag{7.28}$$

We know that the shareholders' benefits before financing and after financing should be the same. With the above optimal financing strategy (D^*, r_b^*), we can solve the value of the firm that has a fixed time debt financing option with Equation (7.23), and the corresponding debt value with Equation (7.16).

7.3 Numerical Methods

In this section, we focus on the numerical algorithms needed to solve the optimal debt financing model (see Section 7.2.2. We first present how to find the numerical value of a leveraged firm and its debt by applying a Semi-Lagrangian method (see Chapter 5 for more details), since they are important to define the payoff functions of debt issuer and debt investors. Next, we describe the three-step algorithm we use to find the numerical solution of a fixed time optimal financing problem.

7.3.1 A Leveraged Firm and Its Debts

As illustrated in the previous section, the leveraged firm Model (7.8) can be treated as a simple extension of the pure equity financed firm, but with different cash flows, dividend redistribution policy, and the stopping boundaries. Therefore, we can solve the model numerically by simply extending the Semi-Lagrangian method (5.20) with the new boundary conditions as defined in the model, see Figure 7.1 for a pictorial representation of the domain and boundary conditions.

The debt value cannot be solved independently because it requires the operational strategy of the firm as an input to the model. Therefore, we need to first solve the firm value v and find the operational strategy, namely the optimal dividend payment boundary $(\tilde{R}, \tilde{A})$ and the abandonment boundary (R^*, A^*) at each time step. Once these functions have been found, we can then address the debt value u by using the appropriate boundary conditions, see Figure 7.2 for a pictorial representation. It should be noted here that there are no free boundaries in the debt value system, we just need to solve the PDE with the given operational strategy $\tilde{d}$ and τ.

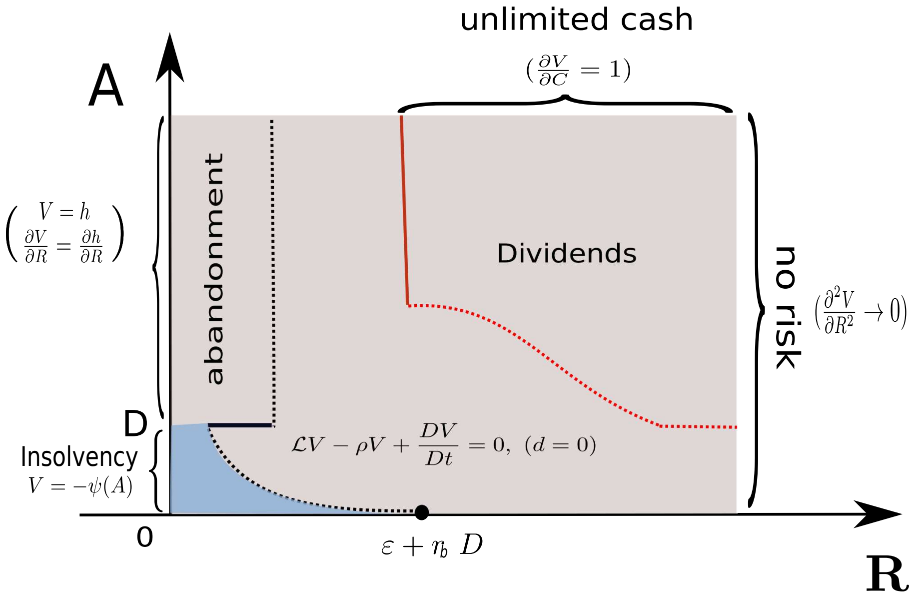


Figure 7.1: A schematic sketch to show the calculation domain and boundary layouts for a leveraged firm that has a fixed level of debts D and coupon rate r_b. Here, R is the normalised annual revenue; A is the asset; ε is the fixed operating costs; $r_b D$ denotes the coupon rate per unit time; $\Psi(A)$ denotes the insolvency cost; the red line denotes the dividend payment boundary; the black line denotes the stopping boundary; the light blue region denotes the insolvency region.

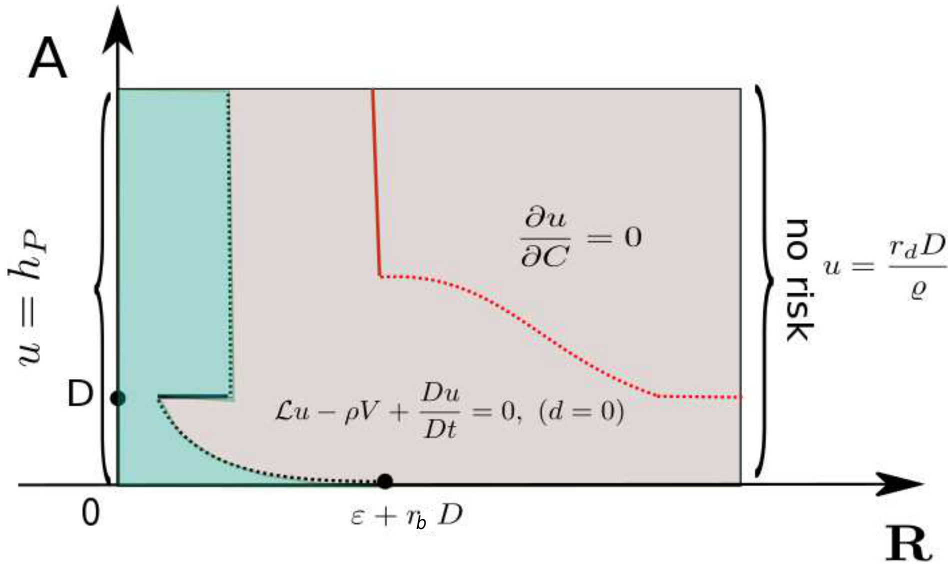


Figure 7.2: A schematic sketch to show the calculation domain and boundary layouts for the debt value model with the given corporate operational strategy. The red line is the solved dividend payment boundary; the black line is the solved stopping boundary; the light green area is the firm's stopping region.

We can summarize the algorithm to solve v and u as follows:

Leveraged Firm Solver: $LFsolver(D, r_b)$ (7.29)

Set $v_0 = E$ and $u_0 = D$ as initial value

/// loop time dimension until convergence

for $t = 1 : t_m$ **do**

Solve v_t and find boundaries $(\tilde{R}, \tilde{A})$ and (R^*, A^*) with Semi-Lagrangian methods.

Solve u_t with $(\tilde{R}, \tilde{A})$ and (R^*, A^*)

if $|v_t - v_{t-1}| < TOL$ **then**

Stop the for loop;

end if

end for

7.3.2 An Optimal Debt Financing Firm

The key steps that we use to derive the optimal fixed-time debt financing problem give us clear clues about how to solve it numerically. We now present how to solve the optimal fixed time financing problem based on the numerical algorithms we use to solve the leveraged firm model and its debt value. Our algorithm can be summarized as follows,

Optimal Debt Financing at Start Time Solver (7.30)

Step one: find payoff V^F and V^D for given (R, A)

for Loop debts level $D \in \mathcal{D}$ **do**

  for Loop intereste rate $r_b \in \mathcal{R}$ **do**

    call solver $(v(D, r_b), u(D, r_b)) = LFsolver(D, r_b)$ (see Definition (7.29))

    find $\mathbf{V^F}$ and $\mathbf{V^D}$ based on $(v(D, r_b), u(D, r_b))$;

  end for

end for

Step two: find $r_b^*(D)$ for given (R, A)

for Loop debts level $D \in \mathcal{D}$ **do**

  for Loop coupon rate $r_b \in \nabla_\lfloor$ **do**

    Find $r_b^*(D) = \text{argmin}_{r_b}\{V^D = 0\}$

  end for

end for

Step three: find D^* for each (R, A)

for Loop debts level $D \in \mathcal{D}$ **do**

  Find $D^* = \text{argmax}_{r_b*}\{V^F = 0\}$

end for

7.4 Solution and Analysis

In this section, we show numerical results for the optimal fixed-time debt financing problems. We are interested in how the debt financing flexibility reduces a firm's liquidity risk and also how it affects its valuation and operation strategy. We are also interested in how to find the market coupon rate of a particular corporate debt, based on which, the firm can make optimal financing decisions.

We assume a linear instantaneous financing cost function $\Psi(D)$,

$$\Psi(D) = \psi_0 + \psi D, \tag{7.31}$$

and a set of parameter values (see Table 7.1). To simply the explanation of the results, we normalize the parameters based on the operational cost $\varepsilon = 1$ (My^{-1}), so that we do not need to explain the unit of each variable.

Parameters	$\mu(y^{-1})$	σ	ρ	ϱ	$\varepsilon(My^{-1})$	$r(y^{-1})$	$\psi_0(M)$	ψ
Value	0.03	0.25	0.05	0.03	1	0.03	0.1	0.05

Table 7.1: A basic set of parameter values for numerical solutions. Here, μ and σ are the growth rate and volatility of the annualised revenue (see Equation (5.1)); ρ is the market discounted rate (used in model (5.10)); ϱ is the market discounted rate for debt investors (see (7.16), we assumed it equals the risk free interest rate); ε is the annualised running cost (defined in Equation (5.3)); r is the short term interest rate (used in Equation (5.3)); $\psi_0(M)$ and ψ are the coefficients required by agency cost Function (7.31).

7.4.1 The Value of a Leveraged Firm and Its Debts

As illustrated in the mathematical model section, before solving an optimal debt financing problem, it is essential to value a leveraged firm and its debts, in order to address the payoff matrix of exercising a debt financing option. Also, we are interested in the corporate motivations of holding debts from the angle of reducing the liquidity risks. To give some intuition, we consider a firm that has a fixed level of debts ($D = 2$) with a constant coupon rate ($r_b = 0.1$), and then presents its valuation and operational strategy by comparing it with another firm that has no option to refinance with debts.

Figure 7.3 plots the value of different firms against the initial equity at a fixed value R referenced by the net cash flow (NCF) -0.05 and 0.05 at the given value of R. According to this figure, we can see that the leveraged firm value line and the non-leveraged one cross each other at some point. More precisely, when the initial equity is less than these cross points, the leveraged firm has higher value, whereas, when the equity level is greater than this level, it is lower. This is because holding debts increases the firm's liquid assets, so it reduces the liquidity risk, yet at the same time, it increases the firm's operational costs, which reduces the corporate value (the shareholders' benefits). When the firm has very low initial cash assets, the benefits of holding debts surpass the costs, therefore the leveraged firm has higher value than the equity financed corporation. When the firm has enough initial cash assets, it seems it has no need to hold any debts because the costs of doing so exceed the return. This observation is very important since it gives us an intuitive explanation why a firm should consider a debt financing from the angle of reducing liquidity risk.

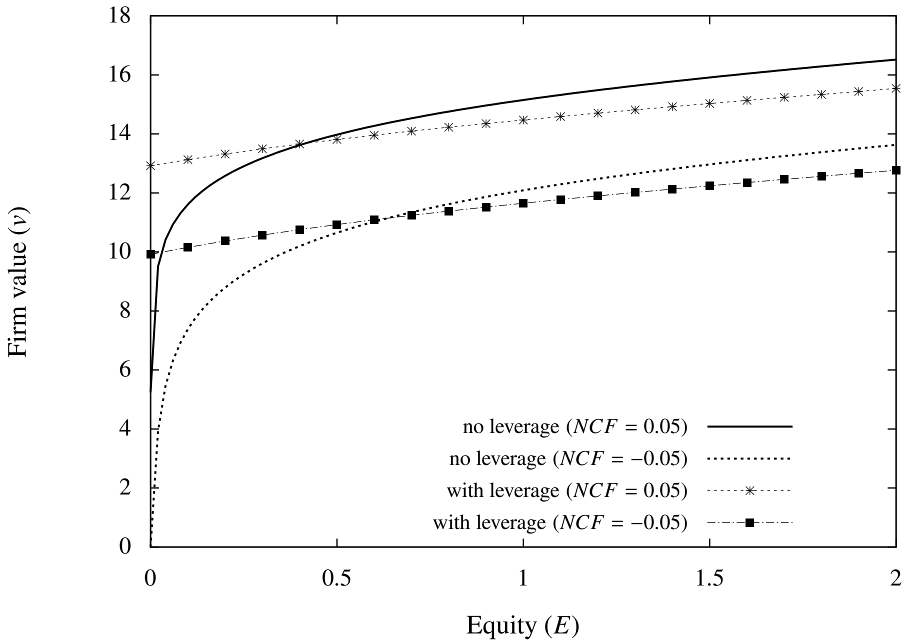


Figure 7.3: A comparison of firm value between a leveraged company and a non-leveraged company, where their net cash flows (NCF) are assumed to be 0.05 and -0.05.

Next we investigate what happens to the operational strategy if we assume dividend payments are not allowed when the firm has a negative cash flow. Figures 7.4 and 7.5 present the operational strategy for a leveraged and non-leveraged businesses, respectively. By comparing these two figures, we can see that the leveraged corporate dividend payment boundary locates at a higher level of revenue compared with that for a non-leveraged corporation, due to the extra running cost (debt repayments). It is interesting that when a leveraged business has negative equity ($A < D$), it prefers to wait than to exercise its abandonment option (see the abandonment line in Figure 7.5). This is because, without strict regulation, the shareholders can transfer risks from themselves to the debt holders, particularly in a negative equity case.

The market value of the debts issued can be found by calculating the expected discounted debt repayments under the risk neutral measure. Figure 7.6 presents how the debt value varies with different revenues, for a firm that has no cash assets, one-month cash assets, half year cash assets, one-year cash assets and ten years cash assets to pay the operating costs. From this figure, we can see that the market value of corporate debts grows as the firm has more cash assets since more cash assets mean less

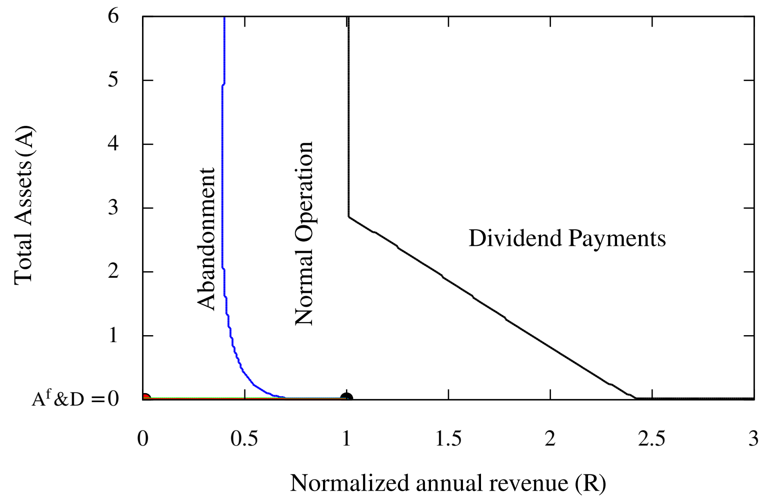


Figure 7.4: A plot to show the optimal operational strategy of a non-leveraged firm within Revenue-Asset (R_t, A_t) space, where both the fixed assets A^f and the debt level D are zero.

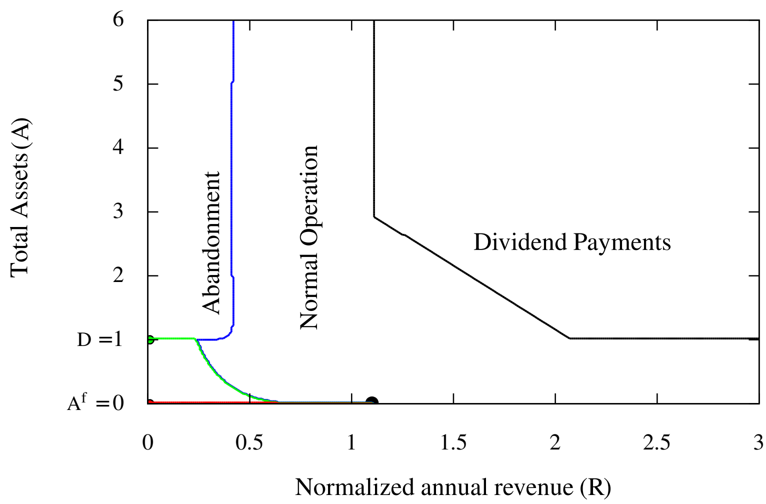


Figure 7.5: A plot to show the optimal operational strategy of a leveraged firm within Revenue-Asset (R_t, A_t) space, where both the fixed assets $A^f = 0$, the debt level $D = 1$ and the coupon rate $r_b = 0.1$.

liquidity risk. However, the debt value does not monotonously increase with corporate income. More precisely, when the firm has a negative equity with $A < D = 1$, the debt value first decreases and then rises along the revenue dimension. This result is consistent with the operational strategy of a leveraged firm. In a negative equity situation, the shareholders prefer to take risks and wait for recovery instead of abandoning the business. This is because this strategy helps shareholders get the benefits from their limited liability, transferring the risk to the debt investors.

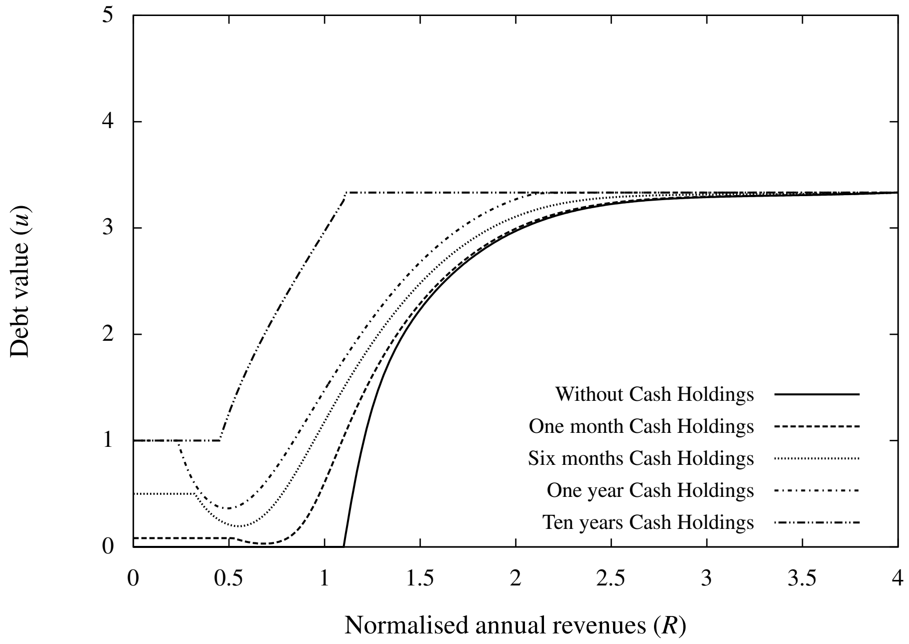


Figure 7.6: The market value of corporate debts along different revenues, where the fixed operational cost is 1, coupon payment ratio $r_b D = 0.1$, and the cash assets $(A = C) = 0, \frac{1}{12}, \frac{1}{2}, 1$ and 10.

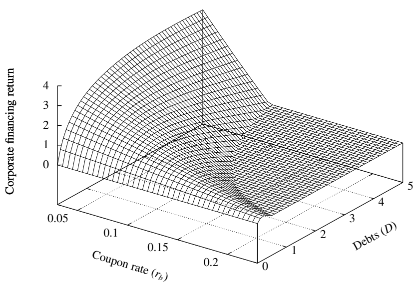
7.4.2 Optimal Debt Financing with Risk-adjusted Coupon Rate

We have so far presented a firm's optimal financing amount based on a fixed debt coupon rate. We now want to show how to find the market debt coupon rate that reflects a firm's different types of risks, and how this feeds back into how a corporation makes financing decisions. In order to solve this problem, we follow Algorithm (7.30) in Section 7.3.2.

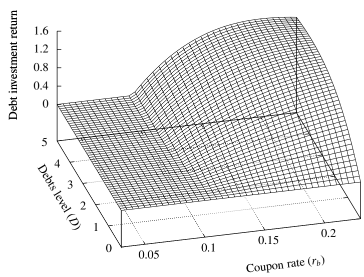
Step one: Financing payoff analysis for a firm and its debt investors

We assume a firm starts its business from the operational state $R_t = 0.95$ and $A_t = E_t = 0.2$. The firm has a feasible financing amount interval $D \in [0, 5]$, and its debt investors has a feasible coupon rate interval $r_b \in [0, 0.26]$. With these settings, we can solve the payoff matrix of the debt issuer and investors based on the definitions (7.25) and (7.26).

Figures 7.7(a) and 7.7(b) plot out the payoff surfaces of the debt issuer and investors on the all possible financing regimes (D, r_b). According to these two figures, we can see that each of the surfaces in the whole $D \times r_b$ space can be divided into two different regions: the positive payoff region and the non-positive payoff region. As illustrated in the mathematics section, the non-positive payoff region represents cases that participants cannot get any benefits if they implement the corresponding financing strategy. To show more details, we generate a firm payoff Figure 7.8 by selecting $r_b = 0.03, 0.06, 0.09, 0.12$ and 0.15, and a debt payoff Figure 7.9 by choosing financing amount $D = 1, 2, 3, 4$ and 5.



(a) Corporate financing return



(b) Debt investment return

Figure 7.7: The returns of a firm and its debt investors within the 'Debts - Coupon Rate' space (D, r_b), where the firm has initial state $R = 0.95$ and $A = E = 0.2$.

Figure 7.8 shows the payoff value of the debt issuer against different financing amount for selected coupon rate $r_b = 0.03, 0.06, 0.09, 0.12$ and 0.15. According to this figure, we can see that there is an optimal financing amount for each given debt coupon rate, which can maximise the firm's payoff (see the solid circle on each line). Also, this optimal financing amount increases with the decrease of coupon rate. This is because raising a reasonable amount of debts can help the firm increase liquid assets and hedge future risks. However, holding too high a level of debt can overburden the firm's normal operation with high costs, which in turn, reduces the firm's future value.

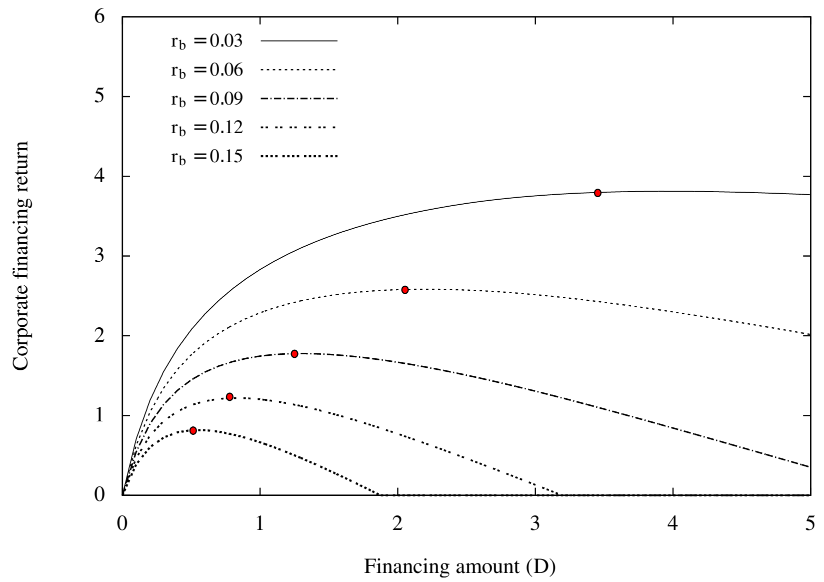


Figure 7.8: A 2-D plot to show the corporate financing return against the financing amount D, where the coupon rate takes value $r_b = 0.03, 0.06, 0.09, 0.12$ and 0.15 in turn. The peak of each line is marked by a solid circle.

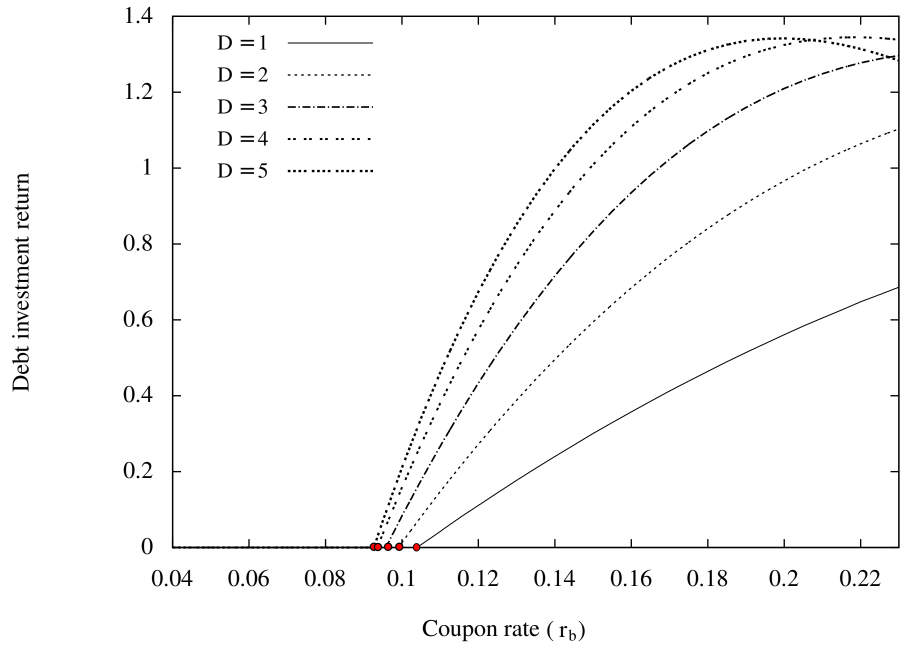


Figure 7.9: A 2-D plot to show the debt investors' return against the coupon rate r_b, where financing amount takes value $D = 1, 2, 3, 4$ and 5. The solid circles at the bottom represent the corresponding coupon rates where the payoff of debt investors equals zero with given debt financing levels.

Figure 7.9 presents the payoff value for the bank against the possible risk-adjusted coupon rate by fixing the financing amount to be $D = 1, 2, 3, 4$ and 5. The solid circles mark the corresponding coupon rate where the bank's payoff equals zero. For each given financing amount, this zero payoff coupon rate is the real market interest (or risk-adjusted coupon rate). We can see from the figure, the debt investors can get the highest payoff when the firm decides to finance 5 units of debts with a coupon rate around $r_b = 0.18$, instead of $r_b = r_b^m$. This makes sense because as we have just explained, too high a coupon rate has a negative effect on the firm value which is reflected in an increase of the firm's default risk.

The results in the above two Figures (7.8 and 7.9) show that the payoff of both debt investors and issuer do not always monotonically change with D and r_b. This observation gives us strong motivation to find out an optimal financing regime, which can be of benefit to both of the participants.

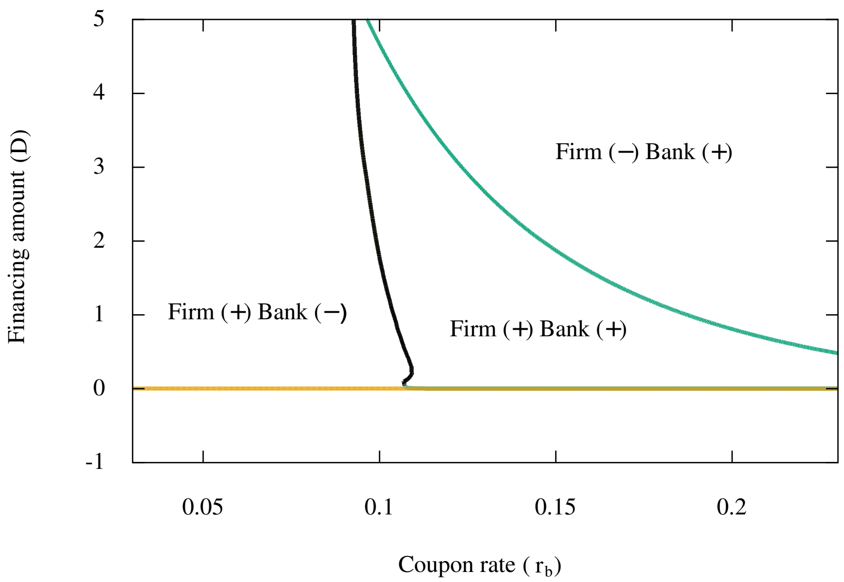


Figure 7.10: A plot to show the sign of returns for both participants within the (D, r_b) space, where the firm has the operational state $(R = 0.95, E = 0.2)$.

Step two: Market chosen coupon rate

According to the results we have presented so far, we can see that not every pair of financing situation (D, r_b) will generate positive returns for both participants. Figure 7.10 summarises all the possible states that the debt issuer and investors can take, where (+) means the party has a positive return value and (−) denotes it has a zero return value. We can see that the whole (D, r_b) space is divided into three different regions, Firm(+)Bank(-), Firm(-)Bank(+) and Firm(+)Bank(+). Since in the regions Firm(+)Bank(-) and Firm(-)Bank(+), at least one participant has a non-positive return, therefore, a successful debt offering can only happen in the region Firm(+)Bank(+). These results also help us understand what should be the market coupon rate. To explain, suppose the capital market is a buyer's market, which means there are enough funds available but limited business opportunities. Since there are competitions between debt investors, in such a market, they would only ask for a debt coupon rate, which can just cover the investment risks, i.e. the internal rate of return given by Equation (7.27). As shown in Figure 7.10, these coupon rates for each financing level are summarised by the black boundary line. Therefore, in the following contents, we are going to find the optimal financing amount with given risk-adjusted interested rate along the black boundary line. To show more information about how this type of 'interest-boundary' varies with different operational state, we generate Figures 7.11 by taking different R as examples.

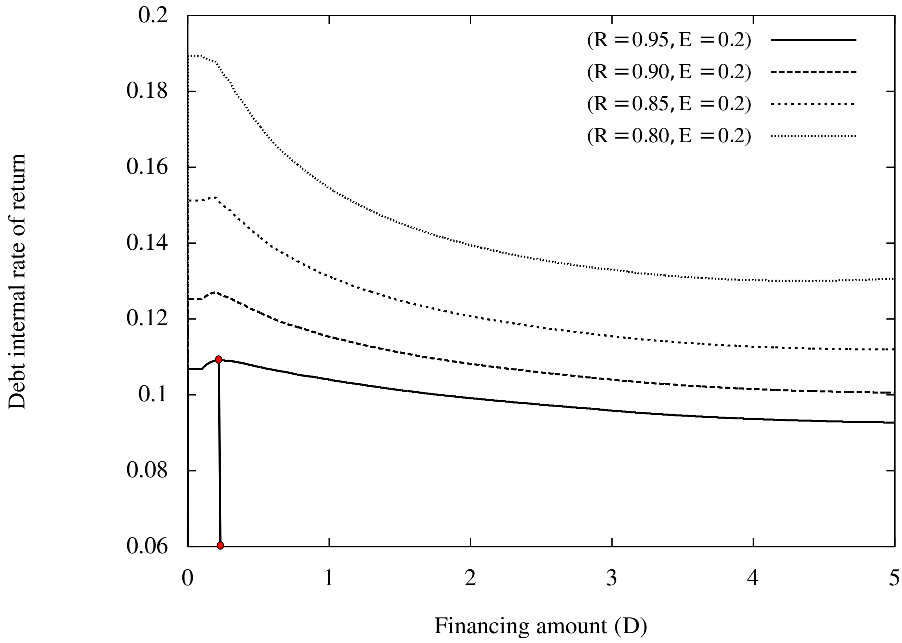


Figure 7.11: The debt investors' internal rate of return against financing amount for a firm that has initial operational states, $(R = 0.95, E = 0.2)$, $(R = 0.9, E = 0.2)$, $(R = 0.85, E = 0.2)$ and $(R = 0.8, E = 0.2)$. The solid circle at the bottom marks the optimal financing amount that maximises the debt investors' payoff.

Figure 7.11 shows the debt investors' internal rate of return against each possible financing level, when the firm has initial operational state, $(R = 0.95, E = 0.2)$, $(R = 0.9, E = 0.2)$, $(R = 0.85, E = 0.2)$ and $(R = 0.8, E = 0.2)$. The solid line represented the case when the firm has the operational state $(R = 0.95, E = 0.2)$ and the solid circle at the bottom marks the optimal financing amount that maximises the debt investors' payoff. However, since the debt investor can only choose a coupon rate after the debt issuer's decision of the financing amount, this financing amount marked by the solid circle is not the solution of the optimal debt financing problem.

In the following contents, we are going to find the optimal financing amount for the firm based on the risk-adjusted interested rate along the zero payoff line of the bank for a given operational regime (R, E). To simplify the explanation, we mainly focus on the case when the firm has the operational state $(R = 0.95, E = 0.2)$ to show how to find the optimal financing amount for the firm with a risk-adjusted coupon rate.

Step three: Optimal financing amount with market chosen coupon rate
With the above risk-adjusted coupon rate (market debt coupon rate), a firm can make financing decisions by balancing the benefits and costs of raising each particular level of debts. Therefore, we can find the optimal debt level and corporate value via Equations (7.23) and (7.28).

Figures 7.12 and 7.13 present the optimal financing strategy (D^*, r_b^*) of this optimal financing game. Figure 7.12 shows the firm payoff value against its possible financing amount based on the corresponding IRR for cases that the firm starts from the operation states $(R = 0.95, E = 0.2)$, $(R = 0.9, E = 0.2)$, $(R = 0.85, E = 0.2)$ and $(R = 0.8, E = 0.2)$, and Figure 7.13 is a review of Figure 7.11. Here, we take the firm's operational state $(R = 0.95, E = 0.2)$ as an example to show how to find the optimal financing strategy (D^*, r_b^*) based on these two figures.

In Figure 7.12, the solid line represents the firm's return of financing in the case $(R = 0.95, E = 0.2)$, and the solid star at the bottom shows the optimal financing amount, i.e. $D^* = 1.15$. With this optimal financing amount, we can find the optimal financing coupon rate according to the solution shown in Figure 7.13 (see the star on the y-axis). By combing these two figures, we can get that the optimal financing regime for both participants is $(D^*, r_b^*) = (1.15, 0.112)$ (marked by the stars at the bottom of Figures 7.12 and 7.13).

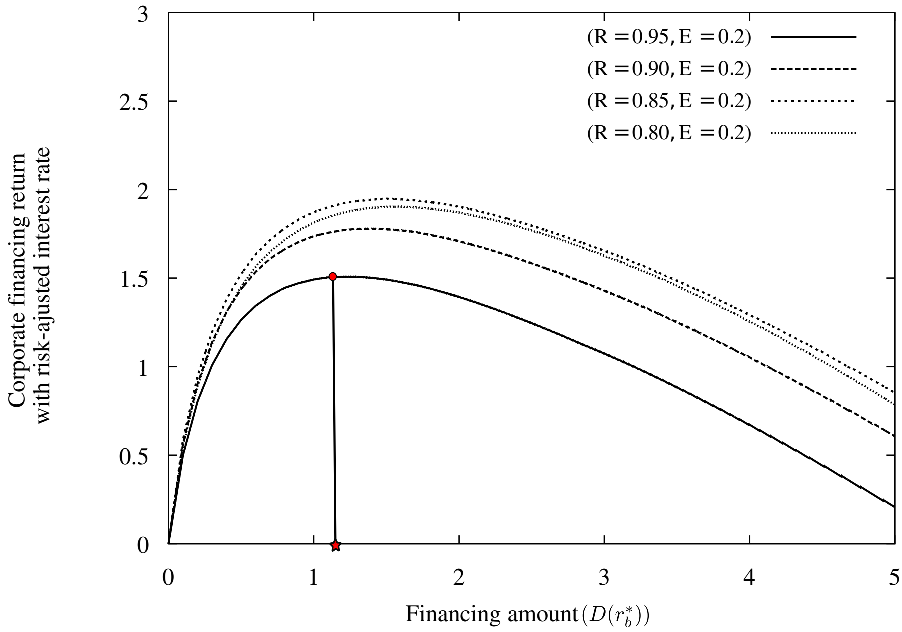


Figure 7.12: Corporate financing return on different financing amount, where the firm has initial operational states $(R = 0.95, E = 0.2)$, $(R = 0.9, E = 0.2)$, $(R = 0.85, E = 0.2)$ and $(R = 0.8, E = 0.2)$.

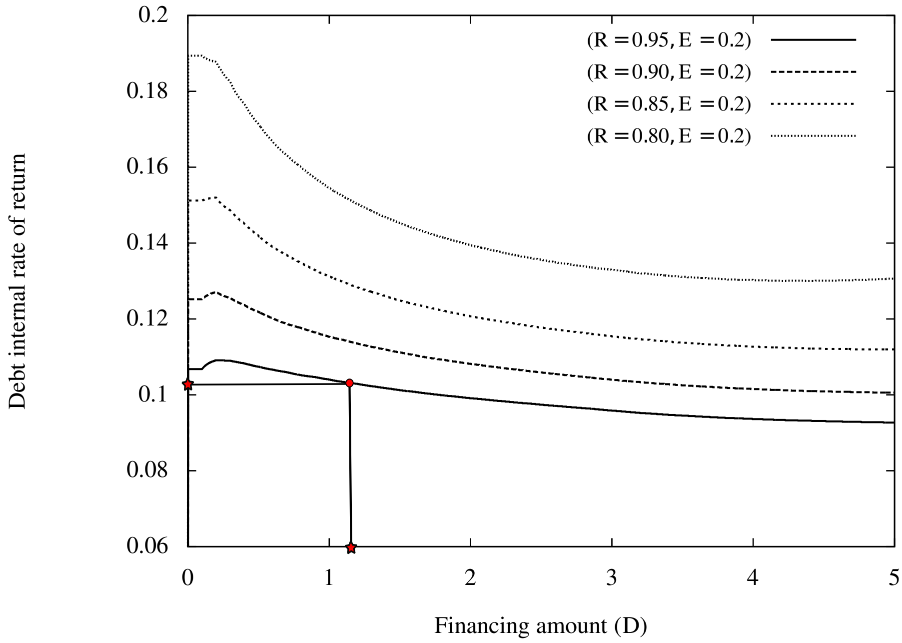


Figure 7.13: The debt investors' internal rate of return against financing amount for a firm that has initial operational states, $(R = 0.95, E = 0.2)$, $(R = 0.9, E = 0.2)$, $(R = 0.85, E = 0.2)$ and $(R = 0.8, E = 0.2)$.

As a supplement, we generate another set of results for the optimal financing problem by varying the firm's equity level E. Figure 7.14 shows the firm payoff value against its possible financing amount based on the corresponding IRR for cases that it starts from the operation states ($R = 0.8, E = 0.2$), ($R = 0.8, E = 0.4$) and ($R = 0.8, E = 0.8$). According to this figure, we can see the firm's payoff value augments sharply at low financing amount and then turn to a peak. After that, it goes down slowly. This means the marginal value of one unit of debt to the firm decreasing with the financing amount compared with the financing costs (r_b). Thus, a firm can always find an optimal financing amount in a buyer's market. Figure 7.15 shows the debt investors' internal rate of return against each possible financing level, when the firm has initial operational state, ($R = 0.8, E = 0.2$), ($R = 0.8, E = 0.4$), and ($R = 0.8, E = 0.8$).

According to this figure, we can see that the IRR of debts decreases with the increase of firm profitability and cash holdings. However, it is not monotonically varying as the firm increases the financing level. This solution is in accordance with the real business case. To explain, when the firm borrows only a little money compared with its cash holdings, say $D = 0.1$, it has a high probability of covering the repayments with hardly any risk, since $E > D$. When the firm borrows a large amount of money from banks, the more capital the firm raises, the fewer liquidity risks it faces, so that investors would like to charge a lower interest return.

By taking the firm operational state ($R = 0.8, E = 0.2$) as example, the optimal financing strategy is given by $(D^*, r_b^*) = (1.55, 0.145)$. By comparing the optimal financing strategy ($(D^*, r_b^*) = (1.55, 0.145)$) in this case with the optimal financing strategy ($(D^*, r_b^*) = (1.15, 0.112)$) when ($R = 0.95, E = 0.2$), we can see that the firm chooses to finance more when the revenue is lower, and the debt investors charges higher coupon rate due to the possible higher risk. Further, our solution provide more insight into how the optimal financing strategy and break-even coupon rate varies with a firm's cash holdings, which was missing in the literature for example Glover and Hambusch (2012) and Bolton et al. (2014b).

7.5 Conclusion

In this chapter, we illustrated why it is very difficult to model and solve an unlimited-debt-financing problem, particularly when the debt coupon rate is determined by the

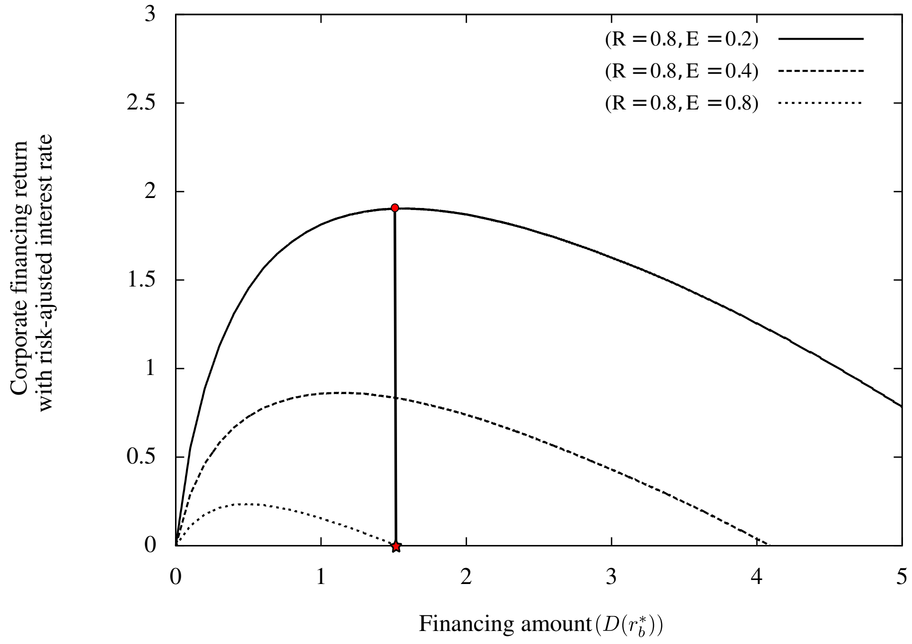


Figure 7.14: The corporate financing return against financing amount D, where the firm has the initial operational regimes $(R = 0.8, E = 0.2)$, $(R = 0.8, E = 0.4)$ and $(R = 0.8, E = 0.8)$.

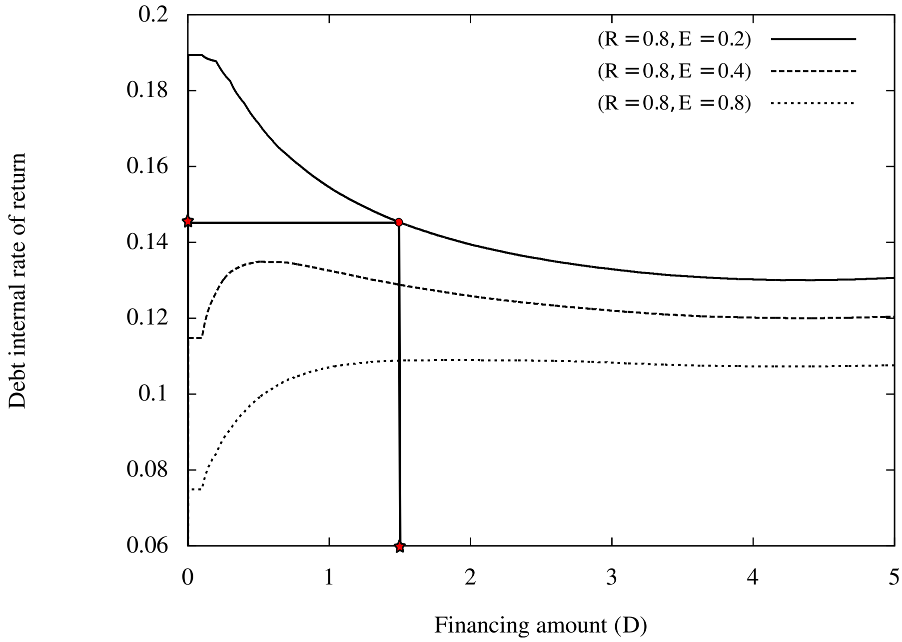


Figure 7.15: The debt investors' internal rate of return against financing amount for a firm that has initial operational states, $(R = 0.8, E = 0.2)$, $(R = 0.8, E = 0.4)$ and $(R = 0.8, E = 0.8)$.

markets understanding of the company's potential risks. As a simplification, we studied fixed-time debt-financing, and we designed a bargaining game structure to study the problem. This structure allows the firm to find the optimal debt-financing amount when the debt investors ask for a fair market coupon rate according to the investment risk that comes from the firm's future operation.

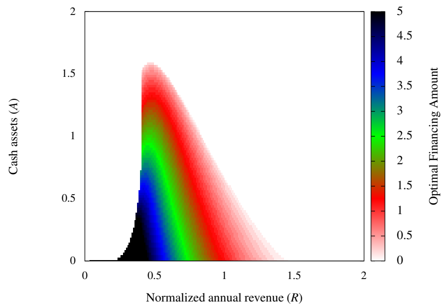
According to the solution and analysis, we found that allowing the firm the flexibility to issue debt means that they are able to increase the firm value by reducing the effects of financial constraints, yet the benefit of issuing debt is strongly limited by the financing costs. For the debt investors, the return they can expect not only depends on the firm's current operational state but also depends on the debt-financing level since it affects the firm's future default risk. The debt investors can benefit from the debt financing, too, when this action can significantly reduce the corporate financial constraints. However, by increasing the financing level, the benefits are surpassed by the potential default risk due to the increase in the leverage ratio and the associated operational costs. We also found that when the firm is in a non-profitable state and it holds hardly any cash assets, the shareholders are very likely to over-distribute the dividends, if there is no good corporate governance and regulation. In this situation, the corporate risks continuously transfer from the shareholders to the debt investors, whereas the benefits are opposite, moving from the debt investors to the shareholders.

It should be noted, in practice, that the computation time to get reasonably accurate results is the main restriction to the practical use of our model. It takes around a *week* to solve the fixed-time debt-financing model. Therefore, to some extent, people still cannot generate real-time solutions even using this simplified model. This is also the main reason why we leave the more complex models, i.e. the optimal debt financing at an optimal stopping time, for future study. In the next chapter, we move our focus to how this financial flexibility affects corporate investment decisions.

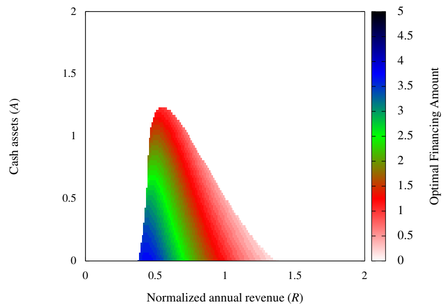
7.6 Appendix – The Optimal Debt Financing V.S. The Optimal Equity Financing

We have compared the valuation and operational strategy of a leveraged and non-leveraged firm. In this appendix, we assume the firm can finance once with either debt or equity at the start time, then, we compare the optimal financing amount with debt or equity on each operational state (R, A). We assume both the equity and debt financing option have linear cost functions defined by Equation (7.31), and simply show the following set of results. More results can be generated by implementing our model illustrated in this chapter.

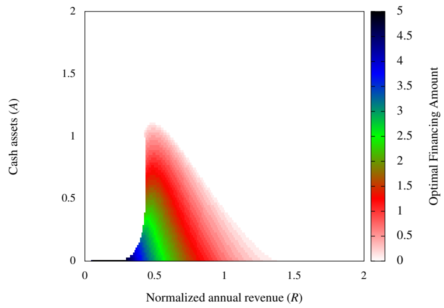
Figure 7.16 presents the optimal financing amount for each operational state (R, A) in six different six cases. The first four are debt financing with different coupon rates $(r_b = 0.1, 0.15)$ and financing costs, and the last two are equity financing with different financing costs, respectively. According to Figure (a) and (c), we can see that the optimal financing amount is negatively correlated with the coupon rate. The agency cost of financing significantly affects the scale of the risk transfer between the shareholders and the debt holders, which we can see clearly in Figure (a) and (b). What's more, when comparing Figures $(a)(b)$ with $(e)(f)$, we can see that the firm will choose a relevantly low level of debt financing amount compared to that when equity financing is available. This is because, a firm that finances with debt has to pay extra operational costs, the promised coupon interest, whilst choosing equity financing results in a loss of ownership which is offset by reduced risk.



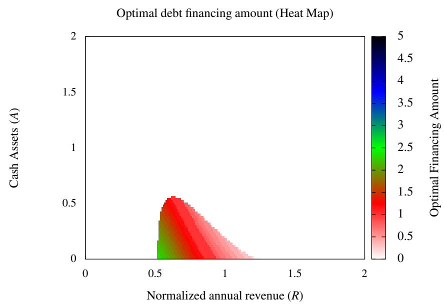
(a) Debt financing ($r_b = 0.1, \psi_0 = \psi = 0$)



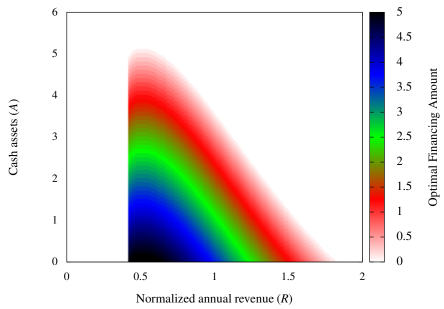
(b) Debt financing (r_b=0.1, ψ_0=0.1, ψ=0.05)



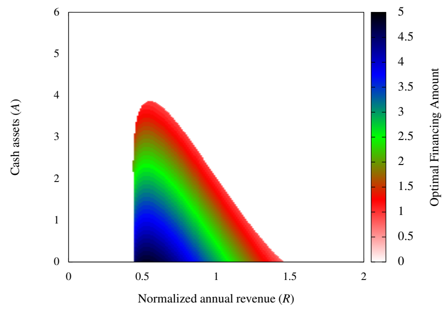
(c) Debt financing (r_b=0.15, ψ_0=ψ=0)



(d) Debt financing (r_b=0.15, ψ_0=0.1, ψ=0.05)



(e) Equity financing ($\psi_0 = \psi = 0$)



(f) Equity financing (ψ_0=0.1, ψ=0.05)

Figure 7.16: The optimal financing amount for a firm that has a fixed time financing option with either debt or equity. The white colour represents the lowest financing amount and black colour represents the highest level. We assume both debt and equity financing have the same linear agency cost function with $\psi_0 = 0.1$ and $\psi = 0.05$. Figure (a) shows the optimal debt financing amount with $r_b = 0.1$ but no agency costs; Figure (b) shows the optimal debt financing amount with $r_b = 0.15$ and agency costs ($\psi_0 = 0.1$ and $\psi = 0.05$); Figure (c) shows the optimal debt financing amount with $r_b = 0.15$ but no agency costs; Figure (d) shows the optimal debt financing amount with $r_b = 0.1$ and agency costs ($\psi_0 = 0.1$ and $\psi = 0.05$); Figure (e) shows the optimal equity financing with no agency financing cost; Figure (f) shows the optimal equity financing with agency cost ($\psi_0 = 0.1$ and $\psi = 0.05$).

Chapter 8

Optimal Investment Decisions with Cash Constraints

8.1 Introduction

As illustrated in Brealey et al. (2010), the value of a firm mainly comes from the asset side of the balance sheet. From this perspective, a firm's cash flow management, financing decisions and other operations should be organised to support investment. To make an optimal investment decision within a cash constrained firm is particularly difficult. This is because this type of firm has limited liquid financial resources thereby restricting their management flexibility. As a consequence, any other operational decisions that are linked with cash constraints might later affect investment decisions. Simply put, a firm cannot invest in a new project if it can neither raise enough internal cash nor external capital (new equity or debts) to afford the investment cost, and these financing resources depend on the cash management decisions previously made (dividend redistribution). Alternatively, if the new project were to be financed by issuing debt, the debt level will re-frame the firm's capital structure thereby altering the investment risks.

Recently, several academics have sought to model the optimal investment problem for a cash constrained firm within the optimal stochastic control framework. Asvanunt et al. (2009) structured a model to investigate the interaction of cash balance and investment opportunities for a debt holding firm. In their paper, they considered a firm that holds a growth option, and the exercise cost of the growth option can be financed by cash or costly equity issuance. Importantly, they assumed the entire firm asset follows a stochastic process. Based on this assumption, they investigated the interaction of cash accumulation and growth option and concluded that an increase in the cash balance does not significantly raise the firm value in the absence of the growth option, and it adds significant value to the firm only a growth option is present. This is quite different compared with our results. Bolton et al. (2011) and Décamps et al. (2011) studied optimal investment from internal cash holdings with the assumption that the cash assets follow a continuous process. Later, Bolton et al. (2014b) revisited the model for an initial investment with only equity financing. Hugonnier et al. (2014) looked at the same problem only now assuming that the firm needs to search for investors in order to raise funds, and that the successful meeting of parties follows a Poisson-type process. However, they do not consider the investment with debt issuance or other mixed financing resources. Although these papers make an excellent attempt to illustrate the effects of financial constraints on investment decisions, their models can only reveal part of the real situation faced by firm managers. The main contribution of our model is to frame all of these considerations in one system and to capture the effect of being able to optimally choose the source of funding, something we feel is lacking from current literature.

Previously in this thesis, we have studied the effects of the cash holdings (cash constraints) on corporate abandonment, dividend distribution, equity financing and debt financing decisions. We will combine all these studies together in this chapter to investigate the optimal investment problem, using an Expansion Option as an example. To comprehensively analyse the financial constraints, we assume the firm can expand the project with three possible funding resources: internally retained cash, equity or debts. In the first part of the chapter, we lay down the mathematical framework for the problem. Later, in the numerical solution section, we present and analyse the results with three example cases: expansion with self-holding cash; expansion with either internal cash or external equity; expansion with both internal cash and external debts.

8.2 Mathematical Model

8.2.1 Problem Setting

Consider a purely equity-based firm that holds a certain amount of cash to operate a project, which is supposed to generate uncertain revenues over a fixed period of time. To maximize the shareholders' benefits, we assume the firm has the flexibility to redistribute the excess cash as dividends or to abandon the project entirely, where these options have identical settings as those illustrated in Chapter 5. In this chapter, we now further assume the firm has an extra flexibility to expand its productivity by paying a one-off sunk cost. To pay the expansion cost whilst hedging against future operational uncertainty, we assume the firm can raise funds from three possible resources: free internal retained earnings (or cash holdings), costly external equity, or debts. Since in this chapter, we mainly focus on how different financing resources affect the expansion decision, we assume a consistent dividend payment policy: the firm can only distribute excess cash as the dividend when it has a positive cash flow. This setting will also help us reduce the effects of the shareholders-bondholders conflict when valuing a leveraged business. Taking account of this new type of option that is available to the manager, we now derive the mathematical framework for such an optimal expansion problem.

8.2.2 Stochastic Framework

To keep consistency, we inherit the notations from previous chapters. In addition, we use $\mathring{(\cdot)}$ to denote the corresponding parameters in the after-expansion phase, for example, $\mathring{A}$ denotes the asset level after expansion, and $\mathring{R}$ denotes the revenue after expansion.

Suppose a firm starts a project with an initial operational state (t, R_t, A_t), where there are no fixed assets and no current debts (so we have $A_t = C_t = E_t$). It operates the project to maximise its shareholders' benefits, $V(t, R_t, A_t)$ by optimally choosing the expansion time τ_e ($\tau_e \in [t, T]$), paying out dividends d, and deciding the best closure time τ ($\tau \in [t, T]$). We assume the firm has to pay the I amount of money to exercise the expansion flexibility. After expansion, the firm can augment its productivity to q times the original level, but the corresponding fixed operational cost will also increase from ε to $\varepsilon(q)$ ($\varepsilon(q) > \varepsilon$), although not necessarily in a linear way (economies of scale etc.). We now show how this option affects the corporate revenues and cash assets.

The Change of Revenue Process: Suppose the firm originally generate p units of products with a market price S. After expansion, the firm's productivity increases q times its original level without changing the market price. Therefore, the firm revenues and fixed operational cost in the before-expansion and in the after-expansion always satisfy the following relation,

	before expansion	after expansion		relation
Revenues:	$R = pS$	$\mathring{R} = q \times pS$	$\Rightarrow$	$\mathring{R} = qR$
Cost:	ε	$\varepsilon(q)$		$\varepsilon(q) > \varepsilon$

The Change of Cash Assets: To pay for the investment, the firm must first raise its cash levels (C_s for $s \in [t, T]$) to at least that of the investment cost ($C_s \geq I$). We assume the firm decides to expand the business at time τ_e ($\tau_e \in [t, T]$). Then, the cash required may come from a variety of sources, and here, we denote C_{τ_e} as the amount from the internal cash holdings at the expansion time, which is costless, E^F as the amount from an equity financing with cost $\Psi^e(E^F)$, and D^F as the amount that comes from debt issued at a cost $\Psi^d(D^F)$, where $\Psi^e(0) = \Psi^d(0) = 0$. For simplicity, we assume the firm issues debts with a constant coupon rate r_b and face value D^F. Therefore, if we include all these funding possibilities, we have the cash assets just after expansion $\mathring{C}_{\tau_e}$ as,

$$\mathring{C}_{\tau_e} = C_{\tau_e} + E^F - \Psi^e(E^F) + D^F - \Psi^d(D^F), \tag{8.1}$$

where,

$$\begin{cases} \mathring{C}_{\tau_e} \geq I, \\ C_{\tau_e}, E^F - \Psi^e(E^F) \textbf{ and } D^F - \Psi^d(D^F) \geq 0. \end{cases}$$

To model a firm that has such an Expansion Option, we must consider two different operating phases, the before-expansion phase and the after-expansion phase, and the economic connections between these two phases. Thus, the firm's operating timeline and possible decisions can be schematically summarised in Figure 8.1, where $\mathring{(\cdot)}$ denotes the corresponding parameters in after-expansion phase.

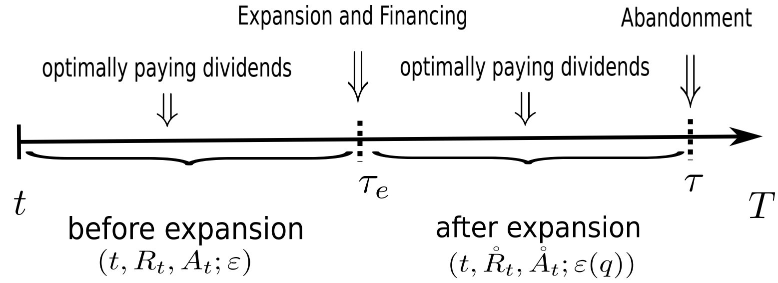


Figure 8.1: A schematic sketch to show a firm's operating timeline and possible operational decisions, where τ_e and τ are optimal expansion and abandonment time and the expansion time cannot be later than the abandonment time ($\tau \geq \tau_e$).

Based on Figure 8.1, we now show how to value such a firm by analysing its operation in reverse chronological order.

After-expanding Phase: Suppose the firm expands its business at time τ_e. Since the firm has only one chance to finance and expand its business, by combining with the explanation in Figure 8.1, we can see that after exercising this flexibility, the firm can be treated as a leveraged business that has no financing and investment opportunities, but it has new revenues, assets, capital structure and operational cost. Therefore, for any given financing strategy (E^F, D^F), we can define the firm value in after-expanding phase as $v(t, \mathring{R}_t, \mathring{A}_t; \varepsilon(q), E^F, D^F, r_b)$, where $v(\cdot)$ is the firm model defined by Equation (7.8), $\mathring{R}_t, \mathring{A}_t$ denotes the new revenue and assets after expansion, and $(\varepsilon(q), E^F, D^F, r_b)$ are used to uniquely define the total operational cost $\mathring{\varepsilon}$,

$$\mathring{\varepsilon} = \varepsilon(q) + r_b D^F. \tag{8.2}$$

The Economic Connections between the Before and After Expansion Phases: As soon as a decision is implemented to invest, there is an alteration on both the value of the firm and its balance sheet, so we must consider all these aspects in turn.

Denote ${\tau_e}^-$ as the time just before expanding, and ${\tau_e}^+$ as the time just after expanding. According to Equation (8.1), the firm's balance sheet, from ${\tau_e}^-$ to ${\tau_e}^+$, has the following changes,

$$\begin{cases} \textbf{Just before expansion:} & A_{\tau_e^-} = E_{\tau_e^-} = C_{\tau_e^-}, \\ \textbf{Just after expansion:} & \mathring{A}_{\tau_e^+} = \underbrace{\mathring{C}_{\tau_e}}_{\mathring{C}_{\tau_e} = C_{\tau_e^-} + E^F - \Psi^e(E^F) + D^F - \Psi^d(D^F)} - I = D^F + \mathring{E}_{\tau_e^+}. \end{cases} \tag{8.3}$$

We also know that the shareholders' benefits from ${\tau_e}^-$ to ${\tau_e}^+$ should stay the same, otherwise there will be arbitrage opportunities. Therefore, for any given financing strategy (E^F, D^F), we have *the original shareholders' value* (firm value) at just before-expanding time $V(\tau_e^-, R_{\tau_e^-}, A_{\tau_e^-}; \varepsilon(q), E^F, D^F, r_b)$ and the value at just after-expanding time $v(\tau_e^+, \mathring{R}_{\tau_e^+}, \mathring{A}_{\tau_e^+}; \varepsilon(q), E^F, D^F, r_b)$ satisfies the following equation,

$$\begin{aligned} & V(\tau_e^-, R_{\tau_e^-}, A_{\tau_e^-}; \varepsilon(q), E^F, D^F, r_b) \\ = & v(\tau_e^+, \mathring{R}_{\tau_e^+}, \mathring{A}_{\tau_e^+}; \varepsilon(q), E^F, D^F, r_b) - E^F \\ = & v(\tau_e^+, qR, A_{\tau_e^+} + E^F + D^F - \Psi^e(E^F) - \Psi^d(D^F) - I; \varepsilon(q), E^F, D^F, r_b) - E^F. \end{aligned} \tag{8.4}$$

Here, the value function v is defined by the Model (7.8), and V is the firm value that we are going to address in the following contents. Given this relationship, now we can derive the firm value at its before-expanding phase.

Before-expanding Phase: According to Figure 8.1, the firm value in the before-expansion phase depends on three parts: the expected dividend payment before the expansion and abandonment, the payoff from exercising the Expansion Option, or the payoff from exercising the abandonment option. The latter two parts are mutually exclusive.

If the abandonment decision exercises before the expansion at time τ ($\tau \leq \tau_e$), the stopping payoff can be defined by Equation (5.8). When the firm expands the business before abandonment at time τ_e ($\tau_e \leq \tau$), the instant payoff that shareholders will obtain is the firm value at just before-expanding value with an optimal financing strategy, which is given by,

$$\textbf{Payoff of expansion} := \max_{E^F, D^F} \big[v(t, \mathring{R}, \mathring{A}; \varepsilon(q), E^F, D^F, r_b) - E^F \big]. \tag{8.5}$$

Therefore, we can define the firm value $V(t, R_t, A_t)$ in the before-expanding phase as,

$$
\begin{aligned}
V(t, R_t, A_t) = \sup_{d,\tau,\tau_e} E\bigg[& \int_t^{\tau_e \wedge \tau} \big(de^{-\rho(s-t)}\big) dt \\
& + e^{-\tau_e \rho} \mathbb{1}_{\{\tau > \tau_e\}} \max_{E^F, D^F} \big[v(\tau_e^+, \mathring{R}, \mathring{A}; \varepsilon(q), E^F, D^F, r_b) - E^F \big], \\
& + e^{-\tau\rho} \mathbb{1}_{\{\tau_e \geq \tau\}} h(\tau, R_\tau, A_\tau) \bigg],
\end{aligned} \tag{8.6}
$$

where, $V(\tau_e^-, R, A; \varepsilon(q), E^F, D^F, r_b)$ is defined by Equation (8.4) and $h(\tau, R_\tau, A_\tau)$ denotes the abandonment payoff used in Equation (5.9).

8.2.3 HJB Variational Inequality

To solve the original optimal expansion firm model (8.6), we can use the following HJB Variation Inequality,

$$
\begin{aligned}
\min\bigg\{ & \frac{\partial V}{\partial t} + \mathcal{L}V - \rho V + \sup_{d \in [0,\infty)} \bigg(\big(R + rC - \varepsilon - d\big) \frac{\partial V}{\partial A} + d \bigg), \\
& V - \max_{E^F, D^F} \big[v(\tau, \mathring{R}, \mathring{A}; \varepsilon(q), E^F, D^F, r_b) - E^F \big], \\
& V - h \bigg\} = 0,
\end{aligned} \tag{8.7}
$$

subject to the boundary conditions,

$$
\begin{cases}
V = E & \text{on} \quad t = T, \qquad (8.8) \\
V = \max_{E^F, D^F} \big\{ v(t, \mathring{R}, \mathring{A}; \varepsilon(q), E^F, D^F, r_b) - E^F \big\} & \text{as} \quad R \to \infty. \qquad (8.9)
\end{cases}
$$

Here the linear operator is defined by Equation (4.8). We now display the explanations of each boundary condition as follows:

1. The termination boundary condition (see Equation (8.8))
   At the natural end of the project life, the firm has no options to exercise, thus the value of the project to shareholders is the net equity.
2. The boundary condition as $R \to \infty$, (see Equation (8.9))
   When the firm is very profitable ($R \to \infty$), we assume the frictional cost of financing is smaller than the benefits of expansion, therefore, the firm will immediately expand the productivity when R is very large. So, we have,

$$
V(t, R, A) = \max_{E^F, D^F} \big\{ v(\tau_e^+, \mathring{R}, \mathring{A}; \varepsilon(q), E^F, D^F, r_b) - E^F \big\}.
$$

The optimal expansion model with multiple financing resources (8.7) has many inputs and considerations, expansion ratio (q), expansion cost (I), equity financing cost (Ψ^e), debt financing cost Ψ^d. It is not efficient for us to study and understand the problem directly with so many considerations. To give a progressive explanation of the expansion problem, we carry out our analysis with three basic cases: the investment with internal cash, the investment with cash and new equity and the investment with cash and debts. We now illustrate the numerical methods that used to solve these cases based on the Semi-Lagrangian Methods (see Chapter 3).

8.3 Numerical Methods

Compared with models in the previous chapters, the main difficulty of the optimal expansion model is how to find the optimal expansion decisions by optimally doing regime switching between the before-expanding phase and the after-expanding phase. Taking the advantages of Dynamic Programming Principle and PDE approaches, the global optimal expansion decision can be obtained by locally choosing the optimal operational strategy: continuation, abandonment and expansion. Thus, we can implement the discretization formula of the Algorithm (5.20) to find the numerical approximation of the continuation value, and design new algorithms to find the local optimal operational strategy.

8.3.1 Expansion with Accumulated Cash

We assume the firm can expand the business only when it accumulates enough cash for the expansion cost I, and there are no external capital available in this case. Thus, the expansion payoff function (8.5) can be reformulated as,

$$V(\tau_e, R, A; \varepsilon(q)) = v(\tau_e, qR, A - I; \varepsilon(q)) \quad \text{as} \quad A \geq I. \tag{8.10}$$

Since the firm can only expand the business when the self-holding cash over the expansion cost ($A_s = C_s \geq I$), the optimal expansion boundary shall include two parts: the solid barrier due to the investment cost, and the optimal expansion boundary via comparing different operational strategy.

Figure 8.2 schematically shows the calculation domain and boundary layouts for the cash financed expansion model. We know a firm has no incentive to expand the business when it is losing money. Therefore, when R is low, the firm only considers when it is the optimal abandonment time. As for dividend policy, here, we assume the firm pays no dividends when the cash flow is negative, so that a fair comparison can be made against the expansion decision when the firm can finance it with cash holdings and debt issuance. These two settings are very similar to that shown in Figure 5.1.

We particularly want to show the boundary condition when $R \to \infty$. Since the firm has no financing flexibility, when $C < I$, it has no option to expand the business even expanding the business can benefit the share investors. In this case, we set $\frac{\partial^2 V}{\partial R^2} \to 0$, which means the firm value linearly increase on R dimension. When $C \geq I$, the firm exercise its expansion option when R is very large. So, we have,

$$V(t, R, A) = \alpha q R + \beta \varepsilon(q) + A - I, \tag{8.11}$$

where α and β can be defined based on Equations (4.16) and (4.18).

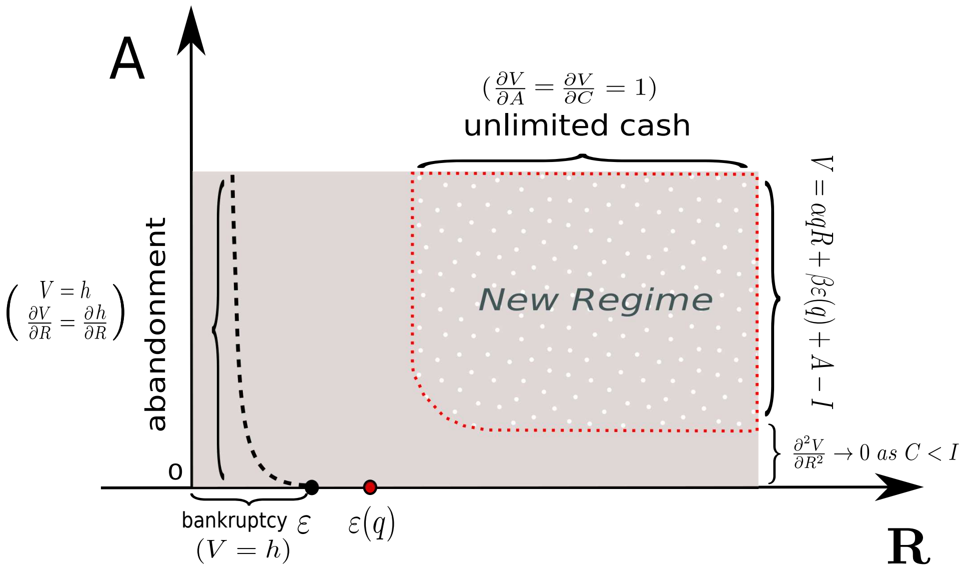


Figure 8.2: A schematic sketch to show the calculation domain and boundary layouts for a self-financing-expansion option. Here, R is the normalised annual revenue; A is the cash assets; I represents the sunk investment cost; ε is the operational cost of before expansion; $\varepsilon(q)$ is the operational cost after expansion; The red dotted line represents the expansion boundary; the black dash line represents the abandonment boundary; Other boundary conditions as shown in the sketch.

Based on these equations and boundary conditions, we design the following algorithm to solve the cash financed expansion problem.

Expansion with Cash (8.12)

for $k = K \to 0$ (calculate backward in time dimension) **do**

  solve continuation value $CV(t_k, R, A)$ for all the value at all states (R, A) at time t_k based on the SLM discretization (5.20).

  for $j = 0 \to J$ and $i = 0 \to I$ (loop A and R dimensions) **do**

    if $A_j > I$ (check expansion) **then**

      solve $v^*_{i,j} = v(t_k, qR_i, A_j - I; \varepsilon(q))$ (based on payoff 8.10)

      $v_{i,j} = \max(CV_{i,j}, v^*_{i,j})$

    end if

  end for

end for

It should be mentioned that the optimal expansion boundaries probably consist two parts: the fixed boundary defined by the investment cost, and the free boundary generated by the optional exercising strategy. To find this combined exercising threshold, PSOR algorithms are not the best choice via our test (it is about 10 times slower than the Algorithm (8.12)). Therefore, we borrow the idea of variational inequality and find the expansion boundary based on information generated in the previous time step (explicit finite difference methods). We tested that the solution generated with Algorithm (8.12) and the solution with this algorithm convergence to that generated with PSOR Algorithm (Here, we set grid size ratio $\frac{\Delta t}{\Delta R} < 0.1$). Based on the new algorithm, we can use the 'if' condition to find the fixed boundary and the variational inequality to address the free boundary in each time interval.

8.3.2 Expansion with Mixed Internal Cash and External Equity

Suppose the firm can raise an optimal amount of new equity to pay the expansion cost and hedge the future operational risks. Thus, we can rewrite the payoff function as,

$$V(\tau_e, R, A; \varepsilon(q), E^F) = \max_{E^F}\{v(\tau_e, qR, A + E^F - \Psi^e(E^F) - I; \varepsilon(q)) - E^F\},$$
$$\text{when} \quad A + E^F - \Psi^e(E^F) \geq I. \tag{8.13}$$

We denote E^F_{max} as the upper limit of the new equity that the firm can raise. In order to test the effects of up-limit financing amount, we consider two cases: limited financing amounts, where $E^F_{max} \leq I$ and unlimited financing amounts where $E^F_{max} < \infty$. With these settings, we can solve the optimal cash-equity financed expansion problem with the following algorithm in (8.14).

Expansion with Cash and Equity (8.14)

for $k = K \to 0$ (calculate backward in time dimension) **do**
  solve continuation value $CV(t_k, R, A)$ for all the value at all states (R, A) at time t_k based on the SLM discretization (5.20).
  for $j = 0 \to J$ and $i = 0 \to I$ (loop A and R) **do**
    for $E^F \in [0, E^F_{max}]$ **do**
      Define $A_{local} = A_j + E^F - \Psi^e(E^F)$
      if $A_{local} > I$ (check expansion) **then**
        solve $v_{i,j}(E^F) = v(t_k, qR_i, A_{local} - I; \varepsilon(q), E^F)$ (see Equation 8.13)
      end if
    end for
    Define $v^*_{i,j} = \max_{E^F \in [0, E^F_{max}]}\{v_{i,j}(E^F)\}$
    $v_{i,j} = \max(CV_{i,j}, v^*_{i,j})$
  end for
end for

8.3.3 Expansion with Mixed Internal Cash and External Debts

As discussed in the optimal debt financing chapter, the debt financing cost comes mainly from two aspects: the interest payment (depends on the payback period, coupon rate and financing amount) and the agency cost (depends on the financing amount). To focus on the analysis of effects of financing cost, we simply assume here, the firm can only issue utmost I amount of debts to pay the expansion cost and hedge the future operational risks. Denote the upper limit of the debt financing amount as D^F_{max}. We only consider the case ($D^F_{max} < I$) and particularly study how the coupon rate affects the optimal financing-expansion time.

With the optimal debt financing flexibility, we can rewrite the expansion payoff function as,

$$\begin{aligned} &V(\tau_e, R, A; \varepsilon(q), D^F, r_b) = \max_{D^F}\{v(\tau_e, qR, A + D^F - \Psi^d(D^F) - I; \varepsilon(q), D^F, r_b)\}, \\ &\text{when} \;\; A + D^F - \Psi^d(D^F) \geq I. \end{aligned} \tag{8.15}$$

Based on this payoff, we can design the algorithm as follows in (8.16),

Expansion with cash and Debts (8.16)

for $k = K \to 0$ (calculate backward in time dimension) **do**
  solve continuation value $CV(t_k, R, A)$ for all the value at all states (R, A) at time t_k based on the SLM discretization (5.20).
  for $j = 0 \to J$ and $i = 0 \to I$ (loop A and R) **do**
    for $D^F \in [0, D^F_{max}]$ **do**
      Define $A_{local} = A_j + D^F - \Psi^d(D^F)$
      if $A_{local} > I$ (check expansion) **then**
        solve $v_{i,j}(D^F) = v(t_k, qR_i, A_{local} - I; \varepsilon(q), D^F, r_b)$ (see Equation 8.15)
      end if
    end for
    Define $v^*_{i,j} = \max_{D^F \in [0, D^F_{max}]} \{v_{i,j}(D^F)\}$
    $v_{i,j} = \max\{CV_{i,j}, v^*_{i,j}\}$
  end for
end for

8.4 Solution and Analysis

In this section, we present numerical solutions of the three fundamental cases: the optimal expansion with cash holdings, the optimal expansion with equity and cash, and the optimal expansion with debts and cash. In the first case, we focus on how the cash holdings affect the exercise of Expansion Option, and how the corresponding parameters affect the exercising boundaries, i.e. σ, I and $\varepsilon(q)$. In the second case, we present how the equity financing relaxes the cash constraints, and particularly test the financing cost (economic frictions) effects on the financing amount and expansion time. In the final case, we are interested in how the debt financing interacts with the cash constraints, and give particular study to the difference between the equity financing and debt financing.

Before carrying out these numerical solutions, we define several important definitions and functions that used to find and explain numerical results. We use the **no-expansion** firm to define a firm that has no expansion option; **self-financing-expansion** firm to define a firm that has an expansion option but has no any financing flexibility; **equity-financing-expansion** firm to define a firm that has both expansion option and equity financing flexibility; and **debt-financing-expansion** firm to define a firm that has both expansion option and debt financing flexibility. Further, we define the Expansion Option Value (EOV) as the difference of a firm value with Expansion Options and with no expansion flexibility,

$$EOV(t, R, A) = V(t, R, A) - v(t, R, A), \tag{8.17}$$

where, $V(t, R, A)$ is the expansion-firm value defined by Model (8.7), and $v(t, R, A)$ is the non-expansion firm value given by Model (5.19). We assume the financing activity has linear agency cost functions, where the equity financing cost is given by,

$$\Psi^e(E^F) = \psi_0^e + \psi_1^e E^F, \tag{8.18}$$

and, the debt financing cost is given by,

$$\Psi^d(D^F) = \psi_0^d + \psi_1^d D^F. \tag{8.19}$$

We also assume a set of parameter values as shown in the Table 8.1. Without specification, our numerical solutions are based on these parameter values.

Parameters	$\mu(y^{-1})$	σ	ρ	$\varepsilon(My^{-1})$	$r(y^{-1})$	q	$\varepsilon(q)(My^{-1})$
Value	0.02	0.25	0.05	1	0	1.5	1.5
Parameters	$\psi_0^e(M)$	ψ_1^e	$\psi_0^d(M)$	ψ_1^d	r_b	$I(M)$	$T(y)$
Value	0.1	0	0.1	0	0.05	1	80

Table 8.1: A basic set of parameter values for numerical solutions. Here, μ, σ, ρ and r have the same meaning as illustrated in the previous chapters; ε is the original annualised running cost (defined in Equation (5.3)); $\varepsilon(q)$ is the new operational cost after expansion; ψ_0^e, ψ_1^e, ψ_0^d, ψ_1^d are coefficients defined in Equations 8.18 and 8.19; r_b is the coupon rate of the debts; I is the sunk cost of expansion option; T is the expected life-time of the project.

To simply the explanation of the results, we normalize the parameters based on the operational cost $\varepsilon = 1$ (My^{-1}), so that we do not need to explain the unit of each variable. With these settings, we now show the numerical solution of each case.

8.4.1 Expansion with Accumulated Cash

Trade-off of different operational strategies

The Expansion Option illustrated in Section 8.3.1 gives the firm flexibility to expand its productivity to q times the original level. To achieve this, the firm has to pay a one-off expenditure I and then operate the business with a higher running cost $\epsilon(q)$. In order to intuitively to show the tradeoff that firm managers face in choosing an optimal decision, we summarise the Return on Equity (ROE) (see Definition (5.22)) for three different operational cases at the initial time in Figure 8.3: the ROE of the case when the firm has never exercised the expansion option; the ROE of the case when the firm exercises the option now; and the ROE of the case when the firm holds the option for later exercise. Here, we assume the firm has an operational cost $\varepsilon = 1$, expansion cost $I = 1$, and it holds $A = C = 10$ cash assets which are enough to pay the expansion cost and hedge against the future operational uncertainty.

According to this figure, we can see that the firm's optimal operational strategy strongly depends on its revenue levels: a high level of revenue shows it is optimal to expand the business, whereas a very low level of revenue suggests it is optimal to abandon the business. When the revenue is around the fixed operational cost, the optimal operational strategy is to operate normally and keep the expansion option open. This comparison figure intuitively shows the pros and cons of three different operational cases when there is enough cash. We now show how the firm's optimal operational regimes vary with cash assets.

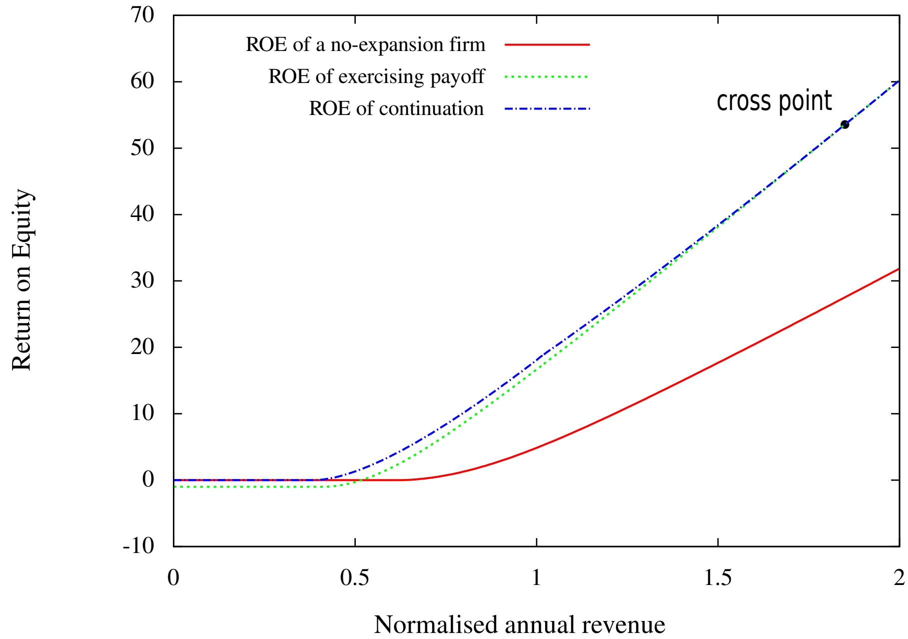


Figure 8.3: A comparison of Return on Equity (ROE) between three different operational cases: The ROE of the case when the firm does not hold the expansion option (the solid line); The ROE of the case when the firm exercises the option now (the dotted line); and the ROE of the case when the firm holds the option for later exercise (the dot-dashed line). The cross point marks the revenue level where the firm's continuation ROE equals the exercising payoff ROE. Here, we assume Cash Assets $A = C = 10$, $\varepsilon = 1$, and $t = 0$.

Value of Expansion Option and the optimal operational regimes

Figure 8.4 shows how the Value of Expansion Option varies along the revenue dimension by taking cash assets $A = 0.5, 1$, and 1.5 as examples. According to this figure, we can see that the benefits of expanding the business increases as revenue increases. What is more, a higher cash holdings level relates to higher option value, particularly when the firm not in a profitable situation (revenue is not that high). This is reasonable since more cash means fewer limitations to exercising the expansion option, and more reserves to hedge against future operational risks. We carry on this discussion by combining with the firm's optimal operational regimes, see Figure 8.5.

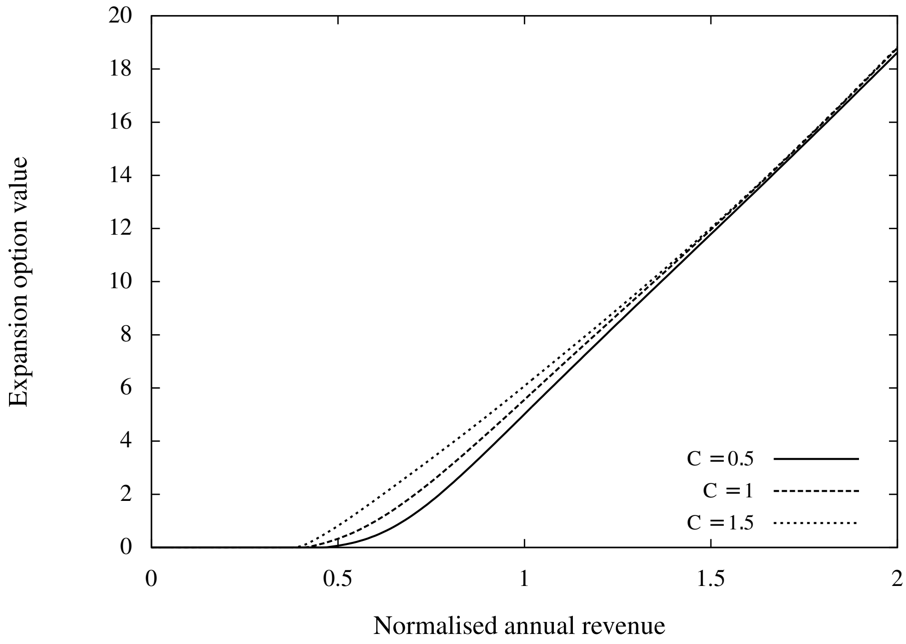


Figure 8.4: A plot to show the effects of the cash holdings on the the Expansion Option Value. Here, cash assets takes $A = C = 0.5, 1$ and 1.5.

Figure 8.5 summarizes the firm's optimal operational regimes within the Revenue-Asset $(R - A)$ space. It is clear that the whole space is divided into four different regions: the abandonment region, the normal operating region, the dividend payment region and the expansion region. The colourful lines between these regions denote the exercising boundaries of different operational regimes. Here, we are particularly interested in the expansion and dividend boundaries.

The expansion boundary is composed with two pieces, the horizontal line and the non-strict vertical line. The horizontal line is a solid barrier defined by the sunk cost I, here $I = 1$. The non-strict vertical line is a solved free boundary, over which, the firm immediately expands the business. It is interesting to see that this piece of the boundary is not strictly vertical. With the growth of cash levels, the expanding boundary continuous drifts left and then turns to a vertical line. To explain this observation, we consider the following example to show why the firm prefers to exercise the expansion option at a higher level of revenue when cash is just above I, rather than exercise the option at the same level of revenue like when the firm has a lot of cash assets.

When the firm has only $A = 1.2$, its optimal expanding boundary is $R^* \approx 1.7$, which is higher than the expansion boundary at $A = 10$ ($R^* = 1.6$). This is because, after expansion, the firm has only $A - I = 0.2$ units of cash assets, which is only enough to pay $\frac{A-I}{\varepsilon(q)} = 1.6$ years operational cost. Therefore, the firm prefers to exercise its expansion option at a higher level of R (the business is more profitable) when the A is relevantly low. This boundary shows that the cash constraints significantly limits the exercising flexibility of the expansion option. Thus, the classical Real Option approaches provide misleading decisions and over-valued solution, due to ignoring the cash constraints.

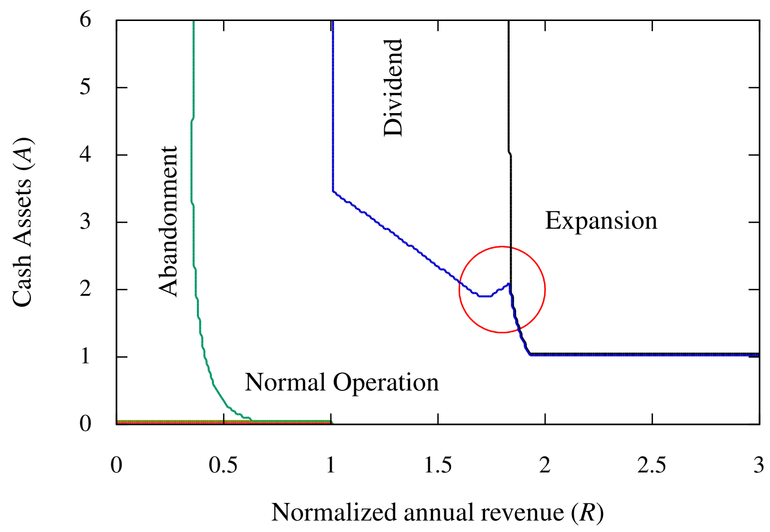


Figure 8.5: The optimal operational regimes for a self-financing-expansion firm, where the original operational cost $\varepsilon = 1$, after expansion operational cost $\varepsilon(q) = 1.5$, the expansion sunk cost $I = 1$.

Another interesting observation is the non-monotonic characteristics of dividend payment boundary. We can see from this figure, the dividend payment boundary no longer monotonically varies with the increase of revenue, instead, it moves downward first and then upward until meeting the expansion boundary (see the circled region of Figure 8.5). This is because, when the firm operates in a state which is near the expansion boundary, it prefers to wait and accumulate cash in order to switch to the expansion regime. However, when the firm is far from the expansion region, the motivation decreases, so that we can see the non-monotonicity varies on the dividend payment boundary.

In order to show the interaction between the expansion decision and other operational decisions, we superpose the operational regimes of a non-expansion firm with that of a self-financing-expansion firm, as shown in Figure 8.6. Here, the thin lines represent the boundaries of the non-expansion firm, while the thick lines represent the boundaries of the self-financing-expansion firm. We can see that the abandonment boundary of a self-financing-expansion firm significantly drifts left compared with that of a non-expansion firm. This is because the expansion flexibility gives the firm an opportunity to increase shareholders' benefits by increasing the productivity. Therefore, even the current revenue is relevantly low, an expansion-firm prefers to wait than abandon the business immediately. The dividend payment boundary of a self-financing-expansion firm locates upper than that of a non-expansion firm. This solution is reasonable since the marginal value of cash for a self-financing-expansion firm is higher than that of a no-expansion firm at the same level of cash assets.

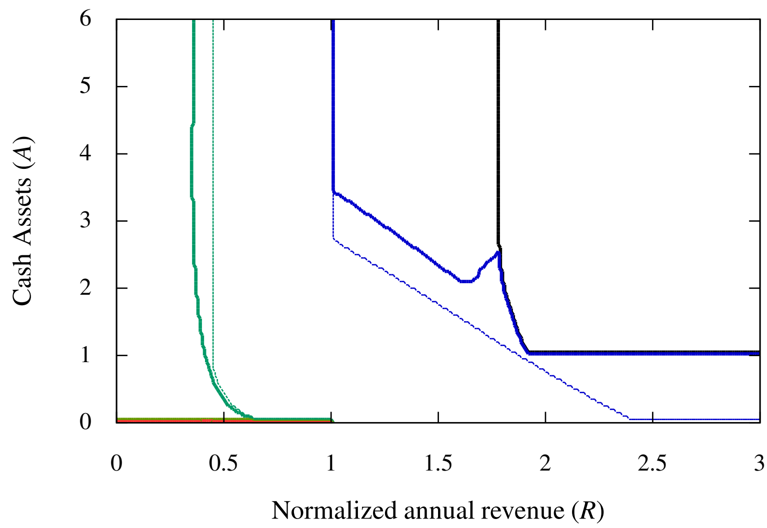


Figure 8.6: Comparisons of the operational strategy between a firm that has no expansion option and one that has expansion option, where the thin lines represent the boundaries of non-expansion firm, while the thick lines represent the boundaries of the self-financing-expansion firm.

Volatility effects on expansion option

Volatility (σ) is an important measure of uncertainty in the classical Real Option Analysis. It is generally believed to have a positive correlation with the option value (see Dixit, 1994), which means that an option has higher value when the underlying project has greater uncertainty. We are interested in here how the expansion option varies with different volatility.

Figure 8.7 shows how the value of Expansion Option varies with different σ. It is interesting to see, the Expansion Option value does not monotonically increase as σ goes up. When the firm's revenue is low, the volatility has a positive effect on the option value, whereas when the revenue is high, the volatility has an opposite effect on expansion option value. The explanation of this results is not straightforward. Essentially, the value showed in Figure 8.7 includes three types of flexibilities: abandonment, dividend redistribution and expansion. Abandonment option hedges the downside risk, so its value positive correlated with the project risk. However, the benefits of expansion option come from the upside growth when the revenue is high. Suppose the firm currently operates at a high level of revenue, bigger volatility of the revenue means higher probability that it moves to lower levels. This leads to a high probability of losing the expansion benefits and being left with high running costs. Based on this understanding, it is not strange to see a 'teeter-totter' type of volatility effects on Expansion Option value. Brosch (2008) presented the similar conclusion in his study of portfolios of real options.

Figures 8.8 gives an overall view how the firm's operational regimes vary with different revenue volatility (σ). According to these figures, we can see that when σ increases, the firm is more prudent to abandon the business and exercise the expansion option, so that the abandonment boundary drifts left and the expansion boundary drifts right. As a result of the movement of these two boundaries, the dividend payment region expands wider as the volatility increases. The observation of the effects of σ on the operating regimes is consistent with that on the expansion option value.

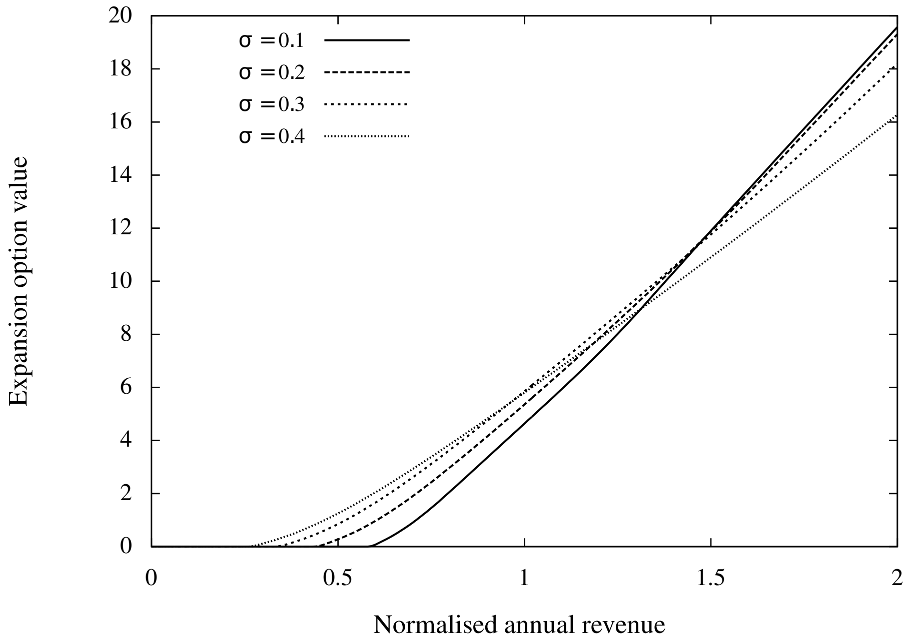
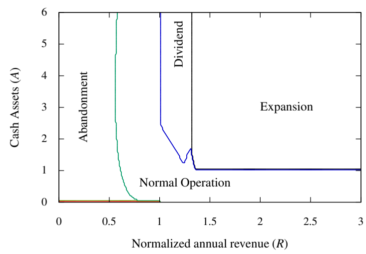
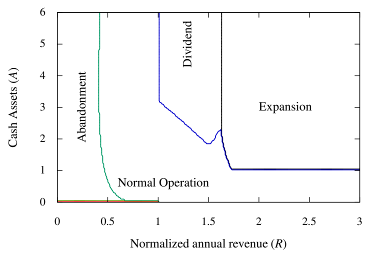


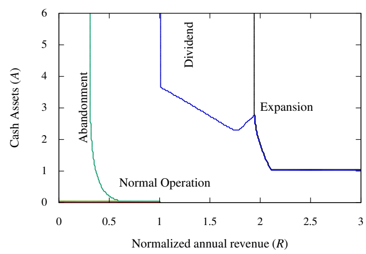
Figure 8.7: The volatility effects on the value of self-financing-expansion option, where we set $\epsilon = 1$, $\epsilon(q) = 1.5$ and $\sigma = 0.1, 0.2$, 0.3 and 0.4.



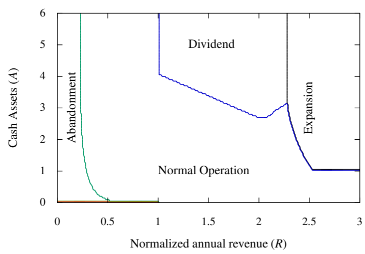
(a) Operational regimes for $\sigma = 0.1$



(b) Operational regimes for $\sigma = 0.2$



(c) Operational regimes for $\sigma = 0.3$



(d) Operational regimes for $\sigma = 0.4$

Figure 8.8: The volatility effects on operational strategy, were σ takes $0.1, 0.2, 0.3$ and 0.4 as examples.

The explicit expansion and implicit operational cost test

The cost of exercising of an expansion option comes from two aspects: the one-off sunk cost (the explicit economic friction) and the implicit cost generated by the larger operational expenses. We now show how these two types of costs affect a firm's expansion decision.

Figure 8.9 shows how the sunk cost influences the expansion option value, and Figures 8.10 summarize how the operational regimes varies with different I. According to these figures, we can see that the growth of expansion cost I significantly decreases the option value, particularly when the revenue is high. What is more, the increase of expansion costs leads to an up-movement of the solid piece of expansion boundary and a right-movement of the non-strict vertical expanding boundary, also it reduces the firm's redistributing of the dividends. Also, it is interesting to see that the firm's abandonment boundary is not sensitive to the change of the value of I. This is because the abandonment cases generally happen when the revenue is low and the expansion cases generally happen when the revenue is high. In addition, these two regions are mutually exclusive, thus the normal operational region and dividend distribution region naturally buffer the effects from the expansion region as I varies.

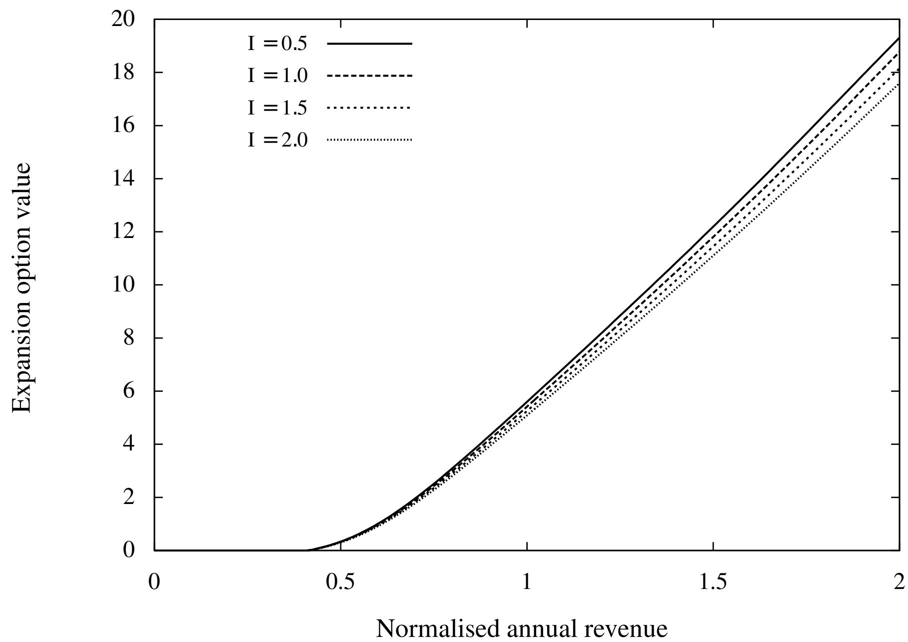
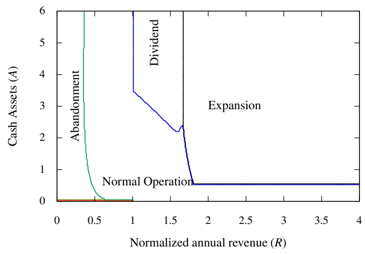
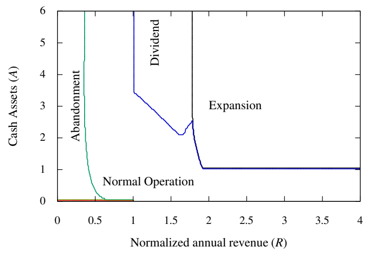


Figure 8.9: A plot to show the effects of sunk cost I on the expansion option value, where the cash assets takes value $A = 1$.

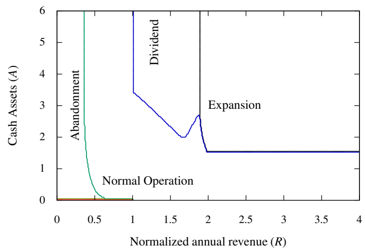
Apart from the explicit sunk cost, we are also interested in how the new fixed operational cost $\varepsilon(q)$ affects the expansion decision. Figure 8.11 presents how the expansion option value varies against revenues by taking $\varepsilon(q) = 1.3, 1.4, 1.5$ and 1.6. It is clear that the expansion option value is very sensitive to $\varepsilon(q)$, particularly when the net



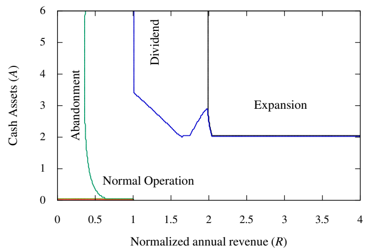
(a) Operational regimes for $I = 0.5$



(b) Operational regimes for $I = 1$



(c) Operational regimes for $I = 1.5$



(d) Operational regimes for $I = 2$

Figure 8.10: The effects of sunk cost I on the operational strategy. Here, we fix $\varepsilon = 1$, $\varepsilon(q) = 1.5$ and take $I = 0.5, 1, 1.5$ and 2 as examples.

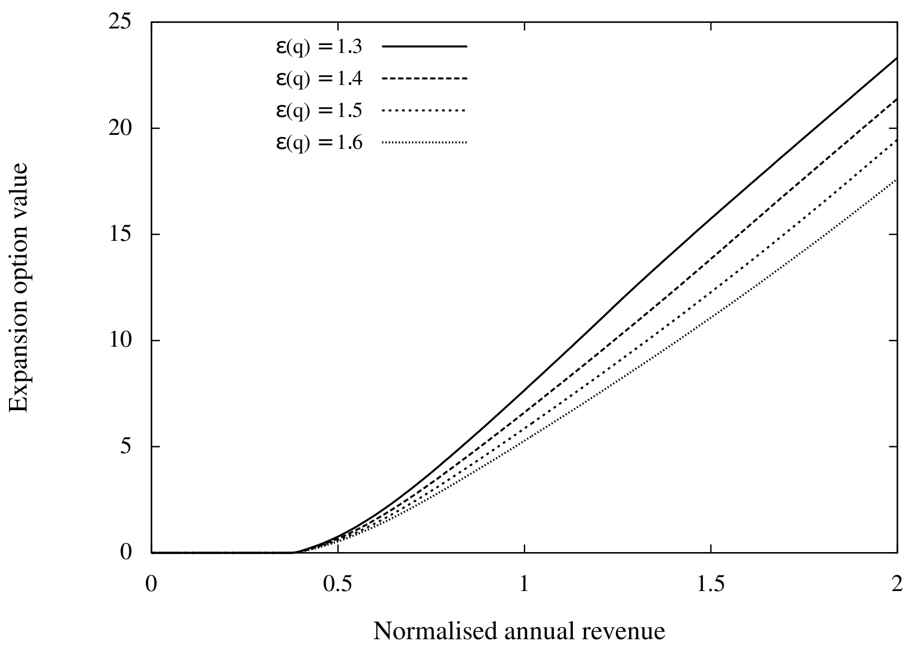
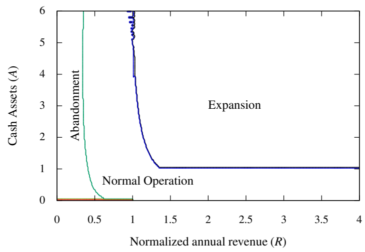
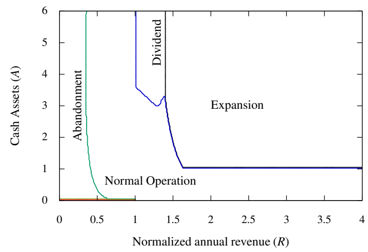


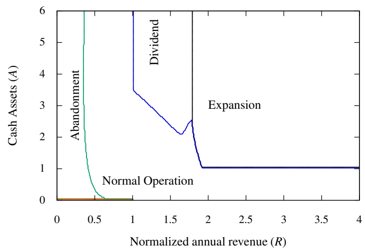
Figure 8.11: The effects of after-expansion operational cost $\varepsilon(q)$ on the value of self-financing expansion option. Here, we fix $\varepsilon = 1$, $q = 1.5$ and $I = 1$, and take new operational cost $\varepsilon(q) = 1.3, 1.4, 1.5$ and 1.6 as examples.



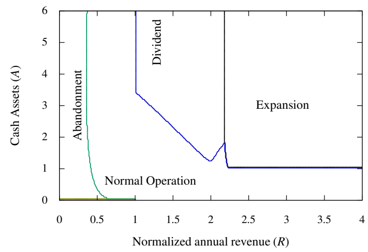
(a) Operational regimes for $\varepsilon(q) = 1.3$



(b) Operational regimes for $\varepsilon(q) = 1.4$



(c) Operational regimes for $\varepsilon(q) = 1.5$



(d) Operational regimes for $\varepsilon(q) = 1.6$

Figure 8.12: The effects of after-expansion operational cost $\varepsilon(q)$ on the operational strategy. Here, we fix $\varepsilon = 1$, $q = 1.5$ and $I = 1$, and take new operational cost $\varepsilon(q) = 1.3, 1.4, 1.5$ and 1.6 as examples.

cash flow is positive. Figures 8.12 present how the operational regimes vary with new fixed operating cost. According to these four figures, we can see that the increase of new fixed operating cost doesn't change the abandonment boundaries that much. However, the expanding boundaries drift left and dividend boundaries drift upward as $\varepsilon(q)$ increases.

8.4.2 Expansion with Cash and Equity

In the previous subsection, we have presented the significant effects of cash limitations on a firm's expansion decision. Here, we study how a firm's equity financing flexibility reduces the cash limitations. We assume the firm has an option to issue new shares to support the expansion decision, and the equity financing option can only be exercised when the expansion option is exercised. The numerical results and tests in this subsection are mainly based on the parameter values defined in Table 8.1, if there is no particular specification.

Case one: Financing no more than expansion cost

Suppose the firm has a constant financing cost, i.e. $(\psi_0^e, \psi_1^e) = (0.1, 0)$. Figure 8.13 illustrated the operating regimes of an equity-financing-expansion firm when the maximum financing amount E_{max}^F cannot surpass I. We can see from this figure that the whole Revenue-Asset space is divided into five different regions: the abandonment region, the normal operational region, the dividend payment region, the self-financing-expansion region and the equity-financing expansion region. However, if a firm starts the business from some state in the normal operational region, then it can only end up at one state in one of the normal operation, abandonment region, dividend payment region or financing expansion region. This is because the tunnel between the self-financing-expansion region and normal operation region is blocked by the equity-financing expansion region.

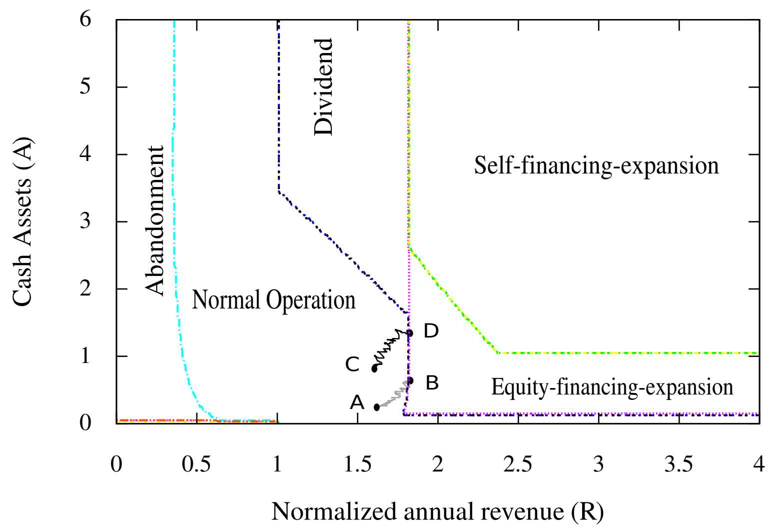


Figure 8.13: The operational regimes of an equity-financing-expansion firm, where we assume a constant financing cost $(\psi_0^e, \psi_1^e) = (0.1, 0)$ and the maximum financing amount cannot be larger than the sunk cost I. Other parameter values are based on the definitions of Table 8.1.

To show more details of the optimal financing amount, we generate Figure 8.14 based on the same parameter values. According to this figure, we can see that a cash constrained firm has a preference to finance more than the required amount to pay the expansion cost, i.e. $E^F + C > I$. To explain this observation with more details, we take the following two cases as examples to illustrate a firm's optional financing amount.

Case one: Suppose the firm starts from an operational state, point A $(1.6, 0.1)$ and by following the path $A \to B$ it arrives at the expansion boundary point B $(1.75, 0.5)$. According to the numerical results of optimal equity financing in Figure 8.14, we can see that the firm will immediately raise new equity $E^F = 1$ to expand the business. After paying out of the agency cost $\Psi^e(E^F) = 0.1$ and expansion cost $I = 1$, the firm switch its operational regimes from point B immediately to the new stage $(1.75, 0.4)$. It is interesting that the firm raises $E^F = 1$ such that $E^F + C > I$. This is because the extra cash assets could be used to hedge the future operational risk. *Case two:* Suppose the firm starts from the operational state, point C $(1.6, 0.8)$ and by following the path $C \to D$ it arrives at the expansion boundary point D $(1.75, 1.2)$, where $C = 1.2 > I = 1$. According to the numerical results of the optimal financing amount in Figure 8.14, we can see that the firm still raises new equity $E^F = 1$ to expand the business. After paying out of the agency cost $\Psi^(E^F) = 0.1$ and expansion cost $I = 1$, the firm switch its operational regimes from point D immediately to the new stage $(1.75, 0.7)$. This case also shows that a cash constrained firm has the incentive to raise extra money in the expansion decision as the cash buffers.

The solution presented here is very different compared with the previous study given by Bolton et al. (2014b). In their work, they assume a firm can only finance external capital to pay out expansion cost.

We superpose the Figure 8.13 and 8.5 as a new Figure 8.15 to show how the external financing flexibility affect a firm's expansion decision, where the thin lines represent the operational regimes in the Figure 8.5 and the thick lines represent the operational regimes in Figure 8.13. According to the comparison, we can see that the solid barrier (horizontal line near the bottom) of the expansion boundary moves downwards after including the financing cost, which means that the cash limitations are significantly reduced. However, this barrier is still higher than zero. This is because the firm has a constant financing cost $\psi_0^e = 0.1$, and it cannot finance over $I = 1$. Therefore, when the firm holds zero cash assets, it cannot raise enough cash to pay the sunk cost

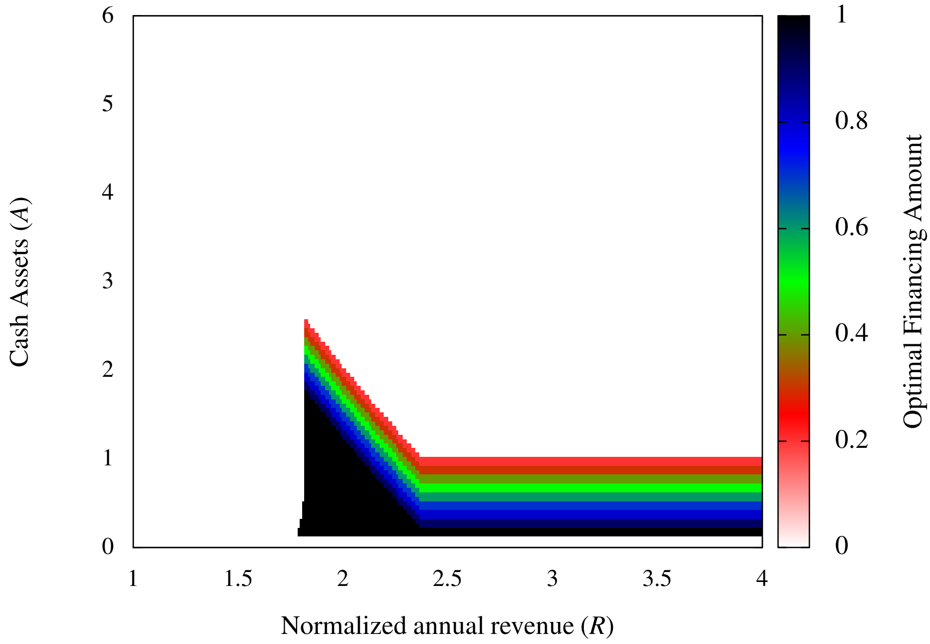


Figure 8.14: The optimal financing amount for an equity-financing-expansion firm with a bounded optimal financing amount $E^F_{max} \leq I$, where the financing cost function is based on the coefficients $(\psi^e_0, \psi^e_1) = (0.1, 0)$. This figure is linked with operational regime Figure 8.13.

$I > E^F - \Psi^e(E^F)$. The non-strict vertical expansion drifts right due to the effects of non-zero financing cost. And the dividend payment boundary reformed and moved downside. This is because financing flexibility reduce the cash constraints, therefore a firm doesn't need to exercise the expansion option via accumulating cash.

Case two: Financing unlimited equity

To show how the limitations of financing amount affect the firm's decision, Figures 8.16 and 8.17 present the optimal operational regimes and financing amount when the firm has an unbounded equity financing option.

According to Figure 8.16(b), we can see that the firm decides to finance and expand the business even it has zero cash assets, and Figure 8.17(b) shows that the optimal financing amount is much higher than the expansion cost I. For example, when the firm holds zero cash with revenue $R = 1.5$, it raised 4 units of cash to exercise the expansion option and increase the liquidity. This observation is consistent with our

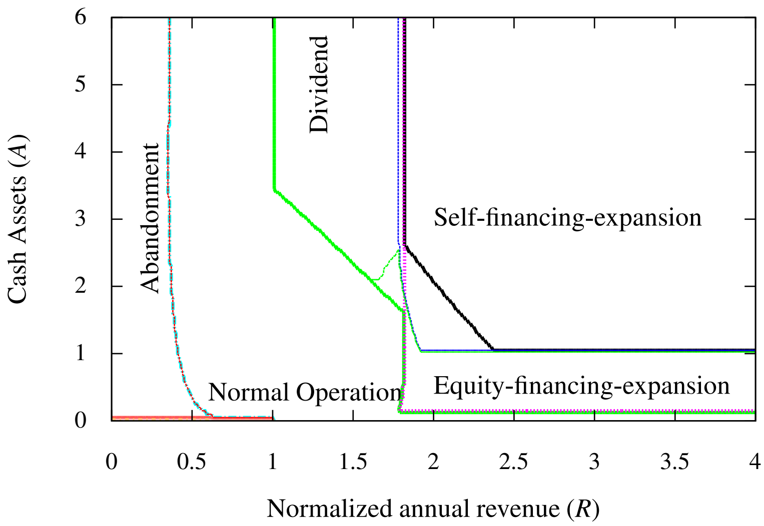
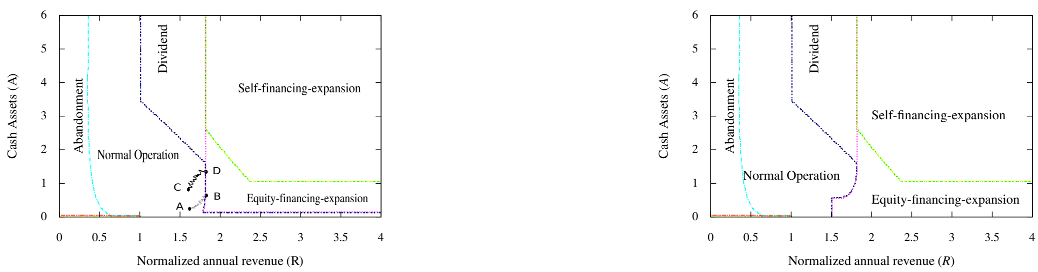


Figure 8.15: Comparisons of the operational strategy between a self-financing-expansion firm and an equity-financing-expansion firm, where the thin lines represent the operational boundaries of the first type of firm, while the thick lines represent the operational boundaries of the second type of firm. This figure is a combination of Figure 8.13 and Figure 8.5.

previous analysis. The motivation to raise an excess amount of capital comes from two aspects: to capture the expansion opportunity when self-holding cash is not enough, and to reduce the future liquidity risks by preparing enough liquid assets. We now show how the financing cost affects the financing-expansion decision.



(a) Operational Regimes for $E^F_{max} \leq I$

(b) Operational Regimes for $E^F_{max} < \infty$

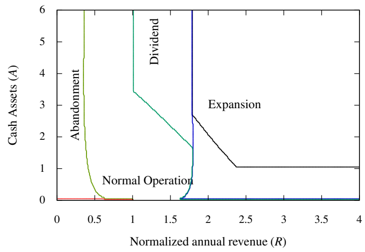
Figure 8.16: A comparison to show how the upper limit of the financing amount affects the operational strategy, where we take $E^F_{max} \leq I$ and $E^F_{max} < \infty$ two cases as examples (E^F_{max} denotes the upper limit of the financing amount).

(a) Financing amount for $E^F_{max} \leq I$ (b) Financing amount for $E^F_{max} < \infty$

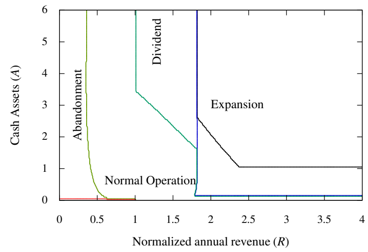
Figure 8.17: A comparison to show how the upper limit of the financing amount affects the operational financing amount, where we take $E^F_{max} \leq I$ and $E^F_{max} < \infty$ two cases as examples.

The effects of frictional cost of equity financing on expansion decision

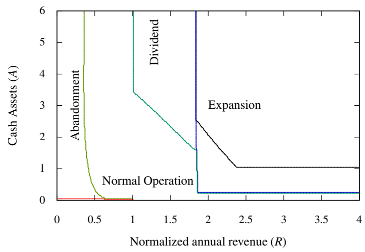
Figures 8.18, 8.19, 8.20 and 8.21 present how the financing cost affects the firm's expansion decision (expansion boundary and optimal financing amount) by varying ψ^e_0 and ψ^e_1 independently. According to these figures, we can see that the financing cost significantly changes the financing-expansion boundaries when cash assets $A < I$, however, it does not affect the abandonment boundary and dividend boundary too greatly. More precisely, an increasing of either constant financing cost or proportional financing cost leads to a right movement of expansion boundary, and a decrease of the optimal financing amount. These observations suggest that the friction cost should be considered in an optimal equity-financing investment problem, therefore, the previous study of optimal investment problems, to some extent, provide misleading results (see Anderson and Carverhill (2012)).



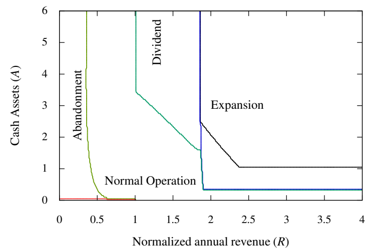
(a) Operational regimes for $\psi_0^e = 0$



(b) Operational regimes for $\psi_0^e = 0.1$

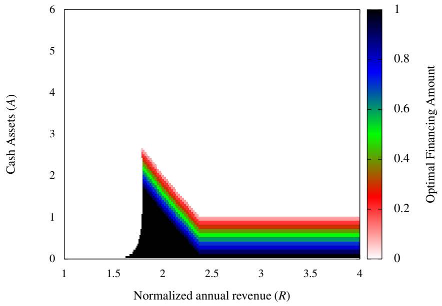


(c) Operational regimes for $\psi_0^e = 0.2$

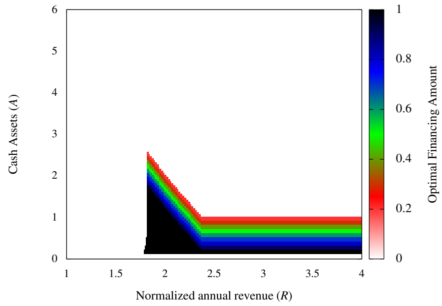


(d) Operational regimes for $\psi_0^e = 0.3$

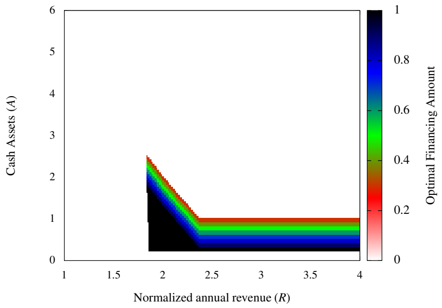
Figure 8.18: The effects of fixed financing cost on expansion decisions by fixing $\psi_1^e = 0$ and varying $\psi_0^e = 0, 0.1, 0.2$ and 0.3 as examples. Here, we take $\epsilon = 1$, $q = 1.5$ and $I = 1$, and new fixed operational cost $\varepsilon(q) = 1.5$.



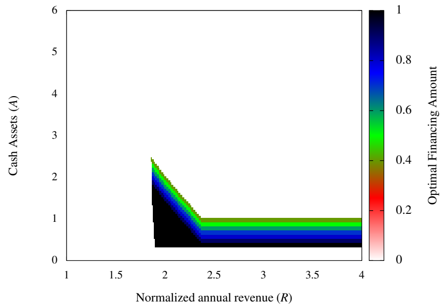
(a) Optimal financing amount for $\psi_0^e = 0$



(b) Optimal financing amount for $\psi_0^e = 0.1$

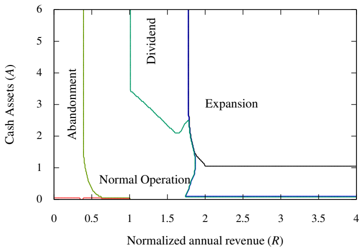


(c) Optimal financing amount for $\psi_0^e = 0.2$

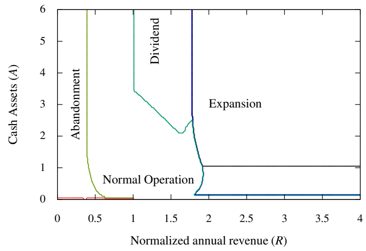


(d) Optimal financing amount for $\psi_0^e = 0.3$

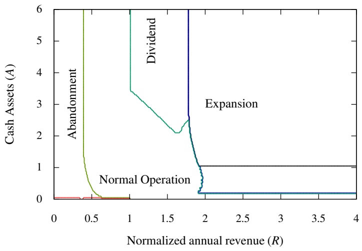
Figure 8.19: The effects of fixed financing cost on optimal financing amount in the equity-financing-expansion decision by fixing $\psi_1^e = 0$ and taking $\psi_0^e = 0, 0.1, 0.2$ and 0.3 as examples. Here, we take $\epsilon = 1$, $q = 1.5$ and $I = 1$, and new fixed operational cost $\varepsilon(q) = 1.5$.



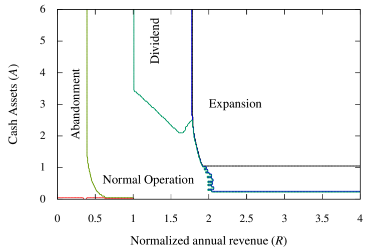
(a) Operational regimes for $\psi_1^e = 0.05$



(b) Operational regimes for $\psi_1^e = 0.10$

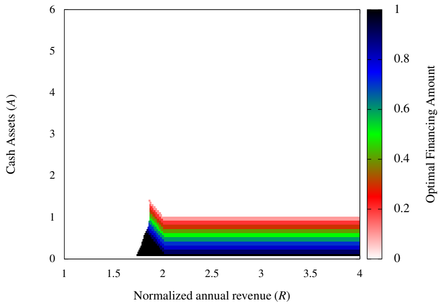


(c) Operational regimes for $\psi_1^e = 0.15$

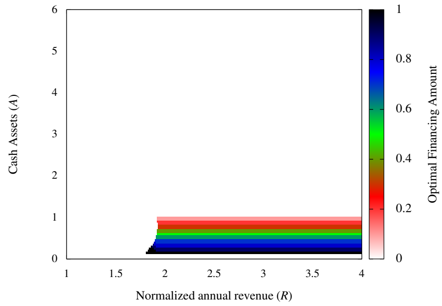


(d) Operational regimes for $\psi_1^e = 0.20$

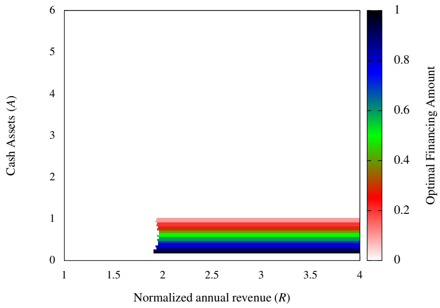
Figure 8.20: The effects of percentage financing cost on expansion decisions by fixing $\psi_0^e = 0.5$ and taking $\psi_1^e = 0.05, 0.1, 0.15$ and 0.2 as examples. Here, we take $\epsilon = 1$, $q = 1.5$ and $I = 1$, and new fixed operational cost $\varepsilon(q) = 1.5$.



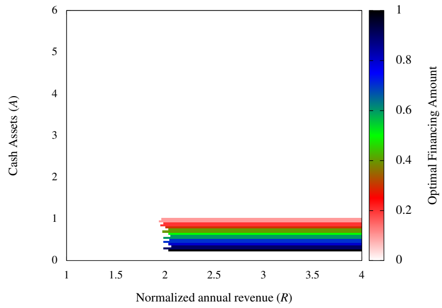
(a) Operational regimes for $\psi_1^e = 0.05$



(b) Operational regimes for $\psi_1^e = 0.10$



(c) Operational regimes for $\psi_1^e = 0.15$



(d) Operational regimes for $\psi_1^e = 0.20$

Figure 8.21: The effects of percentage financing cost on optimal financing amount by fixing $\psi_0^e = 0.5$ and taking $\psi_1^e = 0.05, 0.1, 0.15$ and 0.20 as examples. Here, we take $\epsilon = 1$, $q = 1.5$ and $I = 1$, and new fixed operational cost $\varepsilon(q) = 1.5$.

8.4.3 Optimal Expansion with Cash and Debts

In this subsection, we assume the firm has the flexibility to issue debts to finance the expansion decision. Compared with equity financing, debt financing has two possible costs: the explicit agency expenses and the implicit cost - promised coupon repayments. The second type of the cost mainly depends on three factors, the repayment years, coupon rate and debt amount. Therefore, in this chapter, we will focus on how these criteria affect the debt-financing-expansion decision. Our numerical results can supplement the research in Bolton et al. (2014b), where Bolton did not distinguish the financing sources: debt financing and equity financing.

Suppose all the raised debt has an expiration time from the issuing date to the life end of the project. All the issued debts have a constant coupon rate, which doesn't vary with debt amount and maturity. The parameter values are taken from the Table 8.1. With these setting and assumptions, we can solve the Model (8.6) and get the numerical solutions as follows.

Figure 8.22 shows the optimal operational strategy of a debt-financing-expansion firm at the initial time by taking $r_b = 0.05$. It is interesting to see that the firm chooses to expand the business only with the self-holding cash at the initial time. This solution is not difficult to understand, we use the following example to show the reason. Suppose the firm borrows one unit of cash for eighty years (here, $T = 80$), the total coupon payments are $1 \times e^{(0.05\times 80)} = 54.6$. Since the total payments are 54 times as much as the principle, the firm has no motivation to issue debts for expansion. We now check the firm's optimal financing time and financing amounts.

Figure 8.23(a) shows the optimal debt-financing-expansion time (the time to the expiration date of the project) along the revenue dimension by taking cash assets $A = 0.3, 0.6$ and 0.9 as examples. According to this figure, we can see that the firm's optimal exercising time strongly depends on how many years left to the project expiration time. It (the time to the expiration) first goes up and then goes down as revenue increases, and increases with the increasing of Assets level. These solutions are not difficult to understand. When the revenue is relevantly low, the motivations to

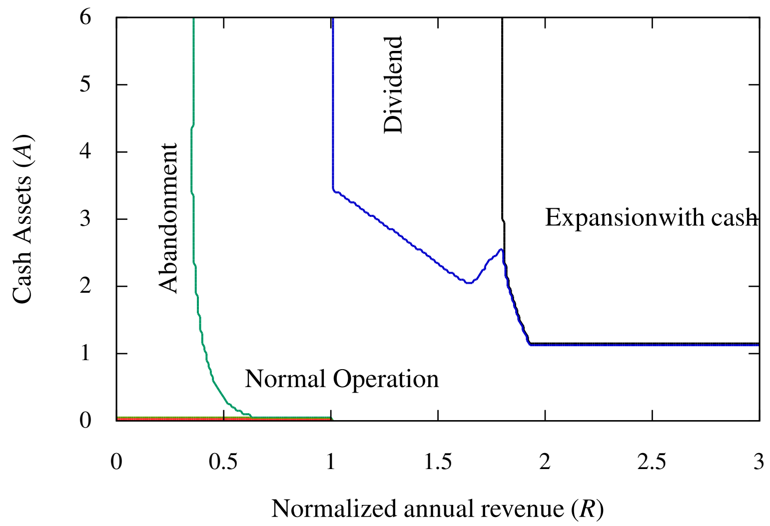
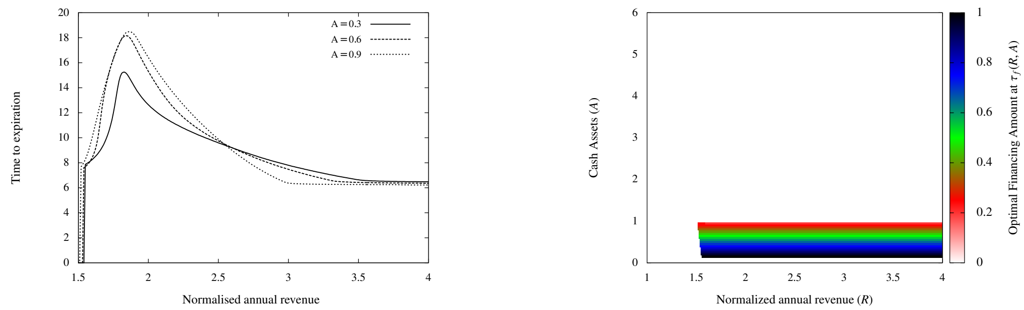


Figure 8.22: The operational regimes of a debt-financing-expansion firm, where we assume a constant financing cost $(\psi_0^d, \psi_1^d) = (0.1, 0)$ and upper limit of the financing amount $D_{max}^F \leq I$.

issue debts come from two aspects: to hedge the future operational risk and to expand the business immediately. However, in cases when revenue is high, the motivation to finance is only to support the expansion decision. Since waiting and accumulating cash to finance expansion is costless compared with debt financing, a firm that has a higher level of cash assets have a preference to wait than to finance external capital.

Figure 8.23(b) shows the corresponding optimal financing amount in $R - A$ space, when the debt-financing-expansion option is exercised. It is interesting to see that the optimal financing amount does not vary with the revenue, and it is even not strongly related to the optimal financing time. The firm always chooses a debt level, with which it is just able to afford the expansion costs I, such that $D - \Psi(D) + A > I$. This is because the firm has the flexibility to choose both financing time and financing amount, and to implement the flexibility on financing time is more frictionless than that on financing amount, particularly when there is agency cost.

We are particularly interested in how the coupon rate affects the firm's debt-financing-expansion decision. By assuming $A = 0.9$, Figure 8.24 shows how the optimal financing time varies along the revenue by implementing different coupon rate $r_b =$



(a) Financing time (b) Financing amount

Figure 8.23: The optimal financing time and the financing amount for a debt-financing-expansion firm. Here, we take cash assets $A = 0.3, 0.6$ and 0.9 as examples to show the characteristics of the financing time, and present the optimal financing amount at every operational state (R, A).

$0.05, 0.07, 0.09$ and 0.11. According to this figure, we can see that the optimal financing time monotonically decreases as r_b increases, particularly when the revenue is relevantly low. This is because higher coupon rate r_b means more repayments. Thus, a firm that has to issue a higher coupon rate debt would like to wait and accumulate cash to finance the expansion decision than via raising expansive external capital, which results in a shorter optimal financing time.

Figures 8.25 compares how different coupon rate affects the optimal financing amount. It is interesting to see that the financing amount region shrinks as r_b increases. However, the optimal financing amount has no significant change for different coupon rate. The reason is that the firm has a preference to find an optimal operational strategy via controlling the financing time, instead of financing amount, since increasing financing amount is more costly.

8.5 Conclusion

In this chapter, we studied how the financial constraints affect a firm's expansion decision. We derived a general optimal regime switching model for a firm's expansion flexibility, where we allow the firm to finance with internal cash, external equity and debts. Particularly, we present the numerical solutions of the optimal expansion with self-holding cash, optimal expansion with equity financing and optimal expansion with

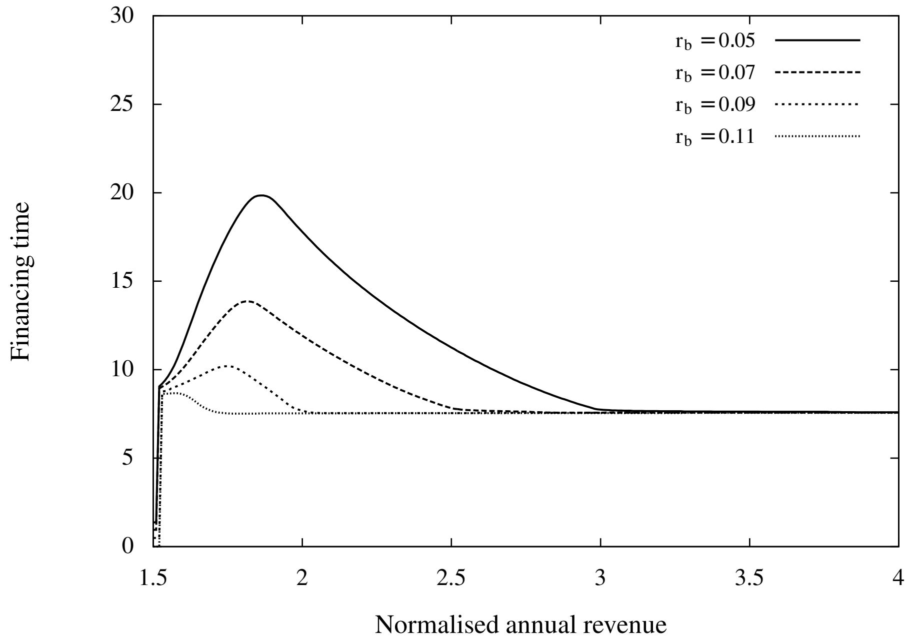
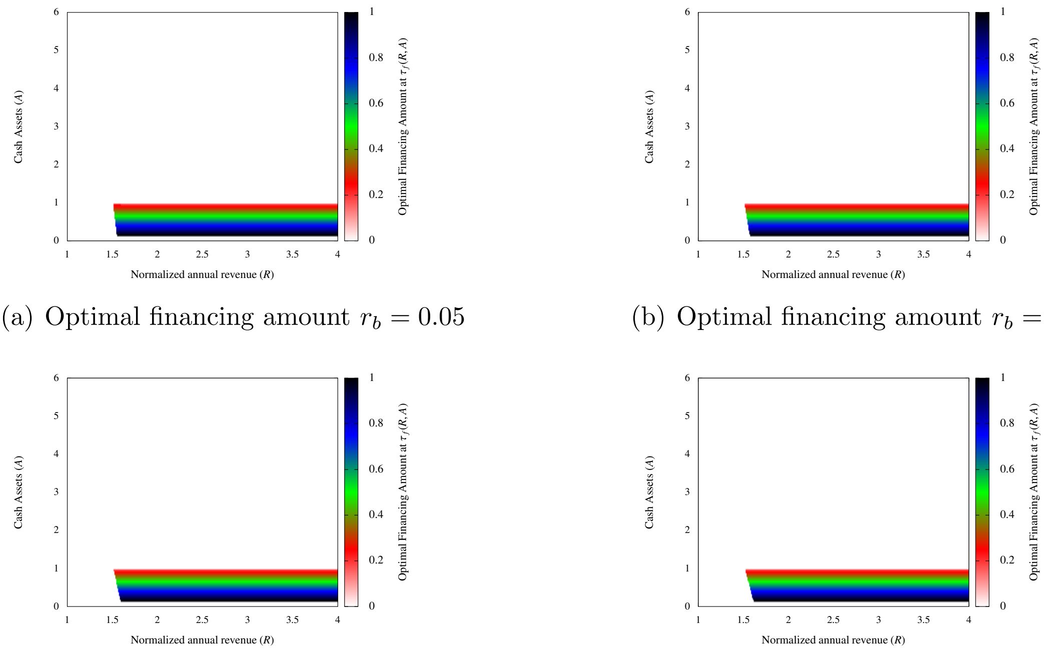


Figure 8.24: The effects of coupon rate on the optimal financing time (the time to the expiration of the project) for a debt-financing-expansion firm by fixing $A = 0.9$ and taking coupon rate $r_b = 0.05, 0.07, 0.09$ and 0.11 as examples. Here, we assume a constant financing cost $(\psi_0^d, \psi_1^d) = (0.1, 0)$ and limited maximum financing amount $D_{max}^F \leq I$.

debt financing. Based on our analysis, we found that the financial constraints have significant effects on expansion flexibility. People not only need to finance to pay the expansion cost but also need to prepare extra cash for the future operation. Both equity and debt financing can reduce the financial constraints on the investment opportunity, however, the external financing cost reduces the benefits. When a firm has the flexibility to raise external capital, it is potentially more active in exercising the Expansion Option, this is particularly significant when the firm is allowed to finance with equity. In addition to this, the features of interdependency between different operational flexibilities were well presented to support the empirical observations shown in the introduction. One more interesting observation is the non-monotonic effect of volatility on the multiple interactive options. We showed that an increase of volatility can reduce the value of expansion when the firm is in the very profitable situation.



(a) Optimal financing amount $r_b = 0.05$

(b) Optimal financing amount $r_b = 0.07$

(c) Optimal financing amount $r_b = 0.09$

(d) Optimal financing amount $r_b = 0.11$

Figure 8.25: The effects of coupon rate on the optimal financing amount for a debt-financing-expansion firm by taking $r_b = 0.03, 0.05, 0.07$ and 0.09 as examples. Here, we fix $(\psi_0^d, \psi_1^d) = (0.1, 0)$, $A = 0.9$, $\epsilon = 1$, $q = 1.5$ and $I = 1$, and new fixed operational cost $\varepsilon(q) = 1.5$.

Chapter 9

Conclusions

This thesis explored the significant effects of cash on corporate valuation and operational strategy. We provided an extendable mathematical framework for multiple interdependent management flexibilities. It captures several important features, namely the optimal abandonment decision, dividend redistribution decision, equity financing decision, debt financing decision, investment decisions. We also presented how to find numerical solutions for each problem based on the PSOR and Semi-Lagrangian Methods. Here, we show the main findings and discuss avenues for future research.

9.1 Main Results and Contributions

Classically, investment under uncertainty problems can be valued via Real Option approaches with the assumption that there are no cash constraints (see Dixit (1994) and Myers and Majd (2001)). Recent researches from both theoretical and empirical perspectives show that cash constraints play a significant role in corporate decisions (see theory work Faulkender and Wang (2006) and Kisser (2013); and empirical work Denis and Sibilkov (2009) Fresard (2010) and Nikolov and Whited (2014)). The first study in Chapter 4 relaxes the assumption and investigated how the cash holdings affect a firm's abandonment decisions. According to our analysis, we found that classical Real

Options generally overvalued a firm's decision due to the ignorance of the cash constraints. We showed that a firm should abandonment its business at a higher threshold (revenue level) compared with the classical Real Option solution when the firm has limited cash holdings. The effects of financial constraints will be expanded when the volatility increase and discount rate decrease. Our research and results are consistent with other work (for examples, Décamps and Villeneuve (2013) and Hugonnier et al. (2014)).

In corporate finance studies, the shareholders benefits are believed to be the expected dividend payment, thus the dividend payout policy significantly affects the shareholders' long term return. Recent studies on optimal dividend payment model generally assumes that the firm itself to be a stochastic process (see Décamps, Jean-Paul and Villeneuve, Stéphane (2007), Belhaj (2010), Jiang and Pistorius (2012) and Chevalier et al. (2013)). This assumption fails to involve the interaction between the dividend policy with the firm's liquidity risk (see the discussion in Gryglewicz (2011)). The research in Chapter 5 focuses on how to value the shareholders' benefits based on the firm's revenue and cash holdings. Particularly, we explicitly present how the marginal value of cash affect the firm's dividend payout policy. The contribution of our work can be summarized as follows: We showed an efficient way to solve the path depended model for this stochastic control and optimal stopping boundary; We found that the optimal dividend payment threshold is more sensitive to the revenue volatility when the firm holds low level of cash; the dividend payment boundary on the cash holding dimension is positively correlated with the firm's profit level and time to expiration, and it is negative correlated with the firm's discount rate.

The external financing flexibility is a significant factor in studying a firm's liquidity risk (see the discussion particularly in Fama and French (1998), Gamba and Triantis (2008) and Rapp et al. (2014)). We investigated the optimal financing problems from two aspects: the equity financing and debt financing, which supplement the literature. More precisely, we studied three types of equity financing options by gradually increasing the financing flexibility; we showed that a firm that has no flexibility to choose a financing time would raise an amount of money at the initial time such that the marginal value of increasing one more unit of capital equals the marginal cost of doing so. This observation is quite different with existing studies (see Gamba and Triantis (2008), Hugonnier et al. (2014) and Hugonnier et al. (2014)). In the debt financing problem, we particular discussed how to find the market coupon rate (one important financing cost) via a game structure. According to our analysis, we showed that the firm can reduce its cash constraints via debt financing option and this flexibility increases the firm value to some extend. In addition, the benefit of issuing debt is strongly limited by the financing costs. For the debt investors, the return they can expect not only depends on the firm's current operational state, but also depends on the debt-financing level, since it affects the firm's future default risk. The debt investors can benefit from the debt financing, too, when this action can significantly reduce the corporate cash constraints. However, with the increasing financing level, the benefits are counteracted by the potential default risk due to the increasing leverage ratio and operational costs. We also found that when the firm is in a low liquid asset state, the shareholders are very likely to over-distribute the dividends due to their limited liability. In this situation, the corporate risks continuously transfer from the shareholders to the debt investors, whereas the benefits are opposite, moving from the debt investors to the shareholders.

The analysis of Chapter 8 confirmed that cash constraints do reduce expansion flexibility when there are expansion costs. Both equity and debt financing can reduce the cash constraints on the investment opportunity, however, the economic frictions (financing cost) will reduce the benefits. When a firm has the flexibility to raise external capital, it is potentially more active in exercising the Expansion Option, which is particularly significant when the expansion decision is financed with equity. These observations are consistent with the current literature on optimal investment (see Brealey et al., 2010; Vollert, 2012; Chen et al., 2012; Bolton et al., 2014b). Apart from the above observations, we found that a cash constrained firm would finance more than the required amount to cover the expansion cost, since they need extra cash as liquid buffers to hedge future operational risk. In addition to this, the features of interdependency between different operational flexibilities were well presented to support the empirical observations shown in the introduction. One more interesting observation is the non-monotonic effect of volatility on the multiple interactive options. We showed that an increase of volatility can reduce the value of expansion when the firm is in the very profitable situation.

9.2 Future Studies

The thesis presented a framework that can be extended for more corporate finance topics. Focusing on the setup of each topic we discussed, there are some insightful problems left to be further investigated by future work.

The first natural extension is to adapt our modelling to different types of revenue processes. We know the stochastic features of a firm's revenue describes the uncertainty of industry. In our work, the firm's revenue was assumed to be a Geometric Brownian Motion. This type of revenue only models the business cases when the expected income of the firm grows exponentially. However, from practice perspective, there are other attractive stochastic processes, for example, the log-normal Mean Reversion Processes (see Glover and Hambusch (2012)), which are widely accepted in the energy industry, and the Geometric Brownian Motion with Compounded Poisson Jumps (see Belhaj (2010)), which were recommended when the distribution of the revenue has a fat tail, and also other stochastic processes that are designed for particular requirements. Implementing different stochastic processes makes it more attractive to practitioners.

Next, more realistic factors can be involved to relax our assumptions on the models, for instance, the tax ratios, nonlinear cost functions, and liquidation cost. These settings do not change the main structure of the model, but they offer more flexibility so that we would be able to investigate different aspects such as the Trade-off Theory, the Pecking-Order Theory and the Agency Conflicts[1] under considerations of cash constraints. The recent studies of these basic financial theory can be found in Anderson and Carverhill (2012) and J. Glover and Hambusch (2014). They assume the firm follows stochastic process, which is not able to analyse the cash flow effects and liquidity risk effects on the corporate decisions. The extension of our model would allows people to have a deep insight from these aspects.

[1]The fundamental definition of these theory can be found in the book Brealey et al. (2010)

Also, in Chapter 7, we did not completely solve the dynamic optimal debt financing problems due to the limitation of computational costs. For example, we did not consider the optimal choosing of debt expiration time in our study. People who are interested can extend our study for the dynamic optimal debt financing problems. To achieve this, other more efficient numerical techniques or an alternate mathematical approach should be applied in order to reduce the calculation costs.

Lastly, one could extend our model to solve a specific problem generated by a certain industrial sector. For example, we have implemented and extended the framework of the optimal equity financing firm in Chapter 6 to value a deposit-taking bank and studied how the regulation affects a firm's default risk.

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