Characterizing minimally flat symmetric hypergraphs

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Abstract

In [2] we gave necessary conditions for a symmetric d-picture (i.e., a symmetric realization of an incidence structure in $\mathbb{R}^d$) to be minimally flat, that is, to be non-liftable to a polyhedral scene without having redundant constraints. These conditions imply very simply stated restrictions on the number of those structural components of the picture that are fixed by the elements of its symmetry group. In this paper we show that these conditions on the fixed structural components, together with the standard non-symmetric counts, are also sufficient for a plane picture which is generic with three-fold rotational symmetry $\mathcal{C}_3$ to be minimally flat. This combinatorial characterization of minimally flat $\mathcal{C}_3$-generic pictures is obtained via a new inductive construction scheme for symmetric sparse hypergraphs. We also give a sufficient condition for sharpness of pictures with $\mathcal{C}_3$ symmetry.

Keywords: incidence structure, picture, polyhedral scene, lifting, symmetry, sparse hypergraph

1. Introduction

1.1. Background and motivation

The vertical projection of a spatial polyhedral scene with flat faces yields a straight line drawing of the corresponding incidence structure in the projection plane. Conversely, given an incidence structure S and a straight line

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drawing of S in the plane, one may ask whether this drawing can be 'lifted' to a polyhedral scene, i.e., whether it is the vertical projection of a spatial polyhedral scene. This is a well studied question in Discrete Geometry which has some beautiful connections to areas such as Geometric Rigidity Theory and Polytope Theory [6, 7, 8, 9, 10]. Moreover, this problem has important applications in Artificial Intelligence, Computer Vision and Robotics [4, 5].

A fundamental result in Scene Analysis is Whiteley's combinatorial characterization of all incidence structures which are 'minimally flat' if realized generically in the plane, where a realization of an incidence structure is called minimally flat if it is non-liftable to a spatial polyhedral scene, but the removal of any incidence yields a liftable structure. This characterization was conjectured by Sugihara in 1984 [3] and proved by Whiteley in 1989 [7], and it is given in terms of sparsity counts on the number of vertices, faces and incidences of the given incidence structure.

Since symmetry is ubiquitous in both man-made structures and structures found in nature, it is natural to consider the impact of symmetry on the liftability properties of straight line drawings of incidence structures. Recently, we used methods from group representation theory to derive additional necessary conditions for a symmetric realization of an incidence structure to be minimally flat [2]. These conditions can be formulated in a very simple way in terms of the numbers of vertices, faces and incidences that are fixed under the various symmetries of the structure. We conjectured in [2] that these added conditions, together with the standard Sugihara-Whiteley counts are also sufficient for a symmetric incidence structure to be minimally flat, provided that it is realized generically with the given symmetry group.

In this paper we verify this conjecture for the symmetry group $\mathcal{C}_3$ which is generated by a three-fold rotation (i.e., a rotation by 120 degrees) in the plane. This result is obtained via a new symmetry-adapted recursive construction for symmetric sparse hypergraphs. Moreover, we give a sufficient condition for $\mathcal{C}_3$-symmetric generic incidence structures to lift to a sharp polyhedral scene (i.e., a scene where each pair of faces sharing a vertex lie in separate planes). Finally, we provide some observations regarding extensions of these results to other symmetry groups in the plane.

1.2. Basic definitions

A *(polyhedral) incidence structure* S is an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subseteq V \times F$.

A $(d-1)$*-picture* is an incidence structure S together with a corresponding location map $r : V \to \mathbb{R}^{d-1}$, $r_i = (x_i, y_i, \ldots, w_i)^T$, and is denoted by $S(r)$.

A *d-scene* $S(p, P)$ is an incidence structure $S = (V, F; I)$ together with a pair of location maps, $p : V \to \mathbb{R}^d$, $p_i = (x_i, \ldots, w_i, z_i)^T$, and $P : F \to \mathbb{R}^d$, $P^j = (A^j \ldots, C^j, D^j)^T$, such that for each $(i, j) \in I$ we have $A^j x_i + \ldots + C^j w_i + z_i + D^j = 0$. (We assume that no hyperplane is vertical, i.e., is parallel to the vector $(0, \ldots, 0, 1)^T$.)

A *lifting* of a $(d-1)$-picture $S(r)$ is a d-scene $S(p, P)$, with the vertical projection $\Pi(p) = r$. That is, if $p_i = (x_i, \ldots, w_i, z_i)^T$, then $r_i = (x_i, \ldots, w_i)^T = \Pi(p_i)$.

A lifting $S(p, P)$ is *trivial* if all the faces lie in the same plane. Further, $S(p, P)$ is *folded* (or *non-trivial*) if some pair of faces have different planes, and is *sharp* if each pair of faces sharing a vertex have distinct planes. A picture is called *sharp* if it has a sharp lifting. Moreover, a picture which has no non-trivial lifting is called *flat* (or *trivial*). A picture with a non-trivial lifting is called *foldable*.

The *lifting matrix* for a picture $S(r)$ is the $|I| \times (|V| + d|F|)$ coefficient matrix $M(S, r)$ of the system of equations for liftings of a picture $S(r)$: For each $(i, j) \in I$, we have the equation $A^j x_i + B^j y_i + \ldots + C^j w_i + z_i + D^j = 0$, where the variables are ordered as $[\ldots, z_i, \ldots; \ldots, A^j, B^j, \ldots, D^j, \ldots]$. That is the row corresponding to $(i, j) \in I$ is:

$$(i,j)\quad \left[\ \underbrace{0\ldots0\ \ \overset{i}{1}\ \ 0\ldots0}_{|V|}\ \middle\|\ \underbrace{0\ldots0\ \ \overbrace{r_i\ \ 1}^{j}\ \ 0\ldots0}_{d|F|}\ \right]$$

A $(d-1)$-picture $S(r)$ is called *generic* if for every $r' : V \to \mathbb{R}^{d-1}$, the rank of every square submatrix of the lifting matrix $M(S, r)$ is greater than or equal to the rank of the corresponding submatrix of $M(S, r')$. So in particular, $M(S, r)$ has maximum rank among all lifting matrices $M(S, r')$.

Theorem 1.1 (Picture Theorem). *[7, 9] A generic $(d-1)$-picture of an incidence structure $S = (V, F; I)$ with at least two faces has a sharp lifting, unique up to lifting equivalence, if and only if $|I| = |V| + d|F| - (d+1)$ and $|I'| \leq |V'| + d|F'| - (d+1)$ for all subsets I' of incidences inducing the vertex set $V' \subseteq V$ and face set $F' \subseteq F$ with $|F'| \geq 2$.*

A generic picture of S has independent rows in the lifting matrix if and only if for all non-empty subsets I' of incidences, we have $|I'| \leq |V'| + d|F'| - d$.

It follows from the Picture Theorem that a generic $(d-1)$-picture of an incidence structure $S = (V, F; I)$ is flat with independent rows in the lifting matrix if and only if $|I| = |V| + d|F| - d$ and $|I'| \leq |V'| + d|F'| - d$ for all non-empty subsets I' of incidences. The removal of any incidence from such a picture results in a foldable picture, and hence we call such a picture *minimally flat*. We may think of the minimally flat generic pictures $S(r)$ as the bases of the row matroid of the lifting matrix $M(\tilde{S}, r)$ of a flat picture $(\tilde{S}, r)$ on an incidence structure $\tilde{S} = (V, F; \tilde{I})$ with $I \subseteq \tilde{I}$. Therefore, the Picture Theorem may be considered as an analog of the celebrated Laman's Theorem in Geometric Rigidity Theory, which gives a description of the bases of the generic rigidity matroid of a rigid graph in dimension 2 [9, 10].

1.3. Symmetric incidence structures and pictures

An *automorphism* of an incidence structure $S = (V, F; I)$ is a pair $\alpha = (\pi, \sigma)$, where π is a permutation of V and σ is a permutation of F such that $(v, f) \in I$ if and only if $(\pi(v), \sigma(f)) \in I$ for all $v \in V$ and $f \in F$. For simplicity, we will write $\alpha(v)$ for $\pi(v)$ and $\alpha(f)$ for $\sigma(f)$.

The automorphisms of S form a group under composition, denoted $\mathrm{Aut}(S)$. An *action* of a group Γ on S is a group homomorphism $\theta : \Gamma \to \mathrm{Aut}(S)$. The incidence structure S is called Γ-*symmetric* (with respect to θ) if there is such an action. For simplicity, if θ is clear from the context, we will sometimes denote the automorphism $\theta(\gamma)$ simply by γ.

Let Γ be an abstract group, and let S be a Γ-symmetric incidence structure (with respect to θ). Further, suppose there exists a group representation $\tau : \Gamma \to O(\mathbb{R}^{d-1})$. Then we say that a picture $S(r)$ is Γ-*symmetric* (with respect to θ and τ) if

$$\tau(\gamma)(r_i) = r_{\theta(\gamma)(i)} \text{ for all } i \in V \text{ and all } \gamma \in \Gamma. \tag{1}$$

In this case we also say that $\tau(\Gamma) = \{\tau(\gamma) | \, \gamma \in \Gamma\}$ is a *symmetry group* of $S(r)$, and each element of $\tau(\Gamma)$ is called a *symmetry operation* of $S(r)$.

Let Γ be a group, and let $S(r)$ be a Γ-symmetric $(d-1)$-picture (with respect to θ and τ). Then $S(r)$ is said to be Γ-*generic* if for every Γ-symmetric $(d-1)$-picture $S(r')$ (with respect to θ and τ), the rank of every square submatrix of the lifting matrix $M(S, r)$ is greater than or equal to the rank of the corresponding submatrix of $M(S, r')$. Clearly, the set of all Γ-generic realizations of S is a dense and open subset of all Γ-symmetric realizations of S. Moreover, all Γ-generic realizations of S share the same lifting properties. We say that S is Γ-*generically (minimally) flat* in $\mathbb{R}^{d-1}$ if all Γ-generic realizations of S in $\mathbb{R}^{d-1}$ are (minimally) flat.

$|V_n(S)|$, $|F_n(S)|$, and $|I_n(S)|$ denote the numbers of vertices, faces, and incidences of S that are fixed by an n-fold rotation C_n, $n \geq 2$, respectively.

Similarly, $|V_s(S)|$, $|F_s(S)|$, and $|I_s(S)|$ denote the numbers of vertices, faces, and incidences that are fixed by a reflection s. The incidence structure S may be dropped from this notation if it is clear from the context.

In the following, $\mathcal{C}_n$ ($\mathcal{C}_s$) denotes the group generated by an n-fold rotation (reflection, respectively). We also use the notation $\mathcal{C}_3 = \{id, \gamma, \gamma^2\}$, that is, γ denotes a three-fold rotation.

2. Symmetry extended counting rule for 2-pictures

The following theorem gives necessary conditions for a $(d-1)$-picture to be minimally flat.

Theorem 2.1. *[2] Let $S(r)$ be a $(d-1)$-picture which is Γ-symmetric with respect to θ and τ. If $S(r)$ is minimally flat, then we have*

$$\chi(P_I) = \chi(P_V \oplus (\hat{\tau} \otimes P_F)) - \chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})}). \tag{2}$$

We refer the reader to [2] for further details.

From Theorem 2.1 we obtain the following necessary conditions for a Γ-symmetric 2-picture (with respect to θ and τ) to be minimally flat:

identity: $$|I| = |V| + 3|F| - 3 \tag{3}$$

half-turn: $$|I_2| = |V_2| - |F_2| + 1 \tag{4}$$

reflection: $$|I_s| = |V_s| + |F_s| - 1 \tag{5}$$

n-fold rotation, $n > 2$: $$|I_n| = |V_n| + (|F_n| - 1)\Big(1 + 2\cos\frac{2\pi}{n}\Big) \tag{6}$$

where a given equation applies when the corresponding symmetry operation is present in $\tau(\Gamma)$. We will call (3), (4), (5) and (6) the *symmetry extended counting rule*. In [2] we conjectured that the symmetry extended counting rule together with the standard (non-symmetric) sparsity condition is sufficient for a symmetric picture to be minimally flat.

Conjecture 2.2. *[2] A Γ-generic $(d-1)$-picture $S(r)$ is minimally flat if and only if*

(i) $|I| = |V| + d|F| - d$ *and* $|I'| \leq |V'| + d|F'| - d$ *for all nonempty subsets* I' *of incidences with* $|F'| \geq 2$*;*

(ii) S *satisfies the conditions for* Γ *in the symmetry extended counting rule;*

(iii) For every subset I' of I which induces a Γ'-symmetric incidence structure S' with $|I'| = |V'| + d|F'| - d$ (where $\Gamma' \subseteq \Gamma$), S' satisfies the conditions for Γ' in the symmetry extended counting rule.

In the present paper we prove Conjecture 2.2 for $d = 3$ and $\Gamma = \mathcal{C}_3$. For the group $\mathcal{C}_3$ the symmetry extended counting rule simplifies to (3) and to

$$|I_3| = |V_3| \tag{7}$$

which is the special case of (6) for $n = 3$. Note that for $\Gamma = \mathcal{C}_3$ condition (ii) implies (iii). There are two cases. First, if $|V_3(S)| = 0$ then $|I_3(S)| = 0$ must hold, and this implies that $|V_3(S')| = |I_3(S')| = 0$ for every $\mathcal{C}_3$-symmetric substructure. If $|V_3(S')| = 1$, then $|I_3(S')| = 1$, because if for a $\mathcal{C}_3$-symmetric substructure S' which contains the fixed vertex, $|I'| = |V'| + 3|F'| - 3$ holds, then S' must contain the fixed incidence, too. Thus S' satisfies the symmetry-extended counting rule in both cases.

The following example shows that there exist incidence structures that are minimally flat in the generic setting but become foldable if realized as $\mathcal{C}_3$-symmetric pictures. Let $V = \{v_*, v_0, \ldots, v_{11}\}$ and let F have two different types of faces. The fixed faces are $f_i = \{v_*, v_i, v_{i+4}, v_{i+8}\}$ for $0 \leq i \leq 3$ and the rest of the faces have the form $g_j = \{v_{2j-2}, v_{2j-1}, v_{2j}, v_{2j+1}\}$ for $1 \leq j \leq 6$, where we compute modulo 12. This example is minimally flat when realized as a generic picture by Theorem 1.1 but does not satisfy (7), and hence is $\mathcal{C}_3$-symmetrically foldable.

3. Constructive characterization of $\mathcal{C}_3$-tight hypergraphs

In order to characterize minimally flat $\mathcal{C}_3$-symmetric incidence structures in the plane we will first reduce the problem to the special case where every face of the incidence structure S is incident with exactly four vertices.

3.1. Notation

We first introduce some basic notation. Let $H = (V, F)$ be a hypergraph. For a set $X \subseteq V$ let $H[X]$ denote the subhypergraph induced by the set X. The number of hyperedges in $H[X]$ is denoted by $e_H(X)$. The degree of a vertex v in V (i.e., the number of hyperedges in F incident with v) is denoted by $d_H(v)$. More generally, for a subset $W \subseteq V$, we define $d_H(W)$ to be the number of hyperedges in F that are incident with at least one vertex in W. The set of neighbours of v in a hypergraph H is denoted by $N_H(v)$. $d(z, v)$ denotes the number of hyperedges containing both z and v. The *deficiency* of $X \subseteq V$ in a hypergraph H is the value $|X| - 3 - e_H(X)$, and is denoted by $\mathrm{def}_H(X)$. If $|X| \leq 3$, then the deficiency of X is simply $|X| - 3$. The subscripts may be omitted if the hypergraph is clear from the context.

For a Γ-symmetric picture (H, r) and $v \in V$, $\gamma \in \Gamma$, γv denotes the vertex $u \in V$ for which $\tau(\gamma)(r(v)) = r(u)$. Similarly, for $X \subseteq V$ $\gamma X = \{\gamma v : v \in X\}$, while for $X \subseteq V$ and $\Gamma' \subseteq \Gamma$, $\Gamma' X = \{\gamma v : v \in X, \gamma \in \Gamma'\}$.

Throughout this section v_0 denotes the fixed vertex and f_0 denotes the fixed hyperedge. (Note that v_0 and f_0 may not exist).

3.2. Symmetric derived 4-hypergraphs

Let $S = (V, F; I)$ be a $\mathcal{C}_3$-symmetric incidence structure. We may assume that each face has at least 4 vertices, because any face with fewer vertices does not give rise to any constraints on the possible liftings of a 2-dimensional picture $S(r)$, and hence may be discarded. We will define the *$\mathcal{C}_3$-symmetric 4-hypergraph* $H_3(S) = (V, \bigcup \mathcal{C}_3 E_j)$ of S as follows. Fix an ordering of the vertex orbits of S under the $\mathcal{C}_3$ action. Choose a representative element from every face orbit. For a representative element $f_j \in F$ on the vertices $v_1, v_2, v_3, \ldots, v_{3+m}$ which is not a fixed face, the set E_j consists of the edges v_1, v_2, v_3, v_{3+k} for $1 \leq k \leq m$. If f_j is a fixed face then we can assume that it contains $\mathcal{C}_3 v_1$ where v_1 is not a fixed vertex. In this case E_j consists of $|f_j| - 3$ 4-tuples of the form $\{\mathcal{C}_3 v_1, v_i\}$ for every $v_i \in f_j - \mathcal{C}_3 v_1$.

Lemma 3.1. *Let S be a $\mathcal{C}_3$-symmetric incidence structure. The following are equivalent:*

(i) *S satisfies $|I| = |V| + 3|F| - 3$, $|I'| \leq |V'| + 3|F'| - 3$ for every subset of incidences $|I'|$ with at least one face and $|I_3(S)| = |V_3(S)|$.*

(ii) *$H_3(S)$ is $(1, 3)$-tight*[1] *and $|I_3(H_3(S))| = |V_3(H_3(S))|$.*

Proof: The number of fixed vertices and fixed incidences does not change during the modification, and hence $|I_3(S)| = |V_3(S)|$ holds if and only if $|I_3(H_3(S))| = |V_3(H_3(S))|$ holds. It follows from a simple calculation that the conditions $|I| = |V| + 3|F| - 3$, $|I'| \leq |V'| + 3|F'| - 3$ for every subset of incidences $|I'|$ with at least one face are equivalent to $|I(H_3(S))| = |V(H_3(S))| + 3|F(H_3(S))| - 3$, $|I(H')| \leq |V(H')| + 3|F(H')| - 3$ for every subhypergraph H' of $H_3(S)$ with at least one hyperedge. Then using

[1] A hypergraph $H = (V, F)$ is called $(1, 3)$-*sparse* or *sparse* for short if $|F'| \leq |V(F')| - 3$ for every $\emptyset \neq F' \subseteq F$. H is called $(1, 3)$-*tight* or simply *tight* if it is (1,3)-sparse and satisfies $|F| = |V| - 3$.

the fact that $H_3(S)$ is 4-uniform we get that the latter is equivalent to the $(1,3)$-tightness of $H_3(S)$. $\square$

We will say that a 4-uniform hypergraph H is $\mathcal{C}_3$-*tight* if it is (1,3)-tight and satisfies $|I_3(H)| = |V_3(H)|$.

3.3. A constructive characterization for 4-uniform $(1,3)$*-tight hypergraphs*

In this section we define the operations used for constructing (non-symmetric) 4-uniform $(1,3)$-tight hypergraphs and summarize the results in [1].

Let $H = (V, E)$ be a 4-uniform hypergraph and let $v \in V$ be a vertex with $d(v) \geq j$. The *j-extension operation at vertex v* picks j hyperedges $e_1, e_2, ..., e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size 4 incident with both v and z, and replaces e_i by $e_i - v + z$ for all $1 \leq i \leq j$. Thus the new vertex z has degree $j+1$ in the extended hypergraph. Note that a 0-extension operation simply adds a new vertex z and a new hyperedge of size 4 incident with z.

The inverse of the j-extension operation can be described as follows. Let z be a vertex with $d(z) = j+1$ and let v be a neighbour of z with $d(z, v) = 1$. Let $e, e_1, ..., e_j$ be the edges incident with z, where e is the edge which is incident with v, too. The *j-reduction operation at vertex z with neighbour v* deletes e and replaces e_i by $e_i - z + v$ for all $1 \leq i \leq j$. A j-reduction is called *admissible* if the hypergraph obtained as the result of the j-reduction is (1,3)-sparse. See Figure 1 for examples.

Theorem 3.2. *[1] Let $H = (V, E)$ be a 4-uniform hypergraph. H is (1,3)-tight if and only if it can be obtained from a single hyperedge of size four by a sequence of j-extensions, where $0 \leq j \leq 2$.*

We shall also use the next lemma which is the key in the proof of Theorem 3.2.

Lemma 3.3. *[1] Let $H = (V, E)$ be a $(1,3)$-tight 4-uniform hypergraph and let $z \in V$ be a vertex with $d(z) = j$ for some $1 \leq j \leq 3$. Then there is an admissible j-reduction at z.*

3.4. Preliminary lemmas

Lemma 3.4. *If H is a $(1,3)$-tight 4-uniform hypergraph then H has at least four vertices with degree at most three. Furthermore if there are exactly four vertices with degree at most three, then they must have degree one.*

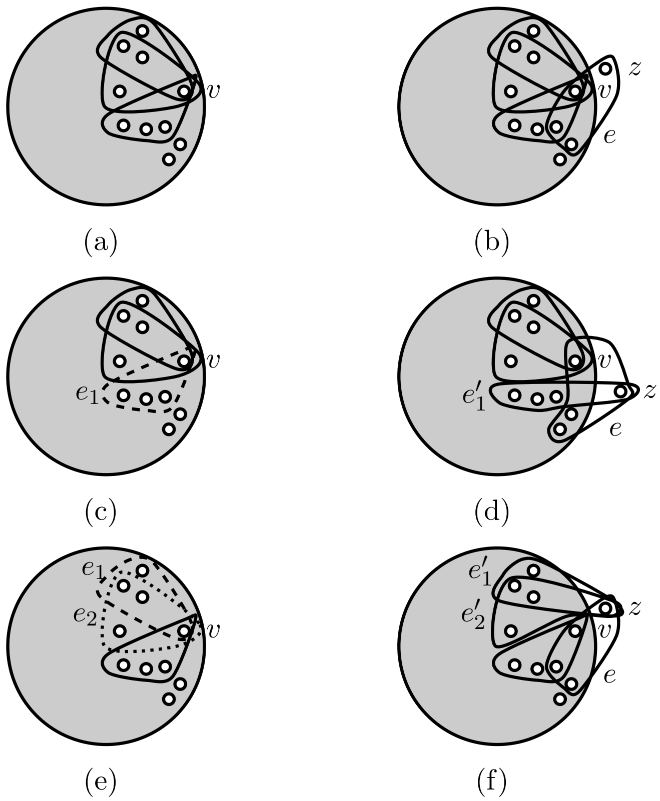

Figure 1: Extensions performed at a degree three vertex v of a 4-uniform hypergraph H. When there are no hyperedges chosen (a) the 0-extension adds a new vertex z and a new hyperedge incident with both v and z (b). Let the unique chosen hyperedge be e_1, denoted with a dashed line (c). The 1-extension adds a new vertex z, replaces v with z in e_1 and leaves the rest of the hyperedges incident with v unchanged. It also adds a hyperedge e incident with both v and z (d). When there are two chosen hyperedges e_1 and e_2, denoted by dashed and dotted lines, respectively (e), the 2-extension adds a new vertex z, replaces v with z in e_1 and e_2, leaves the rest of the hyperedges unchanged and adds a new hyperedge e incident with both v and z (f). Note that the two vertices of e different from v and z can be arbitrary; they may or may not be adjacent with v in H.

Proof: Since H is 4-uniform with $|V|-3$ edges, the sum of degrees in H is $4|V|-12$. Every vertex is incident with at least one hyperedge, by (1,3)-tightness. If there are at most three vertices with degree at most three, then the total degree in H is at least $4(|V|-3)+3$, which is a contradiction. From a similar simple calculation the second part of the statement also follows. □

Lemma 3.5. *The* def *function is submodular, that is,* $\mathrm{def}(X) + \mathrm{def}(Y) \geq \mathrm{def}(X \cup Y) + \mathrm{def}(X \cap Y)$ *for every* $X, Y \subseteq V$.

From now on we will assume that H is $\mathcal{C}_3$-tight. The next lemma follows immediately from Lemma 3.5.

Lemma 3.6. *For* $X \subseteq V$, *we have*

$$\mathrm{def}(\mathcal{C}_3 X) \leq 3\mathrm{def}(X) - \mathrm{def}(X \cap \gamma X) - \mathrm{def}(\gamma^2 X \cap (X \cup \gamma X)).$$

Proof: By Lemma 3.5 we have

$$\mathrm{def}(X \cup \gamma X) \leq \mathrm{def}(X) + \mathrm{def}(\gamma X) - \mathrm{def}(X \cap \gamma X).$$

Applying Lemma 3.5 again (to $\gamma^2 X$ and $X \cup \gamma X$), it follows from the symmetry of H that

$$\begin{aligned} \mathrm{def}(\mathcal{C}_3 X) &\leq \mathrm{def}(\gamma^2 X) + \mathrm{def}(X \cup \gamma X) - \mathrm{def}(\gamma^2 X \cap (X \cup \gamma X)) \\ &\leq 3\mathrm{def}(X) - \mathrm{def}(X \cap \gamma X) - \mathrm{def}(\gamma^2 X \cap (X \cup \gamma X)) \end{aligned}$$

as claimed. $\square$

Lemma 3.7. *Suppose that* $X \subseteq V$ *is such that* $\mathrm{def}(X \cap \gamma X) \geq \mathrm{def}(X)$ *and* $\mathrm{def}(Z) \geq \mathrm{def}(X)$ *for any* $Z \supseteq X$. *Then* $\mathrm{def}(X) = \mathrm{def}(\mathcal{C}_3 X)$.

Proof: By the symmetry of H, Lemma 3.5, and the conditions of the lemma we obtain:

$$2\mathrm{def}(X) = \mathrm{def}(X) + \mathrm{def}(\gamma X) \geq \mathrm{def}(X \cup \gamma X) + \mathrm{def}(X \cap \gamma X) \geq 2\mathrm{def}(X).$$

This implies that $\mathrm{def}(X \cup \gamma X) = \mathrm{def}(X)$. Furthermore,

$$2\mathrm{def}(X) = 2\mathrm{def}(X \cup \gamma X) = \mathrm{def}(X \cup \gamma X) + \mathrm{def}(\gamma X \cup \gamma^2 X) \geq$$

$$\geq \mathrm{def}(\mathcal{C}_3 X) + \mathrm{def}((X \cup \gamma X) \cap (\gamma X \cup \gamma^2 X)) \geq 2\mathrm{def}(X)$$

from which $\mathrm{def}(X) = \mathrm{def}(\mathcal{C}_3 X)$ follows. $\square$

Lemma 3.8. $\mathrm{def}(\mathcal{C}_3 X) \equiv 0, 1 \pmod 3$ *for every* $X \subseteq V$.

Proof: By definition, $|\mathcal{C}_3 X| - 3 - e(\mathcal{C}_3 X) = \mathrm{def}(\mathcal{C}_3 X)$. The $\mathcal{C}_3$ symmetry implies $|\mathcal{C}_3 X| \equiv 0, 1 \pmod 3$ and $e(\mathcal{C}_3 X) \equiv 0, 1 \pmod 3$. But $e(\mathcal{C}_3 X) \equiv 1 \pmod 3$ implies $|\mathcal{C}_3 X| \equiv 1 \pmod 3$, and hence $\mathrm{def}(\mathcal{C}_3 X) \equiv 2 \pmod 3$ is not possible. $\square$

3.5. Reducing low degree vertices

In this section we will define symmetric reductions for $\mathcal{C}_3$-tight hypergraphs. We will also prove that a $\mathcal{C}_3$-symmetric reduction that preserves sparsity always exists.

From now on we will suppose that $|V| \geq 7$. (There are three non-isomorphic $\mathcal{C}_3$-tight hypergraphs with $|V| \leq 6$. We will discuss these 'base graphs' later in Section 3.5.4.) Let $u \in V$ be a vertex not incident with f_0. Suppose that $d(u, v) = 1$ for some $v \in V - \mathcal{C}_3 u$. Reduce u on v then reduce γu on γv and then $\gamma^2 u$ on $\gamma^2 v$. This operation (that consists of three successive reductions) will be called a *symmetric reduction*. We will say that we *reduce $\mathcal{C}_3 u$ on $\mathcal{C}_3 v$*. If the resulting hypergraph H' is (1,3)-sparse then the symmetric reduction is called *admissible*. The inverse operations of symmetric reductions will be called *symmetric extensions*.

Lemma 3.9. *Let H be a $\mathcal{C}_3$-symmetric 4-uniform hypergraph and let $u \in V$ be a vertex not incident with f_0. The hypergraph H' obtained from a $\mathcal{C}_3$-symmetric reduction is $\mathcal{C}_3$-symmetric.*

Proof: It suffices to show that for every hyperedge $f \in E(H')$ we have $\gamma f, \gamma^2 f \in E(H')$. This is clearly true for every hyperedge in $E(H) \cap E(H')$.

If an edge $f_1 \in E(H)$ is incident with both u and v then $f_1, \gamma f_1, \gamma^2 f_1$ are deleted during the reductions. If an edge $f_2 \in E(H)$ is incident with u but is not incident with v, then in f_2 the vertex u (γu and $\gamma^2 u$) is replaced with v (γv and $\gamma^2 v$, respectively), and it is not difficult to see that $\gamma f_2', \gamma^2 f_2' \in E(H')$ holds. □

The main result of this section is that we can always find a symmetric set of three vertices for which an admissible symmetric reduction exists. Our first task is to find a vertex u with $d(u) \leq 3$ that is not incident with f_0. By Lemma 3.4 the vertices of f_0 are the only vertices with degree at most three if and only if they all have degree one. But then H has four vertices only. Hence we can always find an appropriate u if $|V| > 6$.

Lemma 3.10. *If $d(\mathcal{C}_3 u) = 3$ then there is an admissible symmetric reduction at $\mathcal{C}_3 u$.*

Proof: The result of an arbitrary symmetric reduction is the deletion of $\mathcal{C}_3 u$ together with the incident hyperedges. This reduction is clearly admissible. □

Note that if $d(\mathcal{C}_3 u) = 3$ then either $d(u) = 1$ or $d(u) = 2$ and there is a hyperedge incident with both u and γu. Lemma 3.10 covers both of these cases.

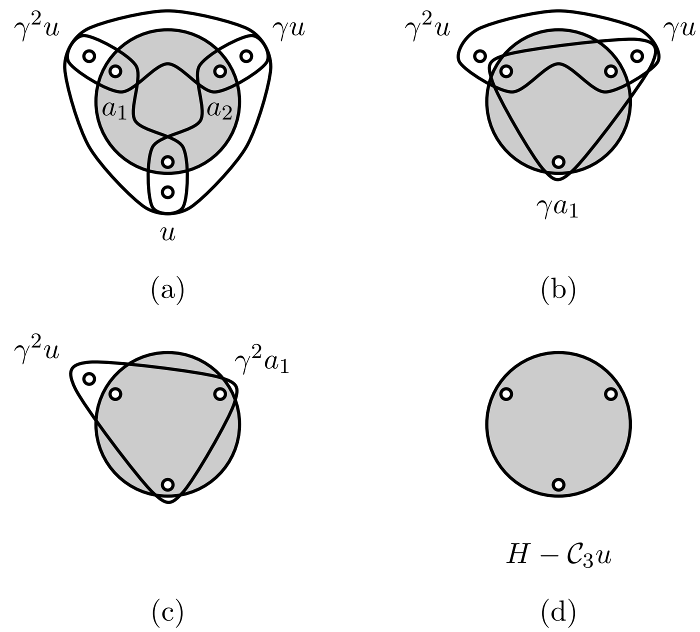

Figure 2: Example for a symmetric admissible reduction for the case $d(u) = 2$ and $d(\mathcal{C}_3 u) = 3$. The hypergraph H is shown in (a). There are two possible ways to reduce u. We can either reduce it on a_1 or on a_2; the figure shows the former. We first reduce u on a_1 (b), then γu on γa_1 (c) and finally $\gamma^2 u$ on $\gamma^2 a_1$ (d) which gives the hypergraph $H - \mathcal{C}_3 u$. The first two reductions are 1-reductions and the third one is a 0-reduction.

3.5.1. Blockers for symmetric reductions

By Lemma 3.10, there is an admissible symmetric reduction if $d(\mathcal{C}_3 u) = 3$. From now on we will focus on the cases $d(\mathcal{C}_3 u) = 6$ or 9. These imply $2 \leq d(u) \leq 3$. We will denote the hyperedges incident with u by e_1, e_2 (and e_3 if $d(u) = 3$) and we will also use the notation $\hat{e}_j = e_j - u$.

Let $a_1, \ldots, a_l$ denote the neighbours of u in $V - \mathcal{C}_3 u$ for which $d(u, a_i) = 1$, $1 \leq i \leq l$. First, we show that $d(u) \leq 3$ implies $l \geq 1$.

This statement is trivial for $d(u) = 1$. To see that it holds for $d(u) = 2$, we consider three distinct cases. First, if $\{u, \gamma u, \gamma^2 u, w\} \in F$ with $w \in V - \mathcal{C}_3 u$, then we obtain a contradiction to $d(u) = 2$. (Recall that u is not incident with f_0.) Second, if $\{u, w_1, w_2, w_3\} \in F$ with $w_i \in V - \mathcal{C}_3 u$ for $i = 1, 2, 3$, then the statement is clearly true. Third, if $\{u, \gamma u, w_1, w_2\} \in F$ with $w_i \in V - \mathcal{C}_3 u$ for $i = 1, 2$ then, by symmetry, $\{\gamma^2 u, u, \gamma^2 w_1, \gamma^2 w_2\} \in F$. However, $\{w_1, w_2\} \neq \{\gamma^2 w_1, \gamma^2 w_2\}$ and hence the statement again holds. Similarly, we may show that the statement also holds for $d(u) = 3$. Again we consider three cases. First, if $\{u, \gamma u, \gamma^2 u, w\} \in F$ with $w \in V - \mathcal{C}_3 u$, then the statement clearly holds. Second, if $\{u, w_1, w_2, w_3\} \in F$ with $w_i \in V - \mathcal{C}_3 u$ for $i = 1, 2, 3$, then the statement also holds, for otherwise the $(1, 3)$-sparsity is violated by the three hyperedges incident with u. Third, if the first two cases don't occur, then $\{u, \gamma u, w_1, w_2\} \in F$ and, by symmetry, $\{\gamma^2 u, u, \gamma^2 w_1, \gamma^2 w_2\} \in F$ with $w_i \in V - \mathcal{C}_3 u$ for $i = 1, 2$. If $w_1, w_2, \gamma^2 w_1$ and $\gamma^2 w_2$ are all distinct, then the statement clearly holds. Otherwise, two of these vertices are the same, say $w_1 = \gamma^2 w_2$, in which case the three distinct vertices are $\mathcal{C}_3 w_1$. Suppose without loss of generality that the third hyperedge containing u is of the form $\{u, \gamma u, x, y\}$. Then we are done unless $\{x, y\} = \{\gamma w_1, \gamma^2 w_1\}$. However, in this case we also have $\{\gamma^2 u, u, \gamma^2 x, \gamma^2 y\} \in F$, contradicting $d(u) = 3$.

In the following, we will use the notation $N_1(u) = \{a_1, \dots, a_l\}$. We will prove that for every u with $d(u) \leq 3$, there exists an index i for which the reduction of $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ yields a (1,3)-tight hypergraph.

The reduced hypergraph H' is not (1,3)-sparse if and only if there is a set of hyperedges $F \subseteq E(H') - E(H)$ for which $V(F) \subseteq X \subseteq V(H) - \mathcal{C}_3 u$ with some $\mathrm{def}(X) \leq |F| - 1$. We will call such a set X a *blocker for the symmetric reduction* or a *blocker* for short. Now we describe the blockers for symmetric reductions. The blocker of a_i will be denoted by X_i.

We will divide blockers into three groups to simplify the discussion. Let X_i be a blocker of a_i, i.e., a blocker for the symmetric reduction of $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$. We may assume that $a_i \in X_i$, because $\mathcal{C}_3 a_i \cap X_i \neq \emptyset$ must hold, and if $a_i \notin X_i$, then we can replace X_i with γX_i or $\gamma^2 X_i$ to obtain a blocker that contains a_i.

Vertices u and γu may or may not share a hyperedge. First suppose that there is no hyperedge incident to both u and γu. In this case we cannot reduce $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ if and only if one of the three following cases occurs.

(i) After reducing u on a_i the resulting hypergraph H_1 has a vertex set that violates sparsity and does not contain γu and $\gamma^2 u$. Such a vertex set is a blocker and will be called a *type 1* blocker.

(ii) There is no type 1 blocker and after the reduction of γu on γa_i in H_1 the resulting hypergraph H_2 has a vertex set that violates sparsity and does not contain $\gamma^2 u$. Such a vertex set is also a blocker which will be called a *type 2* blocker.

(iii) There is no type 1 or type 2 blocker but after the reduction of $\gamma^2 u$ on $\gamma^2 a_i$ in H_2 the resulting hypergraph has a vertex set that violates sparsity. Such a set is also a blocker and will be called a *type 3* blocker.

It follows from the definitions of type 1, 2 and 3 blockers that if X is a type 2 (or type 3) blocker, then X must contain the vertex set of at least one (at least two) previously reduced hyperedge(s).

Consider first the case $d(u) = 2$. In this case there are three different types of blockers. Let e_t for $1 \leq t \leq 2$ be the edge not incident with a_i. If X_i is type 1 then $\mathrm{def}(X_i) = 0$ and $\hat{e}_t + a_i \subseteq X_i$. If X_i is type 2 then $\mathrm{def}(X_i) = 1$ and $(\hat{e}_t + a_i) \cup \gamma(\hat{e}_t + a_i) \subseteq X_i$. Finally, if X_i is type 3 then $\mathrm{def}(X_i) = 2$ and $\mathcal{C}_3(\hat{e}_t + a_i) \subseteq X_i$.

Now suppose that $d(u) = 3$. This implies that $d(\mathcal{C}_3 u) = 9$. To simplify notation we will assume that $a_i \in e_1$.

If X_i is type 1, then $\mathrm{def}(X_i) = 0$ or 1. In the former case, $\hat{e}_t + a_i \subseteq X_i$ for some $2 \leq t \leq 3$, and in the latter case, $\hat{e}_1 \cup \hat{e}_2 + a_i \subseteq X_i$.

If X_i is type 2, then $1 \leq \mathrm{def}(X_i) \leq 3$. We have that $a_i, \gamma a_i \in X_i$, and X_i contains at least one of the sets $\hat{e}_2$ and $\hat{e}_3$, at least one of $\gamma\hat{e}_2$ and $\gamma\hat{e}_3$, and at least $\mathrm{def}(X_i) + 1$ of these four vertex sets. There are two kinds of type 2 blockers that will play an important role in the proofs. The first one has $\mathrm{def}(X_i) = 1$ and $\hat{e}_t \cup \gamma\hat{e}_t \subseteq X_i$ for some $2 \leq t \leq 3$. We will call such an X_i a *type 2a blocker*. If $\mathrm{def}(X_i) = 1$ and $\hat{e}_t \cup \gamma\hat{e}_s \subseteq X_i$ for $\{s, t\} = \{2, 3\}$ then X_i is called a *type 2b blocker*.

Finally, if X_i is type 3, then $2 \leq \mathrm{def}(X_i) \leq 5$. We have that $\mathcal{C}_3 a_i \subseteq X_i$, and X_i contains at least one of the sets $\hat{e}_2$ and $\hat{e}_3$, at least one of $\gamma\hat{e}_2$ and $\gamma\hat{e}_3$, and at least one of $\gamma^2\hat{e}_2$ and $\gamma^2\hat{e}_3$. X_i contains at least $\mathrm{def}(X_i) + 1$ of these six vertex sets.

In the final case, we have $d(u) = 3$, and u and γu share an edge. $d(u) \leq 3$ implies that u and γu cannot share more than one edge. In this case, instead of e_1, e_2, e_3 we will use a different notation for the edges incident with u. Let f be the unique edge incident with both u and γu, and so the edges incident with u are $f, \gamma^2 f, g$ for some $g \in F$. We will use the notation $\hat{f} = f - u - \gamma u$ and $\hat{g} = g - u$. If $\hat{f} \cap \gamma\hat{f} = \emptyset$ then $(\hat{f} \cup \gamma\hat{f}) \cap N_1(u) \neq \emptyset$ and in this case we will reduce $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ for some $a_i \in \hat{f} \cup \gamma\hat{f}$. If $\hat{f} \cap \gamma\hat{f} \neq \emptyset$ then either $f = \{u, \gamma u, w, \gamma w\}$ or $f = \{u, \gamma u, w, v_0\}$ for some $w \in V - v_0$. In this case the (1,3)-sparsity implies that $g \cap N_1(u) \neq \emptyset$ and we will reduce $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ for some $a_i \in \hat{g}$.

We will apply the same method as in the case before, that is, we will reduce u on some of its neighbours $a_i \in N_1(u)$, then reduce γu on γa_i, and finally reduce $\gamma^2 u$ on $\gamma^2 a_i$. Note that the first reduction is a 2-reduction but the other ones may be 1-reductions. In either case this sequence of three operations results in adding exactly three hyperedges to $H - \mathcal{C}_3 u$ in a symmetric way. If $a_i \in \hat{f}$ then let $h_i = \hat{g} + a_i$, and if $a_i \in \hat{g}$ then let $h_i = \hat{f} + a_i + \gamma a_i$. The three new hyperedges are $\mathcal{C}_3 h_i$.

If the reduction is not admissible then again we have three types of blockers. $X_i \subseteq V - \mathcal{C}_3 u$ is a blocker of a_i if one of the following holds:

(i) $h_i \subseteq X_i$ and $\mathrm{def}(X_i) = 0$;

(ii) $h_i \cup \gamma h_i \subseteq X_i$ and $\mathrm{def}(X_i) = 1$;

(iii) $\mathcal{C}_3 h_i \subseteq X_i$ and $\mathrm{def}(X_i) = 2$.

If $u, \gamma u$ share an edge then we will call these blockers type 1, 2 and 3, respectively.

We shall also use the following property of (1,3)-sparse symmetric hypergraphs throughout this section. Let $U \subseteq V - \mathcal{C}_3 u$ be a vertex set. If $U + u$ spans k edges incident with u then $\mathrm{def}(U) \geq k - 1$ and if $d_{U+\mathcal{C}_3 u}(\mathcal{C}_3 u) = l$ then $\mathrm{def}(U) \geq l - 3$. We will call this the $(*)$ property.

To see that the $(*)$ property holds, note that $\mathrm{def}(U + u) \geq 0$, and hence

$$\begin{aligned}\mathrm{def}(U+u) &= |U+u| - 3 - e_H(U+u)\\ &= |U| + 1 - 3 - (e_H(U) + k)\\ &= \mathrm{def}(U) - k + 1 \geq 0.\end{aligned}$$

Similarly, $\mathrm{def}(U + \mathcal{C}_3 u) \geq 0$, and hence

$$\begin{aligned}\mathrm{def}(U+\mathcal{C}_3 u) &= |U+\mathcal{C}_3 u| - 3 - e_H(U+\mathcal{C}_3 u)\\ &= |U| + 3 - 3 - (e_H(U) + l)\\ &= \mathrm{def}(U) + 3 - l \geq 0.\end{aligned}$$

From now on we will suppose that a_i has a blocker X_i for every $1 \leq i \leq l$ and X_i will be a blocker with the smallest possible deficiency among blockers of a_i. Note that it follows from the definition of type 1, 2 and 3 blockers that if X_i is type h then there is no type k blocker of a_i with $k < h$.

3.5.2. Case $d(u) = 2$ and $d(\mathcal{C}_3 u) = 6$

Lemma 3.11. *If $d(\mathcal{C}_3 u) = 6$ and $d(u) = 2$ then there is an admissible symmetric reduction at $\mathcal{C}_3 u$.*

Proof: Suppose for a contradiction that there is no symmetric reduction at $\mathcal{C}_3 u$. Then there is a blocker X_i for every a_i.

First we will show that every X_i is type 1 or type 2. Suppose for a contradiction that X_i is type 3. By our assumption $\mathrm{def}(Y) \geq 2$ for every

$Y \supseteq X_i$ and $\operatorname{def}(X_i \cap \gamma X_i) \geq 2$ and hence we can use Lemma 3.7. We get that $\operatorname{def}(\mathcal{C}_3 X_i) = 2$, which contradicts Lemma 3.8. Thus, X_i is type 1 or type 2, as we claimed.

Now suppose that X_i is type 2 for some $1 \leq i \leq l$. If $X_i \cap \gamma X_i$ is tight then it is a type 1 blocker of a_i which is not possible. Thus, we must have $\operatorname{def}(X_i \cap \gamma X_i) \geq 1$. We can again use Lemma 3.7 to obtain $\operatorname{def}(\mathcal{C}_3 X_i) = 1$.

By the assumption on $d(u)$ and $d(\mathcal{C}_3 u)$, u and γu do not share a hyperedge and hence Lemma 3.3 can be used to see that it is not possible that every blocker is type 1. Therefore, we can assume that X_1 is type 2. Assume further that $a_1 \in e_1$. Suppose first that X_j is type 1 for every $a_j \in e_2$. Then, by Lemma 3.5, $\bigcup_{a_j \in e_2} X_j$ is a tight set and contains every neighbour of u, which contradicts the $(*)$ property. It follows that there must be an $a_2 \in e_2$ for which X_2 is type 2. Consider the sets $\mathcal{C}_3 X_1$ and $\mathcal{C}_3 X_2$. We have $\operatorname{def}(\mathcal{C}_3 X_1) = \operatorname{def}(\mathcal{C}_3 X_2) = 1$ and $|\mathcal{C}_3 X_1 \cap \mathcal{C}_3 X_2| \geq 4$. This implies $\operatorname{def}(\mathcal{C}_3 X_1 \cup \mathcal{C}_3 X_2) \leq 2$ by Lemma 3.5. Thus, $\mathcal{C}_3 X_1 \cup \mathcal{C}_3 X_2$ violates the $(*)$ property. This completes the proof. $\square$

3.5.3. Case $d(u) = 3$ and $d(\mathcal{C}_3(u)) = 9$

Claim 3.12. *Suppose X_i is type 3. Then $\operatorname{def}(X_i) \leq 4$, and if $\mathcal{C}_3 \hat{e}_t \subseteq X_i$ for some $1 \leq t \leq 3$, then $\operatorname{def}(X_i) \geq 3$.*

Proof: Suppose that there is a type 3 blocker X_i with $\operatorname{def}(X_i) = 5$ or with $\mathcal{C}_3 \hat{e}_t \subseteq X_i$ for some $1 \leq t \leq 3$ and $\operatorname{def}(X_i) = 2$. In both of these cases we can use Lemma 3.7 to deduce that $\operatorname{def}(\mathcal{C}_3 X_i) = \operatorname{def}(X_i)$, which contradicts Lemma 3.8. $\square$

The next claim follows easily from Lemma 3.7.

Claim 3.13. *If X_i is a type 2a blocker then $\operatorname{def}(\mathcal{C}_3 X_i) = 1$.*

Lemma 3.14. *Suppose that $Y \subseteq V$ is such that $\operatorname{def}(\mathcal{C}_3 Y) \leq 4$ and $\mathcal{C}_3(\hat{e}_t \cup \hat{e}_s) \subseteq Y$ for some pair $1 \leq t, s \leq 3$. If $a_i \notin \hat{e}_t \cup \hat{e}_s$ then $\operatorname{def}(\mathcal{C}_3 Y \cup \mathcal{C}_3 X_i) \leq 4$.*

Proof: It suffices to show that $\operatorname{def}(\mathcal{C}_3 Y \cup \mathcal{C}_3 X_i) \leq 5$ because the statement then follows from Lemma 3.8. First we need the following calculation in which Lemma 3.5 is used multiple times along with $\operatorname{def}(X_i) = \operatorname{def}(\gamma X_i) = \operatorname{def}(\gamma^2 X_i)$:

$$\operatorname{def}(\mathcal{C}_3 Y \cup \mathcal{C}_3 X_i) \leq \operatorname{def}(\mathcal{C}_3 Y \cup X_i \cup \gamma X_i) + \operatorname{def}(X_i) - \operatorname{def}((\mathcal{C}_3 Y \cup X_i \cup \gamma X_i) \cap \gamma^2 X_i)$$

$$\leq \operatorname{def}(\mathcal{C}_3 Y \cup X_i) + 2\operatorname{def}(X_i) - \operatorname{def}((\mathcal{C}_3 Y \cup X_i) \cap \gamma X_i) - \operatorname{def}((\mathcal{C}_3 Y \cup X_i \cup \gamma X_i) \cap \gamma^2 X_i)$$

$$\leq \operatorname{def}(\mathcal{C}_3 Y) + 3\operatorname{def}(X_i) - \operatorname{def}(\mathcal{C}_3 Y \cap X_i) - \operatorname{def}((\mathcal{C}_3 Y \cup X_i) \cap \gamma X_i) - \operatorname{def}((\mathcal{C}_3 Y \cup X_i \cup \gamma X_i) \cap \gamma^2 X_i).$$

By the $(*)$ property, $\operatorname{def}(\mathcal{C}_3 Y \cap X_i) + \operatorname{def}((\mathcal{C}_3 Y \cup X_i) \cap \gamma X_i) + \operatorname{def}((\mathcal{C}_3 Y \cup X_i \cup \gamma X_i) \cap \gamma^2 X_i) \geq 3$ must hold, and hence the statement follows if $\operatorname{def}(X_i) \leq 1$.

Claim 3.15. *If* $\operatorname{def}(X_i) \geq 2$ *then* $\operatorname{def}(\mathcal{C}_3 X_i) \leq 4$ *holds.*

Proof: Again, it suffices to show that $\operatorname{def}(\mathcal{C}_3 X_i) \leq 5$. We split the proof into several cases. In each case we will use Lemma 3.6 and the fact that X_i is a blocker with the smallest deficiency.

If X_i is type 2 and $\operatorname{def}(X_i) = 2$ ($\operatorname{def}(X_i) = 3$), then $\operatorname{def}(X_i \cap \gamma X_i) \geq 1$ and $\operatorname{def}(\gamma^2 X_i \cap (X_i \cup \gamma X_i)) \geq 1$ ($\operatorname{def}(X_i \cap \gamma X_i) \geq 2$ and $\operatorname{def}(\gamma^2 X_i \cap (X_i \cup \gamma X_i)) \geq 2$) otherwise they are type 1 blockers. Then $\operatorname{def}(\mathcal{C}_3 X_i) \leq 6-1-1$ ($\operatorname{def}(\mathcal{C}_3 X_i) \leq 9-2-2$) follows.

Now suppose that X_i is type 3. If $\operatorname{def}(X_i) = 2$, then similarly to the previous case we have $\operatorname{def}(\mathcal{C}_3 X_i) \leq 6-1-1$. If $\operatorname{def}(X_i) = 3$, then there are two cases. We can assume in the first case that X_i contains $\mathcal{C}_3 \hat{e}_t, \hat{e}_s$ and then $\operatorname{def}(X_i \cap \gamma X_i) \geq 3$ and $\operatorname{def}(\gamma^2 X_i \cap (X_i \cup \gamma X_i)) \geq 3$ because there is no type 3 blocker with deficiency 2 by our assumption. Thus $\operatorname{def}(\mathcal{C}_3 X_i) \leq 9-3-3$ holds. In the second case we can assume that X_i contains $\hat{e}_t, \hat{e}_s, \gamma \hat{e}_s, \gamma^2 \hat{e}_t$ and $\operatorname{def}(X_i \cap \gamma X_i) \geq 2$ and $\operatorname{def}(\gamma^2 X_i \cap (X_i \cup \gamma X_i)) \geq 2$ because there is no type 2 blocker. Hence $\operatorname{def}(\mathcal{C}_3 X_i) \leq 9-2-2$.

Finally, if $\operatorname{def}(X_i) = 4$, then $\operatorname{def}(\mathcal{C}_3 X_i) \leq 12-4-4$ because blockers with smaller deficiency cannot exist. By Claim 3.12, $\operatorname{def}(X_i) = 5$ is not possible. This completes the proof. $\square$

If $\operatorname{def}(X_i) \geq 2$, then $\operatorname{def}(\mathcal{C}_3 X_i) \leq 4$, by Claim 3.15. In this case $\operatorname{def}(\mathcal{C}_3 Y \cap \mathcal{C}_3 X_j) \geq 3$ follows from the $(*)$ property. Then, by Lemma 3.5, $\operatorname{def}(\mathcal{C}_3 Y \cup \mathcal{C}_3 X_j) \leq 4+4-3$, and the proof is complete. $\square$

Lemma 3.16. *Suppose there is a set* $Y \subseteq V$ *with* $\operatorname{def}(\mathcal{C}_3 Y) \leq 4$ *and* $\mathcal{C}_3(\hat{e}_t \cup \hat{e}_s) \subseteq Y$ *for some pair* $1 \leq t, s \leq 3$*. Then* H *is not (1,3)-sparse.*

Proof: By the $(*)$ property, for a vertex set Z with $N(u) \subseteq Z$, $\operatorname{def}(Z) \geq 6$ holds. Hence there must be a vertex $v \in N(u) \setminus \mathcal{C}_3 Y$ and then $v = a_j$ for some $1 \leq j \leq l$. Using Lemma 3.14 multiple times if necessary we get that $\operatorname{def}(\mathcal{C}_3 Y \bigcup_{j: a_j \in N_1(u) \setminus \mathcal{C}_3 Y} \mathcal{C}_3 X_j) \leq 4$, and hence the set $\mathcal{C}_3 Y \bigcup_{j: a_j \in N_1(u) \setminus \mathcal{C}_3 Y} \mathcal{C}_3 X_j$ violates sparsity by the $(*)$ property. $\square$

Lemma 3.17. *For every blocker* X_i*, we have* $\operatorname{def}(X_i) \leq 1$*. Further, if* $\operatorname{def}(X_i) = 1$*, then* X_i *is type 2.*

Proof: Suppose for a contradiction that $\operatorname{def}(X_i) \geq 2$ for some $1 \leq i \leq l$ with $a_i \in \hat{e_1}$. Then $\operatorname{def}(\mathcal{C}_3 X_i) \leq 4$, by Claim 3.15. If X_i is type 2, then it must contain at least three of the vertex sets $\hat{e_2}, \hat{e_3}, \gamma\hat{e_2}, \gamma\hat{e_3}$. Thus, $\mathcal{C}_3(\hat{e_2} \cup \hat{e_3}) \subseteq \mathcal{C}_3 X_i$. We can use a similar argument to deduce that $\mathcal{C}_3(\hat{e_2} \cup \hat{e_3}) \subseteq \mathcal{C}_3 X_i$ if X_i is type 3 with $\operatorname{def}(X_i) \geq 3$, as at least one of $\hat{e}_t, \gamma\hat{e}_t, \gamma^2\hat{e}_t$ is contained in X_i for $t = 2, 3$. If X_i is type 3 with $\operatorname{def}(X_i) = 2$, then by Claim 3.12 we get that X_i does not contain $\mathcal{C}_3\hat{e}_t$ for any $t \in \{2, 3\}$. Thus $\mathcal{C}_3 X_i$ contains both $\mathcal{C}_3\hat{e}_2$ and $\mathcal{C}_3\hat{e}_3$. Then, using Lemma 3.16, we get a contradiction.

To prove the second part of the statement, suppose that X_i is type 1 with $\operatorname{def}(X_i) = 1$. Suppose $a_i \in \hat{e}_1$. The $(*)$ property implies that every vertex set Z with $\hat{e_2} \cup \hat{e_3} \subseteq Z$ has $\operatorname{def}(Z) \geq 1$. Thus if X_j is type 1 for every $a_j \in \hat{e}_1$, $i \neq j$, then, using Lemma 3.5, it can easily be seen that $\operatorname{def}(\bigcup_{j:a_j \in \hat{e}_1} X_j) \leq 1$, which is a contradiction, as $N(u) \subseteq \bigcup_{j:a_j \in \hat{e}_1} X_j$ and then by the $(*)$ property $\operatorname{def}(\bigcup_{j:a_j \in \hat{e}_1} X_j) \geq 2$ holds. Hence there is some $a_k \in \hat{e}_1$ with a type 2 blocker X_k. If X_k is type 2a, then $\operatorname{def}(\mathcal{C}_3 X_k) = 1$, by Claim 3.13. Using Lemma 3.5 we get $\operatorname{def}(\mathcal{C}_3(X_i \cup X_k)) \leq 4$, which contradicts Lemma 3.16. Thus X_k is type 2b and we can assume that $\hat{e}_2 \cup \gamma\hat{e}_3 \subseteq X_k$. Consider the set $X_i \cup X_k$. By Lemma 3.5, $\operatorname{def}(X_i \cup X_k) \leq 2$. As $\gamma^2\hat{e}_3 \cup \hat{e}_3 \cup \{X_k, \gamma^2 X_k\} \subseteq ((X_i \cup X_k) \cup \gamma(X_i \cup X_k)) \cap \gamma^2(X_i \cup X_k)$ and X_k has no type 1 blocker, $\operatorname{def}(((X_i \cup X_k) \cup \gamma(X_i \cup X_k)) \cap \gamma^2(X_i \cup X_k)) \geq 1$. Hence, using Lemmas 3.6 and 3.8, $\operatorname{def}(\mathcal{C}_3(X_i \cup X_k)) \leq 4$, and again we get a contradiction using Lemma 3.16. This completes the proof. $\square$

We have shown so far that every blocker has to be a tight type 1 blocker, a type 2a blocker or a type 2b blocker. We shall also use the following lemma.

Lemma 3.18. *Suppose that a_i is such that $a_i \in \hat{e}_j$, $\hat{e}_k \subseteq X_i$ and $|\hat{e}_j \cap \hat{e}_k| \geq 1$. Then X_i is not type 2a.*

Proof: Suppose for a contradiction that X_i is type 2a. Then $\operatorname{def}(\mathcal{C}_3 X_i) = 1$ by Claim 3.13. Thus $\operatorname{def}(\mathcal{C}_3(X_i \cup \hat{e}_j)) \leq 4$, which is a contradiction by Lemma 3.16. $\square$

Now we will show that if there is a blocker X_i of a_i for every $1 \leq i \leq l$ then H cannot be $(1,3)$-sparse.

Lemma 3.19. *If $d(u) = 3$, and u and γu do not share a hyperedge, then there is an admissible symmetric reduction at $\mathcal{C}_3 u$.*

Proof: Suppose for a contradiction that there is no admissible symmetric reduction at $\mathcal{C}_3 u$. Then a_i has a blocker for every $1 \leq i \leq l$. By Lemma 3.17, X_i is a tight type 1 or type 2a or type 2b blocker for every $1 \leq i \leq l$.

By Lemma 3.3, there is a (non-symmetric) admissible reduction at u, and hence we may assume that a_1 has no type 1 blocker. Thus, X_1 is a type 2 blocker.

Case 1: Suppose first that every blocker is either type 1 or type 2a.

Claim 3.20. *Suppose that X_j is type 1 or type 2a for every $a_j \in \hat{e}_s$ and $\hat{e}_s \cup \hat{e}_t \subseteq \bigcup_{a_j \in \hat{e}_s} X_j$. Then $\hat{e}_t \not\subseteq \bigcap_{a_j \in \hat{e}_s} X_j$ for any $t \neq s$.*

Proof: Suppose the contrary for a contradiction. If X_i is type 2a for some $a_i \in \hat{e}_s$, then $\operatorname{def}(\mathcal{C}_3 X_i) \leq 1$ by Claim 3.13. We can easily deduce that $\operatorname{def}(\bigcup_{a_j \in \hat{e}_s} \mathcal{C}_3 X_j) \leq 3$ using Lemma 3.5 (as adding type 2a blockers increases the deficiency by at most 1, and adding the three symmetric copies of tight type 1 blockers does not increase the deficiency), and then we get a contradiction using Lemma 3.16. If X_j is type 1 for every j, then by Lemma 3.5 we have $\operatorname{def}(\bigcup_{a_j \in \hat{e}_s} X_j) = 0$, which contradicts sparsity by the $(*)$ property. □

We first claim that $|\hat{e}_t \cap N_1(u)| \leq 2$ for every $1 \leq t \leq 3$. Suppose for a contradiction that $\hat{e}_1 \cap (\hat{e}_2 \cup \hat{e}_3) = \emptyset$. Then $\hat{e}_1 = \{a_i, a_j, a_k\}$ for some triple $1 \leq i, j, k \leq l$. It follows from Claim 3.20 that $X_i \cap X_j \cap X_k \supseteq \hat{e}_t$ is not possible if $t \in \{2, 3\}$. Hence we can assume that $X_i \not\supseteq \hat{e}_2$ and $X_j \not\supseteq \hat{e}_3$. Assume $X_k \supseteq \hat{e}_2$. $\hat{e}_1 \cap (\hat{e}_2 \cup \hat{e}_3) = \emptyset$ implies $|\hat{e}_2 \cap N_1(u)| \geq 1$, and hence $a_m \in \hat{e}_2$ for some $1 \leq m \leq l$. If $X_m \supseteq \hat{e}_1$, then we claim that $X_j \cup X_k \cup X_m$ violates sparsity. If X_j, X_k, X_m are type 1 blockers, then $X_j \cup X_k \cup X_m$ is tight by Lemma 3.5 and hence violates sparsity. If at least one of X_j, X_k, X_m is type 2a, then $\operatorname{def}(\mathcal{C}_3(X_j \cup X_k \cup X_m)) \leq 3$ by Lemma 3.5 and Claim 3.21 and the $(*)$ property, which violates sparsity by Lemma 3.14.

Hence $X_m \supseteq \hat{e}_3$ for every $a_m \in \hat{e}_2$. But this contradicts Claim 3.20. We deduce that $|\hat{e}_j \cap N_1(u)| \leq 2$ for every $1 \leq j \leq 3$, which implies $|N(u)| \leq 7$. There is a type 2a blocker, and hence $\hat{e}_j \cap \hat{e}_k = \emptyset$ for some pair $1 \leq j, k \leq 3$ by Lemma 3.18. This implies $|N(u)| \geq 6$.

If $|N(u)| = 6$ then $l \geq 3$. Consider the sets X_1, X_2. X_1 is type 2a by our assumption, and hence $\operatorname{def}(\mathcal{C}_3 X_1) = 1$ by Claim 3.13. If $N(u) \subseteq X_1 \cup X_2$, then $|X_1 \cap X_2| \geq 2$. Hence, if X_2 is type 1, then $\operatorname{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 4$, and if X_2 is type 2a, then $\operatorname{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 2$ by Lemma 3.5. In both of these cases we get a contradiction using the $(*)$ property. If $|N(u) \cap (X_1 \cup X_2)| = 5$, then $|X_1 \cap X_2| \geq 3$. If X_2 is type 1, then $\operatorname{def}(\mathcal{C}_3(X_1 \cup X_2)) = 1$, and hence $\operatorname{def}(\mathcal{C}_3(N(u) \cup X_1 \cup X_2)) \leq 4$, while if X_2 is type 2a, then $\operatorname{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 2$, and hence $\operatorname{def}(\mathcal{C}_3(N(u) \cup X_1 \cup X_2)) \leq 5$ by Lemma 3.5. These contradict sparsity by the $(*)$ property.

The remaining case is $|N(u)| = 7$ and $|\hat{e}_j \cap N_1(u)| \leq 2$ for every $1 \leq j \leq 3$. The only possible configuration is $\hat{e}_1 \cap \hat{e}_2 \cap \hat{e}_3 = \emptyset$ and $|\hat{e}_1 \cap \hat{e}_2| = |\hat{e}_2 \cap \hat{e}_3| = 1$

as $\hat{e}_j \cap \hat{e}_k = \emptyset$ for some pair $1 \leq j, k \leq 3$. Thus, $|\hat{e}_2 \cap N_1(u)| = 1$ and $|\hat{e}_1 \cap N_1(u)| = |\hat{e}_3 \cap N_1(u)| = 2$. If $a_1 \in \hat{e}_2$ then we get a contradiction by Lemma 3.18. Using the same lemma, we can assume without loss of generality that $a_1 \in \hat{e}_1$ and $\hat{e}_3 \subseteq X_1$. By Claim 3.20, there is an $a_j \in \hat{e}_3$ with $\hat{e}_1 \subseteq X_j$. If X_j is type 1, then $\mathrm{def}(\mathcal{C}_3(X_1 \cup X_j)) \leq 4$ by Lemma 3.5, and if X_j is type 2a, then $\mathrm{def}(\mathcal{C}_3(X_1 \cup X_j)) \leq 2$. Both lead to a contradiction by Lemma 3.16.
Case 2: It remains to consider the case where X_1 is a type 2b blocker. We may assume that $\hat{e}_3 \cup \gamma\hat{e}_2 \subseteq X_1$ and $a_1 \in \hat{e}_1$.

Claim 3.21. *If X_i is type 2b, then $X_i \cap \gamma X_i = \gamma a_i$.*

Proof: Clearly $\gamma a_i \in X_i \cap \gamma X_i$. Suppose that $|X_i \cap \gamma X_i| \geq 2$ for a type 2b blocker X_i. This implies $\mathrm{def}(X_i \cap \gamma X_i) \leq -1$ and also $|\gamma^2 X_i \cap (X_i \cap \gamma X_i)| \geq 2$ from which $\mathrm{def}(\gamma^2 X_i \cap (X_i \cap \gamma X_i)| \geq 2) \leq -1$ follows. Then, by Lemma 3.6, $\mathrm{def}(\mathcal{C}_3 X_i) \leq 3 + 1 + 1$, and hence $\mathrm{def}(\mathcal{C}_3 X_i) \leq 4$ by Lemma 3.8. Then we get a contradiction using Lemma 3.14. □

We have $\hat{e}_2 \cap \hat{e}_3 = \emptyset$ by Claim 3.21. We first claim that there is a vertex $a_2 \in \hat{e}_2$ for which X_2 is type 2b. Suppose that X_j is type 1 or type 2a for every vertex $a_j \in \hat{e}_2 \cap N_1(u)$. If $\hat{e}_1 \subseteq X_j$ for every $X_j \in N_1(u) \cap \hat{e}_2$ then, by $\hat{e}_2 \cap \hat{e}_3 = \emptyset$ and Claim 3.20, there must be an $a_k \in \hat{e}_2$ for which $\hat{e}_3 \subseteq X_k$. But if X_k is type 1, then by Lemma 3.5 $X_1 \cup X_k$ is a type 2b blocker of a_1, which contradicts Claim 3.21. Otherwise X_k is type 2a, and then by Claim 3.13 $\mathrm{def}(\mathcal{C}_3 X_k) = 1$, and hence $\mathrm{def}(\mathcal{C}_3(X_1 \cup X_k)) \leq 4$ by Lemma 3.5, which contradicts Lemma 3.16. Hence there is a vertex, say $a_2 \in \hat{e}_2$, for which X_2 is type 2b.

Then, using Claim 3.21 for X_2, we can conclude that the sets $\hat{e}_1, \hat{e}_3$ are also disjoint. Repeating the above argument it follows that $\hat{e}_3$ also contains a vertex with a type 2b blocker, $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are pairwise disjoint and hence $|N_1(u)| = 9$. It also follows from the argument above that every blocker must be type 2b.

Now suppose that $\hat{e}_2 \cup \gamma\hat{e}_3 \subseteq X_4$ for some type 2b blocker X_4 with $a_4 \in \hat{e}_1$. Then $\mathrm{def}(X_1 \cup \gamma X_4) \leq 2$ by Lemma 3.5. By the $(*)$ property $\mathrm{def}((X_1 \cup \gamma X_4) \cap \gamma(X_1 \cup \gamma X_4)) \geq 1$, and $\mathrm{def}(\gamma^2(X_1 \cup \gamma X_4) \cap ((X_1 \cup \gamma X_4) \cup \gamma(X_1 \cup \gamma X_4))) \geq 2$ follows. This implies $\mathrm{def}(\mathcal{C}_3(X_1 \cup X_4)) \leq 3$ by Lemma 3.6, which contradicts Lemma 3.16. Hence $\hat{e}_3 \cup \gamma\hat{e}_2 \subseteq X_1 \cap X_4 \cap X_7$, with the notation $\hat{e}_1 = \{a_1, a_4, a_7\}$.

Next we will show that $\hat{e}_3 \cup \gamma\hat{e}_1 \subseteq X_2$ holds for $a_2 \in \hat{e}_2$. Suppose that $\hat{e}_1 \cup \gamma\hat{e}_3 \subseteq X_2$. We will show that $\mathrm{def}(\mathcal{C}_3(X_1 \cup \gamma^2 X_2)) = \mathrm{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 4$ to get a contradiction by Lemma 3.6. $\mathrm{def}((X_1 \cup \gamma^2 X_2) \cap \gamma(X_1 \cup \gamma^2 X_2)) \geq 0$

and $\mathrm{def}(\gamma^2(X_1 \cup \gamma^2 X_2) \cap ((X_1 \cup \gamma^2 X_2) \cup \gamma(X_1 \cup \gamma^2 X_2))) \geq 2$ must hold otherwise $\gamma^2(X_1 \cup \gamma^2 X_2) \cap (X_1 \cup (\gamma^2 X_2) \cup \gamma(X_1 \cup \gamma^2 X_2))$ would be a type 2b blocker which contradicts Claim 3.21. Then we can use Lemma 3.6 for $\mathcal{C}_3(X_1 \cup \gamma^2 X_2)$ to get $\mathrm{def}(\mathcal{C}_3(X_1 \cup X_2)) \leq 4$.

Hence $\hat{e}_1 \cup \gamma\hat{e}_3 \subseteq X_2$ is the only possible case. Consider $\mathcal{C}_3 X_2 \cup X_1 \cup \gamma X_4 \cup \gamma^2 X_7$ which contains $\mathcal{C}_3 N(u)$. We will prove that this set violates sparsity. We have $\mathrm{def}(X_2 \cup \gamma X_4) \leq 2$, and by adding the sets $\gamma X_2, \gamma^2 X_7, \gamma^2 X_2$ (in this order), we can easily conclude that $\mathrm{def}(X_2 \cup \gamma X_4 \cup \gamma X_2 \cup \gamma^2 X_7 \cup \gamma^2 X_2) \leq 5$ from Lemma 3.5, because each set intersects the union of the previous ones in at least three vertices. We have $\hat{e}_3 + a_1 \subseteq (X_2 \cup \gamma X_4 \cup \gamma X_2 \cup \gamma^2 X_7 \cup \gamma^2 X_2) \cap X_1$, and hence $\mathrm{def}(\mathcal{C}_3 X_2 \cup X_1 \cup \gamma X_4 \cup \gamma^2 X_7) \leq 5$ which contradicts the $(*)$ property.

In each case we got a contradiction, and hence we can always perform a symmetric reduction, as we claimed. □

3.5.4. Case $d(u) = 3$ and $d(\mathcal{C}_3(u)) = 6$

Lemma 3.22. *Suppose that $d(u) = 3$ and that the hyperedges incident with u are $f, \gamma^2 f, g$. Then there is a vertex $a_i \in N_1(u)$ such the reduction of $\mathcal{C}_3 u$ on $\mathcal{C}_3 a_i$ is admissible. Moreover, a_i can be chosen such that if $\hat{f} \cap \gamma^2 \hat{f} = \emptyset$ then $a_i \in \hat{f} \cup \gamma^2 \hat{f}$, and if $\hat{f} \cap \gamma^2 \hat{f} \neq \emptyset$ then $a_i \in \hat{g}$.*

Proof: Suppose for a contradiction that there is a blocker X_i for every $1 \leq i \leq l$. First we claim that there are no type 3 blockers. Suppose for a contradiction that there is a type 3 blocker X_i. Assume that X_i is maximal among the type 3 blockers of a_i. We can then use Lemma 3.7 to prove that $\mathrm{def}(\mathcal{C}_3 X_i) = 2$. But this is a contradiction by Lemma 3.8 and thus X_i cannot be type 3.

Suppose first that $\hat{f} \cap \gamma^2 \hat{f} = \emptyset$. Recall that in this case the reduced edges have the form $\hat{g} + a_i$. If there is some $a_i \in \hat{f} \cup \gamma^2 \hat{f}$ for which X_i is type 2, then let X_i be maximal again. Then by Lemma 3.7 $\mathrm{def}(\mathcal{C}_3 X_i) = 1$. If there is a vertex $a_j \notin \mathcal{C}_3 X_i$ then consider a maximal X_j. If X_j is type 2 then we get $\mathrm{def}(\mathcal{C}_3(X_i \cup X_j)) \leq 1$ by Lemmas 3.5 and 3.8. If X_j is type 1 then adding $X_j, \gamma X_j, \gamma^2 X_j$ one by one to $\mathcal{C}_3 X_i$ using Lemma 3.5 we get again $\mathrm{def}(\mathcal{C}_3(X_i \cup X_j)) \leq 1$. In both cases we get a contradiction as then it follows that X_i is not maximal. Consider the case where every possible reduction has a type 1 blocker. Then there is a type 1 blocker X_i for every $a_i \in \hat{f} \cup \gamma^2 \hat{f}$ with $\hat{g} \subseteq X_i$. Let Y be the union of these blockers. By Lemma 3.5 Y is tight and by Lemma 3.7 $\mathrm{def}(\mathcal{C}_3 Y) \leq 1$. On the other hand $N(u) - \{\gamma u, \gamma^2 u\} \subseteq Y$, and hence $\mathrm{def}(\mathcal{C}_3 Y) \geq 3$ must hold by the $(*)$ property. These together give a contradiction which completes the proof of this case.

Now suppose that $\hat{f} \cap \gamma^2 \hat{f} \neq \emptyset$. The reduced edges have the form $\hat{f} + a_i + \gamma a_i$ in this case. Consider the blockers X_i for every $a_i \in \hat{g}$. If X_i is type 2 then we can use the same argument as in the previous case to deduce $\mathrm{def}(\mathcal{C}_3 X_i) = 1$. If X_i is type 1 then from Lemma 3.7 $\mathrm{def}(\mathcal{C}_3 X_i) \leq 1$ follows. Thus, using Lemmas 3.5 and 3.8 multiple times we have $\mathrm{def}(\mathcal{C}_3(\bigcup_{i:a_i \in \hat{g}} X_i)) \leq 1$, which contradicts the $(*)$ property. This completes the proof. $\square$

If we combine the results of Lemmas 3.10, 3.11, 3.19 and 3.22 we get the following:

Theorem 3.23. *Let $H = (V, F)$ be a $\mathcal{C}_3$-tight hypergraph with $|V| > 6$. Let $u \in V$ be a vertex with $d(u) \leq 3$ not incident to f_0. Then there is a $\mathcal{C}_3$-symmetric admissible reduction at $\mathcal{C}_3 u$.*

There are three non-isomorphic $\mathcal{C}_3$-tight hypergraphs with $|V| \leq 6$. H_4 is the smallest possible hypergraph with these properties; it has four vertices and one hyperedge and satisfies $|I_3(H_4)| = |V_3(H_4)| = 1$. The hypergraph can also have six vertices and three hyperedges. Hence we have two vertex orbits, $\mathcal{C}_3 v_1$ and $\mathcal{C}_3 v_2$. There are two possible hypergraphs with these properties. For the first one, which we will denote by H_6, $F = \mathcal{C}_3\{\mathcal{C}_3 v_1 + v_2\}$, and for the second one, which we will denote by H_6', $F = \mathcal{C}_3\{v_1, v_2, \gamma v_1, \gamma v_2\}$. They satisfy $|I_3(H_6)| = |V_3(H_6)| = 0$ and $|I_3(H_6')| = |V_3(H_6')| = 0$.

We will call H_4, H_6 and H_6' the *base graphs*. As a corollary of the above observations and Theorem 3.24 we get the main result of this section:

Theorem 3.24. *$H = (V, F)$ is a $\mathcal{C}_3$-tight hypergraph if and only if it can be obtained from one of the base graphs with a sequence of symmetric j-extensions for $0 \leq j \leq 2$.*

4. Characterization of $\mathcal{C}_3$-generic minimally flat 4-uniform hypergraphs

4.1. j-extensions preserve independence in the lifting matrix

Recall that the *j-extension operation at vertex v* picks j hyperedges $e_1, e_2, ..., e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size 4 incident with both v and z, and replaces e_i by $e_i - v + z$ for all $1 \leq i \leq j$. The lifting matrix for a picture $S(r)$ is the $|I| \times (|V| + d|F|)$

coefficient matrix $M(S, r)$ in which the row correspoding to $(i, j) \in I$ is:

$$(i,j)\quad \underbrace{\begin{array}{|ccc}\hline & i & \\ 0\ldots0 & 1 & 0\ldots0 \\ \hline\end{array}}_{|V|}\Big\|\underbrace{\begin{array}{cccc|}\hline 0\ldots0 & \multicolumn{2}{c}{\overbrace{r_i \quad 1}^{j}} & 0\ldots0 \\ \hline\end{array}}_{d|F|}$$

In this section we show that if H is a 4-uniform hypergraph with an independent 2-picture and H' is obtained from H by a j-extension for some $j \geq 0$, then H' also has an independent 2-picture.

Theorem 4.1. *Let (H, r) be an independent 2-picture, where $H = (V, F)$ is a 4-uniform hypergraph, and $r : V \to \mathbb{R}^2$ is a location map. Let $H' = (V', F')$ be the hypergraph obtained from H by performing a j-extension at $v \in V$ such that $V' = V + z$ and $\{a, b, v, z\} \in F'$. Put $r(z) = r(v)$. If $r(a), r(b), r(v)$ do not lie on a line, then (H', r) is an independent 2-picture.*

Proof: (H, r) is an independent 2-picture if and only if the rows of $M(H, r)$ are independent. We have to show that the rows of $M(H', r)$ are also independent. $M(H', r)$ is given by

$$\begin{array}{c|ccccc|cccccc|}
 & z & v & a & b & & & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{e_i'}} & & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{e}} \\ \hline
 & & & & \star & & & & \star & & & \\
(z, e_i') & 1 & 0 & 0 & 0 & 0\ldots0 & 0\ldots0 & r(z) & 1 & 0\ldots0 & \underline{0} & 0 \\
 & & & & \star & & & & \star & & & \\ \hline
(z, e) & 1 & 0 & 0 & 0 & & & & & & r(z) & 1 \\
(v, e) & 0 & 1 & 0 & 0 & 0 & & & 0 & & r(v) & 1 \\
(a, e) & 0 & 0 & 1 & 0 & & & & & & r(a) & 1 \\
(b, e) & 0 & 0 & 0 & 1 & & & & & & r(b) & 1 \\ \hline
\end{array}$$

$M(H', r)$ can be constructed from $M(H, r)$ as follows. First, add 4 zero columns, one of which corresponds to z and the rest of them correspond to e. Clearly, this operation results in a row-independent matrix. Then add the rows of incidences $(v, e), (a, e), (b, e)$. The rows of the matrix obtained are independent since $r(a), r(b), r(v)$ do not lie on a line. Then adding the row of (z, e) preserves the independence because no other row has a non-zero element in the first column.

Now, observe that what is left is to modify the rows corresponding to the incidences (v, e_i) for $0 \leq i \leq j$. We can obtain the desired row of (z, e_i') by subtracting the row of (v, e) and adding the row of (z, e). These operations also preserve independence. This completes the proof. $\square$

4.2. Minimally flat C_3-symmetric 4-uniform hypergraphs

First we shall see that the lifting matrices corresponding to the base graphs have full rank. Observe that for H_4 the first four columns of $M(H,r)$ form an identity matrix and hence its rows are independent.

If H is isomorphic to H_6 or H_6', then we will construct a row-independent $\mathcal{C}_3$-symmetric realization using Theorem 4.1. Let $r(v_1) \neq (0,0)$ be arbitrary, and place $\mathcal{C}_3 v_1$ symmetrically.

For H_6, start with the hyperedge $\{\mathcal{C}_3 v_i, v_2\}$ and put $r(v_1) = r(v_2)$. Then add $\{\mathcal{C}_3 v_i, \gamma v_2\}$ with $r(\gamma v_1) = r(\gamma v_2)$, and finally add $\{\mathcal{C}_3 v_i, \gamma^2 v_2\}$ with $r(\gamma^2 v_1) = r(\gamma^2 v_2)$. This realization is row-independent by Theorem 4.1.

For H_6', we put $r(v_2) = r(\gamma v_1)$ (and then $r(\gamma v_2) = r(\gamma^2 v_1)$, $r(\gamma^2 v_2) = r(v_1)$). We start again with the hyperedge $\{\mathcal{C}_3 v_i, \gamma v_2\}$. Then apply a 1-extension at $\gamma^2 v_1$. This results in deleting the only edge and adding $\{v_1, v_2, \gamma v_1, \gamma v_2\}$ and $\{\mathcal{C}_3 v_i, \gamma v_2\}$. After one more 1-extension at v_1 we obtain hypergraph H_6'. Both of these extensions satisfy the conditions of Theorem 4.1, and hence we can conclude that this realization is row-independent.

We shall prove that the symmetric extensions defined in Section 3.5 preserve the row-independence of the lifting matrix.

Lemma 4.2. *Every $\mathcal{C}_3$-symmetric extension preserves the ($\mathcal{C}_3$-generic) independence of the rows of the lifting matrix.*

Proof: Let H be $\mathcal{C}_3$-generically independent, and let $H(r)$ be $\mathcal{C}_3$-generic. Then without loss of generality we may assume that no three vertices of $H(r)$ are collinear (by slightly perturbing the vertices if necessary). Recall that symmetric extensions consist of three consecutive extensions performed such that the resulting graph is symmetric. It suffices to show that in all three extensions, $r(a), r(b), r(v)$ do not lie on a line. The statement then follows from Theorem 4.1 since the existence of a $\mathcal{C}_3$-symmetric independent picture $H'(r)$ implies that H' is $\mathcal{C}_3$-generically independent.

If we apply three j-extensions on H such that no pair of the three new vertices $\mathcal{C}_3 u$ share an edge, then, for the first of the three j-extensions, we have $\{a, b, v\} \in V(H)$, and hence $r(a), r(b)$ and $r(v)$ cannot be collinear. Similarly, for the second and third j-extension, the vertices $\{a, b, v\}$ are again in H (as images of the original set under γ and γ^2, respectively) and hence $r(a), r(b)$ and $r(v)$ can again not be collinear.

If u and γu share an edge, then, for the first of the three j-extensions, we again have $\{a, b, v\} \in V(H)$, and hence a, b and v cannot be collinear. Moreover, for the second and third j-extension, the vertices $\{a, b, v\}$ can again be chosen to be the images of the original set under γ and γ^2, respectively. This completes the proof. $\square$

As a corollary we obtain the following.

Theorem 4.3. *A $\mathcal{C}_3$-symmetric 4-uniform hypergraph H is $\mathcal{C}_3$-generically minimally flat if and only if it is $\mathcal{C}_3$-tight.*

4.3. Minimally flat $\mathcal{C}_3$-symmetric incidence structures

Theorem 4.4. *A $\mathcal{C}_3$-symmetric incidence structure $S = (V, F; I)$ is $\mathcal{C}_3$-generically minimally flat if and only if $|I| = |V| + 3|F| - 3$, $|I'| \leq |V'| + 3|F'| - 3$ for every subset of incidences $|I'|$ with at least one face and $|I_3(S)| = |V_3(S)|$.*

Proof: By Lemma 3.1, S satisfies the conditions of the theorem if and only if $H_3(S)$ is $\mathcal{C}_3$-tight.

Consider the edge set E_j that corresponds to the face $f_j = \{v_1, v_2, v_3, \ldots, v_{m+3}\}$. (See Section 3.2 for definitions.) Put $\{v_1, v_2, v_3, v_i\} = e_i$ for every $4 \leq i \leq m+3$. We will delete vertex v_i from the hyperedge e_i and add v_i to e_4 for every $5 \leq i \leq m+3$ successively. Thus, at the end of the process, we obtain the face f_j and $m-1$ copies of the face $\{v_1, v_2, v_3\}$ from the set E_j. We will observe that moving every vertex results in an independent structure. The faces of size three can then be deleted.

Let $M_{i,l}$ denote the row of $M(S, r)$ corresponding to the incidence between v_i and e_l. By our assumption, r is symmetry-generic, and hence r_i is in the affine span of r_1, r_2, r_3 for every $4 \leq i \leq m+3$. Equivalently, there are coefficients $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}$ with $\alpha_{1i}(r_1, 1) + \alpha_{2i}(r_2, 1) + \alpha_{3i}(r_3, 1) = (r_i, 1)$.

The next matrix shows the rows of $M(S, r)$ corresponding to the edges e_4 and e_i.

$$
\begin{array}{c|cccccc|cccc|l}
 & v_1 & v_2 & v_3 & v_4 & v_i & & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{e_4}} & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{e_i}} & \\
\hline
(v_1, e_4) & 1 & 0 & 0 & 0 & 0 & & r_1 & 1 & & & = M_{1,4} \\
(v_2, e_4) & 0 & 1 & 0 & 0 & 0 & 0 & r_2 & 1 & & 0 & = M_{2,4} \\
(v_3, e_4) & 0 & 0 & 1 & 0 & 0 & & r_3 & 1 & & & = M_{3,4} \\
(v_4, e_4) & 0 & 0 & 0 & 1 & 0 & & r_4 & 1 & & & = M_{4,4} \\
\hline
(v_1, e_i) & 1 & 0 & 0 & 0 & 0 & & & & r_1 & 1 & = M_{1,i} \\
(v_2, e_i) & 0 & 1 & 0 & 0 & 0 & 0 & & 0 & r_2 & 1 & = M_{2,i} \\
(v_3, e_i) & 0 & 0 & 1 & 0 & 0 & & & & r_3 & 1 & = M_{3,i} \\
(v_i, e_i) & 0 & 0 & 0 & 0 & 1 & & & & r_i & 1 & = M_{i,i} \\
\hline
\end{array}
$$

If we replace the row $M_{i,i}$ with $M_{i,i} + \sum_{k=1}^{3} \alpha_{ki}(M_{k,4} - M_{k,i})$, then we get the following:

$$\begin{array}{c|cccccc|ccc|}
 & v_1 & v_2 & v_3 & v_4 & v_i & & \overbrace{\hphantom{r_i \quad 1}}^{e_4} & & \overbrace{\hphantom{0 \quad 0}}^{e_i} \\ \hline
(v_i, e_4) & 0 & 0 & 0 & 0 & 1 & 0\ldots0 & r_i \quad 1 & 0\ldots0 & 0 \quad 0 \\ \hline
\end{array}$$

The rows of the resulting matrix are linearly independent and it corresponds to the incidence structure in which v_i is deleted from e_i and is added to e_4. Observe that we only used the fact that v_1, v_2, v_3 are in general position and are contained in both e_4 and e_i. Hence the above argument also works for fixed edges and edges with larger cardinality. Now we delete the rows $M_{1,i}M_{2,i}M_{3,i}$, and then the columns corresponding to e_i only contain zeros, so that the deletion of these columns also preserves independence.

We can apply the same method for every $1 \leq j \leq |V| - 3$ to construct S and see that it is independent. This completes the proof. $\square$

5. Sharp $\mathcal{C}_3$-symmetric pictures

In this section we give a sufficient condition for sharpness of $\mathcal{C}_3$-generic pictures. We will first need the definition of deficiency for *incidence structures*. For $S = (V, F; I)$ the *deficiency* of a vertex set $X \subseteq V$ is defined by $\mathrm{def}(X) = |X| + 3f(X) - i(X)$ where $f(X)$ and $i(X)$ denote the number of faces and the number of incidences in $S[X]$. We may apply the same proof method as for Lemmas 3.7 and 3.8 to obtain the following two results:

Lemma 5.1. *Suppose that $X \subseteq V$ is such that $\mathrm{def}(X \cap \gamma X) \geq \mathrm{def}(X)$ and $\mathrm{def}(Z) \geq \mathrm{def}(X)$ for any $Z \supseteq X$. Then $\mathrm{def}(X) = \mathrm{def}(\mathcal{C}_3 X)$.*

Lemma 5.2. $\mathrm{def}(\mathcal{C}_3 X) \equiv 0, 1 \pmod 3$ *for every $X \subseteq V$.*

As a corollary of Theorem 4.4 we get a sufficient condition for independence of $\mathcal{C}_3$-generic incidence structures.

Corollary 5.3. *Suppose that for the $\mathcal{C}_3$-symmetric incidence structure $S = (V, F; I)$ we have $|I'| \leq |V'| + 3|F'| - 3$ for every subset of incidences $|I'|$ with at least one face and $|I_3(S)| = |V_3(S)|$. Then S is $\mathcal{C}_3$-generically independent.*

Proof: We may assume that $|I| \leq |V| + 3|F| - 4$, for otherwise S is $\mathcal{C}_3$-generically minimally flat and hence independent by Theorem 4.4. We will prove that S is a substructure of a $\mathcal{C}_3$-generically minimally flat incidence structure. Observe that by the symmetry of S and $|I_3(S)| = |V_3(S)|$, $|I| \leq |V| + 3|F| - 6$ must hold. Hence there is an incidence $(v, f) \notin I$ with $v \neq v_0$ for which the (non-symmetric) 2-picture $S_1(r)$, where $S_1 = (V, F, I + (f, v))$, does not violate the sparsity condition.

Consider the symmetric $\mathcal{C}_3$-generic structure $S_2(r)$, where $S_2 = (V, F, I_2)$ and $I_2 = I + \mathcal{C}_3(f, v)$. Note that $|I_3(S_2)| = |V_3(S_2)|$. Suppose that $S_2(r)$ is not independent, that is, there is a substructure $S' = (V', F', I')$ of S_2 with $|I'| > |V'| + 3|F'| - 3$. Let S' be minimal. By the sparsity of S_1 this can happen in two ways. The first case is $f, \gamma f \in F'$, $\mathrm{def}_S(V') = 1$, and the second case is $\mathcal{C}_3 f \in F'$, $\mathrm{def}_S(V') = 2$. In the second case we get a contradiction using Lemmas 5.1 and 5.2.

The first case can only occur if $v_0 \in V'$ and $f_0 \notin F'$. If for every vertex $v \notin f$, $v \neq v_0$, we can find such a substructure, then it is not difficult to see that the union of these substructures violate sparsity. Thus S_2 is independent for some $v \notin f$, $v \neq v_0$, and so is its substructure S. $\square$

Theorem 5.4. *Let $S = (V, F, I)$ be a $\mathcal{C}_3$-symmetric incidence structure with $|I'| \leq |V'| + 3|F'| - 4$ for every substructure of S with at least two faces.*

(i) If $|V_3(S)| = 0$ then S is $\mathcal{C}_3$-generically sharp.

(ii) If $|V_3(S)| = |I_3(S)| = 1$ and $|I'| \leq |V'| + 3|F'| - 6$ holds for every $\mathcal{C}_3$-symmetric substructure of S with at least two faces, then S is $\mathcal{C}_3$-generically sharp.

Proof: Without loss of generality we may assume that every face contains at least four vertices. Let $S(r)$ be a $\mathcal{C}_3$-generic 2-picture. First we would like to show that for every pair (f_1, f_2) of its faces, there is a lifting $S(p, P)$ in which f_1 and f_2 lie in a different plane.

Note that two faces f_1, f_2 cannot have the same plane in a lifting if there is a vertex $u \in f_2 - f_1$ which is not in the plane of f_1. It follows from the sparsity condition of the theorem and from the assumption that every face contains at least four vertices that $f_2 - f_1 \neq \emptyset$. Hence for every pair $f_1, f_2 \in F$, there is a vertex u for which $S_1 = (V, F, I + (f_1, u))$ satisfies $|I'| \leq |V'| + 3|F'| - 3$. We shall see that the structure $S_2 = (V, F, I + \mathcal{C}_3(f_1, u))$ satisfies the same count. Note that in case (i) it follows easily that S satisfies $|I'| \leq |V'| + 3|F'| - 6$ for every $\mathcal{C}_3$-symmetric substructure with at least two faces.

Suppose for a contradiction that there is a substructure $S' = (V', F', I')$ of S_2 with $|I'| > |V'| + 3|F'| - 3$ and let S' be minimal. Then we may assume that $f_1, \gamma f_1 \in F'$ and $\mathrm{def}_S(V') = 1$ or $\mathcal{C}_3 f_1 \in F'$ and $\mathrm{def}_S(V') = 2$. Both of these lead to a contradiction using Lemma 5.1. Then, by Corollary 5.3, $S_2(r)$ is independent, and so is $S_1(r)$.

Hence the dimension of the solution space of the matrix $M(S, r)$ is larger than that of $M(S_1, r)$. Thus, it must contain a solution which places u out of the plane of f_1, that is, f_1, f_2 are not in the same plane. Such a solution exists for every pair $f_1, f_2 \in F$. An appropriate linear combination of these solutions gives a sharp lifting. This completes the proof. □

6. Concluding remarks

In this paper we characterized $\mathcal{C}_3$-generically minimally flat incidence structures. However, we note that there are flat $\mathcal{C}_3$-symmetric incidence structures without a spanning minimally flat $\mathcal{C}_3$-symmetric substructure. Hence our result does not give a complete characterization for $\mathcal{C}_3$-generically flat (but not necessarily minimally flat) incidence structures. (Consider the following example: $V = \{v_0, v_1, \ldots, v_6\}$ and $F = \{\{v_0, v_i, v_j, v_k\} : 1 \leq i, j, k \leq 6\}$. This structure is clearly $\mathcal{C}_3$-generically flat, but it does not have a minimally flat spanning incidence structure with $\mathcal{C}_3$ symmetry, because none of its spanning substructures satisfy (7).) Finding a characterization for the class of $\mathcal{C}_3$-generically flat incidence structures is a direction for future research.

We also gave a sufficient condition for sharpness of $\mathcal{C}_3$-symmetric 2-pictures. However, the problem of giving a full characterization remains open.

It is of course also natural to try to prove a constructive characterization for Γ-generically minimally flat structures with respect to other symmetry groups $\tau(\Gamma)$. However, our proof for $\mathcal{C}_3$ cannot directly be transferred to these other groups. In particular, considering symmetric derived 4-uniform hypergraphs is in general not useful for symmetry groups different from $\mathcal{C}_3$. In other words, a statement equivalent to Lemma 3.1 does in general not exist.

For example, consider the half-turn symmetry group $\mathcal{C}_2$ and the $\mathcal{C}_2$-symmetric incidence structure $S = (V, F; I)$, where $V = \{v_0, \mathcal{C}_2 v_1, \mathcal{C}_2 v_2\}$ and $F = \{\{v_0, \mathcal{C}_2 v_1, \mathcal{C}_2 v_2\}\}$. Clearly, S is $\mathcal{C}_2$-generically minimally flat and satisfies all the necessary conditions for minimal flatness. There are two possibilities to construct a $\mathcal{C}_2$-symmetric derived 4-uniform hypergraph of S. Both hypergraphs have two hyperedges which share three collinear vertices, and hence are foldable. Also, neither of them satisfies (4).

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