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Functional inequalities on manifolds with non-convex boundary

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Abstract In this article, new curvature conditions are introduced to establish functional inequalities including gradient estimates, Harnack inequalities and transportation-cost inequalities on manifolds with non-convex boundary.

Keywords Ricci curvature, gradient inequality, log-Sobolev inequality, geometric flow

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1 Introduction

Let (M, g) be a complete and connected Riemannian manifold of dimension $d \geqslant 2$, with Riemannian distance ρ, boundary ∂M and inward pointing unit normal vector N. Define the second fundamental form of the boundary by

$$\mathrm{II}(X,Y) = -\langle \nabla_X N, Y\rangle, \quad X, Y \in T_x\partial M, \quad x \in \partial M$$

where $T\partial M$ denotes the tangent bundle of ∂M. In order to study non-convex boundaries we will perform a conformal change of metric such that the boundary is convex under the new metric. In particular, we will use the fact that if

$$\mathscr{D} := \{\phi \in C_b^2(M)\colon\ \inf \phi = 1, \mathrm{II} \geqslant -N \log \phi\}$$

and $\phi \in \mathscr{D}$ then the boundary ∂M is convex under the metric $\phi^{-2}g$ (see [16, Theorem 1.2.5]).

Given a C^1-vector field Z on M, consider the elliptic operator $L := \Delta + Z$ and let X_t^x be a reflecting L-diffusion process starting from $X_0^x = x$. Then X_t^x solves the Stratonovich equation

$$\mathrm{d}X_t^x = \sqrt{2}u_t^x \circ \mathrm{d}B_t + Z(X_t^x)\,\mathrm{d}t + N(X_t^x)\,\mathrm{d}l_t^x, \quad X_0^x = x$$

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where u_t^x is the horizontal lift of X_t^x to the orthonormal frame bundle $\mathrm{O}(M)$ with $\pi(u_0^x) = x$, B_t is a standard $\mathbb{R}^d$-valued Brownian motion defined on a complete naturally filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and l_t^x is a continuous adapted nondecreasing and nonnegative process which increases only on $\{t \geqslant 0\colon X_t^x \in \partial M\}$. The process l_t^x is the local time of X_t^x on ∂M.

We assume that X_t^x is non-explosive for each $x \in M$. Then the diffusion process X_t^x gives rise to the Neumann semigroup P_t which solves the diffusion equation $(\partial_t - L)P_t = 0$ with Neumann boundary condition $NP_t = 0$. Furthermore $P_t f(x) = \mathbb{E}[f(X_t^x)]$ for each $f \in C_b(M)$.

In [7], Hsu found a probabilistic formula for $\nabla P_t f$ for compact manifolds with boundary, which he used to derive a gradient estimate. Feng-Yu Wang extended it to the non-compact case [16, Theorem 3.2.1] under the assumption that $|\nabla P_{\boldsymbol{\cdot}} f|$ is uniformly bounded on $[0,t]\times M$. Wang's formula is given below by Theorem 2.1. In [16, Proposition3.2.7], he proved that if

$$\mathrm{Ric}^Z := \mathrm{Ric} - \nabla Z \geqslant K$$

for some $K \in C(M)$ and if there exists $\phi \in \mathcal{D}$ such that

$$\tilde{K}_\phi := \inf_M \left\{ \phi^2 K + \frac{1}{2} L\phi^2 - |\nabla \phi^2|\,|Z| - (d-2)|\nabla\phi|^2 \right\} > -\infty \tag{1.1}$$

then $|\nabla P_{\boldsymbol{\cdot}} f|$ is uniformly bounded on $[0,t]\times M$ by an expression involving the constant $\tilde{K}_\phi$.

In this article, we revisit this problem using coupling methods. In particular, we prove (see Theorem 2.2) that if there exists $\phi \in \mathcal{D}$ and a constant K_ϕ such that

$$\mathrm{Ric}^Z + L\log\phi - 2|\nabla \log \phi|^2 \geqslant K_\phi$$

then $|\nabla P_{\boldsymbol{\cdot}} f|$ is uniformly bounded on $[0,t]\times M$ and our upper bound improves that of [16, Proposition 3.2.7]. We construct a suitable function ϕ in Proposition 3.3, under the assumption that there exist non-negative constants σ and θ such that $-\sigma \leqslant \mathrm{II} \leqslant \theta$ and a positive constant r_0 such that on $\partial_{r_0} M := \{x \in M\colon \rho_\partial(x) \leqslant r_0\}$ the function ρ_∂ is smooth, the norm of Z is bounded and $\mathrm{Sect} \leqslant k$ for some positive constant k.

F.-Y. Wang also considered Harnack and transportation-cost inequalities on manifolds with boundary [16]. We reconsider these problems too and find that the curvature conditions used to establish these inequalities can also be weakened and simplified. It is worth mentioning that we find a transportation-cost inequality on the path space of the reflecting diffusion process which (see Theorem 2.8) recovers the results for the convex boundary case, making this aspect of the theory of functional inequalities on path space complete.

Let us now describe the organization of this article. In Section 2, we prove the gradient estimates, Harnack inequalities and transportation-cost inequalities for the Neumann semigroup via coupling methods. In Section 3, we construct a function ϕ which satisfies the new curvature conditions.

2 Functional inequalities

2.1 Gradient estimates

A derivative formula for $P_t f$ that does not involve derivatives of f is typically called a Bismut formula (see [4,5]). The Bismut formula we introduce is of a type due originally to Thalmaier [9]. As mentioned in the introduction, Hsu [7] found this type of formula for compact manifolds with boundary. The following formula for manifolds with boundary, due to F.-Y. Wang [16, Theorem 3.2.1], does not require compactness. See also [1] for recent work on probabilistic representations of the derivative of Neumann semigroups.

Theorem 2.1. *Let $t > 0$ and $u_0 \in \mathrm{O}_x(M)$ be fixed. Suppose $K \in C(M)$ and $\sigma \in C(\partial M)$ are such that $\mathrm{Ric}_Z \geqslant K$ and $\mathrm{II} \geqslant \sigma$. Assume that*

$$\sup_{s\in[0,t]} \mathbb{E}^x \left[\exp\left(-\int_0^s K(X_r)\,\mathrm{d}r - \int_0^s \sigma(X_r)\,\mathrm{d}l_r \right) \right] < \infty.$$

Then there exists a progressively measurable process $\{Q_s\}_{s\in[0,t]}$ *on* $\mathbb{R}^d\otimes\mathbb{R}^d$ *such that*

$$Q_0=I,\quad \|Q_s\|\leqslant \exp\left(-\int_0^s K(X_r)\,\mathrm{d}r-\int_0^s\sigma(X_r)\,\mathrm{d}l_r\right),\quad s\in[0,t]$$

and for any $f\in C_b^1(M)$ *such that* $\nabla P_{\boldsymbol{\cdot}} f$ *is bounded on* $[0,t]\times M$*, for any* $h\in C^1([0,t])$ *with* $h(0)=0$ *and* $h(t)=1$*, we have*

$$u_0^{-1}\nabla P_t f(x)=\mathbb{E}^x\left[Q_t u_t^{-1}\nabla f(X_t)\right]=\frac{1}{\sqrt{2}}\,\mathbb{E}^x\left[f(X_t)\int_0^t \dot h(s)Q_s\,\mathrm{d}B_s\right].$$

In order to use this formula it is necessary to check the uniform boundedness of $\nabla P_{\boldsymbol{\cdot}} f$ on $[0,t]\times M$. In [16, Proposition 3.2.7], F.-Y. Wang did so using a conformal change of metric such that under the new metric the boundary is convex, and by then making a time change of the L-diffusion process X_t. Here, we use coupling methods to study this problem again and obtain improved upper bounds.

Theorem 2.2. *If there exist* $\phi\in\mathscr{D}$ *and a constant* K_ϕ *such that*

$$\mathrm{Ric}^Z+L\log\phi-2|\nabla\log\phi|^2\geqslant K_\phi \tag{2.1}$$

then for all $f\in C^1(M)$ *such that* f *is constant outside a compact set,*

$$|\nabla P_t f|\leqslant\|\phi\|_\infty\|\nabla f\|_\infty\,\mathrm{e}^{-K_\phi t},\quad t>0.$$

Proof. We start with a conformal change of the metric g. Since $\phi\in\mathscr{D}$, the boundary ∂M is convex under the metric $g':=\phi^{-2}g$. Let Δ' and ∇' be the Laplacian and gradient operator associated with the metric g'. Then

$$L=\phi^{-2}\left(\Delta'+\phi^2\left(Z+(d-2)\nabla\log\phi\right)\right)=\phi^{-2}\left(\Delta'+Z'\right) \tag{2.2}$$

where $Z':=\phi^2\left(Z+(d-2)\nabla\log\phi\right)$. For the process X_t generated by L, viewed as a process on (M,g'), denoting by d_I the Itô differential, it follows that

$$\mathrm{d}_I X_t=\sqrt{2}\phi^{-1}(X_t)u_t\,\mathrm{d}B_t+\phi^{-2}(X_t)Z'(X_t)\,\mathrm{d}t+N'(X_t)\,\mathrm{d}l_t,\quad X_0=x \tag{2.3}$$

where B_t is the Brownian motion and the lift u_t and boundary local time l_t are defined now with respect to the metric g'. Recall that in local coordinates, the Itô differential of a continuous semimartingale X_t on M is given (see [6] or [2]) by

$$(\mathrm{d}_I X_t)^k=\mathrm{d}X_t^k+\frac{1}{2}\sum_{i,j=1}^d {\Gamma'}_{ij}^k(X_t)\,\mathrm{d}\langle X^i,X^j\rangle_t,\quad 1\leqslant k\leqslant d$$

where ${\Gamma'}_{ij}^k$ are the Christoffel symbols of g'. Similarly, let Y_t solve

$$\mathrm{d}_I Y_t=\sqrt{2}\phi^{-1}(Y_t)(1_{\{(X_t,Y_t)\notin\mathrm{cut}\}}P'_{X_t,Y_t}u_t\,\mathrm{d}B_t+1_{\{(X_t,Y_t)\in\mathrm{cut}\}}\tilde u_t\,\mathrm{d}B'_t)+\phi^{-2}(Y_t)Z'(Y_t)\,\mathrm{d}t+N'(Y_t)\,\mathrm{d}\tilde l_t$$

with $Y_0=y$, lift $\tilde u_t$ and boundary local time $\tilde l_t$, where $\mathrm{cut}\subset M\times M$ denotes the set of cut points and where B'_t is a Brownian motion independent of B_t (see [16, Theorem 3.2.5] or [11, Section 2.1]). Now, for $(x,y)\notin\mathrm{cut}$ and $x\neq y$, define

$$I_Z^\phi(x,y):=\sum_{i=1}^d(U_i)^2\rho'(x,y)+\left\langle\phi^{-2}(y)Z'(y),\nabla'\rho'(x,\cdot)(y)\right\rangle'+\left\langle\phi^{-2}(x)Z'(x),\nabla'\rho'(\cdot,y)(x)\right\rangle'$$

where $\{U_i\}_{i=1}^d$ are vector fields on $M\times M$ such that $\nabla' U_i(x,y)=0$ and

$$U_i(x,y)=(\phi^{-1}(x)V_i,\phi^{-1}(y)\,P'_{x,y}V_i),\quad 1\leqslant i\leqslant d$$

for $\{V_i\}_{i=1}^d$ a g'-orthonormal basis of T_xM. Here $P'_{x,y}$ denotes parallel displacement from x to y with respect to the metric g'. Denote by ρ' the distance function for the metric g'. Since the boundary ∂M is convex under g', that is $N'\rho'|_{\partial M} \leqslant 0$, by a slight modification of the proof of [11, Theorem 2.1.6], we have the following result which is similar to [16, Theorem 3.2.5]: if there exists $J \in C(M \times M)$ such that $J \geqslant I_Z^\phi$ outside the cut locus and $D(M)$, where $D(M) := \{(x,x) : x \in M\}$, then

$$\mathrm{d}\rho'(X_t, Y_t) \leqslant \sqrt{2}\left(\phi^{-1}(X_t) - \phi^{-1}(Y_t)\right)\mathrm{d}b_t + J(X_t, Y_t)\,\mathrm{d}t \tag{2.4}$$

up to the coupling time $\tau := \inf\{t \geqslant 0 : X_t = Y_t\}$, where b_t is a one-dimensional Brownian motion. From this, it now suffices for us to estimate the term I_Z^ϕ. Write $\rho' = \rho'(x,y)$ and for a minimizing g'-geodesic γ with $\gamma(0) = x$ and $\gamma(\rho') = y$ let

$$J_i(s) = \phi^{-1}(\gamma(s))\, P'_{\gamma(0),\gamma(s)}V_i, \quad 1 \leqslant i \leqslant d$$

where $J_i(0) = \phi^{-1}(x)V_i$ and $J_i(\rho') = \phi^{-1}(y)P'_{x,y}V_i$. Since $P'_{\gamma(0),\gamma(s)}V_i$ are parallel vector fields along γ with respect to the metric g', we have that for $(x,y) \notin \mathrm{cut} \cup D(M)$, that

$$\begin{aligned}
&\sum_{i=1}^d (U_i)^2 \rho'(x,y)\\
&\leqslant \sum_{i=1}^d \int_0^{\rho'} \left\{ |\nabla'_{\dot\gamma} J_i|'^2 - \langle R'(\dot\gamma, J_i)J_i, \dot\gamma\rangle' \right\}(s)\,\mathrm{d}s\\
&= d\int_0^{\rho'} \phi^{-2}(\gamma(s))\langle \nabla \log\phi(\gamma(s)), \dot\gamma(s)\rangle^2\,\mathrm{d}s - \int_0^{\rho'} \phi^{-2}(\gamma(s))\mathrm{Ric}'(\dot\gamma(s), \dot\gamma(s))\,\mathrm{d}s.
\end{aligned} \tag{2.5}$$

On the other hand

$$\begin{aligned}
&\phi^{-2}(x)\,\langle Z'(x), \nabla'\rho'(\cdot, y)(x)\rangle' + \phi^{-2}(y)\,\langle Z'(y), \nabla'\rho'(x, \cdot)(y)\rangle'\\
&= \int_0^{\rho'} \frac{\mathrm{d}}{\mathrm{d}s}\left\{\phi^{-2}(\gamma(s))\,\langle Z'(\gamma(s)), \dot\gamma(s)\rangle'\right\}\,\mathrm{d}s\\
&= \int_0^{\rho'} \phi^{-2}(\gamma(s))\,\left\langle (\nabla'_{\dot\gamma}Z')\circ\gamma, \dot\gamma\right\rangle'(s)\,\mathrm{d}s\\
&\quad - 2\int_0^{\rho'} \phi^{-2}(\gamma(s))\langle\nabla\log\phi(\gamma(s)), \dot\gamma(s)\rangle\,\langle Z'(\gamma(s)), \dot\gamma(s)\rangle'\,\mathrm{d}s.
\end{aligned} \tag{2.6}$$

Moreover

$$\langle Z'(\gamma(s)), \dot\gamma(s)\rangle' = \langle Z, \dot\gamma(s)\rangle + (d-2)\langle\nabla\log\phi, \dot\gamma(s)\rangle.$$

Combining this with (2.5) and (2.6), we have

$$\begin{aligned}
I_Z^\phi(x,y) \leqslant &-\int_0^{\rho'} \phi^{-2}(\gamma(s))((\mathrm{Ric}^Z)'(\dot\gamma(s), \dot\gamma(s)) + (d-4)\langle\nabla\log\phi, \dot\gamma(s)\rangle^2\,\mathrm{d}s\\
&- \int_0^{\rho'} 2\langle\nabla\log\phi, \dot\gamma(s)\rangle\langle Z, \dot\gamma(s)\rangle)\,\mathrm{d}s.
\end{aligned} \tag{2.7}$$

By [3, Theorem 1.159], in which the Laplacian differs from our's by a negative sign, we know that

$$\begin{aligned}
(\mathrm{Ric}^Z)'(\dot\gamma, \dot\gamma) &= \mathrm{Ric}'(\dot\gamma, \dot\gamma) - \langle\nabla'_{\dot\gamma}Z', \dot\gamma\rangle'\\
&= \mathrm{Ric}^Z(\dot\gamma, \dot\gamma) + \frac{1}{2}L\phi^2 - 2\,\langle\nabla\log\phi, \dot\gamma\rangle\,\langle Z, \dot\gamma\rangle - (d-2)\,\langle\dot\gamma, \nabla\log\phi\rangle^2 - 2|\nabla\phi|^2
\end{aligned}$$

and, noting that $|\dot\gamma| = \phi$, we thus have

$$(\mathrm{Ric}^Z)'(\dot\gamma(s), \dot\gamma(s)) + (d-4)\langle\nabla\log\phi, \dot\gamma(s)\rangle^2 + 2\langle\nabla\log\phi, \dot\gamma(s)\rangle\langle Z, \dot\gamma(s)\rangle$$

$$
\begin{aligned}
&= \mathrm{Ric}^Z(\dot\gamma(s),\dot\gamma(s)) + \frac{1}{2}L\phi^2 - 2\left\langle \dot\gamma, \nabla\log\phi\right\rangle^2 - 2|\nabla\phi|^2\\
&\geqslant \mathrm{Ric}^Z(\dot\gamma(s),\dot\gamma(s)) + \frac{1}{2}L\phi^2 - 4|\nabla\phi|^2\\
&= \mathrm{Ric}^Z(\dot\gamma(s),\dot\gamma(s)) + \phi^2 L\log\phi - 2|\nabla\phi|^2. \qquad (2.8)
\end{aligned}
$$

Consequently, using the condition (2.1), letting $X_t = Y_t$ after coupling time, and then combining (2.7) with (2.8) and (2.4), we arrive at

$$
\mathrm{d}\rho'(X_t,Y_t) \leqslant \sqrt{2}(\phi^{-1}(X_t) - \phi^{-1}(Y_t))\,\mathrm{d}b_t - K_\phi\rho'(X_t,Y_t)\,\mathrm{d}t. \qquad (2.9)
$$

From this, we know that

$$
\mathbb{E}^{(x,y)}\left[\rho'(X_t,Y_t)\right] \leqslant \mathrm{e}^{-K_\phi t}\,\rho'(x,y).
$$

Then, observing that $\rho' \leqslant \rho \leqslant \|\phi\|_\infty\rho'$, we have

$$
\begin{aligned}
|\nabla P_t f|(x) &= \lim_{y\to x}\left|\frac{P_tf(x)-P_tf(y)}{\rho(x,y)}\right|\\
&= \lim_{y\to x}\left|\mathbb{E}^{(x,y)}\left[\frac{f(X_t)-f(Y_t)}{\rho(X_t,Y_t)}\,\frac{\rho(X_t,Y_t)}{\rho'(X_t,Y_t)}\,\frac{\rho'(X_t,Y_t)}{\rho'(x,y)}\,\frac{\rho'(x,y)}{\rho(x,y)}\right]\right|\\
&\leqslant \|\phi\|_\infty\|\nabla f\|_\infty\,\mathrm{e}^{-K_\phi t}
\end{aligned}
$$

which completes the proof. □

Remark 2.3. (i) Since $(U_d)^2\rho' \neq 0$, it was indeed necessary to account for this quantity in inequality (2.5), correcting the proof of [16, Theorem 3.4.6].

(ii) Compared with the proof of [16, Theorem 3.4.6], our choice of vector field J_i yields a simpler result.

(iii) In [17], a certain technical assumption which was used to ensure the uniformly boundedness of $|\nabla P_\cdot f|$ on $[0,t]\times M$ is no longer needed in the results.

The following results remove the additional condition in [16, Corollary 3.6.5 (1)] and [17, Corollary 1.2 (1)] to ensure the uniform boundedness of $|\nabla P_\cdot f|$ on $[0,t]\times M$ and give a another proof of an extension of these inequalities to L^p forms for $p>1$:

Theorem 2.4. *If there exists $\phi\in\mathscr{D}$ such that for $p>1$ the inequality*

$$
\mathrm{Ric}^Z + L\log\phi - p|\nabla\log\phi|^2 \geqslant K_{\phi,p} \qquad (2.10)
$$

holds, then for $t>0$ and $f\in C_b^1(M)$,

$$
|\nabla P_t f| \leqslant \frac{1}{\phi}\,\mathrm{e}^{-K_{\phi,p}t}\left(P_t(\phi|\nabla f|)^{p/(p-1)}\right)^{(p-1)/p}.
$$

Proof. The lower bound (2.10) implies $\mathrm{Ric}^Z + L\log\phi - 2|\nabla\log\phi|^2$ is bounded below (since $\phi\in\mathscr{D}$ implies $|\nabla\log\phi|$ bounded). By Theorem 2.2, it follows that $|\nabla P_\cdot f|$ is bounded on $[0,t]\times M$. Furthermore

$$
\mathrm{Ric}^Z \geqslant K_{\phi,p} - L\log\phi + p|\nabla\log\phi|^2 = K_{\phi,p} + \frac{1}{p}\phi^p L\phi^{-p} \quad \text{and} \quad \mathrm{II} \geqslant -N\log\phi
$$

and so, by Theorem 2.1, there exists $\{Q_s\}_{s\in[0,t]}$ such that

$$
\|Q_t\| \leqslant \exp\left(-K_{\phi,p}t - \frac{1}{p}\int_0^t \phi^p L\phi^{-p}(X_s)\,\mathrm{d}s + \int_0^t N\log\phi(X_s)\,\mathrm{d}l_s\right) \qquad (2.11)
$$

with

$$
|\nabla P_t f|^p \leqslant \left(P_t(\phi|\nabla f|)^{p/(p-1)}\right)^{p-1}\mathbb{E}\left[\phi^{-p}(X_t)\|Q_t\|^p\right]. \qquad (2.12)
$$

It therefore suffices to give the upper bound estimate of the following term:

$$\mathbb{E}\left[\phi^{-p}(X_t)\exp\left(-\int_0^t \phi^p L\phi^{-p}(X_s)\,\mathrm{d}s + p\int_0^t N\log\phi(X_s)\,\mathrm{d}l_s\right)\right].$$

To this end, by the Itô formula, it is easy to see that

$$\begin{aligned}\mathrm{d}\phi^{-p}(X_t) &= \langle\nabla\phi^{-p}(X_t), u_t\,\mathrm{d}B_t\rangle + L\phi^{-p}(X_t)\,\mathrm{d}t + N\phi^{-p}(X_t)\,\mathrm{d}l_t\\ &= \langle\nabla\phi^{-p}(X_t), u_t\,\mathrm{d}B_t\rangle - p\phi^{-p}(X_t)\left(-\frac{1}{p}\phi^p L\phi^{-p}(X_t)\,\mathrm{d}t + N\log\phi(X_t)\,\mathrm{d}l_t\right).\end{aligned}$$

So

$$M_t = \phi^{-p}(X_t)\exp\left(-\int_0^t \phi^p(X_s)L\phi^{-p}(X_s)\,\mathrm{d}s + p\int_0^t N\log\phi(X_s)\,\mathrm{d}l_s\right)$$

is a positive local martingale. Thus

$$\mathbb{E}\left[\phi^{-p}(X_t)\exp\left(-\int_0^t \phi^p(X_s)L\phi^{-p}(X_s)\,\mathrm{d}s + p\int_0^t N\log\phi(X_s)\,\mathrm{d}l_s\right)\right] \leqslant \phi^{-p}(x).$$

Combining this with (2.11) and (2.12) completes the proof. □

Corollary 2.5. *If there exists $\phi \in \mathscr{D}$ such that for $p > 1$ the inequality*

$$\mathrm{Ric}^Z + L\log\phi - p|\nabla\log\phi|^2 \geqslant K_{\phi,p}$$

holds, then for $t > 0$ and $f \in C_b^1(M)$,

$$|\nabla P_t f| \leqslant \|\phi\|_\infty\,\mathrm{e}^{-K_{\phi,p}t}(P_t|\nabla f|^{p/(p-1)})^{(p-1)/p};$$

and for $f \in \mathscr{B}_b(M)$ and $t > 0$,

$$|\nabla P_t f|^2 \leqslant \|\phi\|_\infty^2\,\frac{K_{\phi,2}}{\mathrm{e}^{2K_{\phi,2}t}-1}P_t f^2. \tag{2.13}$$

Proof. The first assertion follows from Theorem 2.4 by observing $\phi \geqslant 1$. As Theorem 2.1 can be used under our condition directly, the main idea of the proof of (2.13) is similar to that of [16, Corollary 3.2.8], so we skip it here. □

Note that taking the limit $p \downarrow 1$ in Corollary 2.5 improves Theorem 2.2 by replacing the constant K_ϕ with $K_{\phi,1}$.

2.2 Harnack inequalities

In [16, Theorem 3.4.7] and [14, Theorem 3.1], F.-Y. Wang used a coupling method to obtain dimension free Harnack inequalities and a log-Harnack inequality on manifolds with boundary, assuming $\mathrm{Ric}^Z \geqslant K$ for some $K \in C(M)$ with $\phi \in \mathscr{D}$ such that $\tilde{K}_\phi$ is finite (where the quantity $\tilde{K}_\phi$ is defined as in (1.1)). The coefficient involved in these inequalities is:

$$2\tilde{K}_\phi^- + 4\|\phi Z + (d-2)\nabla\phi\|_\infty\|\nabla\log\phi\|_\infty + 2d\|\nabla\log\phi\|_\infty^2.$$

We now give the following result, weakening the curvature condition, in terms of a different coefficient:

Theorem 2.6. *Assume there exists $\phi \in \mathcal{D}$ such that*

$$\mathrm{Ric}^Z + L\log\phi - 3|\nabla\log\phi|^2 \geqslant K_{\phi,3} \tag{2.14}$$

for some constant $K_{\phi,3}$. Then for $T > 0$, $x, y \in M$, $p > \|\phi\|_\infty^2$ and $f \in C_b^1(M)$, we have

$$\big(P_T f(y)\big)^p \leqslant P_T f^p(x)\exp\left(\frac{\sqrt{p}(\sqrt{p}-1)\,K_{\phi,3}\,\|\phi\|_\infty^2\,\rho^2(x,y)}{8\delta_p(\sqrt{p}-1-\delta_p)(\mathrm{e}^{2K_{\phi,3}T}-1)}\right),$$

where $\delta_p = \max\left\{\|\phi\|_\infty - 1, \frac{\sqrt{p}-1}{2}\right\}$.

Proof. Fix $x,y\in M$ and $T>0$. As in the proof of Theorem 2.2, we consider the process X_t generated by $L=\Delta+Z$ under the metric $g':=\phi^{-2}g$, for which the boundary $(\partial M,g')$ is convex. Let X_t solve equation (2.3) with $X_0=x$. For a strictly positive function $\xi\in C([0,T))$, to be later determined, let Y_t solve

$$\begin{aligned}\mathrm{d}_I Y_t&=\sqrt{2}\phi^{-1}(Y_t)\,1_{\{(X_t,Y_t)\notin\mathrm{cut}\}}P'_{X_t,Y_t}u_t\,\mathrm{d}B_t+\sqrt{2}\phi^{-1}(Y_t)1_{\{(X_t,Y_t)\in\mathrm{cut}\}}\tilde u_t\,\mathrm{d}B'_t\\&\quad+\phi_t^{-2}Z'(Y_t)\,\mathrm{d}t-\frac{\phi^{-1}(Y_t)\rho'(X_t,Y_t)}{\phi^{-1}(X_t)\xi(t)}\nabla'\rho'(X_t,\cdot)(Y_t)\,\mathrm{d}t+N'(Y_t)\,\mathrm{d}\tilde l_t\end{aligned}$$

for $t\in[0,T)$, with $Y_0=y$, where $\tilde l_t$ is the local time of Y_t on ∂M, $\tilde u_t$ is the lift process and B'_t is the Brownian motion independent of B_t from earlier. In the following, we begin with the same argument as in the proof of [16, Theorem 3.4.7]. The different part is how to use our curvature condition to get a new estimate for the radial process $\rho'(X_t,Y_t)$. To this end, as explained in the proof of Theorem 2.2, we may for the sake of conciseness disregard certain technical considerations relating to the cut locus of M. Consider the process (X_t,Y_t) starting from (x,y), which is a well defined continuous process for $t\leqslant T\wedge\zeta$ where ζ is the explosion time of Y_t; that is $\zeta:=\lim_{n\to\infty}\zeta_n$ for $\zeta_n:=\inf\{t>0:\rho'(y,Y_t)\geqslant n\}$. Let

$$\mathrm{d}\tilde B_t=\mathrm{d}B_t+\frac{\rho'(X_t,Y_t)}{\sqrt{2}\xi(t)\phi^{-1}(X_t)}u_t^{-1}\nabla'\rho'(\cdot,Y_t)(X_t)\,\mathrm{d}t,\quad 0\leqslant t<T\wedge\zeta. \tag{2.15}$$

By the Girsanov theorem, for any $s\in(0,T)$ the process B_t is a d-dimensional Brownian motion under the probability measure $R_s\mathbb{P}$ for

$$R_s:=\exp\left[-\int_0^s\frac{\rho'(X_t,Y_t)}{\xi(t)\phi^{-1}(X_t)}\langle\nabla'\rho(\cdot,Y_t)(X_t),u_t\mathrm{d}B_t\rangle'-\frac12\int_0^s\frac{\rho'(X_t,Y_t)^2}{\xi^2\phi^{-2}(X_t)}\mathrm{d}t\right].$$

Let $\mathbb{Q}=R_{T\wedge\zeta}\mathbb{P}$. It has been shown in the proof of [16, Theorem 3.4.7] that $\mathbb{Q}(\zeta=T)=1$ and the coupling is successful up to the time T. In the following, we look at the processes under the new measure $\mathbb{Q}$. Then

$$\begin{aligned}\mathrm{d}_I X_t&=\sqrt{2}\phi^{-1}(X_t)u_t\,\mathrm{d}\tilde B_t+\phi^{-2}(X_t)Z'(X_t)\,\mathrm{d}t-\frac{\rho'(X_t,Y_t)}{\xi(t)}\nabla'\rho'(X_t,\cdot)(Y_t)\,\mathrm{d}t+N'(X_t)\,\mathrm{d}\tilde l_t;\\ \mathrm{d}_I Y_t&=\sqrt{2}\phi^{-1}(Y_t)P'_{X_t,Y_t}u_t\,\mathrm{d}\tilde B_t+\phi^{-2}(Y_t)Z'(Y_t)\,\mathrm{d}t+N'(Y_t)\,\mathrm{d}\tilde l_t,\quad t\leqslant T.\end{aligned}$$

Since

$$\mathrm{Ric}^Z+L\log\phi-2|\nabla\log\phi|^2\geqslant K_{\phi,3}+|\nabla\log\phi|^2,\quad t\in[0,T],$$

by a similar calculation as for (2.8) and (2.9) we find

$$\begin{aligned}\mathrm{d}\rho'(X_t,Y_t)&\leqslant\sqrt{2}(\phi^{-1}(X_t)-\phi^{-1}(Y_t))\left\langle\nabla'\rho'(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde B_t\right\rangle'\\&\quad-\left(\int_0^{\rho'(X_t,Y_t)}(K_{\phi,3}+|\nabla\log\phi|^2)(\gamma(s))\,\mathrm{d}s\right)\mathrm{d}t-\frac{\rho'(X_t,Y_t)}{\xi(t)}\,\mathrm{d}t,\quad 0\leqslant t<T\end{aligned}\tag{2.16}$$

which implies

$$\begin{aligned}\mathrm{d}\frac{\rho'(X_t,Y_t)^2}{\xi(t)}&\leqslant\frac{2\sqrt2}{\xi(t)}\rho'(X_t,Y_t)\left(\phi^{-1}(X_t)-\phi^{-1}(Y_t)\right)\left\langle\nabla'\rho'(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde B_t\right\rangle'\\&\quad-\frac{\rho'(X_t,Y_t)^2}{\xi^2(t)}\left(\dot\xi(t)+2K_{\phi,3}\xi(t)+2\right)\mathrm{d}t,\quad 0\leqslant t<T,\end{aligned}\tag{2.17}$$

since the positive term coming from the covariation is smaller than the opposite of the negative term involving $|\nabla\log\phi|^2$. Now for $\theta\in(0,2)$ let

$$\xi(t)=(2-\theta)\int_t^T\mathrm{e}^{-2K_{\phi,3}(t-s)}\,\mathrm{d}s,\quad t\in[0,T)$$

so that ξ solves the equation

$$\dot{\xi}(t) + 2K_{\phi,3}\xi(t) + 2 = \theta, \quad t \in [0, T).$$

Combining this with (2.17), we find

$$\mathrm{d}\frac{\rho'(X_t, Y_t)^2}{\xi(t)} \leqslant \frac{2\sqrt{2}}{\xi(t)}\rho'(X_t, Y_t)\left(\phi^{-1}(X_t) - \phi^{-1}(Y_t)\right)\left\langle \nabla'\rho'(\cdot, Y_t)(X_t), u_t \mathrm{d}\tilde{B}_t \right\rangle' - \frac{\rho'(X_t, Y_t)^2}{\xi(t)^2}\theta\,\mathrm{d}t.$$

The remainder of argument is given by the proof of [16, Theorem 3.4.7]. □

2.3 Transportation-cost inequalities

Consider $\mu, \nu \in \mathscr{P}(M)$ where $\mathscr{P}(M)$ denotes the space of all probability measures on M. Recall the L^p-Wasserstein distance between μ and ν is

$$W_p(\mu, \nu) = \inf_{\eta \in \mathscr{C}(\mu, \nu)} \left\{ \int_{M \times M} \rho(x, y)^p \,\mathrm{d}\eta(x, y) \right\}^{1/p}$$

where $\mathscr{C}(\mu, \nu)$ is the set for couplings of μ and ν. When the manifold has no boundary, it is well known that the curvature condition,

$$\mathrm{Ric}^Z \geqslant K \quad \text{for some constant } K$$

is equivalent to

$$W_p(\mu P_t, \nu P_t) \leqslant W_p(\mu, \nu)\,\mathrm{e}^{-Kt}, \quad \mu, \nu \in \mathscr{P}(M),$$

where $\mu P_t \in \mathscr{P}(M)$ is defined by $(\mu P_t)(A) = \mu(P_t 1_A)$ for measurable set A. This equivalence is due to [10] which is extended to the manifolds with convex boundary [15]. Using a coupling method, with the effect of the cut locus accommodated as in the proof of Theorem 2.6, we obtain the following transportation-cost inequality.

Theorem 2.7. *If there exists $\phi \in \mathscr{D}$ and a constant $K_{\phi,3}$ satisfying*

$$\mathrm{Ric}^Z + L\log\phi - 3|\nabla\log\phi|^2 \geqslant K_{\phi,3}$$

then

$$W_2(\mu P_t, \nu P_t) \leqslant \|\phi\|_\infty\,\mathrm{e}^{-K_{\phi,3}t}\,W_2(\mu, \nu).$$

Proof. By [16, Theorem 4.4.2], it suffices to only consider $\mu = \delta_x$ and $\nu = \delta_y$. Let ϕ be a smooth function in $\mathscr{D}$ and recall that $L = \phi^{-2}(\Delta' + Z')$ for the manifold (M, g') as in (2.2), where $g' = \phi^{-2}g$. Let X_t and Y_t solve the following SDEs respectively:

$$\begin{aligned}
\mathrm{d}_I X_t &= \sqrt{2}\phi^{-1}(X_t)u_t\,\mathrm{d}B_t + \phi^{-2}(X_t)Z'(X_t)\,\mathrm{d}t + N'(X_t)\,\mathrm{d}l_t, \quad X_0 = x;\\
\mathrm{d}_I Y_t &= \sqrt{2}\phi^{-1}(Y_t)P'_{X_t,Y_t}u_t\,\mathrm{d}B_t + \phi^{-2}(Y_t)Z'(Y_t)\,\mathrm{d}t + N'(Y_t)\,\mathrm{d}\tilde{l}_t, \quad Y_0 = y.
\end{aligned}$$

Then, as explained in the proof of Theorem 2.2, in which we derived (2.9), we have

$$\begin{aligned}
\mathrm{d}\rho'(X_t, Y_t) \leqslant{}& \sqrt{2}(\phi^{-1}(X_t) - \phi^{-1}(Y_t))\,\langle \nabla'\rho'(\cdot, Y_t)(X_t), u_t\,\mathrm{d}B_t\rangle'\\
&- \left(\int_0^{\rho'(X_t,Y_t)} (\phi^{-2}\mathrm{Ric}^Z(\dot\gamma(s), \dot\gamma(s)) + L\log\phi - 2|\nabla\log\phi|^2)(\gamma(s))\,\mathrm{d}s\right)\mathrm{d}t.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{d}\rho'(X_t, Y_t)^2 &= 2\rho'(X_t, Y_t)\,\mathrm{d}\rho'(X_t, Y_t) + (\phi^{-1}(X_t) - \phi^{-1}(Y_t))^2\,\mathrm{d}t\\
&\leqslant \mathrm{d}\tilde{M}_t + 2\left(\int_0^{\rho'(X_t,Y_t)} \langle \nabla'\phi^{-1}(\gamma(s)), \dot\gamma(s)\rangle'\,\mathrm{d}s\right)^2\mathrm{d}t\\
&\quad - 2\rho'(X_t, Y_t)\int_0^{\rho'(X_t,Y_t)} (\phi^{-2}\mathrm{Ric}^Z(\dot\gamma(s), \dot\gamma(s)) + L\log\phi - 2|\nabla\log\phi|^2)(\gamma(s))\,\mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathrm{d}\tilde{M}_t - 2\rho'(X_t,Y_t)\bigg(\int_0^{\rho'(X_t,Y_t)} (\phi^{-2}(\gamma(s))\mathrm{Ric}^Z(\dot\gamma(s),\dot\gamma(s)) \\
&\quad + L\log\phi(\gamma(s)) - 3|\nabla\log\phi(\gamma(s))|^2)\,\mathrm{d}s\bigg)\,\mathrm{d}t \\
&\leqslant \mathrm{d}\tilde{M}_t - 2K_{\phi,3}\rho'(X_t,Y_t)^2\,\mathrm{d}t,
\end{aligned}
$$

where

$$
\mathrm{d}\tilde{M}_t = 2\sqrt{2}\rho'(X_t,Y_t)(\phi^{-1}(X_t)-\phi^{-1}(Y_t))\,\langle \nabla'\rho'(\cdot,Y_t)(X_t), u_t\,\mathrm{d}B_t\rangle'.
$$

It follows that

$$
\begin{aligned}
W_2(\delta_x P_t,\delta_y P_t)^2 &\leqslant \mathbb{E}^{(x,y)}[\rho(X_t,Y_t)^2] \leqslant \|\phi\|_\infty^2\mathbb{E}^{(x,y)}[\rho'(X_t,Y_t)^2] \\
&\leqslant \|\phi\|_\infty^2\,\mathrm{e}^{-2K_{\phi,3}t}\,\rho'(x,y)^2 \leqslant \|\phi\|_\infty^2\,\mathrm{e}^{-2K_{\phi,3}t}\,\rho(x,y)^2
\end{aligned}
$$

which completes the proof. □

We now investigate Talagrand-type inequalities with respect to the uniform distance on the path space $W^T := C([0,T];M)$ of the (reflecting) diffusion process, for a given $T>0$. Let X_t^μ be the (reflecting if $\partial M\neq\varnothing$) diffusion process generated by L with initial distribution $\mu\in\mathscr{P}(M)$. Let Π_μ^T be the distribution of

$$
X_{[0,T]}^\mu := \{X_t^\mu : t\in[0,T]\},
$$

which is a probability measure on the (free) path space W^T. When $\mu=\delta_x$ we denote $\Pi_{\delta_x}^T=\Pi_x^T$ and $X_{[0,T]}^{\delta_x}=X_{[0,T]}^x$. For any non-negative measurable function F on W^T such that $\Pi_\mu^T(F)=1$, one has

$$
\mu_F^T(\mathrm{d}x) := \Pi_x^T(F)\mu(\mathrm{d}x)\in\mathscr{P}(M). \tag{2.18}
$$

The the uniform distance on W^T is given by

$$
\rho_\infty(\gamma,\eta) := \sup_{t\in[0,T]}\rho(\gamma_t,\eta_t),\quad \gamma,\eta\in W^T.
$$

Let $W_2^{\rho_\infty}$ be the L^2-Wasserstein distance (or L^2-transportation cost) induced by ρ_∞. In general, for any $p\in[1,\infty)$ and two probability measures Π_1,Π_2 on W^T,

$$
W_p^{\rho_\infty}(\Pi_1,\Pi_2) := \inf_{\pi\in\mathscr{C}(\Pi_1,\Pi_2)}\left\{\iint_{W^T\times W^T}\rho_\infty(\gamma,\eta)^p\pi(\mathrm{d}\gamma,\mathrm{d}\eta)\right\}^{1/p}
$$

is the L^p-Wasserstein distance (or L^p-transportation cost) of Π_1 and Π_2, induced by the uniform norm, where $\mathscr{C}(\Pi_1,\Pi_2)$ is the set of all couplings for Π_1 and Π_2. Moreover, for $F\geqslant 0$ with $\Pi_\mu^T(F)=1$, let

$$
\mu_F^T(\mathrm{d}x) = \Pi_x^T(F)\mu(\mathrm{d}x).
$$

The following result improves [15, Theorems 4.1 and 4.2] or [16, Theorems 4.5.3 and 4.5.4]:

Theorem 2.8. *If there exists $\phi\in\mathscr{D}$ and a constant K_ϕ satisfying*

$$
\mathrm{Ric}^Z + L\log\phi - 2|\nabla\log\phi|^2 \geqslant K_\phi
$$

then

(i) *for $F\geqslant 0$, $\Pi_\mu^T(F)=1$ and $\mu\in\mathscr{P}(M)$,*

$$
W_2^{\rho_\infty}(F\Pi_\mu^T,\Pi_{\mu_F^T}^T)^2 \leqslant \frac{2\|\phi\|_\infty^2}{K_\phi}(\mathrm{e}^{2K_\phi^+T}-\mathrm{e}^{2K_\phi^-T})\inf_{R>0}\left\{(1+R^{-1})\exp\left(8(1+R)\|\nabla\log\phi\|_\infty^2\right)\right\}\Pi_\mu^T(F\log F);
$$

(i') *for $F\geqslant 0$, $\Pi_\mu^T(F)=1$ and $\mu\in\mathscr{P}(M)$,*

$$
W_2^{\rho_\infty}(F\Pi_\mu^T,\Pi_{\mu_F^T}^T)^2 \leqslant \frac{2\|\phi\|_\infty^2}{K_\phi}(1-\mathrm{e}^{-2K_\phi T})\inf_{R>0}\left\{(1+R^{-1})\exp\left(8(1+R)\|\nabla\log\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^+T}\right)\right\}\Pi_\mu^T(F\log F);
$$

(ii) *for any* $\mu,\nu\in\mathscr{P}(M)$,

$$W_2^{\rho_\infty}(\Pi_\mu^T,\Pi_\nu^T)\leqslant 2\|\phi\|_\infty\,\mathrm{e}^{(K_\phi^-+\|\nabla\log\phi\|_\infty)T}\,W_2(\mu,\nu).$$

Remark 2.9.

(a) When $\|\nabla\log\phi\|_\infty>0$ and $K_\phi>0$ the upper bound in (i) is better than that in (i').

(b) When the boundary is convex we can choose $\phi\equiv 1$. In this case $\nabla\log\phi=0$ and the estimate in (i') is consistent with [16, Theorem 4.4.2 (2)] for the convex case.

(c) We note that [16, Theorem 4.4.2 (6)] needs to be corrected as follows:

$$W_2^{\rho_\infty}(\Pi_\mu^T,\Pi_\nu^T)\leqslant \mathrm{e}^{K^-T}\,W_2(\mu,\nu),$$

where K is the lower bound of Ricci curvature. It is then consistent with Theorem 2.8 (ii) when $\phi\equiv 1$ and the boundary is convex.

Proof of Theorem 2.8.
(i) Simply denote $X_{[0,T]}^x=X_{[0,T]}$. Let F be a positive bounded measurable function on W^T such that $\inf F>0$ and $\Pi_x^T(F)=1$. Let

$$\mathrm{d}\mathbb{Q}=F(X_{[0,T]})\,\mathrm{d}\mathbb{P}.$$

Since $\mathbb{E}\left[F(X_{[0,T]})\right]=\Pi_\mu^T(F)=1$, $\mathbb{Q}$ is a probability measure on Ω. Then we conclude that there exists a unique $\mathscr{F}_t$-predictable process β_t on $\mathbb{R}^d$ such that

$$F(X_{[0,T]})=\exp\left(\int_0^T\langle\beta_s,\mathrm{d}B_s\rangle-\frac{1}{2}\int_0^T\|\beta_s\|^2\,\mathrm{d}s\right)$$

and

$$\int_0^T\mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\,\mathrm{d}s=2\mathbb{E}\left[F(X_{[0,T]})\log F(X_{[0,T]})\right].\tag{2.19}$$

Then, by the Girsanov theorem, $\tilde{B}_t:=B_t-\int_0^t\beta_s\,\mathrm{d}s$, $t\in[0,T]$ is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}$.

As explained in the proof of [16, Theorem 4.5.3], it suffices to assume $\mu=\delta_x$, $x\in M$. In this case, the desired inequality involves

$$\mu_F^T=\delta_x\quad\text{and}\quad\Pi_\mu^T(F\log F)=\Pi_x^T(F\log F).$$

Since the diffusion coefficients are non-constant, it is convenient to adopt the Itô differential d_I for the Girsanov transformation. So the reflecting L diffusion process X_t can be constructed by solving the Itô SDE

$$\mathrm{d}_IX_t=\sqrt{2}\phi^{-1}(X_t)u_t\,\mathrm{d}B_t+\phi^{-2}(X_t)Z'(X_t)\,\mathrm{d}t+N'(X_t)\,\mathrm{d}l_t,\qquad X_0=x,$$

where B_t is the d-dimensional Brownian motion with natural filtration $\mathscr{F}_t$. Then

$$\mathrm{d}_IX_t=\sqrt{2}\phi^{-1}(X_t)u_t\,\mathrm{d}\tilde{B}_t+\{\phi^{-2}(X_t)Z'(X_t)+\sqrt{2}\phi^{-1}(X_t)u_t\beta_t\}\,\mathrm{d}t+N'(X_t)\,\mathrm{d}l_t,\quad X_0=x\tag{2.20}$$

and let Y_t solve

$$\mathrm{d}_IY_t=\sqrt{2}\phi^{-1}(Y_t)P'_{X_t,Y_t}u_t\,\mathrm{d}\tilde{B}_t+\phi^{-2}(Y_t)Z'(Y_t)\,\mathrm{d}t+N'(Y_t)\,\mathrm{d}\tilde{l}_t,\quad Y_0=x\tag{2.21}$$

where l_t and $\tilde{l}_t$ are the local times of X_t and Y_t on ∂M, respectively. Moreover, for any bounded measurable function G on W^T,

$$\mathbb{E}_{\mathbb{Q}}G(X_{[0,T]}):=\mathbb{E}(FG)(X_{[0,T]})=\Pi_x^T(FG).$$

We conclude that the distribution of $X_{[0,T]}$ under $\mathbb{Q}$ coincides with $F\Pi_x^T$. Therefore,

$$\begin{aligned}W_2^{\rho_\infty}(F\Pi_x^T,\Pi_x^T)^2 &\leqslant \mathbb{E}_{\mathbb{Q}}\rho_\infty(X_{[0,T]},Y_{[0,T]})^2=\mathbb{E}_{\mathbb{Q}}\max_{t\in[0,T]}\rho(X_t,Y_t)^2\\ &\leqslant \|\phi\|_\infty^2\,\mathbb{E}_{\mathbb{Q}}\max_{t\in[0,T]}\rho'(X_t,Y_t)^2.\end{aligned}\tag{2.22}$$

Note that due to the convexity of the boundary,

$$\langle N'(x),\nabla'\rho(\cdot\,,y)(x)\rangle'\leqslant 0,\quad x\in\partial M.$$

From this and equations (2.20) and (2.21), it follows that

$$\mathrm{d}\rho'(X_t,Y_t)\leqslant\sqrt{2}(\phi^{-1}(X_t)-\phi^{-1}(Y_t))\left\langle\nabla'\rho'(\cdot\,,Y_t)(X_t),u_t\,\mathrm{d}\tilde{B}_t\right\rangle'-K_\phi\rho'(X_t,Y_t)\,\mathrm{d}t+\sqrt{2}\|\beta_t\|\,\mathrm{d}t.$$

Defining

$$M_t:=\sqrt{2}\int_0^t\mathrm{e}^{K_\phi s}(\phi^{-1}(X_s)-\phi^{-1}(Y_s))\left\langle\nabla'\rho'(\cdot\,,Y_s)(X_s),u_s\,\mathrm{d}\tilde{B}_s\right\rangle'$$

we have

$$\rho'(X_t,Y_t)\leqslant\mathrm{e}^{-K_\phi t}\left(M_t+\sqrt{2}\int_0^t\mathrm{e}^{K_\phi s}\,\|\beta_s\|\,\mathrm{d}s\right),\quad t\in[0,T].$$

So to prove (i), we will estimate the function

$$h_t=\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}\mathrm{e}^{2K_\phi s}\,\rho'(X_s,Y_s)^2.$$

By the Doob inequality, for any $R>0$, we have

$$\begin{aligned}h_t:&=\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}\mathrm{e}^{2K_\phi s}\,\rho'(X_s,Y_s)^2\\ &\leqslant(1+R)\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}M_s^2+2(1+R^{-1})\max_{s\in[0,t]}\mathbb{E}_{\mathbb{Q}}\left(\int_0^s\mathrm{e}^{K_\phi r}\,\|\beta_r\|\,\mathrm{d}r\right)^2\\ &\leqslant 4(1+R)\mathbb{E}_{\mathbb{Q}}M_t^2+2(1+R^{-1})\int_0^t\mathrm{e}^{2K_\phi s}\,\mathrm{d}s\int_0^t\mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\,\mathrm{d}s\\ &\leqslant 8(1+R)\|\nabla\log\phi\|_\infty^2\int_0^t h_s\,\mathrm{d}s+2(1+R^{-1})\int_0^T\mathrm{e}^{2K_\phi s}\,\mathrm{d}s\int_0^T\mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\,\mathrm{d}s,\quad t\in[0,T].\end{aligned}\tag{2.23}$$

Since $h_0=0$, by using the Gronwall inequality, this inequality further implies

$$h_T\leqslant 2(1+R^{-1})\exp\left(8(1+R)\|\nabla\log\phi\|_\infty^2\right)\int_0^T\mathrm{e}^{2K_\phi s}\,\mathrm{d}s\int_0^T\mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\,\mathrm{d}s\tag{2.24}$$

By (2.19) and (2.24) we thus have

$$\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,T]}\rho'(X_s,Y_s)^2\leqslant 4(1+R^{-1})\exp\left(8(1+R)\|\nabla\log\phi\|_\infty^2\right)\frac{\mathrm{e}^{2K_\phi^+T}-\mathrm{e}^{2K_\phi^-T}}{2K_\phi}\Pi_x^T(F\log F).$$

(i') For this we use the function

$$\tilde{h}_t=\mathrm{e}^{2K_\phi t}\,\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}\rho'(X_s,Y_s)^2.$$

The inequality (2.23) should then be modified as follows:

$$\begin{aligned}\tilde{h}_t&:=\mathrm{e}^{2K_\phi t}\,\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}\rho'(X_s,Y_s)^2\\ &\leqslant\mathrm{e}^{2K_\phi t}(1+R)\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}\mathrm{e}^{-2K_\phi s}\,M_s^2+2\,\mathrm{e}^{2K_\phi t}(1+R^{-1})\max_{s\in[0,t]}\mathbb{E}_{\mathbb{Q}}\left(\int_0^s\mathrm{e}^{-K_\phi(s-r)}\,\|\beta_r\|\,\mathrm{d}r\right)^2\end{aligned}$$

$$
\begin{aligned}
&\leqslant 4(1+R)\,\mathrm{e}^{2K_\phi^+ t}\,\mathbb{E}_{\mathbb{Q}} M_t^2 + 2(1+R^{-1})\int_0^t \mathrm{e}^{2K_\phi r}\,\mathrm{d}r \int_0^t \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\,\mathrm{d}s\\
&\leqslant 8(1+R)\|\nabla\log\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^+ T}\int_0^t \tilde h_s\,\mathrm{d}s + 2(1+R^{-1})\int_0^T \mathrm{e}^{2K_\phi r}\,\mathrm{d}r \int_0^T \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\,\mathrm{d}s, \quad t\in[0,T].
\end{aligned}
$$

Since $\tilde h_0 = 0$, this inequality implies

$$
\tilde h_T \leqslant 2(1+R^{-1})\exp\left(8(1+R)\|\nabla\log\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^+ T}\right)\int_0^T \mathrm{e}^{2K_\phi s}\,\mathrm{d}s \int_0^T \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\,\mathrm{d}s.
$$

We then conclude that

$$
\mathbb{E}_{\mathbb{Q}} \max_{s\in[0,T]} \rho'(X_s,Y_s)^2 \leqslant 4(1+R^{-1})\exp\left(8(1+R)\|\nabla\log\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^+ T}\right)\frac{1-\mathrm{e}^{-2K_\phi T}}{2K_\phi}\Pi_x^T(F\log F).
$$

(ii) Without loss of generality, we consider $\mu=\delta_x$, and $\nu=\delta_y$. Let X_t and Y_t solve the following SDEs, respectively:

$$
\begin{aligned}
&\mathrm{d}_I X_t = \sqrt{2}\phi^{-1}(X_t)u_t\,\mathrm{d}B_t + \phi^{-2}(X_t)Z'(X_t)\,\mathrm{d}t + N'(X_t)\,\mathrm{d}l_t, \quad X_0=x;\\
&\mathrm{d}_I Y_t = \sqrt{2}\phi^{-1}(Y_t)P'_{X_t,Y_t}u_t\,\mathrm{d}B_t + \phi^{-2}(Y_t)Z'(Y_t)\,\mathrm{d}t + N'(Y_t)\,\mathrm{d}\tilde l_t, \quad Y_0=y.
\end{aligned}
$$

Then, as explained in the proof of Theorem 2.2, we have

$$
\begin{aligned}
\mathrm{d}\rho'(X_t,Y_t) \leqslant{}& \sqrt{2}(\phi^{-1}(X_t)-\phi^{-1}(Y_t))\,\langle\nabla'\rho'(\cdot,Y_t)(X_t),u_t\,\mathrm{d}B_t\rangle'\\
&-\left(\int_0^{\rho'(X_t,Y_t)}\big(\phi^{-2}\mathrm{Ric}^Z(\dot\gamma(s),\dot\gamma(s)) + L\log\phi - 2|\nabla\log\phi|^2\big)(\gamma(s))\,\mathrm{d}s\right)\mathrm{d}t.
\end{aligned}
\tag{2.25}
$$

Therefore,

$$
\rho'(X_t,Y_t) \leqslant \mathrm{e}^{-K_\phi t}(\hat M_t + \rho'(x,y)), \quad t\geqslant 0 \tag{2.26}
$$

for

$$
\hat M_t := \sqrt{2}\int_0^t \mathrm{e}^{K_\phi s}(\phi^{-1}(X_s)-\phi^{-1}(Y_s))\,\langle\nabla'\rho(\cdot,Y_s)(X_s),u_s\,\mathrm{d}B_s\rangle'.
$$

Again using the Itô formula, we have

$$
\mathrm{d}\rho'(X_t,Y_t)^2 \leqslant \mathrm{d}\tilde M_t - 2(K_\phi - \|\nabla\log\phi\|_\infty^2)\rho'(X_t,Y_t)^2\,\mathrm{d}t
$$

where

$$
\mathrm{d}\tilde M_t = 2\rho'(X_t,Y_t)(\phi^{-1}(X_t)-\phi^{-1}(Y_t))\,\langle\nabla'\rho'(\cdot,Y_t)(X_t),u_t\,\mathrm{d}B_t\rangle'
$$

which implies

$$
\mathbb{E}\rho'(X_t,Y_t)^2 \leqslant \mathrm{e}^{-2(K_\phi-\|\nabla\log\phi\|_\infty^2)t}\,\rho'(x,y)^2.
$$

Combining this with (2.26) we arrive at

$$
\begin{aligned}
W_2^{\rho_\infty}(\Pi_x^T,\Pi_y^T)^2 &\leqslant \|\phi\|_\infty^2\,\mathbb{E}\max_{t\in[0,T]}\rho'(X_t,Y_t)^2\\
&\leqslant \|\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^- T}\,\mathbb{E}\max_{t\in[0,T]}(\hat M_t+\rho'(x,y))^2\\
&\leqslant 4\|\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^- T}\,\mathbb{E}(\hat M_T+\rho'(x,y))^2\\
&= 4\|\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^- T}\left(\mathbb{E}\hat M_T^2+\rho'(x,y)^2\right)\\
&\leqslant 4\|\phi\|_\infty^2\,\mathrm{e}^{2K_\phi^- T}\left(2\int_0^T \mathrm{e}^{2K_\phi t}\,\|\nabla\log\phi\|_\infty^2\,\mathbb{E}\rho'(X_t,Y_t)^2\,\mathrm{d}t + \rho'(x,y)^2\right)
\end{aligned}
$$

$$\leqslant 4\|\phi\|_\infty^2\, \mathrm{e}^{2(K_\phi^- + \|\nabla \log \phi\|_\infty^2)T}\, \rho'(x,y)^2$$
$$\leqslant 4\|\phi\|_\infty^2\, \mathrm{e}^{2(K_\phi^- + \|\nabla \log \phi\|_\infty^2)T}\, \rho(x,y)^2$$

where the second inequality is due to the Doob inequality. This implies the desired inequality for $\mu = \delta_x$ and $\nu = \delta_y$. □

Corollary 2.10. *If there exists $\phi \in \mathscr{D}$ and a constant K_ϕ satisfying*

$$\mathrm{Ric}^Z + L\log\phi - 2|\nabla \log \phi|^2 \geqslant K_\phi$$

then

(i) *for $F \geqslant 0$, $\Pi_\mu^T(F) = 1$ and $\mu \in \mathscr{P}(M)$,*

$$W_2^{\rho_\infty}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \leqslant \frac{2\|\phi\|_\infty^2}{K_\phi}\left(\mathrm{e}^{2K_\phi^+ T} - \mathrm{e}^{2K_\phi^- T}\right)\exp\left(8\|\nabla\log\phi\|_\infty^2 + 4\sqrt{2}\|\nabla \log\phi\|_\infty\right)\ \Pi_\mu^T(F\log F);$$

(i') *for $F \geqslant 0$, $\Pi_\mu^T(F) = 1$ and $\mu \in \mathscr{P}(M)$,*

$$W_2^{\rho_\infty}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \leqslant \frac{2\|\phi\|_\infty^2}{K_\phi}\left(1 - \mathrm{e}^{-2K_\phi T}\right)\exp\left(8\|\nabla\log\phi\|_\infty^2\, \mathrm{e}^{2K_\phi^+ T} + 4\sqrt{2}\|\nabla \log\phi\|_\infty\, \mathrm{e}^{K_\phi^+ T}\right)\ \Pi_\mu^T(F\log F).$$

Proof. It is easily observed that

$$(1+R^{-1})\exp\left(8(1+R)\|\nabla\log\phi\|_\infty^2\right) \leqslant \exp\left(R^{-1} + 8(1+R)\|\nabla\log\phi\|_\infty^2\right).$$

Taking the infimum about R on the right side above, we arrive at

$$\exp\left(R^{-1} + 8(1+R)\|\nabla\log\phi\|_\infty^2\right) \geqslant \exp\left(8\|\nabla\log\phi\|_\infty^2 + 4\sqrt{2}\|\nabla\log\phi\|_\infty\right)$$

which allows to prove (i). The inequality (i') can be checked in the same way. □

3 New construction of function $\log\phi$

In this section, we give a new construction of a function ϕ which satisfies the conditions of the previous section. To do so, we let ρ_∂ be the Riemannian distance to the boundary ∂M and use a comparison theorem for $\Delta\rho_\partial$ near the boundary, essentially due to [8]. Note that, by using local charts, it is clear that ρ_∂ is smooth in a neighbourhood of ∂M. We call

$$i_\partial := \sup\{r > 0 : \rho_\partial \text{ is smooth on } \{\rho_\partial < r\}\}$$

the injectivity radius of ∂M. Obviously, $i_\partial > 0$ if M is compact, but it could be zero in the non-compact case ($\sup\varnothing = 0$ by convention). As [16, Theorem 1.2.3] we have:

Lemma 3.1. *Let θ, k be constants such that $\mathrm{II} \leqslant \theta$ and $\mathrm{Sect} \leqslant k$. Let*

$$h(t) := \begin{cases} \cos\sqrt{k}t - \frac{\theta}{\sqrt{k}}\sin\sqrt{k}t, & k > 0,\\ 1 - \theta t, & k = 0,\\ \cosh\sqrt{-k}t - \frac{\theta}{\sqrt{-k}}\sinh\sqrt{-k}t, & k < 0 \end{cases} \tag{3.1}$$

for $t \geqslant 0$. Let $h^{-1}(0)$ be the first zero of h (with $h^{-1}(0) := \infty$ if $h(t) > 0$ for all $t \geqslant 0$). Then for any $x \in \mathring{M}$ such that $\rho_\partial(x) \leqslant i_\partial \wedge h^{-1}(0)$ we have

$$\Delta\rho_\partial(x) \geqslant (d-1)\frac{h'}{h}(\rho_\partial(x)). \tag{3.2}$$

Note that if k is positive then

$$h^{-1}(0) = \frac{1}{\sqrt{k}} \arcsin\left(\sqrt{\frac{k}{k+\theta^2}}\right).$$

We now work under the following assumption:

Assumption (A) There exist non-negative constants σ and θ such that $-\sigma \leqslant \mathrm{II} \leqslant \theta$ and a positive constant r_0 such that on $\partial_{r_0} M := \{x \in M : \rho_\partial(x) \leqslant r_0\}$ the function ρ_∂ is smooth, the norm of Z is bounded and $\mathrm{Sect} \leqslant k$ for some positive constant k.

Using this assumption, F.-Y. Wang constructed a function ϕ satisfying $\phi \in \mathscr{D}$ (see [13, p.1436] or [16, Theorem 3.2.9]). Following his construction, we define

$$\log \phi(x) = \frac{\sigma}{\alpha} \int_0^{\rho_\partial(x)} [h(s) - h(r_1)]^{1-d} \mathrm{d}s \int_{s \wedge r_1}^{r_1} [h(u) - h(r_1)]^{d-1} \mathrm{d}u,$$

where $r_1 := r_0 \wedge h^{-1}(0)$ and

$$\alpha := (1 - h(r_1))^{1-d} \int_0^{r_1} [h(s) - h(r_1)]^{d-1} \mathrm{d}s.$$

Then from the proof of [12, Theorem 1.1], we know:

Theorem 3.2. *Suppose that Assumption* (**A**) *holds and* $\mathrm{Ric}^Z \geqslant K$. *Define*

$$K_p = K - \sigma\left(\delta_{r_1}(Z) + \frac{d}{r_1}\right) - p\sigma^2,$$

where

$$\delta_{r_1}(Z) := \sup\left\{|Z(x)| \colon x \in \partial_{r_1} M\right\}. \tag{3.3}$$

Then all results in Section 2 hold by replacing

$$K_\phi,\ K_{\phi,p},\ \|\phi\|_\infty \quad \textit{and} \quad \|\nabla \log \phi\|_\infty$$

with

$$K_2,\ K_p,\ \mathrm{e}^{\sigma d r_1/2} \quad \textit{and} \quad \sigma$$

respectively.

In the following we give a new construction of ϕ by using the function

$$\ell(r) = \begin{cases} \mathrm{e}^{-2} - \mathrm{e}^{-2(1-2r)^{-1}}, & 0 \leqslant r < \frac{1}{2}, \\ \mathrm{e}^{-2}, & r \geqslant \frac{1}{2}. \end{cases}$$

Proposition 3.3. *Suppose that Assumption* (**A**) *holds. Let*

$$H(r) := \frac{\sqrt{k+\theta^2}}{k} \cos\left(\arcsin\left(\sqrt{\frac{k}{k+\theta^2}}\right) - \sqrt{k}\,(r \wedge r_1)\right) - \frac{\theta}{k}.$$

Then the function

$$\log \phi(x) := \frac{1}{2} \sigma\, \mathrm{e}^2\, \ell\left(\frac{H(\rho_\partial(x))}{2H(r_1)}\right) H(r_1) \tag{3.4}$$

satisfies

$$N \log \phi|_{\partial M} = \sigma \geqslant -\mathrm{II}.$$

Moreover,

$$\|\phi\|_\infty \leqslant \mathrm{e}^{\sigma H(r_1)/2}, \quad |\nabla \log \phi| \leqslant \sigma$$

and

$$L \log \phi(x) \geqslant -\sigma\left(d\sqrt{\theta^2 + k} + \delta_{r_1}(Z) + \frac{5}{2H(r_1)}\right).$$

Proof of Proposition 3.3. First it is easy to see that the function ℓ satisfies $\ell \leqslant \mathrm{e}^{-2}$. Differentiating ℓ we obtain

$$\ell'(r) = \begin{cases} \left(\frac{1}{2} - r\right)^{-2} \mathrm{e}^{-\left(\frac{1}{2}-r\right)^{-1}}, & 0 \leqslant r < \frac{1}{2}; \\ 0, & r \geqslant \frac{1}{2} \end{cases}$$

and

$$\ell''(r) = \begin{cases} -2r\left(\frac{1}{2} - r\right)^{-4} \mathrm{e}^{-\left(\frac{1}{2}-r\right)^{-1}}, & 0 \leqslant r < \frac{1}{2}; \\ 0, & r \geqslant \frac{1}{2}. \end{cases}$$

As $\ell'' < 0$ on $[0, 1/2)$, the function ℓ' is at its maximal point when $r = 0$, which implies $0 \leqslant \ell' \leqslant 4\,\mathrm{e}^{-2}$. Using the same method, when $r = \sqrt{3}/6$ the function ℓ'' reaches the minimal value, which implies

$$\ell'' \geqslant -3^{-1/2}(3+\sqrt{3})^4\, \mathrm{e}^{-(3+\sqrt{3})} > -20\,\mathrm{e}^{-2}\,.$$

Using these results, we have

$$N \log \phi|_{\partial M} = \frac{1}{4}\,\mathrm{e}^2\, \sigma \ell'(0) N\rho_\partial = \sigma,$$

and

$$|\nabla \log \phi| = \frac{1}{4}\,\mathrm{e}^2\, \sigma \ell' \left(\frac{H(\rho_\partial)}{2H(r_0)}\right) H'(\rho_\partial) \leqslant \sigma.$$

Moreover, by Lemma 3.1, we have

$$\begin{aligned} L \log \phi &= \frac{1}{4}\,\mathrm{e}^2\, \sigma \left(\ell'\left(\frac{H(\rho_\partial)}{2H(r_0)}\right) h(\rho_\partial) L\rho_\partial + \ell''\left(\frac{H(\rho_\partial)}{2H(r_0)}\right)\frac{h(\rho_\partial)^2}{2H(r_0)} + \ell'\left(\frac{H(\rho_\partial)}{2H(r_0)}\right) h'(\rho_\partial(x))\right) \\ &\geqslant \frac{1}{4}\,\mathrm{e}^2\, \sigma \left(\ell'\left(\frac{H(\rho_\partial)}{2H(r_0)}\right)\left(dh'(\rho_\partial) - \sup_{\partial_{r_0} M} |Z|\right) + \frac{h(\rho_\partial)^2}{2H(r_0)} \ell''\left(\frac{H(\rho_\partial)}{2H(r_0)}\right)\right), \end{aligned}$$

where h is defined as in (3.1) for $k \geqslant 0$. It is easy to calculate that

$$h'(r) \geqslant -\sqrt{\theta^2 + k}.$$

Combining this with properties of ℓ, we conclude that

$$L \log \phi \geqslant -\,\sigma \left(d\sqrt{\theta^2 + k} + \sup\left\{|Z|(x) \colon x \in \partial_{r_0 \wedge h^{-1}(0)} M\right\} + \frac{5}{2H(r_0)}\right)$$

which completes the proof. □

Corollary 3.4. *Suppose that Assumption* ($\mathbf{A}$) *holds and* $\mathrm{Ric}^Z \geqslant K$*. Define*

$$\tilde{K} = K - \sigma \left(d\sqrt{\theta^2 + k} + \delta_{r_1}(Z) + \frac{5}{2H(r_1)}\right),$$

and $\tilde{K}_p = \tilde{K} - p\sigma^2$ *with* $\delta_{r_1}(Z)$ *as defined in* (3.3)*. Then all results in Section 2 hold by replacing*

$$K_{\phi,p}, \ \ \|\phi\|_\infty \ \ \textit{and} \ \ \|\nabla \log \phi\|_\infty$$

with

$$\tilde{K}_p, \ \ \mathrm{e}^{\sigma H(r_1)/2} \ \ \textit{and} \ \ \sigma$$

respectively.

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