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COMMISSION OF THE EUROPEAN COMMUNITIES

REPAIR PROCESSES : FUNDAMENTALS AND COMPUTATION

by

S. GARRIBBA, G. REINA and G. VOLTA

1974

Joint Nuclear Research Centre
Ispra Establishment - Italy

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The code AVACOM-ETARP (Availability Computation-Element Transient and Asymptotic Repair Process) allows computation of availability, of repair density and of failure density accounting any continuous time dependence of failure and restoration rate. Numerical results are shown with regard to a few practical situations.

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ABSTRACT

In reliability engineering and practice an important role is played by those units or components whose life characteristics change with time. The case is herewith considered where change underlies a non-homogeneous Markov model. Simple repair processes then deal with a two-state alternating policy resulting from the superposition of acts of failure and of restoration both showing a continuous aging with time.

Unit's availability can be expressed through linear differential equations or by means of integral equations. The two approaches reveal their equivalence. Asymptotic behavior, repair density and life distributions may be easily obtained. The problems of defining p.d.f. of forward recurrence time, of number of repairs and total safe time are also afforded. In other words, results are sought which may be used for the solution of a series of problems commonly encountered in the practice of repairable systems. These results find a thorough correspondence with developments already offered by the literature for renewal processes. Simple renewal processes would consist of the recurrent superposition of acts of failure and restoration both starting as new after each transition.

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Preface

This study has been carried within the framework of the cooperation agreement no. 034-71-PIPGI between the Euratom JRC of Ispra and the CESNEF (Nuclear Engineering Department) of the Politecnico di Milano. Mr. Reina contribution includes the outcome of his research activity at Euratom JRC from September 1, 1971 to November 30, 1972. As for the CESNEF the research was partially supported through the CNR contract CT 72.01050.07 (Pos. 115.485, Prot. 231487).

Contents

List of symbols

Symbol	Meaning	Section
t	time co-ordinate	1
t_o; $\bar{t}, \bar{\bar{t}}$	initial time; any time $\geq t_o$	
$(\bar{t}, \bar{\bar{t}})$	time interval $\bar{t} \leq t \leq \bar{\bar{t}}$	
$x(t)$	stochastic variable	1
X_j, X_k	state of the unit or event	1
j,k	realization of $x(t)$	1
τ'; τ''	life length to failure; life length to restoration	1,3
$m_{\tau'}$; $m_{\tau''}$	expected life length to failure; to restoration	3
$\sigma_{\tau'}$; $\sigma_{\tau''}$	standard deviation of life length to failure; to restoration	3
λ	failure hazard rate	1,3
μ	restoration hazard rate	1,3
$P_k(t)$	pointwise probability of $\{x(t) \in X_k\}$	2
$P_{k\mid j}(t \mid \bar{t})$	pointwise conditional probability of $\{x(t) \in X_k\}$ given $\{x(\bar{t}) \in X_j\}$	2
R	reliability function	3
S	maintainability function	3
$F = 1-R$	failure (cumulative distribution) function	3
$G = 1-S$	restoration (cumulative distribution) function	3
f	failure probability density function	3
g	restoration probability density function	3
$F^{(n)}$	restoratinn-to-nth failure (cumulative) distribution function	6

$H^{(n)}$	restoration-to-nth restoration (cumulative) distribution function	6
$N_{j \to k}$	number of transitions $X_j \to X_k$	2,6
$M_{j \to k}$	expected number of transitions $X_j \to X_k$	2,4,6
$m_{j \to k}$	(forward) transition density (function) $X_j \to X_k$	2,4,6
n	non-negative integer	
ξ	remaining life to failure	2,6
ζ	remaining life to restoration	2,6
α_n	total on time, $\tau'_1 + \tau'_2 + \ldots + \tau'_n$	1,2,7
β_n	total down time, $\tau''_1 + \tau''_2 + \ldots + \tau''_n$	1,2,7
$\varphi(\bar{t},t)$	total on time in $(\bar{t},t)$	2,7
$\Upsilon(\bar{t},t)$	total down time in $(\bar{t},t)$	2,7
Φ, Ψ	non-negative constants	2,7
$\mathcal{P}\{\ldots\}$	probability of $\{\ldots\}$	
$\mathcal{E}\{\ldots\}$	mathematical expectation of $\{\ldots\}$	
$\mathfrak{D}\{\ldots\}$	dispersion of $\{\ldots\}$	
$O(\ )$	generalized Landau symbol	3
δ_{ij}	Kronecker delta	
$1(\bar{t},t)$	unit step function	
$\delta(t)$	Dirac function	
$\cap$	intersection or product of events	
$\cup$	union or sum of events	
p.d.f.	probability density function	

Sct. 1 Introduction

Let us consider that a unit U entails the stochastic process $\{x(t),\ t_o \leq t < +\infty\}$ where the random variable x(t) takes values in an arbitrary abstract space $\mathbf{X}$. By unit or element, or component we mean an undecomposable part of a system, as well as any device whose reliability characteristics are studied independently of the characteristics of its component parts. Let $\mathbf{X} = X_s \cup X_f$, X_s and X_f representing a mutually exclusive and exhaustive set of events, and define x as follows, $x(t) = j$ with $j = s$ or f, if the unit is in the state X_j at time t. Suppose $x(t_o) = s$. Furthermore, if $x(\bar{t}) = j$ and $x(t) = k$ with $t > \bar{t}$, then say that the unit has made at least one transition $X_j \longrightarrow X_k$ within the interval $(\bar{t}, t)$.

Processes we deal with will assume alternatingly the states X_s, X_f, X_s, X_f, X_s,....Denote τ'_1, τ''_1, τ'_2, τ''_2, τ'_3, ... the times or life lengths spent successively in the states X_s and X_f. $\{\tau'_n\}$ and $\{\tau''_n\}$, with $n=1,2,3,...$, are sequences of non-negative random variables ($\tau'_o = \tau''_o = 0$). If X_s represents the state under work or the safe state of the unit and X_f denotes the state under restoration or the failed state, we may call the instants,

$$(1.1)\qquad t'_n = t_o + \tau_1' + \tau_1'' + \ldots + \tau_n' \ , \quad t'_o = t_o,$$

the failures of the unit and the instants,

$$(1.2)\qquad t''_n = t_o + \tau_1' + \tau_1'' + \ldots + \tau'_n + \tau''_n \ , \quad t''_o = t_o,$$

the restorations. During the intervals $t''_{n-1} \leq t < t'_n$ we think of a failure act or process defined by a failure hazard rate

$$\lambda_n = \lambda_n(t;\ t''_{n-1},\ t'_{n-1},\ \dots,\ t_o) \quad , \tag{1.3$_1$}$$

in such a way that the product λ_n dt is the probability that the unit fails from t to t + dt, given that the unit is safe at time t. On the other hand, during the intervals $t'_n \leqslant t < t''_n$ we consider a restoration act or process characterized by a restoration hazard rate

$$\mu_n = \mu_n\ (t;\ t'_n,\ t''_{n-1},\ \dots,\ t_o). \tag{1.3$_2$}$$

The product $\mu_n$dt is the probability of a restoration occurring from t to t + dt provided that the unit is failed at the instant t. In the mathematical expressions for λ_n and μ_n the dependence upon t may allow for the aging of U, whereas the dependence upon t'_n, t'_{n-1}, t'_{n-1}, etc. may allow for policies whose distinctive characteristics rely on a discrete number of (passed) instants of failure and/or restoration.

A basic role in the theory of non-preventive maintenance is played by those policies where after the failure the unit is replaced by a new one which starts its life taking on all its original properties. Then life lengths of U result independent and equally distributed. The equations

$$\begin{aligned} \lambda_n &= \lambda\,(t - t''_{n-1}), \\ \mu_n &= \mu\,(t - t'_n) \quad , \end{aligned} \tag{1.4}$$

define what is called a simple renewal model or a simple recurrent process.

However, if after each failure the unit is restored to the working state and a progressive aging is admitted both for the failure and the restoration process, equations

$$\begin{aligned} \lambda_n &= \lambda\,(t) \quad , \\ \mu_n &= \mu\,(t) \quad , \end{aligned} \tag{1.5}$$

characterize a non-homogeneous Markov process henceforth referred to as a real-time (or simple) repair process. Conversely, let λ depend upon the total amount of time α the unit is on during its calendar life and let μ depend on the time β the unit is down during the same life. The constitutive equations of an effective-time repair process would then read

$$\begin{aligned} \lambda_n &= \lambda(\alpha) \quad , \quad (\alpha_{n-1} \leqslant \alpha < \alpha_n) \ , \\ \mu_n &= \mu(\beta) \quad , \quad (\beta_{n-1} \leqslant \beta < \beta_n) \ , \end{aligned} \tag{1.6}$$

where $\alpha_n = \tau_1' + \ldots + \tau_n'$, $\beta_n = \tau_1'' + \ldots + \tau_n''$, with $n = 1,2,\ldots$ and $\alpha_o = \beta_o = 0$.

A multitude of different models could be imagined so as to match arrangements occurring in the operation and maintenance practice of engineered systems. In the case of thoroughly constant failure and restoration rates any model entailed by the set (1.3) would obviously degenerate to a two-state homogeneous Markov process. Undoubtedly more intricated processes are needed for interpreting situations commonly found in plant operation,or for describing the behavior of human operators. Mathematical models thus involved may be on their side complicated and almost untreatable by means of direct analytical techniques. Montecarlo method and simulation would then play an essential role. Under many circumstances, however, simple renewal and repair processes represent a sort of starting point for the theory. These two processes can be actually treated in all their analytical aspects so as to throw some light on more intrigued trends and problems.

Renewal processes have found an extensive consideration in the literature concerned with reliability methods and applications [1-4]. But in our knowledge only sparse authors had the will or the need to consider repair processes. Although many results are a direct consequence of the general theory of Markov processes, it is now among the scopes of this study to fill a sort of a gap and try to solve for repair processes quite a few practical problems which have been already afforded as for renewal processes. In other words, the emphasis is on results that can be used to answer specific questions rather than on proofs of theorems under conditions of the largest generality.

Sct. 2 Definitions and nomenclature

Even though we do not believe a comprehensive set of definitions is required for understanding the developments to follow, it may be of some value to present a unified treatment of the various concepts and quantities involved. This treatment will hold whichever the type of the process concerned.

Let us first assume the possibility of a complete knowledge of each statistical variable. Then define the pointwise absolute probability $P_k(t) = \mathcal{P}\{x(t) = k\}$ as the probability that the unit U is in the state X_k at a given instant of time $t \geqslant t_o$. For the two-state unit henceforth considered we have k = s,f as long as U is in its safe or in its failed state, respectively. Since the two events are non-compatible,

$$\sum_k P_k(t) = P_s(t) + P_f(t) = 1 \ . \tag{2.1}$$

We find also convenient to introduce the probability of transition as the pointwise conditional probability $P_{k|j}(t|\bar{t}) = \mathcal{P}\{x(t) = k \mid x(\bar{t}) = j\}$, expressing the probability that U is in the state X_k at t with $t \geqslant \bar{t}$, given that U is in the state X_j at $\bar{t}$. We have $P_k(t) = P_{k|s}(t|t_o) = \mathcal{P}\{x(t) = k|x(t_o) = s\}$. In order to gain generality it is also possible to introduce the notion of higher transition probabilities $P^{n(h\rightarrow i)}_{k|j}(t|\bar{t})$. We then mean the probability that starting from the state X_j we find U in the state X_k after n visits (or jumps, or steps) of the type $X_h \rightarrow X_i$. Unless otherwise stated, situations will be only considered where $n(h\rightarrow i) = n(j \rightarrow k) = n$. Relations of consistency hold as follows

$$\sum_k P_{k|j}(t|\bar{t}) = 1 \ , \qquad P_{k|j}(t|t) = \delta_{jk} \ , \tag{2.2}$$

$$P_k(t) = \sum_j P_j(\bar{t})\, P_{k|j}(t|\bar{t}), \tag{2.3}$$

and

$$(2.4) \qquad \sum_{n}^{0,1,\ldots\infty} P^{n}_{k|j}(t|\bar{t}) = P_{k|j}(t|\bar{t}) \; .$$

Let furthermore $N_{j\to k|j}(\bar{t},t|\bar{t})$ with $\bar{t} \leq t$ denote the number of visits from X_j to X_k during $(\bar{t},t)$ given that the unit enters (or, equivalently, starts from) the state X_j at $\bar{t}$. Conversely, $N_{j\to k|j^*}(\bar{t},t|\bar{t})$ will denote the number of visits from X_j to X_k within $(\bar{t},t)$, provided that the unit lays in the state X_j at $\bar{t}$. $N_{j\to k|j}$ and $N_{j\to k|j^*}$ are random forward numbers. We may similarly define the backward random numbers of visits $N_{j\to k|k}(\bar{t},t|t)$ and $N_{j\to k|k^*}(\bar{t}, t|t)$. In any case it results $N_{j\to k}(t,t|t) = 0$. With regard to $N_{j\to k|j}$ (parallel considerations would apply to $N_{j\to k|j^*}$, $N_{j\to k|k}$, etc.) probabilities obey rules of the form

$$(2.5) \quad \mathcal{P}\{N_{j\to k|j}(\bar{t},t|\bar{t}) < n\} = \mathcal{P}\{N_{j\to k|j}(\bar{t},t|\bar{t}) \leq n-1\} =$$

$$= 1 - \mathcal{P}\{N_{j\to k|j}(\bar{t},t|\bar{t}) \geq n\} = 1 - \mathcal{P}\{N_{j\to k|j}(\bar{t},t|\bar{t}) > n-1\} \, ,$$

$$(2.6) \quad \mathcal{P}\{N_{j\to k|j}(\bar{t},t|\bar{t}) = n\} = \mathcal{P}\{N_{j\to k|j}(\bar{t},t|\bar{t}) < n+1\} -$$

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$$- \mathcal{P}\{N_{j\to k|j}(\bar{t},t|\bar{t}) \geq n+1\} \, ,$$

and so on. Cumulative distributions (2.6) allow in their turn the computation of the expected number of steps $j\to k$ occuring within a finite interval of time $(\bar{t},t)$,

$$(2.7) \qquad M_{j\to k}(\bar{t},t) \equiv \mathcal{E}\{N_{j\to k}(\bar{t},t)\} =$$

$$= \sum_{n}^{1,\ldots\infty} n \, \mathcal{P}\{N_{j\to k}(\bar{t},t) = n\} \, ,$$

whereas the second-order central moment would result

$$(2.8)\qquad L_{j\to k}(\bar{t},t) \equiv \mathcal{D}\{N_{j\to k}(\bar{t},t)\} = \\ = \sum_{n}^{1,\dots\infty} n^2 \mathcal{P}^2 \{N_{j\to k}(\bar{t},t) = n\} - M^2_{j\to k}(\bar{t},t).$$

The time derivative of (2.7),

$$(2.9)\qquad m_{j\to k}(\bar{t},t) = \frac{\partial M_{j\to k}(\bar{t},t)}{\partial t},$$

is referred to as the (forward) jump density function.

A notable interest is attached to the difference

$$(2.10)\qquad N_{j\to k}(\bar{t},t) - N_{j\to j}(\bar{t},t) = \begin{cases} 1 & \text{if } x(t) \in X_k, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, taking expected values of both sides, we are led to

$$(2.11)\qquad P_{k|j}(t|\bar{t}) = M_{j\to k}(\bar{t},t) - M_{j\to j}(\bar{t},t).$$

If we think of U with $x(\bar{t}) \in X_s$, the sequence of time intervals spent alternatively in X_s and X_f will be written

$$(2.11_1)\qquad \xi(\bar{t}),\ \tau_1''(\bar{t}),\ \tau_2'(\bar{t}),\ \tau_2''(\bar{t}),\ \dots.$$

When $\bar{t} = t'_n$ we would have $\xi(t'_n) = 0$, for $\bar{t} = t_n''$ it would be $\xi(t''_n) = \tau'_1(t_n'')$. On the other hand, thinking of U with $x(\bar{t}) \in X_f$, life lengths are

$$(2.11_2)\qquad \zeta(\bar{t}),\ \tau_1'(\bar{t}),\ \tau_1''(\bar{t}), \tau_2'(\bar{t}),\ \dots.$$

Note, further, that according to Sct. 1, $\zeta(t_o) = 0$, $\xi(t_o) = = \tau_1'(t_o) = \tau_1'$, $\tau''_1(t_o) = \tau''_1, \dots$. Any term or sum of terms in (2.11) is a statistical variable representing a life excerpt of U. Particular life excerpts are the amount of time the unit is on and the amount of time the unit is down during

$(\bar{t},t)$, denoted as $\varphi(\bar{t},t)$ and $\Psi(\bar{t},t)$, respectively. Clearly additive properties hold as follows

$$(2.12_1) \qquad \varphi(\bar{t},t) + \Psi(\bar{t},t) = t - \bar{t} ,$$

$$(2.12_2) \qquad \varphi(\bar{t},t) = \varphi(\bar{t},\bar{\bar{t}}) + \varphi(\bar{\bar{t}},t) , \quad \text{etc.}$$

We may also express simple relations between number of visits and life excerpts. As for $N_{s \to f}$ we have in fact

$$(2.13_1) \qquad \mathcal{P}\{ N_{s \to f|s} (\bar{t},t|\bar{t}) \geq n \} =$$

$$= \mathcal{P}\{ \tau_1'(\bar{t}) + \tau_1''(\bar{t}) + \ldots + \tau'_n(\bar{t}) \leq t-\bar{t} \} ,$$

$$(2.13_2) \qquad \mathcal{P}\{ N_{s \to f|s^*} (\bar{t},t|\bar{t}) \geq n \} =$$

$$= \mathcal{P}\{ \xi(\bar{t}) + \tau_1''(\bar{t}) + \ldots + \tau'_n(\bar{t}) \leq t-\bar{t} \} .$$

Moreover, for conditional probabilities $P^n_{k|j}$ we find

$$(2.14_1) \qquad P^n_{f|s}(t|\bar{t}) = \mathcal{P}\{ \tau'_1(\bar{t}) + \tau_1''(\bar{t}) + \ldots + \tau'_n(\bar{t}) \leq t-\bar{t} <$$

$$< \tau_1'(\bar{t}) + \tau''_1(\bar{t}) + \ldots + \tau'_n(\bar{t}) + \tau''_n(\bar{t}) \} = \mathcal{P}\{ \alpha_n(\bar{t}) + \beta_{n-1}(\bar{t}) \leq$$

$$\leq t-\bar{t} \} - \mathcal{P}\{ \alpha_n(\bar{t}) + \beta_n(\bar{t}) \leq t-\bar{t} \} ,$$

$$(2.14_2) \quad P^n_{s|s}(t|\bar{t}) = \mathcal{P}\{ \tau_1'(\bar{t}) + \tau_1''(\bar{t}) + \ldots + \tau_n'(\bar{t}) + \tau_n''(\bar{t}) \leq t-\bar{t} <$$

$$< \tau_1'(\bar{t}) + \ldots + \tau'_{n+1}(t) = \mathcal{P}\{ \alpha_n(\bar{t}) + \beta_n(\bar{t}) \leq t-\bar{t} \} - \mathcal{P}\{ \alpha_{n+1}(\bar{t}) +$$

$$+ \beta_n(\bar{t}) \leq t-\bar{t} \} ,$$

and so on.

A problem often encountered consists of the determination of $\mathcal{P}\{ \varphi(\bar{t},t) < \Phi \}$ (or $\mathcal{P}\{ \Psi(\bar{t},t) < \Psi \}$), where $t-\bar{t} \geq \Phi > 0$ $(t - \bar{t} \geq \Psi > 0)$.
Because of (2.12_1) we see that

(2.15) $\mathcal{P}\{\Psi(\bar{t},t) \leq \Psi\} = 1 - \mathcal{P}\{\varphi(\bar{t},t) < t-\bar{t}-\Psi\}.$

A useful connection with pointwise availability $P_s(t)$ has the form

(2.16 $$\mathcal{E}\{\varphi(\bar{t},t)\} = \int_{\bar{t}}^{t} P_s(u)\, du = \int_{o}^{t-\bar{t}} \Phi f_\varphi(\Phi)\, d\Phi ,$$

where $f_\varphi(\Phi) = \dfrac{\partial \mathcal{P}\{\varphi < \Phi\}}{\partial \Phi}$.

Finally, if t_Φ denotes the random instant at which the total operating time attains the value Φ we will have

(2.17) $$\mathcal{P}\{\varphi(\bar{t},t) \leq \Phi\} = \mathcal{P}\{t \leq t_\Phi\}.$$

Sct. 3 Irreversible change

In the present section we shall consider both the operation of a unit until its first failure and the maintenance until its first restoration. In other words, failure and restoration are thought of as irreversible or absorbing states; from a formal point of view they will find thoroughly similar analytical treatments.

Suppose that the unit starts to operate at the instant t_o. The probability of failure free operation during (t_o,t) will be denoted

$$(3.1) \qquad R(t_o,t) \equiv P_{s|s}(t|t_o) = \mathcal{P}\{\tau' > t-t_o\} = $$
$$= \mathcal{P}\{N_{s\to f}(t_o,t) = 0\} .$$

$R(t_o,t)$ is called the reliability (function) of the unit. Because of its physical meaning, this function should be real-valued, continuously differentiable and decrease monotonically from $\lim_{t\to t_o^+} R(t_o, t) = 1$ to $\lim_{t\to\infty} R(t_o,t) = 0$. The complementary value to the unit

$$(3.2_1) \qquad F(t_o,t) = 1 - R(t_o,t) \quad , \quad dF = -dR,$$

represents the probability of failure during (t_o, t) or the failure (cumulative distribution) function. The time derivative,

$$(3.2_2) \qquad f(t_o,t) = \frac{dF(t_o,t)}{dt} ,$$

stands for the failure probability density function (p.d.f.)

Now $\lambda(t)\Delta t + O(\Delta t^2)$ is the probability of failure occurring during $(t,t+\Delta t)$, given the absence of failure in (t_o,t). By $O(\Delta^r)$ we will mean a function of Δ which has the property $\lim_{\Delta\to 0^+} \frac{O(\Delta^r)}{\Delta^{r-1}} = 0$ for $r \gtrless 0$. Since the theorem on compound probabilities, in the limit $\Delta t \to dt$ we have

(3.3) $$R(t_o,t)\ \lambda(t)\ dt = d\ (1 - R(t_o,t)) = -\ dR(t_o,\ t).$$

Namely a linear differential equation fully replaceable by the integral form

(3.4) $$R(t_o,\ t) = 1 - \int_{t_o}^{t} R\ (t_o,\ u)\ \lambda(u)\ du\ .$$

In terms of $\lambda(t)$ the solution of (3.3) or (3.4) is

(3.5) $$R(t_o,t) = \exp\ (- \int_{t_o}^{t} \lambda(u)\ du)$$

and the failure density (3.2) results

(3.6) $$f(t_o,t) = \lambda(t)\ R(t_o,\ t)\ .$$

Life length to failure τ' is a statistical variable ranging from t_o to ∞. Its expected value $m_{\tau'}$ and its variance $\sigma^2_{\tau'}$ are obtained as

(3.7$_1$) $$m_{\tau'} \equiv \mathcal{E}\{\tau'\} = \int_{t_o}^{\infty}(t-t_o)f(t_o,t)\ dt = \int_{t_o}^{\infty} R(t_o,t)\,dt - t_o,$$

(3.7$_2$) $$\sigma^2_{\tau'} \equiv \mathfrak{D}\{\tau'\} = \int_{t_o}^{\infty}(t-t_o - m_{\tau'})^2\ f(t_o,\ t)\ dt =$$

$$= 2\int_{t_o}^{\infty}(t-t_o)R(t_o,t)\ dt - (m_{\tau'} + t_o)^2\ .$$

Clearly the expression

(3.8) $$R(\bar{t},t) = \frac{R(t_o,\ t)}{R(t_o,\bar{t}\ )} = \exp\ (-\int_{\bar{t}}^{t} \lambda(u)\ du)\ ,$$

where $t_o \leq \bar{t} \leq t$,

is the probability of absence of failure in $(\bar{t},t)$ given no failure during (t_o,t). Thus $R(\bar{t},t)\ \lambda(t)\ dt$ signify the probability of failure within $(t,t + dt)$ provided the absence of failure in (t_o,t). But the unconditional probability of failure in $(\bar{t},t)$ reads

$$(3.9) \qquad F(\bar{t},t) = \int_{\bar{t}}^{t} f(t_o,u)\, du = R(t_o,\bar{t}) - R(t_o,t) =$$

$$= R\,(t_o,\bar{t}) \left[1 - R\,(\bar{t},t) \right].$$

Dual considerations apply, so to speak, to the absorbing restoration process. Suppose then that the unit, once failed, starts to be restored at the instant t_o. The probability of maintenance free from restoration during (t_o,t) may be called the maintainability(function) of the unit. We write

$$(3.10) \qquad S(t_o,t) \equiv P_{f|f}\,(t\,|t_o) = \mathcal{P}\{\tau'' > t-t_o\} =$$

$$= \mathcal{P}\{ N_{f\to s}\,(t_o,t) = 0 \},$$

admittedly an ever decreasing quantity in (t_o,t) with $\lim_{t\to t_o^+} S(t_o, t) = 1$ and $\lim_{t\to\infty} S(t_o,t) = 0$. The complementary function is

$$(3.11_1) \qquad G(t_o,t) = 1 - S\,(t_o,t) \qquad , \; dG = - dS$$

and the restoration probability density function, assumed continuous for $t \geqslant t_o$, results

$$(3.11_2) \qquad g(t_o,t) = \frac{dG(t_o,t)}{dt} .$$

Let $\mu(t)\,\Delta t + O\,(\Delta t^2)$ denote the probability of restoration in $(t,t+\Delta t)$ given no restoration in (t_o,t). In the limit $\Delta t \to dt$ it follows

$$(3.12) \qquad S(t_o,t)\,\mu(t)\; dt = d(1-S(t_o,t)) = -d\; S(t_o,t) .$$

so that

$$(3.13) \qquad S(t_o,t) = \exp\,(- \int_{t_o}^{t} \mu(u)\; du\,).$$

Analytical expressions for $S(\bar{t},t)$, $G(\bar{t},t)$ could also be easily found. Life length to restoration τ'' is a statistical variable. In complete analogy with τ' we may calculate the expected value

$$(3.14_1) \qquad m_{\tau''} \equiv \mathcal{E}\{\tau''\} = \int_{t_o}^{\infty} S(t_o, t) \; dt - t_o$$

and the variance

$$(3.14_2) \qquad \sigma^2_{\tau''} \equiv \mathcal{D}\{\tau''\} = 2 \int_{t_o}^{\infty} (t-t_o) S(t_o, t) \; dt - (m_{\tau''} + t_o)^2 .$$

Sct. 4 Transition probabilities

Let us refer to a Markov process of first order with a finite number of states and assume conditions as follows [5],

i) to each state X_j there corresponds a non-negative continuous hazard rate function $\gamma_j(t)$ such that

$$(4.1) \qquad P_{j|j}(t+\Delta t|t) = 1 - \gamma_j(t)\Delta t + O(\Delta t^2) \,,$$

ii) to any two different states X_j and X_k there correspond transition probabilities $\pi_{j\to k}(t)$ such that

$$(4.2) \qquad P_{k|j}(t+\Delta t|t) = \gamma_j(t)\,\pi_{j\to k}(t)\,\Delta t + O(\Delta t^2) \,.$$

$\pi_{j\to k}(t)$ are continuous functions in $t \geq t_0$ with $\pi_{j\to j}(t) = 0$ and $\sum_k \pi_{j\to k}(t) = 1$, expressing the conditional probability of a transition of the system from X_j to X_k during $(t, t+\Delta t)$, given that a transition occurred from X_j within the same time interval.

If the passage to the limit for $\Delta t \to 0^+$ holds uniformly in $t > t_0$ we obtain the Kolmogorov's forward system of differential equations

$$(4.3_1) \qquad \frac{\partial P_{k|i}(t|\bar t)}{\partial t} = -\gamma_k(t)\,P_{k|i}(t|\bar t) + \sum_{j\neq k}\gamma_j(t)\,P_{j|i}(t|\bar t)\,\pi_{j\to k}(t) \,.$$

The initial conditions are $P_{k|i}(\bar t|\bar t) = \delta_{ik}$. On the other hand, as long as the passage to the limit for $\Delta \bar t \to 0^+$ holds uniformly in $\bar t > t_0$ we are also given the Kolmogorov's backward system of differential equations

$$(4.3_2) \qquad \frac{\partial P_{k|i}(t|\bar{t})}{\partial \bar{t}} = \gamma_i(\bar{t})\, P_{k|i}(t|\bar{t}) - \gamma_i(\bar{t}) \sum_{j\neq i} \pi_{i\to j}(\bar{t}) \cdot P_{k|j}(t|\bar{t})$$

with the corresponding initial conditions $P_{k|i}(t|t) = \delta_{ik}$ [5,6]. Both systems (4.3) uniquely determine the transition probabilities $P_{k|i}(t|\bar{t})$ and these probabilities result subjected to the Chapman-Kolmogorov relation

$$(4.4) \qquad P_{k|i}\,(t|\bar{t}) = \sum_j P_{j|i}\,(\bar{\bar{t}}|\bar{t})\cdot P_{k|j}\,(t|\bar{\bar{t}}).$$

Now the basic hypotheses underlying a two-state simple repair process are $\gamma_s(t) = \lambda(t)$, $\gamma_f(t) = \mu(t)$, $\pi_{s\to f}(t) = \pi_{f\to s}(t) = 1$, where the hazard functions $\lambda(t)$ and $\mu(t)$ have the same definitions of Sct. 3. Thus, according to (4.3_1), the probability $P_{f|s}\,(t|\bar{t})$ can be assigned as

$$(4.5) \qquad \frac{\partial P_{f|s}(t|\bar{t})}{\partial t} = -\mu(t)\, P_{f|s}(t|\bar{t}) + \lambda(t)\, P_{s|s}(t|\bar{t});$$

a thoroughly similar equation would hold for $P_{f|f}(t|\bar{t})$. Through formula (2.3), knowing the distribution $\{P_i(\bar{t})\}$ with $\bar{t} \geq t_o$, we obtain for the pointwise unavailability [4]

$$(4.6) \qquad \frac{d\, P_f(t)}{dt} = -\mu(t)\, P_f(t) + \lambda(t)\, P_s(t).$$

On its turn, equality (2.1) assures the compatibility with the alternative form for the pointwise availability

$$(4.7) \qquad \frac{dP_s(t)}{dt} = -\lambda(t)\, P_s(t) + \mu(t)\, P_f(t) = \mu(t) - (\lambda(t) + \mu(t))P_s(t).$$

Situations where failure is an absorbing state, namely where unit is not repairable, correspond with $\mu = 0$. Conversely, the position $\lambda = 0$ implies restoration as an irreversible process.

More specifically, with the initial condition $P_f(\bar{t})$ the integral of equation (4.6) becomes [4]

$$(4.8) \qquad P_f(t) = P_f(\bar{t}) \exp\left(-\int_{\bar{t}}^{t} (\lambda + \mu)\, du\right) +$$

$$+ \exp\left(-\int_{\bar{t}}^{t} (\lambda + \mu)\, du\right) \int_{\bar{t}}^{t} \lambda(u) \exp\left(\int_{\bar{t}}^{u} (\lambda + \mu)\, dw\right) du =$$

$$= R(\bar{t},t)\, S(\bar{t},t) \left[P_f(\bar{t}) + \int_{\bar{t}}^{t} \frac{\lambda(u)\, du}{R(\bar{t},u)\, S(\bar{t},u)} \right] .$$

On its turn equation (4.7) entails

$$(4.9) \qquad P_s(t) = P_s(\bar{t}) \exp\left(-\int_{\bar{t}}^{t} (\lambda + \mu)\, du\right) +$$

$$+ \exp\left(-\int_{\bar{t}}^{t} (\lambda + \mu)\, du\right) \int_{\bar{t}}^{t} \mu(u) \exp\left(\int_{\bar{t}}^{u} (\lambda + \mu)\, dw\right) du$$

and $P_s(\bar{t}) = 1 - P_f(\bar{t})$.

The next step consists of the calculation of the transition density functions. We have the failure density function of the process

$$(4.10) \qquad m_{s \to f|s}(t|\bar{t}) = P_{s|s}(t|\bar{t})\, \lambda(t)$$

and the restoration (or repair) density

$$(4.11) \qquad m_{f \to s|s}(t|\bar{t}) = P_{f|s}(t|\bar{t})\, \mu(t) = m_{s \to s|s}(t|\bar{t}).$$

Thus (4.5) may be rewritten in the form

$$(4.12) \qquad \frac{\partial P_{f|s}}{\partial t} = m_{s \to f|s} - m_{f \to s|s}.$$

Since the total numbers of failures and of restorations in $(\bar{t},t)$ are defined as

$$(4.13_1) \qquad M_{s \to f|s}(\bar{t},t|\bar{t}) = \int_{\bar{t}}^{t} m_{s \to f|s}(u|\bar{t})\, du,$$

$$(4.13_2) \qquad M_{f \to s|s}(\bar{t}, t|\bar{t}) = \int_{\bar{t}}^{t} m_{f \to s|s}(u|\bar{t})\, du =$$

$$= M_{s \to s|s}(\bar{t}, t|\bar{t})$$

and $P_{f|s}(\bar{t}|\bar{t}) = 0$, the integration of (4.12) gives

$$P_{f|s}(t|\bar{t}) = M_{s \to f|s}(\bar{t},t|\bar{t}) - M_{f \to s|s}(\bar{t},t|\bar{t}). \tag{4.14}$$

As shown by (2.11) this relation holds independently of the type of failure plus restoration policy which has been assumed.

An alternative description of repair processes lies entirely on integral equations. With aim at simplicity, we will henceforth set $\bar{t} = t_o$, $P_s(t_o) = 1$: it does result $P_{s|s}(t|t_o) = P_s(t)$. Then, starting from the last repair onwards, we have

$$\begin{aligned} P_s(t) &= P_s(t_o)\, R(t_o,t) + \int_{t_o}^{t} m_{f \to s|s}(u|t_o) R(u,t)\,du = \\ &= R(t_o,t) + \int_{t_o}^{t} P_f(u)\, \mu(u)\, R(u,t)\, du. \end{aligned} \tag{4.15}$$

This equation should be combined with the one for $P_f(t)$ obtained starting from the last failure,

$$\begin{aligned} P_f(t) &= P_f(t_o)\, S(t_o,t) + \int_{t_o}^{t} m_{s \to f|s}(u|t_o)\, S(u,t)\,du = \\ &= \int_{t_o}^{t} P_s(u)\, \lambda(u)\, S(u,t)\, du. \end{aligned} \tag{4.16}$$

Now equations (4.15) and (4.16) reduce to equation (4.6) or (4.7). In fact differentiation of (4.15) (the same procedure applies to (4.16)) shows

$$\begin{aligned} \frac{dP_s(t)}{dt} &= -\lambda(t)\, R(t_o,t) - \lambda(t) \int_{t_o}^{t} P_f(u)\, \mu(u)\, R(u,t)\, du + \\ &+ \mu(t)\, P_f(t) = -\lambda(t)\, P_s(t) + \mu(t)\, P_f(t). \end{aligned}$$

However, if the integral equation for $P_s(t)$ is written moving from the last failure onwards, we obtain the closed form [7]

$$(4.17) \qquad P_s(t) = R(t_o,t) + \int_{t_o}^{t} \left[m_{s \to f}(u) \int_u^t S(u,w)\,\mu(w) \cdot R(w,t)\,dw \right] du =$$

$$= R(t_o,t) + \int_{t_o}^{t} \left[P_s(u)\,\lambda(u) \int_u^t S(u,w)\,\mu(w)\,R(w,t)\,dw \right] du.$$

Alternatively, for $P_f(t)$ we may move from the last repair onwards thus obtaining

$$(4.18) \qquad P_f(t) = \int_{t_o}^{t} R(t_o,u)\,\lambda(u)\,S(u,t)\,du +$$

$$+ \int_{t_o}^{t} \left[m_{f \to s}(u) \int_u^t R(u,w) .\, \lambda(w)\,S(w,t)\,dw \right] du =$$

$$= \int_{t_o}^{t} R(t_o,u)\,\lambda(u)\,S(u,t)\,du +$$

$$+ \int_{t_o}^{t} \left[P_f(u)\,\mu(u) \int_u^t R(u,w)\,\lambda(w)\,S(w,t)\,dw \right] du .$$

Both equations (4.17) and (4.18) can be shown to contain the differential forms (4.6) and (4.7). For instance after differentiation of (4.17) we have

$$\frac{dP_s(t)}{dt} = -\lambda(t)R(t_o,t) + \int_{t_o}^{t} \left\{ P_s(u)\,\lambda(u) \left[S(u,t)\,\mu(t) - \right.\right.$$

$$\left.\left. - \int_u^t S(u,w)\,\mu(w)(-\lambda(t)R\,(w,t))\,dw \right] \right\} du ,$$

then, through the repeated use of (4.17) and (4.16), we finally end to

$$\frac{dP_s(t)}{dt} = -\lambda(t)\left\{ R(t_o,t) + \int_{t_o}^{t}\left[P_s(u)\,\lambda(u)\int_{u}^{t} S(u,w)\cdot \right.\right.$$

$$\left.\cdot\,\mu(w)\;R(w,t)\;dw\right]du \;+\mu(t)\left\{\int_{t_o}^{t} P_s(u)\lambda(u)S(u,t)\,du\right\} =$$

$$= -\lambda(t)\;P_s(t) + \mu(t)\;P_f(t).$$

Sct. 5 Limiting distributions

Since the number of states of the repair process $\{x(t),\ t_o \le t < +\infty\}$ is finite and every state can be reached from every other state with positive probability, then the repair process is ergodic [5]. The limiting distribution $\{P_k(\infty)\} = \lim_{t\to\infty} \{P_k(t)\}$ is uniquely determined from (4.6) and (4.7). We have $\lim_{t\to\infty} \frac{dP_k}{dt} = 0$ and

$$(5.1) \qquad P_f(\infty) = \lim_{t\to\infty} \frac{\lambda(t)}{\lambda(t)+\mu(t)} \ , \quad P_s(\infty) = \lim_{t\to\infty} \frac{\mu(t)}{\lambda(t)+\mu(t)} \ ,$$

$$m_{s\to f}(\infty) = m_{f\to s}(\infty) \ ,$$

irrespective of $\{P_k(\bar{t})\}$. If the process is homogeneous, then λ and μ are constants independent upon t. Integrals (4.8) and (4.9) become

$$(5.2_1) \quad P_f(t) = P_f(\bar{t}) \exp\left[-(\lambda+\mu)\ (t-\bar{t})\right] + \frac{\lambda}{\lambda+\mu} \cdot \left\{1 - \exp\left[-(\lambda+\mu)\ (t-\bar{t})\right]\right\} ,$$

$$(5.2_2) \quad P_s(t) = P_s(\bar{t}) \exp\left[-(\lambda+\mu)(t-\bar{t})\right] + \frac{\mu}{\lambda+\mu} \cdot \left\{1 - \exp\left[-(\lambda+\mu)\ (t-\bar{t})\right]\right\},$$

and the limits (5.1) result self-evident.

More generally, it is natural to look for asymptotic expressions valid usually as $t - t_o \to \infty$, or occasionally as $t - t_o \to 0$. Let us first refer to values $t \gg t_o$. Having defined the new independent variable $u \equiv \int_{t_o}^{t} (\lambda(w) + \mu(w))\, dw$, equation (4.6) may be written in the form

$$(5.3) \qquad \frac{dP_f(u)}{du} = c\,(u) - P_f(u) \quad ,$$

with $c(u) = \dfrac{\lambda(u)}{\lambda(u) + \mu(u)}$, $P_f(0) = 0$. Now the solution (5.2_1) gives a hint to an asymptotic sequence of the type

$$(5.4_1) \qquad P_f(u) \sim \sum_r^{1,2,\dots} P_{f,r}(u) \left[1-\exp(-u) \right]^r \quad \text{as} \quad \left[1 - \exp(-u) \right] \to 0 \; .$$

Substituting this sequence into (5.3) and equating coefficients of like powers of $\left[1 - \exp(-u) \right]$, we obtain

$$(5.4_2) \qquad P_{f,1}(u) = c(u), \quad P_{f,2}(u) = \frac{1}{2} \frac{dP_{f,1}(u)}{du}, \dots, \quad P_{f,r}(u) = \frac{1}{r} \frac{dP_{f,r-1}(u)}{du}, \dots .$$

Had we considered the first order term of $(5,4_1)$, we would have found

$$(5.5_1) \qquad P_f(t) = \frac{\lambda(t)}{\lambda(t) + \mu(t)} \left\{ 1-\exp\left[- \int_{t_o}^{t} (\lambda(w) + \mu(w)) \, dw \right] \right\} + E \; ,$$

where the error committed E is numerically less than $\hat{E} =$

$= \dfrac{|\dot{\lambda}\mu - \lambda\dot{\mu}|}{(\lambda+\mu)^3} \left\{ 1 - \exp\left[- \int_{t_o}^{t} (\lambda + \mu) \, dw \right] \right\}$. Then, as far as $\exp(-u) \sim 0$ and $\dfrac{|\dot{\lambda}\mu - \lambda\dot{\mu}|}{(\lambda+\mu)^3} \sim 0$, two conditions which under most practical circumstances tend to be effective by taking $t-t_o$ sufficiently large, formula (5.5_2) becomes

$$(5.5_2) \qquad P_f(t) \sim \frac{\lambda(t)}{\lambda(t) + \mu(t)} \quad \text{as} \quad t-t_o \to \infty \; .$$

On the other hand, if it was $\mu = 0$, from equation (4.8) we would obtain simply

$$(5.6) \quad P_f(t) = \int_{t_o}^{t} \lambda(u) \, R(u,t) \, S(u,t) du = \int_{t_o}^{t} \lambda(u) R(u,t) du = F(t_o,t) .$$

This fact suggests the search of an expansion of $P_f(t)$ where $F(t_o,t)$ is the leading term. We may indeed consider the expansion of $S(u,t)$ for small $\int_u^t \mu(w)\, dw \equiv \mathcal{M}$,

$$S(u,t) = 1 - \int_u^t \mu(w)\, dw + O(\mathcal{M}^2) ,$$

and write

$$(5.7_1) \qquad P_f(t) = F(t_o,t) - \int_{t_o}^{t} \lambda(u)R(u,t) \left(\int_u^t \mu(w)\, dw\right) du + O(\mathcal{M}^2).$$

Let us call $\hat{\mu}$ the maximum value taken by $\mu(t)$ in (t_o,t), it results

$$(5.7_2) \qquad P_f(t) \geq F(t_o,t) - \hat{\mu}\int_{t_o}^{t} (t-u)\, \lambda(u)R(u,t)\, du + O(\mathcal{M}^2) \geq$$

$$\geq F(t_o,t)\left[1-\hat{\mu}\,(t-t_o)\right] + O\left((t-t_o)^2\right),$$

so we conclude

$$(5.8) \qquad P_f(t) \sim F(t_o,t) \qquad \text{as} \quad t-t_o \to 0 .$$

Sct. 6 Number of transitions and life distributions

In the present section we explore in more detail the structure of the repair process, that is the nature of the distribution functions associated with some typical occurrences in the operation of the unit. We begin admitting that U enters X_s at $\bar{t}$ and we define the (cumulative) distribution function of first n cycles to failure starting from X_s,

$$(6.1_1) \qquad F^{(n)}(\bar{t},t) = \mathcal{P}\{\tau_1'(\bar{t}) + \tau''_1(\bar{t}) + \ldots + \tau'_n(\bar{t}) \leq t-\bar{t}\} =$$

$$= \mathcal{P}\{N_{s \to f|s}(\bar{t},t|\bar{t}) \geq n\},$$

with $F^{(1)}(\bar{t},t) \equiv F(\bar{t},t)$. We find furthermore convenient to introduce the (cumulative) distribution function of first n cycles to restoration starting from X_s, or, what is equivalent, the distribution of first n repair cycles,

$$(6.1_2) \qquad H^{(n)}(\bar{t},t) = \mathcal{P}\{\tau'_1(\bar{t}) + \tau''_2(\bar{t}) + \ldots + \tau'_n(\bar{t}) + \tau''_n(\bar{t}) \leq t-\bar{t}\} =$$

$$= \mathcal{P}\{N_{f \to s|s}(\bar{t},t|\bar{t}) \geq n\} = \mathcal{P}\{N_{s \to s|s}(\bar{t},t|\bar{t}) \geq n\}.$$

Limiting conditions will hold as follows

$$(6.2_1) \qquad F^{(n)}(\bar{t},t) = H^{(n)}(\bar{t},t), \qquad \lim_{t \to \infty} F^{(n)}(\bar{t},t) =$$

$$= \lim_{t \to \infty} H^{(n)}(\bar{t},t) = 1.$$

It is moreover assumed $\tau'_o(\bar{t}) = \tau''_o(\bar{t}) = 0$ and

$$(6.2_2) \qquad F^{(o)}(\bar{t},t) = H^{(o)}(\bar{t},t) = 1^+(\bar{t},t) = \begin{cases} 0 & \text{for } t \leq \bar{t}, \\ 1 & \text{for } t > \bar{t}, \end{cases}$$

where $1^{+}(\bar{t},t)$ denotes the improper distribution.

The knowledge of (6.1) entails expressions for the expected number of failures and expected number of repairs in $(\bar{t},t)$. In fact it is clear (consult (2.7)) that

$$(6.3_1)\qquad M_{s\to f|s}(\bar{t},t|\bar{t}) = \mathcal{E}\{N_{s\to f|s}(\bar{t},t|\bar{t})\} = \sum_{n}^{1,\dots\infty} n\left[F^{(n)}(\bar{t},t)-F^{(n+1)}(\bar{t},t)\right] = \sum_{n}^{1,\dots\infty} F^{(n)}(\bar{t},t)$$

and

$$(6.3_2)\qquad M_{s\to s|s}(\bar{t},t|\bar{t}) = M_{f\to s|s}(\bar{t},t|\bar{t}) = \mathcal{E}\{N_{f\to s|s}(\bar{t},t|\bar{t})\} = \sum_{n}^{1,\dots\infty} H^{(n)}(\bar{t},t).$$

Moments of higher order could also be easily computed. Since (forward) probability density functions are written as

$$(6.4_1)\qquad f^{(n)}(\bar{t},t) = \frac{\partial F^{(n)}(\bar{t},t)}{\partial t}\ ,\quad h^{(n)}(\bar{t},t) = \frac{\partial H^{(n)}(\bar{t},t)}{\partial t}\ ,$$

$$(6.4_2)\qquad f^{(o)}(\bar{t},t) = h^{(o)}(\bar{t},t) = \delta^{+}(t-\bar{t})\ ,$$

expressions (6.3) lead to

$$(6.5_1)\qquad m_{s\to f|s}(\bar{t},t) = \sum_{n}^{1,\dots\infty} f^{(n)}(\bar{t},t)\ ,$$

$$(6.5_2)\qquad m_{f\to s|s}(\bar{t},t) = \sum_{n}^{1,\dots\infty} h^{(n)}(\bar{t},t)\ .$$

Note that for cases where the unit enters the state X_f at $\bar{t}$, we would have the opportunity of considering two additional functions,

$$(6.6_1) \qquad G^{(n)}(\bar{t},t) = \mathcal{P}\{ \tau''_1(\bar{t}) + \ldots + \tau'_n(\bar{t}) + \tau''_n(\bar{t}) \leq t-\bar{t} \},$$

$$(6.6_2) \qquad K^{(n)}(\bar{t},t) = \mathcal{P}\{ \tau''_1(\bar{t}) + \ldots + \tau''_n(\bar{t}) + \tau'_{n+1}(\bar{t}) \leq t-\bar{t} \},$$

representing respectively the distribution of first n cycles to restoration and first n repairs starting from X_f. Henceforth the (forward) p.d.f.

$$(6.7) \qquad g^{(n)}(\bar{t},t) = \frac{\partial G^{(n)}(\bar{t},t)}{\partial t}, \quad k^{(n)}(\bar{t},t) = \frac{\partial K^{(n)}(\bar{t},t)}{\partial t}.$$

Probability distributions of $\tau'(\bar{t})$ and $\tau''(\bar{t})$ are known (see Sct. 3). Then, since the multiplication theorem of probabilities, we may obtain recurrence relations for $F^{(n)}(\bar{t},t)$, $H^{(n)}(\bar{t},t)$, etc. Let us focus our attention on $F^{(n)}(t_o,t)$ and $H^{(n)}(t_o,t)$. The result is

$$(6.8_1) \qquad F^{(n)}(t_o,t) = \int_{t_o}^{t} \left[h^{(n-1)}(t_o,u) \int_u^t f(u,w)\, dw \right] du =$$

$$= H^{(n-1)}(t_o,t) - \int_{t_o}^{t} h^{(n-1)}(t_o,u)\, R(u,t)\, du,$$

$$(6.8_2) \qquad H^{(n)}(t_o,t) = \int_{t_o}^{t} \left[f^{(n)}(t_o,u) \int_u^t g(u,w)\, dw \right] du =$$

$$= F^{(n)}(t_o,t) - \int_{t_o}^{t} f^{(n)}(t_o,u)\, S(u,t)\, du,$$

where $n=1,2,\ldots$. In terms of probability densities we see that

$$(6.9_1) \qquad f^{(n)}(t_o,t) = \lambda(t) \int_{t_o}^{t} h^{(n-1)}(t_o,u)\, R(u,t)\, du,$$

$$(6.9_2) \qquad h^{(n)}(t_o,t) = \mu(t) \int_{t_o}^{t} f^{(n)}(t_o,u)\, S(u,t)\, du.$$

On the other hand, comparison of (6.8) with (2.14) shows

$$(6.10_1) \qquad P^{n-1}_{s|s}(t_o,t) = \int_{t_o}^{t} h^{(n-1)}(t_o,u)\, R(u,t)\, du,$$

$$(6.10_2) \qquad P^{n}_{f|s}(t_o,t) = \int_{t_o}^{t} f^{(n)}(t_o,u)\, S(u,t)\, du \;.$$

We may therefore assign expressions for the probability density functions of n-th failure life and n-th restoration life ,

$$(6.11_1) \qquad f_{\tau'_n}(t_o,t) = P^{n-1}_{s|s}(t_o,t)\;\lambda(t),$$

$$(6.11_2) \qquad g_{\tau''_n}(t_o,t) = P^{n}_{f|s}(t_o,t)\,\mu(t)\;.$$

When equations (6.9) are summed up as in (6.5),it is obtained

$$(6.12_1) \qquad m_{s\to f|s}(t|t_o) = \lambda(t)\; R(t_o,t) + \lambda(t)\int_{t_o}^{t} m_{f\to s|s}(u|t_o)\cdot R(u,t)\, du,$$

$$(6.12_2) \qquad m_{f\to s|s}(t|t_o) = \mu(t)\int_{t_o}^{t} m_{s\to f|s}(u|t_o)\; S(u,t)\, du,$$

thus recovering (4.15) and (4.16). The use of distributions (6.6)would consent the definition of other types of equations. Noticeable forms are

$$(6.13_1) \qquad F^{(n)}(t_o,t) = \int_{t_o}^{t}\left[f(t_o,u)\int_{u}^{t} K^{(n-1)}(u,w)\, dw\right] du = \int_{t_o}^{t} f(t_o,u)\; K^{(n-1)}(u,t)\; du,$$

$$(6.13_2) \qquad f^{(n)}(t_o,t) = \int_{t_o}^{t} f(t_o,u)\; K^{(n-1)}(u,t)\, du,$$

and so on.

Let us now find an expression for the (forward) interval reliability or the probability that the unit is in X_s at t and operates without failures throughout (t,t+T). Obviously the event we are interested in is the union of all mutually exclusive events [3]

$$e_n = \left\{ \tau_1' + \tau_1'' + \ldots + \tau_n'' \leq t < t + T < \tau_1' + \tau_1'' + \ldots + \tau_n'' + \tau_{n+1} \right\}$$

with n=0,1,.... e have successively

$$(6.14) \qquad \mathcal{P}\left\{ N_{s^* \to f|s}(t,t+T|t_o) = 0 \right\} = \mathcal{P}\left\{ \bigcup_n^{0,1,\ldots\infty} e_n \right\} =$$

$$= \sum_n^{0,1,\ldots\infty} \mathcal{P}\{e_n\} = R(t_o,t+T) + \int_{t_o}^{t} R(u,t+T)\; dM_{s\to s|s}(t_o,u|t_o) =$$

$$= P_{s|s}(t|t_o)\; R(t,t+T).$$

A result which would have been expected since the hypotheses underlying Markov processes. In the limit $T \to 0$ we go back to (4.15). Having defined a new random variable, namely the forward recurrence time to failure $\xi(t)$, we will write

$$(6.15) \qquad \mathcal{P}\left\{ \xi(t) > T \right\} = \mathcal{P}\left\{ N_{s^* \to f|s}(t,t+T|t_o) = 0 \right\}.$$

As opposed to $\xi(t)$, the probability that $x(t) \in X_f$ and U operates without restoration during (t,t+T) is expressed through the forward recurrence time to restoration $\zeta(t)$. Relations hold as follows

$$(6.16) \qquad \mathcal{P}\left\{ \zeta(t) > T \right\} = \mathcal{P}\left\{ N_{f^* \to s|s}(t,t+T|t_o) = 0 \right\} =$$

$$= P_{f|s}(t|t_o)\; S(t,t+T)\;.$$

The problems thus far afforded may be enlarged so as to embrace the evaluation of $\mathcal{P}\{ N_{s^*\to f|s}(t,t+T|t_o) = n \}$ and $\mathcal{P}\{ N_{f^*\to s|s}(t,t+T|t_o) = n \}$. Premise is the knowledge of probability distributions like

$$(6.17)\qquad \mathcal{P}\{ N_{s^*\to f|s}(t,t+T|t_o) > n \} =$$

$$= \mathcal{P}\{ \xi(t) + \tau''_1(t) + \tau'_2(t) + \ldots + \tau'_n(t) \le T \},$$

etc. If we put

$$f_\xi(t,u) = \frac{d\mathcal{P}\{\xi(t) \le u\}}{dt},$$

$$g_\zeta(t,u) = \frac{d\mathcal{P}\{\zeta(t) \le u\}}{du},$$

we may write

$$(6.18_1)\qquad \mathcal{P}\{ N_{s^*\to f|s}(t,t+T|t_o) > n \} =$$

$$= \int_o^T \left[f_\xi(t,u) \int_u^T k^{(n-1)}(t+u,t+w)\,dw \right] du =$$

$$= P_{s|s}(t|t_o)\, F^{(n)}(t,t+T)$$

and

$$(6.18_2)\qquad \mathcal{P}\{ N_{f^*\to s|s}(t,t+T|t_o) > n \} = P_{f|s}(t|t_o)\,G^{(n)}(t,t+T).$$

Two equations whose roots plunge once more into (4.4).

Sct. 7 Distribution of total on time

Our problem is to find the (cumulative) distribution function (2.17)

$$\mathcal{P}\{\varphi(t_o,t)\leq\Phi\} = \mathcal{P}\{t\leq t_\Phi\} \tag{7.1}$$

with $\bar{t}=t_o$, $\Phi\leq t-t_o$.

The event $\varphi(t_o,t)\leq\Phi$ can occur through the following mutually exclusive ways : at the instant t the system is in the state X_f and the time interval (t_o,t) contains n (n=1,2,...) complete X_s intervals with total length $\alpha_n\leq\Phi$, or at the instant t the system is in the state X_s and the time interval (t_o,t) contains n (n=0,1,2,...) complete X_f intervals with total length $\beta_n\geq t-t_o-\Phi$ [8] .

Consequently, with the aid of the total probability theorem, it results

$$\begin{aligned}
\mathcal{P}\{\varphi(t_o,t)\leq\Phi\} &= \mathcal{P}\Big\{\bigcup_n^{1,\dots\infty}(\tau'_1+\tau'_2+\dots+\\
&+\tau'_n\leq\Phi\cap\tau'_1+\tau''_1+\dots+\tau'_n\leq t-t_o<\tau'_1+\tau''_1+\dots+\\
&+\tau'_n+\tau''_n)\Big\}+\mathcal{P}\Big\{\bigcup_n^{o,1,\dots\infty}(\tau''_1+\tau''_2+\dots+\tau''_n\geq t-t_o-\Phi\cap\\
&\cap\tau'_1+\tau''_1+\dots+\tau'_n+\tau''_n\leq t-t_o<\tau'_1+\tau''_1+\dots+\tau'_n+\tau''_n+\tau'_{n+1})\Big\}=\\
&=\sum_n^{1,\dots\infty}\mathcal{P}\{\alpha_n\leq\Phi\cap\alpha_n+\beta_{n-1}\leq t-t_o<\alpha_n+\beta_n\}+\\
&+\sum_n^{o,1,\dots\infty}\mathcal{P}\{\beta_n\geq t-t_o-\Phi\cap\alpha_n+\beta_n\leq t-t_o<\alpha_{n+1}+\beta_n\}.
\end{aligned} \tag{7.2}$$

Now, because of equation (2.14), we may write

$$(7.3_1) \quad \sum_{n}^{1,\dots\infty} \mathcal{P}\{ \alpha_n \le \Phi \cap \alpha_n + \beta_{n-1} \le t-t_0 < \alpha_n + \beta_n \} =$$

$$= \sum_{n}^{0,\dots\infty} \mathcal{P}\{ \alpha_{n+1} \le \Phi \cap \alpha_{n+1} + \beta_n \le t-t_0 \} -$$

$$- \sum_{n}^{1,\dots\infty} \mathcal{P}\{ \alpha_n \le \Phi \cap \alpha_n + \beta_n \le t-t_0 \} ,$$

$$(7.3_2) \quad \sum_{n}^{0,\dots\infty} \mathcal{P}\{ \beta_n \ge t-t_0-\Phi \cap \alpha_n + \beta_n \le t-t_0 < \alpha_{n+1} + \beta_n \} =$$

$$= \sum_{n}^{1,\dots\infty} \mathcal{P}\{ \beta_n \ge t-t_0 - \Phi \cap \alpha_n + \beta_n \le t-t_0 \} -$$

$$- \sum_{n}^{0,\dots\infty} \mathcal{P}\{ \beta_n \ge t-t_0 - \Phi \cap \alpha_{n+1} + \beta_n \le t-t_0 \} .$$

Thus for $0 \le \Phi < t$ we have

$$(7.4) \quad \mathcal{P}\{ \varphi(t_0,t) \le \Phi \} =$$

$$= \sum_{n}^{0,\dots\infty} \Big[\mathcal{P}\{ \alpha_{n+1} \le \Phi \cap \alpha_{n+1} + \beta_n \le t-t_0 \} -$$

$$- \mathcal{P}\{ \beta_n \ge t-t_0 - \Phi \cap \alpha_{n+1} + \beta_n \le t-t_0 \} \Big] -$$

$$- \sum_{n}^{1,\dots\infty} \Big[\mathcal{P}\{ \alpha_n \le \Phi \cap \alpha_n + \beta_n \le t-t_0 \} -$$

$$- \mathcal{P}\{ \beta_n \ge t-t_0 - \Phi \cap \alpha_n + \beta_n \le t-t_0 \} \Big] .$$

Evidently $\mathcal{P}\{\varphi(t_o,t) \leq t-t_o\} = 1$. Since the theorem on compound probabilities, each term of (7.4) can be separated into two parts. It is obtained formally

$$(7.5)\quad \mathcal{P}\{\alpha_{n+1} \leq \Phi \cap \alpha_{n+1} + \beta_n \leq t-t_o\} =$$

$$= \mathcal{P}\{\alpha_{n+1} \leq \Phi \mid \alpha_{n+1} + \beta_n \leq t-t_o\} \cdot$$

$$\cdot \mathcal{P}\{\alpha_{n+1} + \beta_n \leq t-t_o\} \equiv \mathcal{P}\{\alpha_{n+1} \leq \Phi \mid (t_o,t)\} \cdot$$

$$\cdot \mathcal{P}\{\alpha_{n+1} + \beta_n \leq t-t_o\}$$

and other analogous expressions. We end with

$$(7.6)\quad \mathcal{P}\{\varphi(t_o,t) \leq \Phi\} =$$

$$= \sum_n^{0,\ldots\infty} \left[\mathcal{P}\{\alpha_{n+1} \leq \Phi \mid (t_o,t)\} - \mathcal{P}\{\beta_n \geq t-t_o-\Phi \mid (t_o,t)\}\right] \cdot$$

$$\cdot \mathcal{P}\{\alpha_{n+1} + \beta_n \leq t-t_o\} - \sum_n^{1,\ldots\infty} \left[\mathcal{P}\{\alpha_n \leq \Phi \mid (t_o,t)\} -\right.$$

$$\left. - \mathcal{P}\{\beta_n \geq t-t_o-\Phi \mid (t_o,t)\}\right] \cdot \mathcal{P}\{\alpha_n + \beta_n \leq t-t_o\} .$$

At this point through the equations (6.8) we know how to compute probabilities $\mathcal{P}\{\alpha_{n+1} + \beta_n \leq t-t_o\}$ and $\mathcal{P}\{\alpha_n + \beta_n \leq t-t_o\}$. On the other hand, we may infer the cumulative distributions $\mathcal{P}\{\alpha_n \leq \Phi \mid (t_o,t)\}$ and $\mathcal{P}\{\beta_n \geq t-t_o-\Phi \mid (t_o,t)\}$ from density functions $f_{\tau'_n}$ and $g_{\tau''_n}$ as given by (6.11).

Although computations seem cumbersome we may say that the problem of calculating (7.6) is at least in principle solved.

Sct. 8 Immediate repair

Immediate repair processes correspond with practical situations where the unit after its failure is restored to its working state within such a short interval of time that life to restoration may be neglected and thus set equal to zero.

More explicitly, the state X_f for which $\mu = \infty$ is called a reflecting or unstable state. We have

$$(8.1_1) \qquad G(\bar{t},t) = 1^+(t-\bar{t}) = \begin{cases} 0 & \text{for } t \le \bar{t} \ , \\ 1 & \text{for } t > \bar{t} \ , \end{cases}$$

$$(8.1_2) \qquad g(\bar{t},t) = \delta^+(t-\bar{t}) \ .$$

Hence

$$(8.2) \qquad m_{\tau''} = \sigma_{\tau''} = 0 \ .$$

Pointwise availability and unavailability take the forms

$$(8.3_1) \qquad P_{s|s}(t|t_o) = 1^-(t-t_o) = \begin{cases} 0 & \text{for } t < t_o \ , \\ 1 & \text{for } t \ge t_o \ , \end{cases}$$

$$(8.3_2) \qquad P_{f|s}(t|t_o) = 1 - 1^-(t-t_o) \ ,$$

whereas equations (4.10) and (4.11) reduce themselves to

$$(8.4) \qquad m_{s \to f|s}(t|\bar{t}) = m_{f \to s|s}(t|\bar{t}) = \lambda(t) \ .$$

Clearly, by definitions (6.1) it is

$$(8.5) \qquad f^{(n)}(t_o,t) = h^{(n)}(t_o,t)$$

and recurrence relations (6.9) become

$$(8.6) \qquad f^{(n)}(t_o,t) = \lambda(t) \int_{t_o}^{t} f^{(n-1)}(t_o,u) \ R(u,t) \ du$$

for $n = 1,2,\ldots.$

Now according to (6.5) it is possible to sum up and henceforth obtain

$$(8.7) \qquad m_{s\to f}(t) = \lambda(t)\, R(t_o,t) + \lambda(t)\int_{t_o}^{t} m_{s\to f}(u)R(u,t)\,du,$$

an equation apparently solved by (8.4).

Sct. 9 Numerical computations

Our task is that of computing a series of probability charcteristics of the repair process. They will consist essentially of $P_s(t)$ or $P_f(t)$. It has been shown in fact that many problems can be solved in terms of these two functions. As a premise for numerical computations it is assumed that values of $\lambda(t)$ and $\mu(t)$ are assigned for all $t \geqslant t_o$. The characteristics of the irreversible processes, $R(t_o,t)$ and $S(t_o,t)$, are then be obtained according to the equations of Sct. 3. An alternative starting point may be offered by the initial specification of $R(t_o,t)$ and $S(t_o,t)$ or by the related hazard p.d.f., $f(t_o,t)$ and $g(t_o,t)$. If this is the case, the problem to be solved first is that of inferring the hazard rates $\lambda(t)$ and $\mu(t)$.

In complete generality we may begin with the evaluation of $P_f(t)$ as given by (4.8). When compared with (4.16) or (4.18) this expression has in fact the substantial advantage of consenting direct computations thus avoiding the resort to cumbersome techniques. Henceforth for each time interval $(t,t+\Delta t)$, through the use of a Gauss integration, (4.8) results in a discrete form as follows

$$(9.1) \qquad P_f(t+\Delta t) = P_f(t)\exp\left\{-\frac{\Delta t}{2}\sum_i^{1,\dots m} A_i\left[\lambda(t+u_i^*\,\Delta t)+\right.\right.$$

$$\left.\left.+\mu(t+u_i^*\,\Delta t)\right]\right\} + \frac{\Delta t}{2}\sum_i^{1,\dots m} A_i\;\lambda(t+u_i^*\,\Delta t)\;\exp\left\{-\frac{\Delta t}{2}(1-u_i^*)\right.$$

$$\left.\cdot\sum_j^{1,\dots n} A_j\left[\lambda(t+(u_i^*+u_j^*-u_i^*u_j^*)\Delta t)+\mu(t+(u_i^*+u_j^*-u_i^*u_j^*)\,\Delta t)\right]\right\}+o(\Delta t^2).$$

A and u^* are quantities inherent to the particular method of numerical integration. More specifically, having adopted a six-points Gauss-Legendre formula, we have m=n=6 and $A_i=A_j$,

$u_i^* = u_j^*$ for i=j. In each time segment $(t,t+\Delta t)$ expression (9.1) then requires the calculation of 27 nodal points which implies an estimate of $P_f(t)$ sufficiently accurate for most purposes.

Computations carried on the whole time domain $t \geqslant t_o$ are however faced with two types of difficulties. First, an efficient subdivision of the time axis into sub-intervals $(t,t+\Delta t)$ has to match the actual time shapes of $\lambda(t)$ and $\mu(t)$. The increments Δt should then be established on the ground of some dynamic criterion. An adaptive choice of Δt which grants the fastest convergence is described elsewhere [9] .

On the other hand, in the two extreme cases of small and large values of $t-t_o$ the simple use of (9.1) becomes redundant and unduly time wasting. Computations can be sped up with the help of asymptotic formulae. To this end the time domain (t_o,t) has been divided into three parts. Let $(t_o, t_{ax}^{P(I)})$ denote the initial or unresolved region, $P_f(t)$ is herewith computed from (5.7_2). Time $t_{ax}^{P(I)}$ is fixed according to the error criterion

$$(9.2_1) \qquad \left| P_f - P_{fax} \right| \leqslant \varepsilon P_f \,.$$

Here ε stays for a conveniently small positive quantity and P_{fax} means the asymptotic expression for P_f. If P_{fax} is not too f r from P_f, we may also prescribe

$$(9.2_2) \qquad \left| P_f - P_{fax} \right| \leqslant \varepsilon P_{fax} \,.$$

This signifies that the solution $P_{fax}^{(I)}(t) = F(t_o,t)$ can be accepted for $t_{ax}^{P(I)} \leq \hat{t}$, where

$$\mu(\hat{t}-t_o)\ F(t_o,\hat{t}) = \varepsilon \,.$$

Next let $(t_{ax}^{P(I)}, t_{ax}^{P(II)})$ stand for the transient interval, $P_f(t)$ will be then evaluated as from (9.1). On the contrary in the equilibrium region $(t_{ax}^{P(II)}, \infty)$ use could be made of (5.5). More explicitly we write

$$(9.3) \qquad P_f(t) = \frac{\lambda(t)}{\lambda(t) + \mu(t)} + \delta P_f(t) \equiv P_{fax}^{(II)}(t) + \delta P_f(t) \ ,$$

where

$$\delta P_f(t) \equiv \frac{\lambda}{\lambda + \mu} \exp\left[-\int_{t_o}^{t} (\lambda+\mu)\, dw\right] + E \quad \text{with} \quad |E| \leq \hat{E} =$$

$$= \frac{|\dot{\lambda}\mu - \lambda\dot{\mu}|}{(\lambda + \mu)^3} \left\{1 - \exp - \int_{t_o}^{t} \left[(\lambda+\mu)\, dw\right]\right\}.$$ Agreeing with (9.2_2) the limiting time $t_{ax}^{P(II)}$ is assumed $\sup\left[\hat{t}_1, \hat{t}_2\right]$. We take for $\hat{t}_1$

$$\exp\left[-\int_{t_o}^{\hat{t}_1} (\lambda + \mu)\, dw\right] = R(t_o, \hat{t}_1)\, S(t_o, \hat{t}_1) = r\varepsilon$$

and for $\hat{t}_2$ the largest root of

$$\frac{|\dot{\lambda}\mu - \lambda\dot{\mu}|}{\lambda(\lambda + \mu)^2} = (1-r)\,\varepsilon \ .$$

A proper choice of r, it has to be $0 \leqslant r \leqslant 1$, may help to keep $t_{ax}^{P(II)}$ as low as possible. When λ and μ are constants, we obtain $\hat{t}_2 = t_o$, $r = 1$ and

$$\hat{t}_1 = t_o + \frac{1}{\lambda + \mu} \ln\frac{1}{\varepsilon} = t_o + \frac{m_{\tau'}\, m_{\tau''}}{m_{\tau'} + m_{\tau''}} \ln\frac{1}{\varepsilon} \ .$$

Thus far we always admitted the full knowledge of $\lambda(t)$ and $\mu(t)$. Assume now that we are given the p.d.f. $f(t_o,t)$ and $g(t_o,t)$. We get

$$(9.4)\qquad \lambda(t) = \frac{f(t_o,t)}{1-\int_{t_o}^{t} f(t_o,u)\,du},$$

$$\mu(t) = \frac{g(t_o,t)}{1-\int_{t_o}^{t} g(t_o,u)\,du}$$

and the proofs proceed immediately from (3.3), (3.12). We must however say that the numerator and the denominator of both (9.4) tend to zero as $t \rightarrow \infty$ thus originating for large values of $t-t_o$ a critical form. Again computations are eased up by means of asymptotic expressions. Differentiation of (9.4) gives in fact

$$(9.5_1)\qquad \lambda(t) = -\frac{1}{f(t_o,t)}\frac{df(t_o,t)}{dt} + \frac{1}{\lambda(t)}\frac{d\lambda(t)}{dt} \equiv$$

$$\equiv \lambda_{ax}(t) + \delta\lambda(t),$$

$$(9.5_2)\qquad \mu(t) = \frac{1}{g(t_o,t)}\frac{dg(t_o,t)}{dt} + \frac{1}{\mu(t)}\frac{du\ (t)}{dt} \equiv$$

$$\equiv \mu_{ax}(t) + \delta\mu(t).$$

The time values t_{ax} and t_{ax}, beyond which errors $\delta\lambda(t)$ and $\delta\mu(t)$ are negligible, are determined according to error criteria of the type (9.2). Table 9.1 shows the asymptotic behaviour of a few hazard p.d.f. commonly employed.

Now all these arguments and expedients concur to the numerical code AVACOM -ETARP (Availability Computation-Element Transient and Asymptotic Repair Process). This code, written in FORTRAN IV language, gives computations of $P_f(t)$ for $t \geq t_o$ whichever the type of failure and restoration p.d.f. assumed. $m_{s \rightarrow f}(t)$ and $m_{f \rightarrow s}(t)$ are also deduced through (4,10) and (4.11). The only limiting hypothesis is that distributions are continuous and sufficiently smooth. Since either λ or μ can be the nul function, $R(t_o,t)$ and $S(t_o,t)$ correspond with the positions $\mu = 0$ and $\lambda = 0$, respectively.

Explicit results have been obtained for all the combinations of failure p.d.f. with restoration p.d.f. listed in Table 9.2. Time coordinate, expected life lengths and standard deviations have been formulated in terms of arbitrary time units. Figure 9.1 maps $\log_{10} P_f(t)$. The initial time t_o is placed at t=0. By the ordered couples EX'-EX", EX'-N",..., N'-EX",N'-N",... we mean combinations of failure and restoration, where EX', N', etc. denote failure p.d.f.'s and EX", N", etc. denote restoration p.d.f. 's, respectively. It is seen how the combinations EX'- 0", N'-0", LN'- 0", Γ'-0" and W'-0" which all signify F(0,t) set an upper limit for $P_f(t)$, a fact anticipated in Sct. 5. Figures (9.2) and (9.3) display on the other hand the trend followed by $\log_{10} m_{s \rightarrow f}(t)$ and $\log_{10} m_{f \rightarrow s}(t)$. The combinations EX'- ∞", N'- ∞", LN'- ∞", Γ'- ∞" and W' - ∞" now mean $m_{s \rightarrow f} = m_{f \rightarrow s} = \lambda$, a case occurring when restoration is a reflecting event.

Table 9.1 Asymptotic expressions of the hazard rate for a few commonly used hazard p.d.f.

p.d.f. $f(t_o,t)$ or $g(t_o,t)$	asymptotic hazard rate as $t-t_o$ $\lambda_{ax}(t)$ or $\mu_{ax}(t)$
i) normal	
$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(t-t_o-m)^2}{2\sigma^2}\right]$	$\sim \frac{t-t_o-m}{\sigma^2}$
ii) lognormal	
$\frac{1}{\sigma\sqrt{2\pi}} \frac{1}{t-t_o} \exp\left[-\frac{(\ln(t-t_o)-m)^2}{2\;^2}\right]$	$\sim \frac{\ln(t-t_o)\;\sigma^2-m}{\sigma^2 t}$
iii) gamma	
$\frac{q^r}{\Gamma(r)} (t-t_o)^{r-1} \exp\left[-q(t-t_o)\right]$	$\sim \frac{q(t-t_o)-r+1}{t-t_o}$
iv) Weibull	
$\alpha^{-\beta}\beta(t-t_o)^{\beta-1} \exp\left[-\alpha^{-\beta}(t-t_o)^{\beta}\right]$	$\sim \alpha^{-\beta}\beta(t-t_o)^{\beta-1}$
v) functional gamma	
$\frac{1}{\Gamma(r)} \exp\left[-u(t-t_o)\right] \cdot \frac{d\,u(t-t_o)^r}{dt}$	$\sim \frac{du}{dt} - (r-1)\frac{d\ln u}{dt} - \frac{d}{dt}\ln\frac{du}{dt}$

<u>Table 9.2</u> Failure and restoration probability density functions considered in the numerical computations. (arbitrary time units)

i) failure p.d.f.

$f(t_o,t)$	$m_{\tau'}$	$\sigma_{\tau'}$
exponential (EX')	10^6	10^6
normal (N')	10^6	10^5
lognormal (LN')	10^6	10^5
gamma (Γ')	10^6	10^5
Weibull (W')	10^6	10^5

ii) restoration p.d.f.

$g(t_o,t)$	$m_{\tau''}$	$\sigma_{\tau''}$
exponential (EX")	10^2	10^2
normal (N")	10^2	10
lognormal (LN")	10^2	10
gamma (Γ'')	10^2	10
Weibull (W")	10^2	10
absorbing (O")	∞	-
reflecting (∞'')	0	0

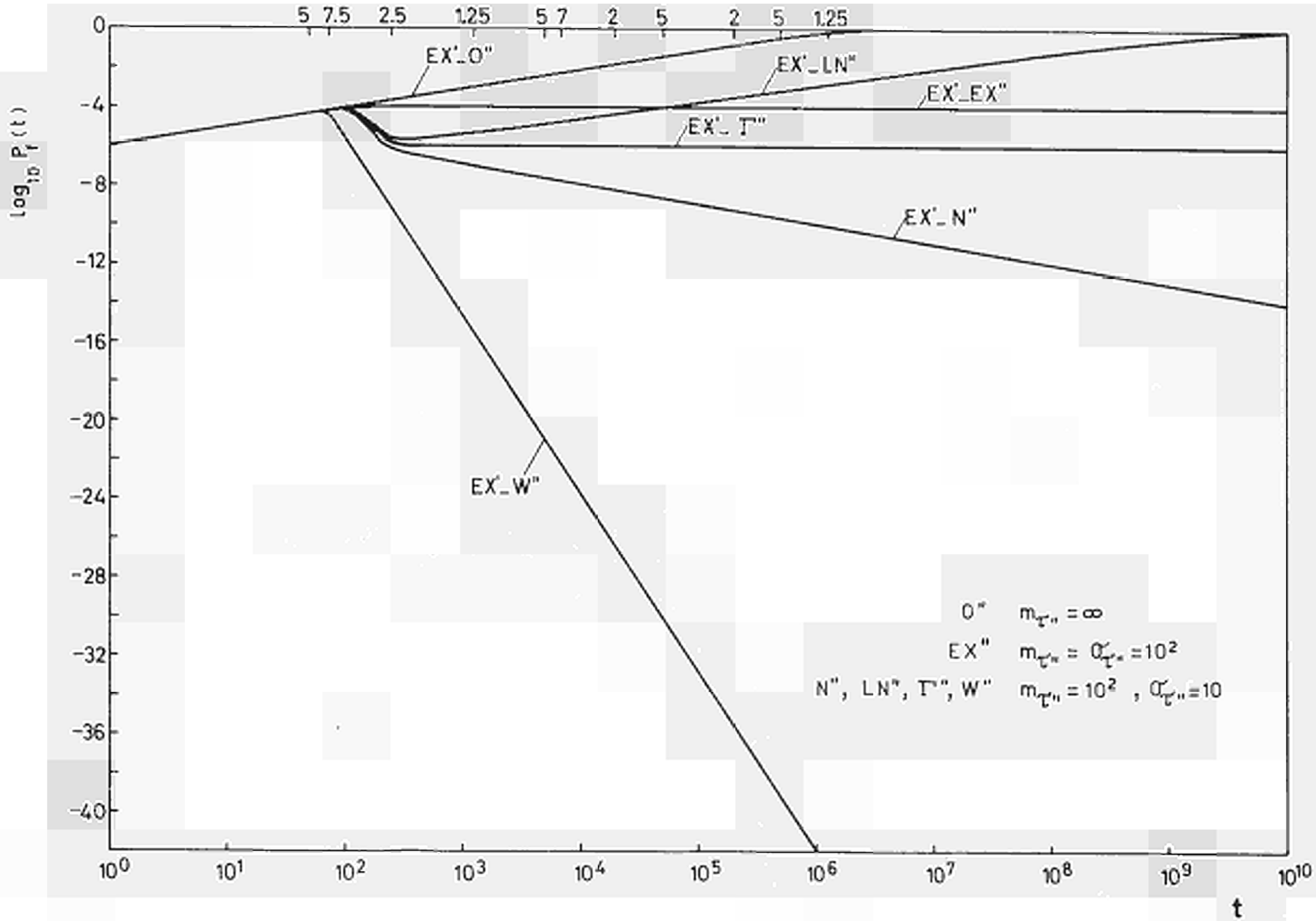

Figure 9.1_1 Unavailability $P_f(t)$ for a repair process. Exponential failure p.d.f. ($m_{\tau'} = \sigma_{\tau'} = 10^6$) combined with various restoration laws.

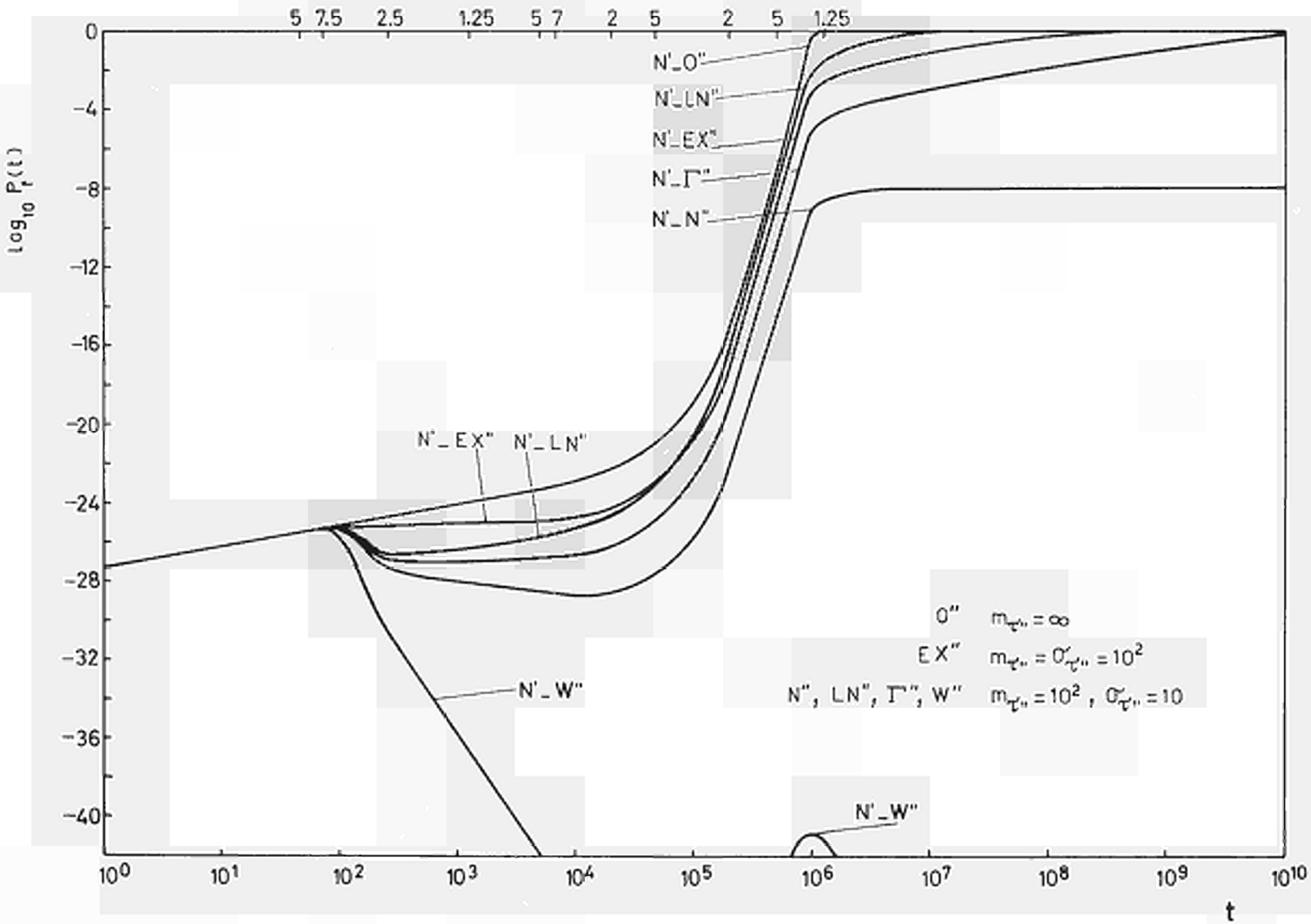

Figure 9.1_2 Unavailability $P_f(t)$ for a repair process. Normal or gaussian failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

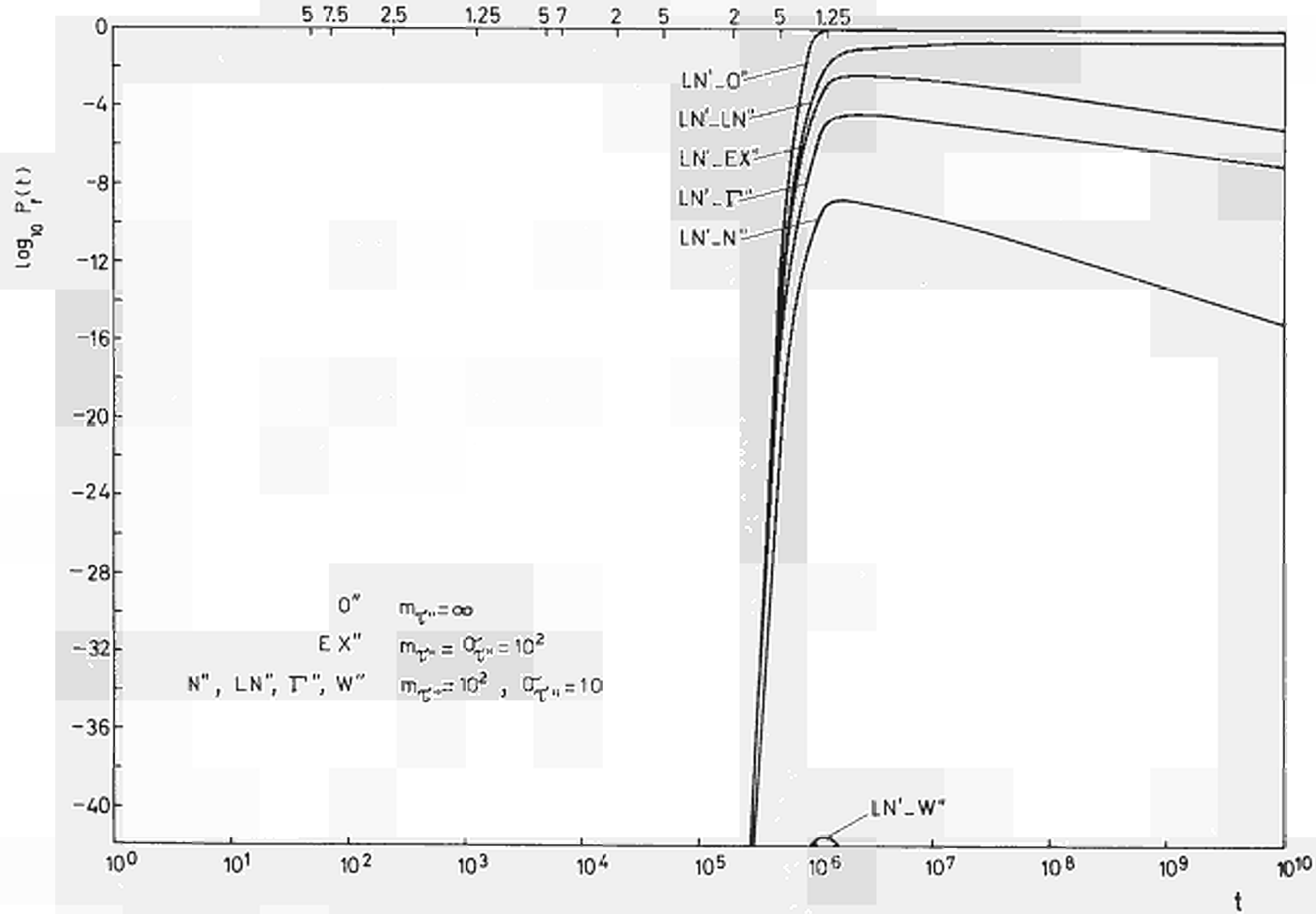

Figure 9.1$_3$ Unavailability $P_f(t)$ for a repair process. Lognormal failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

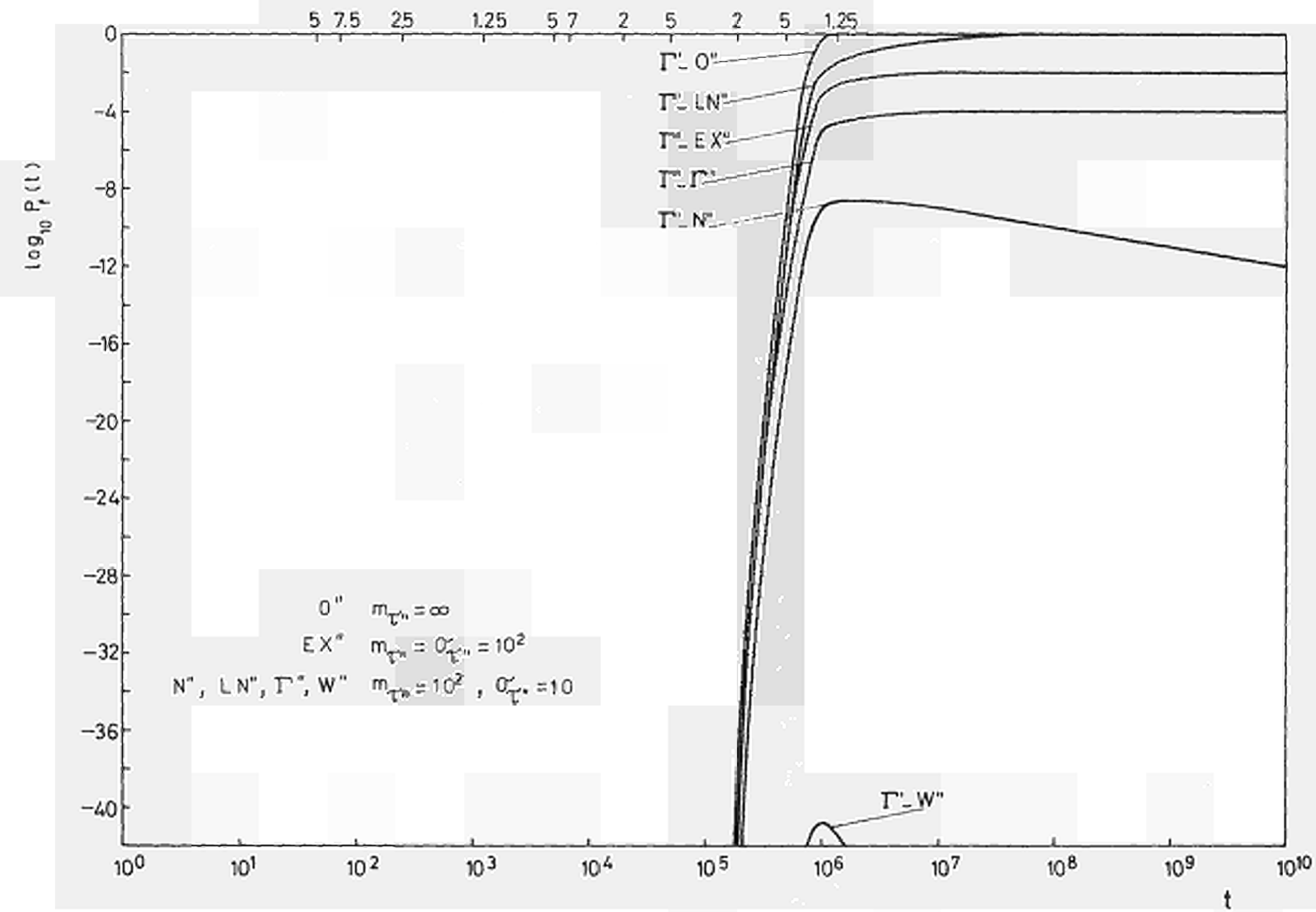

Figure 9.1$_4$ Unavailability $P_f(t)$ for a repair process. Gamma failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

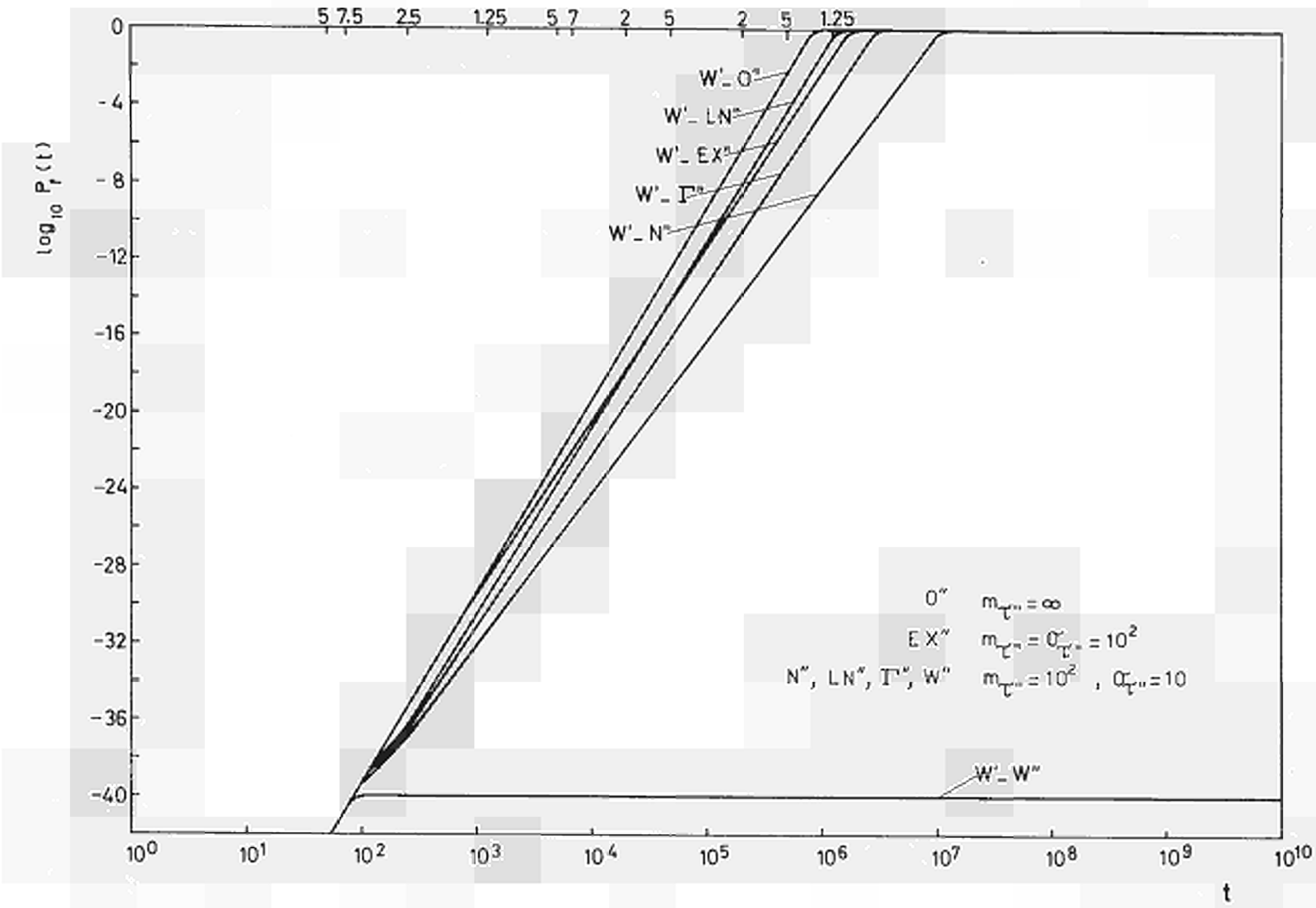

Figure 9.1$_5$ Unavailability $P_f(t)$ for a repair process. Weibull failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

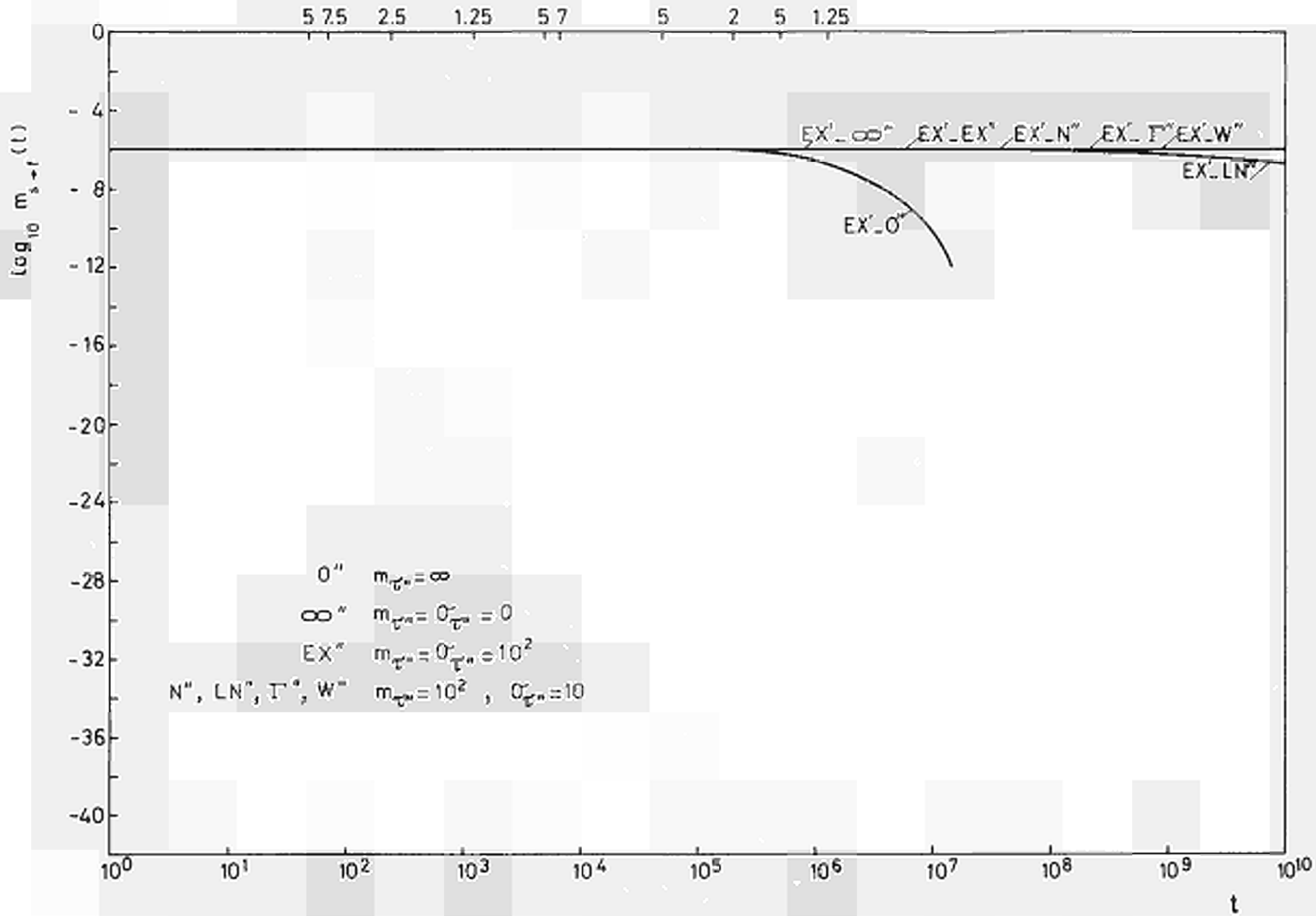

Figure 9.2$_1$ Failure density $m_{s \rightarrow f}(t)$ for a repair process. Exponential failure p.d.f. ($m_{\tau'} = \sigma_{\tau'} = 10^6$) combined with various restoration laws.

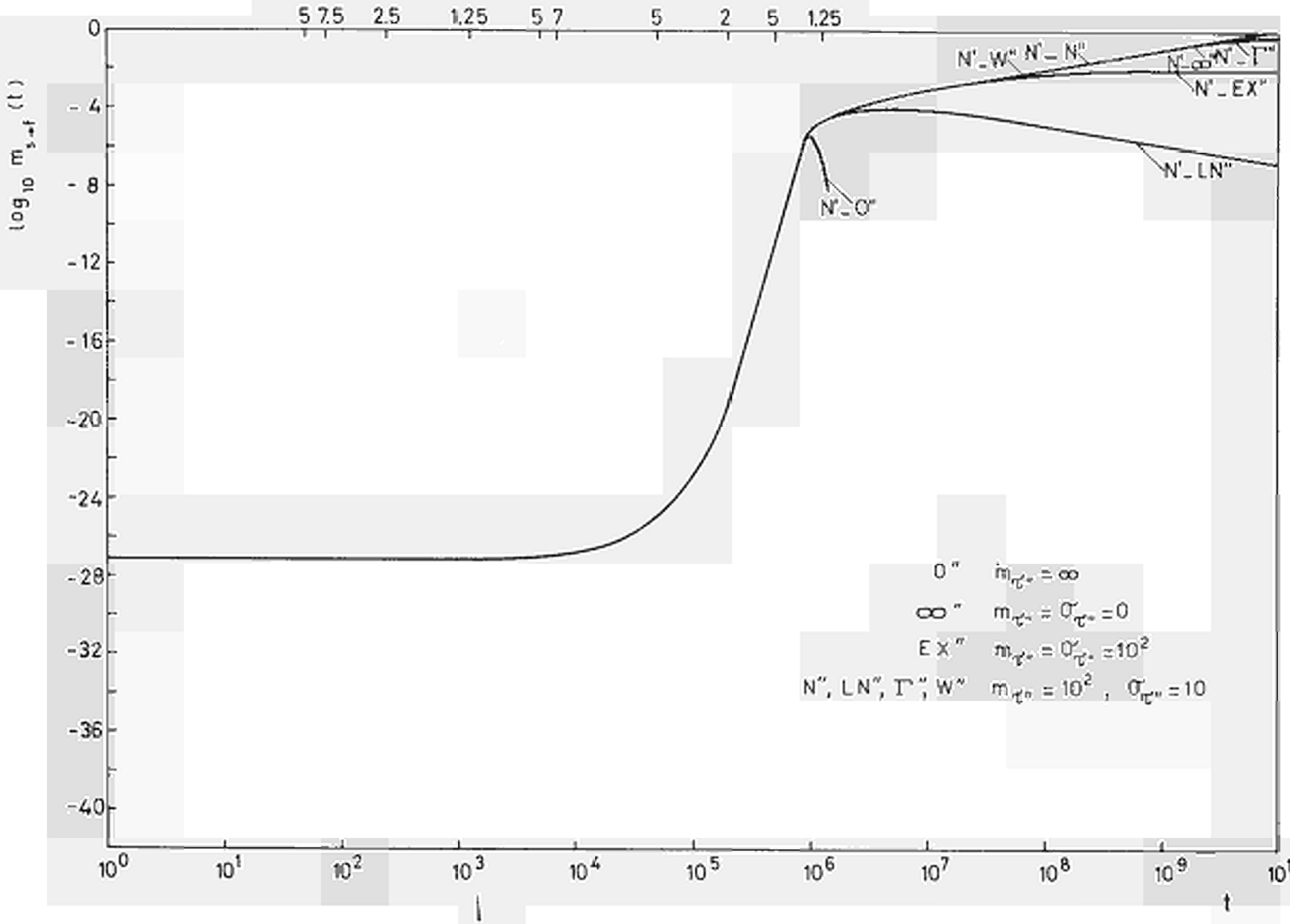

Figure 9.2₂ Failure density $m_{s \to f}(t)$ for a repair process. Normal failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

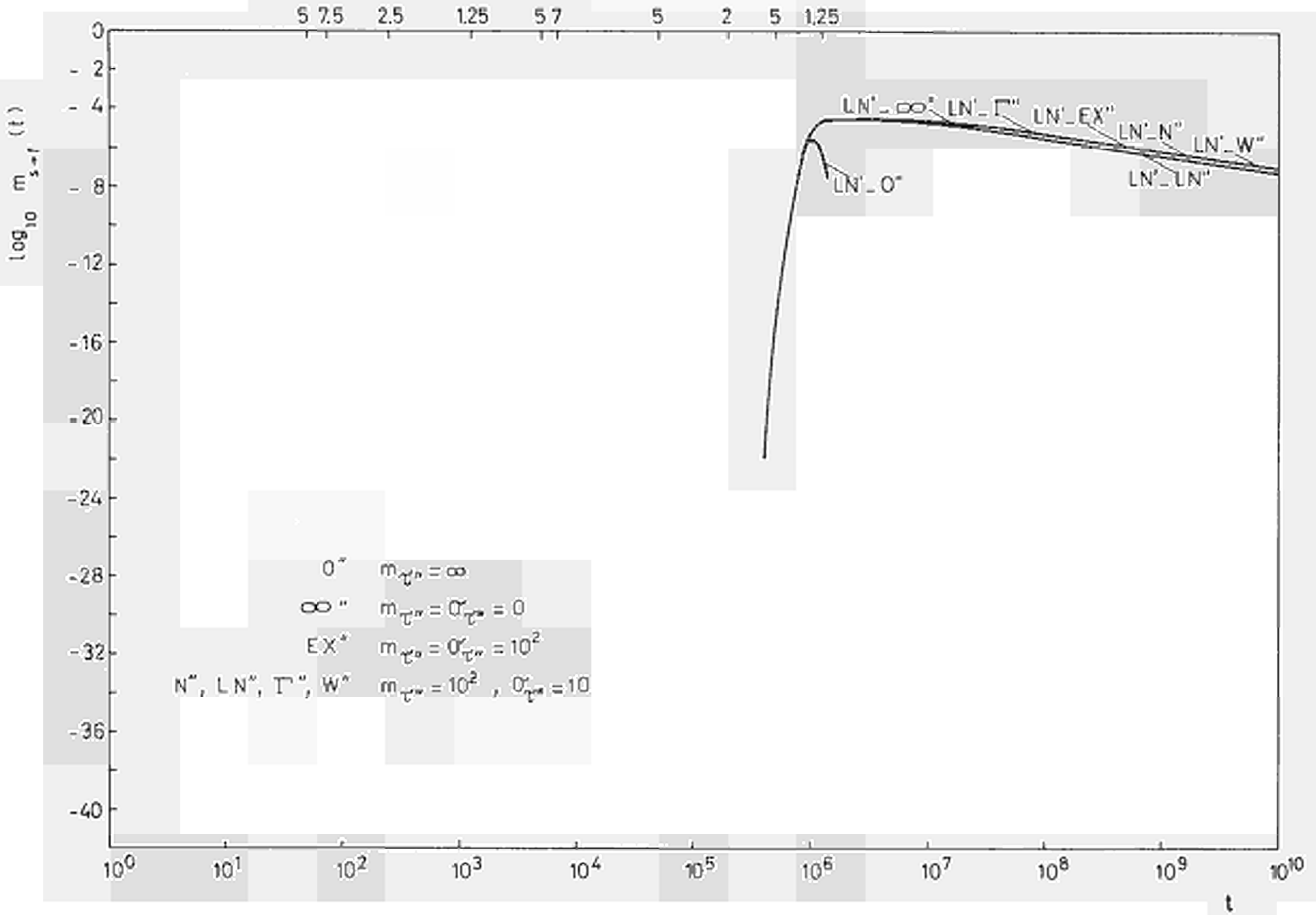

Figure 9.2$_3$ Failure density $m_{s \to f}(t)$ for a repair process. Lognormal failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

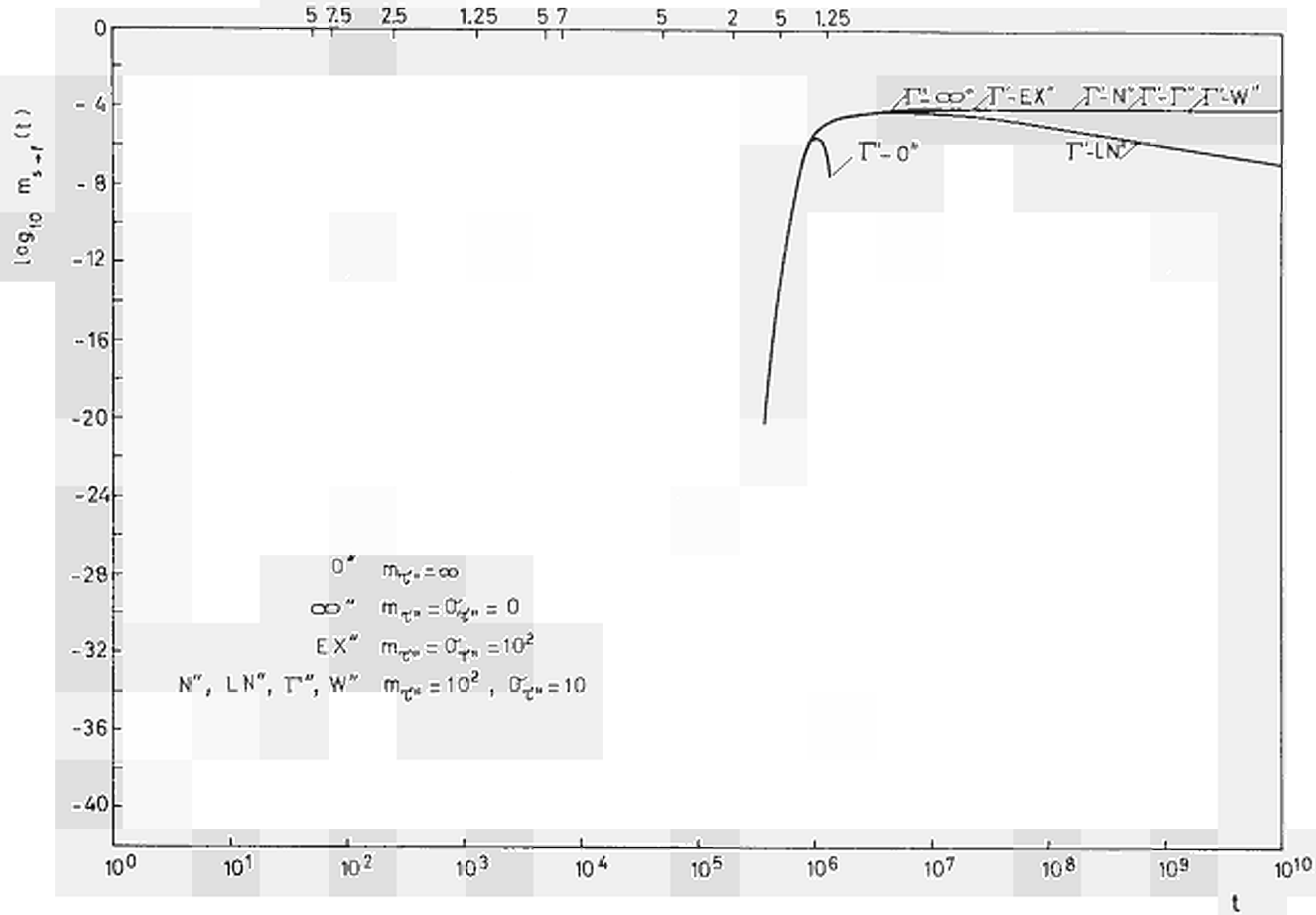

Figure 9.2$_4$ Failure density $m_{s\to f}(t)$ for a repair process. Gamma failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

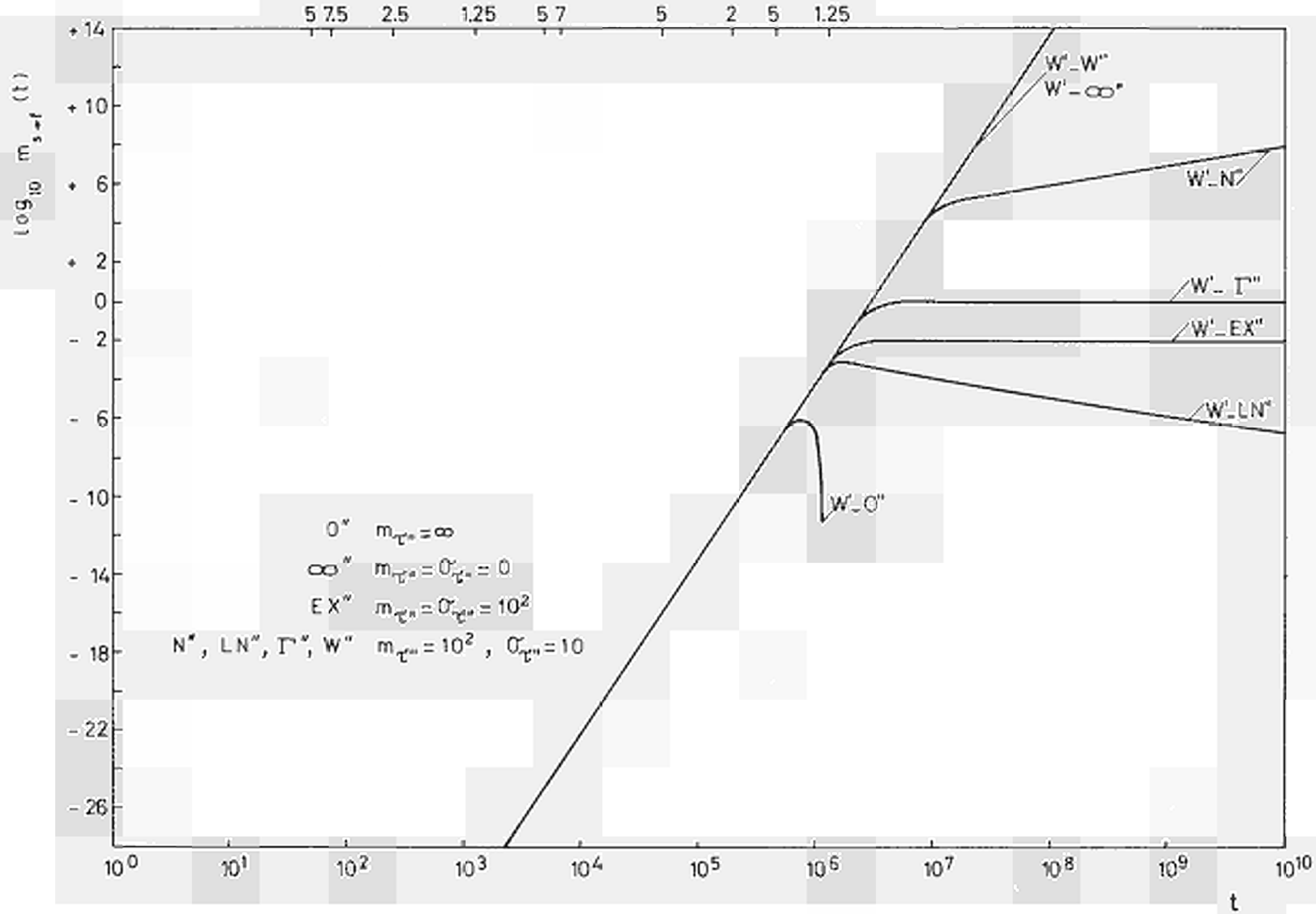

Figure 9.2$_5$ Failure density $m_{s \longrightarrow f}(t)$ for a repair process. Weibull failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

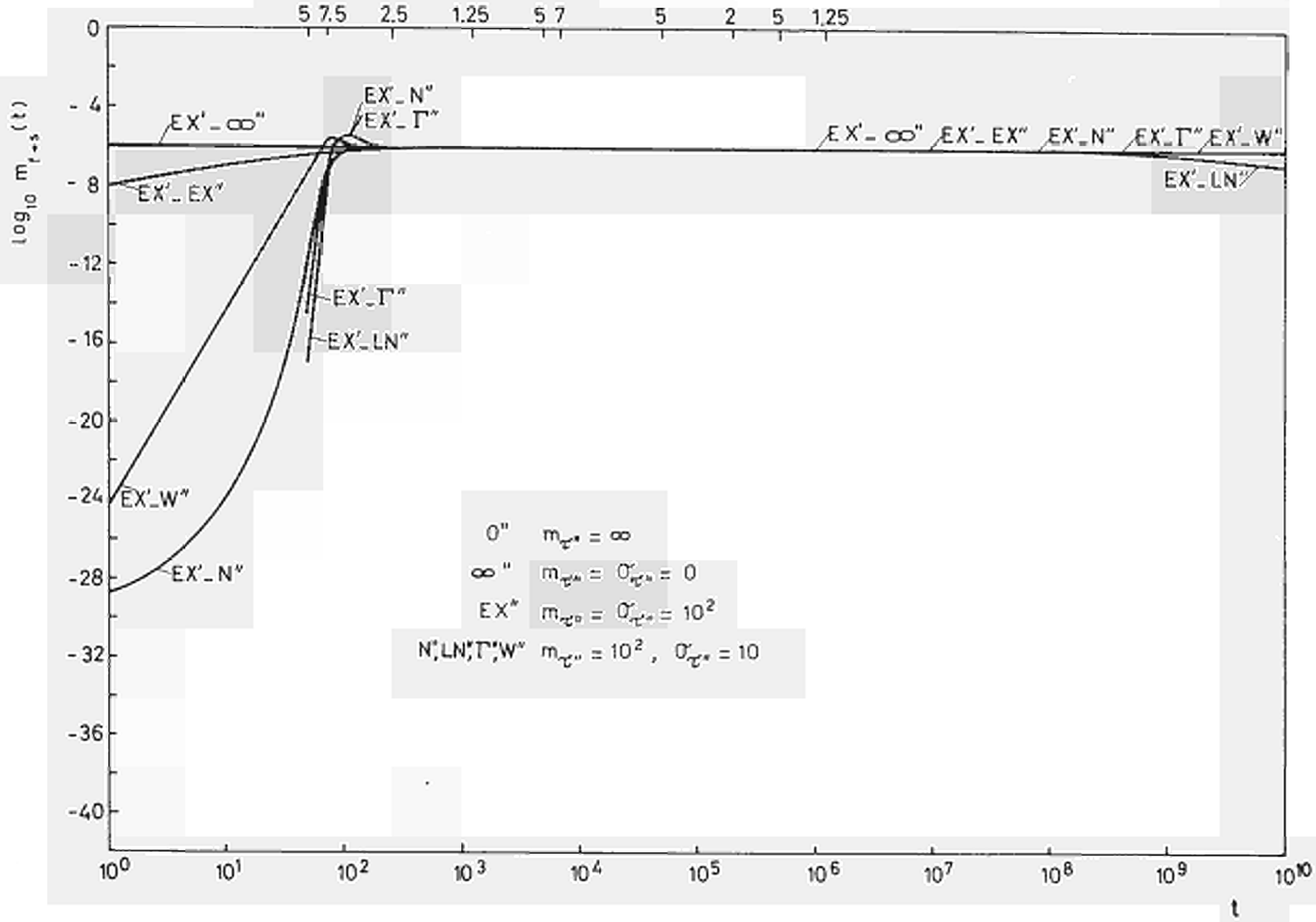

Figure 9.3$_1$ Repair (or restoration density $m_{s \to s}(t) = m_{f \to s}(t)$ for a repair process. Exponential failure p.d.f. ($m_{\tau'} = \sigma_{\tau'} = 10^6$) combined with various restoration laws.

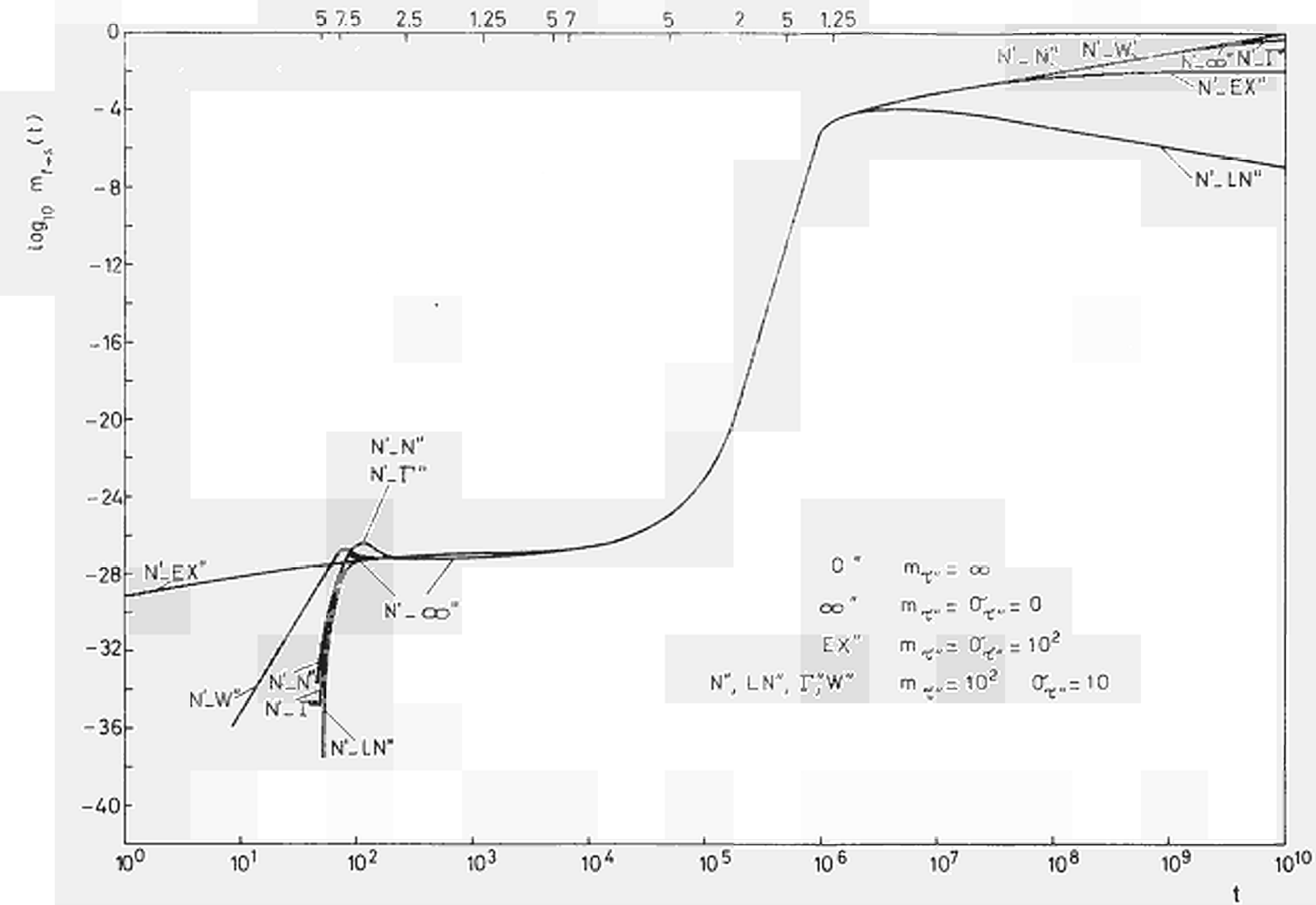

Figure 9.3$_2$ Repair (or restoration) density $m_{s \to s}(t) = m_{f \to s}(t)$ for a repair process. Normal failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

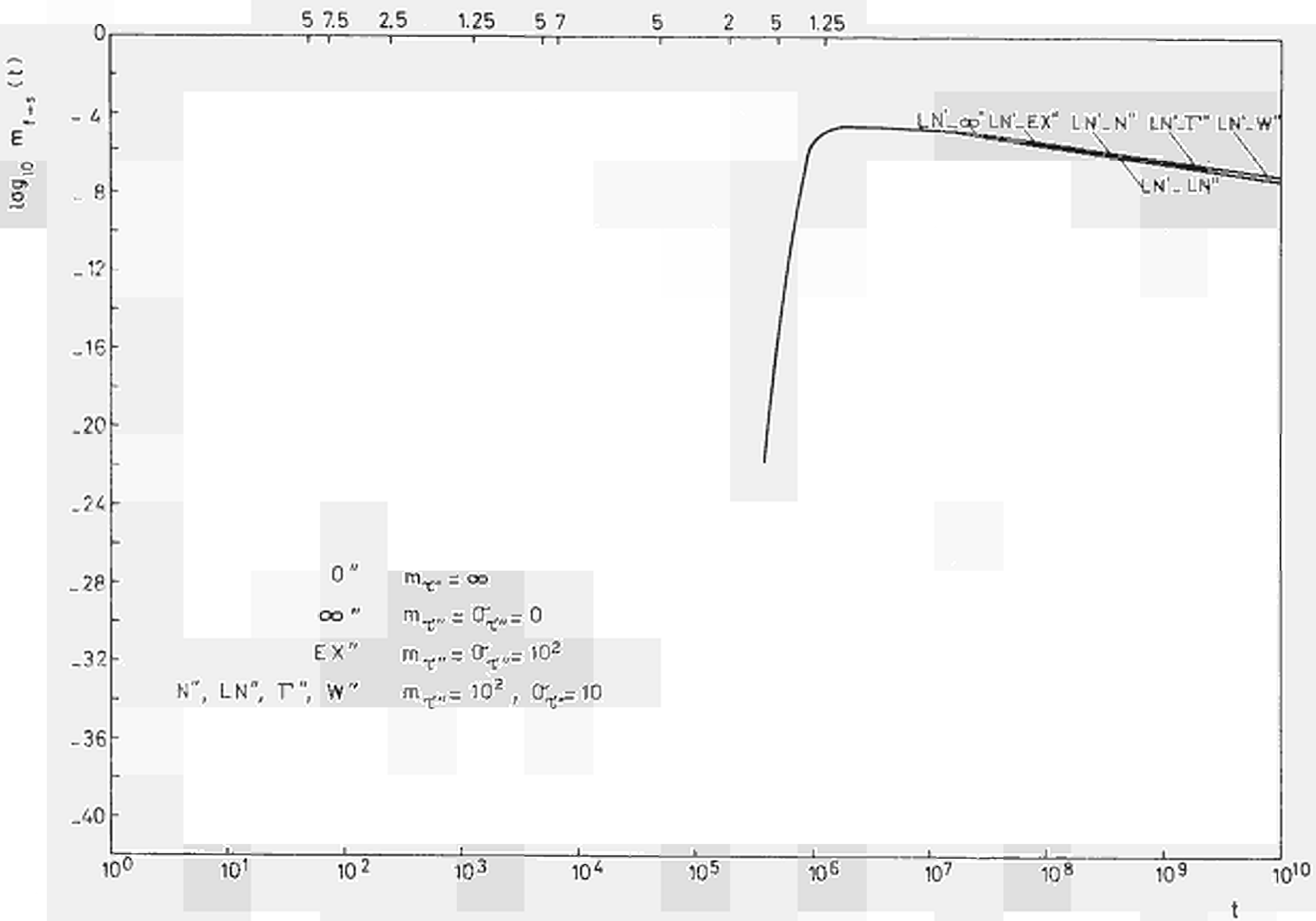

Figure 9.3$_3$ Repair (or restoration) density $m_{s \to s}(t) = m_{f \to s}(t)$ for a repair process. Lognormal failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

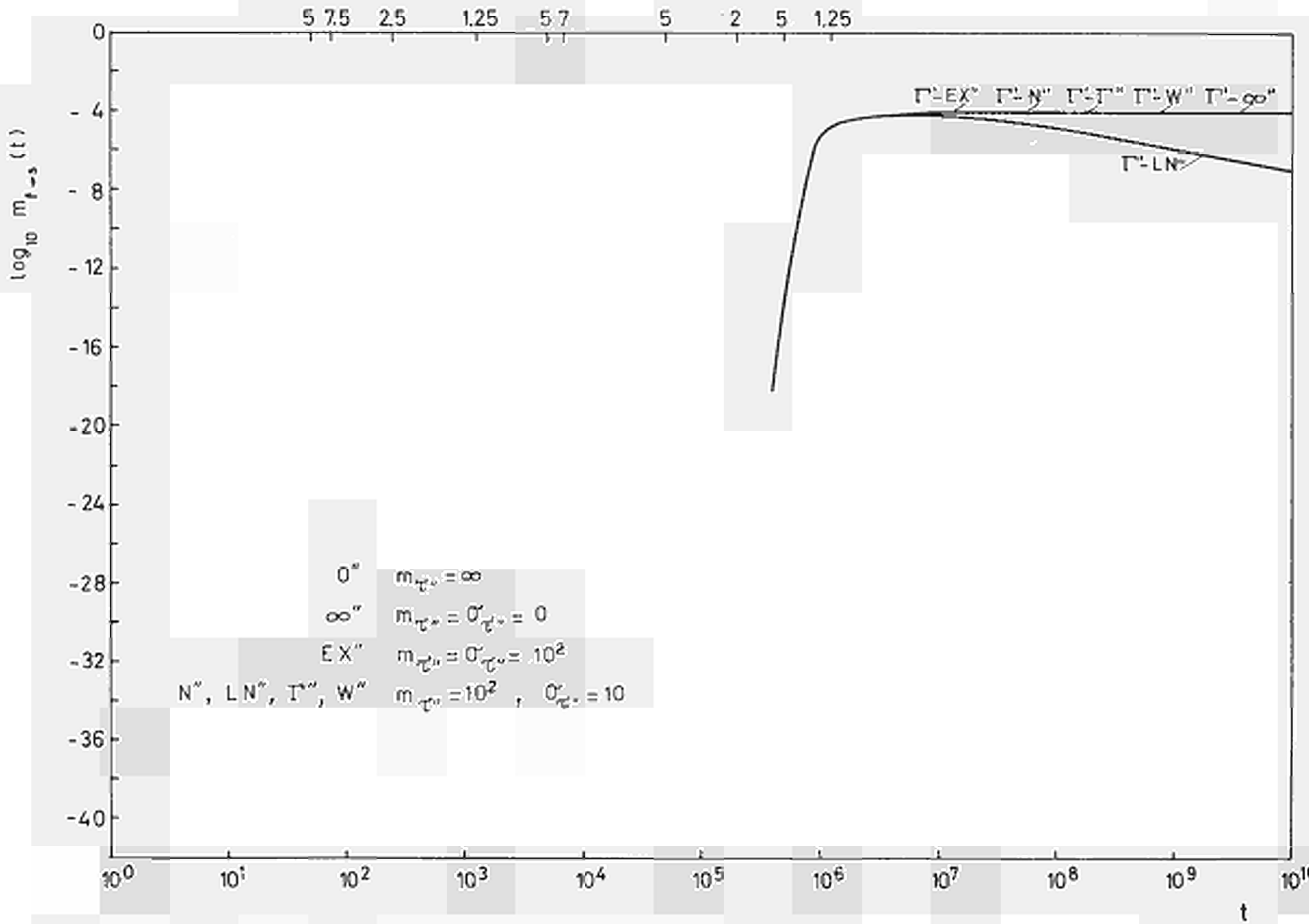

Figure 9.3$_4$ Repair (or restoration) density $m_{s\to s}(t) = m_{f\to s}(t)$ for a repair process. Gamma failure p.d.f. ($m_{\tau'} = 10^6$, $\sigma_{\tau'} = 10^5$) combined with various restoration laws.

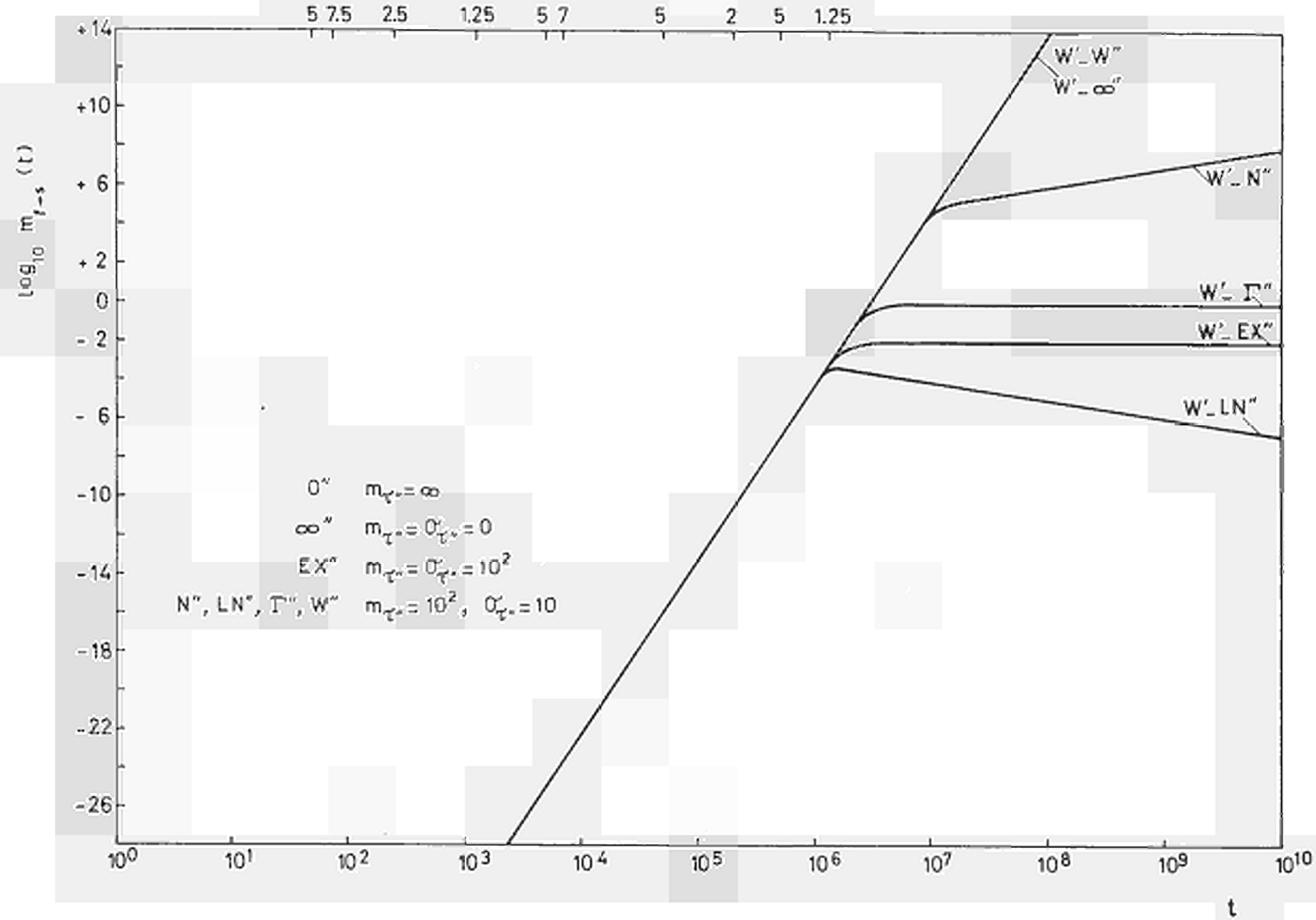

Figure 9.3$_5$ Repair (or restoration) density $m_{s \to s}(t) = m_{f \to s}(t)$ for a repair process. Weibull failure p.d.f. ($m_{\tau'} = 10^6, \sigma_{\tau'} = 10^5$) combined with various restoration laws.

Sct. 10 Final remarks

In the past sections we studied a real time repair process that is a model for the operation of a single unit involving the alternate combination of a failure with a restoration process both processes aging continuously with time. The theory was developped in order to offer a complete and effective basis for affording problems and computations concerned with repairable units or components. On the other hand, we tried to set a thorough analogy with the rather well known theory of simple renewal models.That is with a class of processes where unit after each transition starts as new. Now renewal and repair theory look undistinguishable when hazard p.d.f. are of exponential type. This situation indeed degenerates to the theory of homogeneous Markov processes of first order. Nevertheless, under many practical occurrences hazard rate functions may depend upon time and the non-homogeneous case so ensuing needs an ad hoc consideration.

Comparing renewal and repair theory the most striking and apparent differences may be roughly synthesized as follows. Equations for transition probabilities of repair processes can be always reduced to a differential form. Further, it makes hardly sense to talk of an equilibrium repair process. Properties remote ($t \rightarrow \infty$) from the origin ($t = 0$) are strongly connected with the remote structure of λ and μ. Convolution integrals which express conditional probabilities for renewal processes are substituted by plain products.

The idea of an alternating simple repair process can be generalized. For example we may have $\nu \geqslant 3$ incompatible states or events of the component and a matrix $[\pi_{i \rightarrow j}(t)]$ of transition probabilities, $\pi_{i \rightarrow j}(t)$ with $t \geqslant 0$ specifying the probability that a state X_i is followed on transition by state X_j. A second generalization consists of the case of an effective time repair process. Aging then depends on the time effectively spent by the unit in the state under consideration. Calendar time should be substituted by a random variable σ known variously in the literature as Markov time or stopping time. More explicitly, in the case of a two-events component we would set σ equal to α or β when considering the total time passed in the states X_s or X_f, respectively. To some of these arguments and other fundamental problems, still related with simple repair processes, such as the properties of the distribution of total on time, we think of devoting further special attention.

Appendix Comparison with renewal theory

The previous sections were devoted to an elaboration of basic concepts and methods of simple repair processes. In this appendix we present a brief description of several results obtained for simple renewal processes. Specifically we deal with a two-state, X_s and X_f, semi-Markov recurrent model where all life lengths τ_n' and τ_n'' underly failure and restoration p.d.f. defined as in Sect. 3. In other words, we assume that the unit starts as new after each transition. This hypothesis is expressed through the equality

$$(A.1) \qquad P_{k|j}(t \mid \bar{t}) = P_{k|j}(\bar{\bar{t}} + t - \bar{t} \mid \bar{\bar{t}}) \; ,$$

where $\bar{\bar{t}} \geqslant t_o$. Or, referring to the number of transitions $N_{j \to k}$ and its expected value,

$$(A.2_1) \qquad N_{j \to k \mid j}(\bar{t}, t \mid \bar{t}) = N_{j \to k \mid j}(\bar{\bar{t}}, \bar{\bar{t}} + t - \bar{t} \mid \bar{\bar{t}}) \, ,$$

$$(A.2_2) \qquad M_{j \to k \mid j}(\bar{t}, t \mid \bar{t}) = M_{j \to k \mid j}(\bar{\bar{t}}, \bar{\bar{t}} + t - \bar{t} \mid \bar{\bar{t}}) \; .$$

We may now investigate the form of the equations for transition pointwise probabilities $P_{f|s}(t|\bar{t})$ and $P_{f|s}(t|\bar{t})$.

Intuitively we have

$$(A.3) \qquad P_{f|s}(t \mid \bar{t}) = \int_{\bar{t}}^{t} P_{f|f}(t|u) \; R(\bar{t},u) \, \lambda(u) \; du \; .$$

However, U starts as new at $\bar{t}$, therefore $R(\bar{t},u)\,\lambda(u) = R(t_o, t_o + u - \bar{t}) \; \lambda(t_o + u - \bar{t})$. Taking into account (A.1), equation (A.3) is then rewritten

$$P_{f|s}(t_o+t-\bar{t}\mid t_o) = \int_{\bar{t}}^{t} P_{f|f}(t_o+t-u\mid t_o)\cdot$$

$$\cdot R(t_o,t_o+u-\bar{t})\ \lambda(t_o+u-\bar{t})\ du\ ,$$

or, after variables have been changed to $t^* \equiv t - \bar{t}$ and $u^* = u - \bar{t}$,

$$(A.4_1)\qquad P_{f|s}(t_o+t^*\mid t_o) = \int_{o}^{t^*} P_{f|f}(t_o+t^*-u^*\mid t_o)\cdot$$

$$R(t_o,\ t_o+u^*)\ .\ \lambda(t_o+u^*)\ du^*.$$

An expression which should be combined with the corresponding one

$$(A.4_2)\qquad P_{f|f}(t_o+t^*\mid t_o) = \int_{o}^{t^*} P_{f|s}(t_o+t^*-u^*\mid t_o)\cdot$$

$$\cdot S(t_o,t_o+u^*)\ \mu(t_o+u^*)\ du^* + S(t_o,t_o+t^*)$$

The knowledge of the distribution $\{P_i(t)\}$ with $\bar{t} \geqslant t_o$ would consent the determination of $P_f(t)$. More particularly, if $P_s(t_o) = 1$, we would have simply $P_{f|s}(t_o+t^*\mid t_o)=P_f(t_o+t^*)$.

At this point it seems awkward to maintain quantities referred to an initial time t_o different from zero. With aim at a more direct notation we may take $t_o = 0$, $t^* \equiv t$, $u^* \equiv u$ and restate equations as follows [1,3,4]

$$(A.5_1)\qquad P_{f|s}(t\mid 0) = \int_{0}^{t} P_{f|f}(t-u\mid 0)\ R(0,u)\ \lambda(u)\ du,$$

$$(A.5_2)\qquad P_{f|f}(t\mid 0) = S(0,t) + \int_{0}^{t} P_{f|s}(t-u\mid 0)\ S(0,u)\cdot$$

$$\cdot\mu(u)\ du$$

A thoroughly similar set of equations could be written for $P_{s|f}(t|0)$, $P_{s|s}(t|0)$, then for $P_s(t)$,

(A.6$_1$) $$P_{s|f}(t|0) = \int_0^t P_{s|s}(t-u|o)\; S(o,u)\; u(u)\; du,$$

(A.6$_2$) $$P_{s|s}(t|0) = R(0,t) + \int_0^t P_{s|f}(t-u|0) \cdot R(0,u)\; \lambda(u)\; du.$$

We will not attempt to rigorize this approach and obtain explicit solutions for $P_f(t)$ or $P_s(t)$. Instead, worthy of attention are the asymptotic values [1 - 4],

(A.7) $$P_f(\infty) = \frac{m_{\tau''}}{m_{\tau'} + m_{\tau''}},$$

$$P_s(\infty) = \frac{m_{\tau'}}{m_{\tau'} + m_{\tau''}}.$$

Two limits which coincide with (5.1) for the case of constant λ and μ, in other words when hazard p.d.f. are of a simple exponential type.

The expected number of transitions $X_s \rightarrow X_f$ within (0,t) can be also expressed by means of integral equations. We have [2]

(A.8$_1$) $$M_{s\rightarrow f|s}(0,t|0) = F(0,t) + \int_0^t M_{f\rightarrow f|f}(0,t-u|0) \cdot R(0,u)\; \lambda(u)\; du,$$

(A.8$_2$) $$M_{f\rightarrow f|f}(0,t|0) = \int_0^t M_{s\rightarrow f|s}(0,t-u|0)\; S(o,u)\; \mu(u)\, du = M_{s\rightarrow f|f}(0,t|0).$$

Further, with regard to transitions $X_f \rightarrow X_s$,

$$(A.9_1) \qquad M_{f\rightarrow s|f}(0,t|0) = G(0,t) + \int_0^t M_{s\rightarrow s|s}(0,t-u|0)\cdot S(0,u)\,\mu(u)\,du,$$

$$(A.9_2) \qquad M_{s\rightarrow s|s}(0,t|0) = \int_0^t M_{f\rightarrow s|f}(0,t-u|0)\,R(0,u)\cdot\lambda(u)\,du = M_{f\rightarrow s|f}(0,t|0)\,.$$

It deserves some interest at this point to reconsider equations (2.11),

$$(A.10_1) \qquad P_{f|s}(t|0) = M_{s\rightarrow f|s}(0,t|0) - M_{s\rightarrow s|s}(0,t|0),$$

$$(A.10_2) \qquad P_{s|f}(t|0) = M_{f\rightarrow s|f}(0,t|0) - M_{f\rightarrow f|f}(0,t|0).$$

When (A.8) or (A.9) are introduced into (A.10), we go back to ($A.5_1$) or ($A.6_1$), respectively. But, using a different decomposition of events, in place of (A.8) and (A.9) we may write

$$(A.11_1) \qquad M_{s\rightarrow f|s}(0,t|0) = F(0,t) + \int_0^t M_{s\rightarrow s|s}(0,u|0)\,R(0,t-u)\,\lambda(t-u)\,du,$$

$$(A.11_2) \qquad M_{s\rightarrow s|s}(0,t|0) = \int_0^t M_{s\rightarrow f|s}(0,u|0)\,S(0,t-u)\,\mu(t-u)\,du.$$

Or, owing to a dual treatment,

$$(A.12_1)\qquad M_{f\to s|f}(0,t|0) = G(0,t) + \int_0^t M_{f\to f\ f}(0,u\ 0)$$

$$S(0,t-u)\ \mu\ (t-u)\ du,$$

$$(A.12_2)\qquad M_{f\to f|f}(0,t\ |0) = \int_0^t M_{f\to s\ f}(0,u\ 0)\ R(0,t-u)$$

$$\lambda(t-u)\ du\ .$$

It is seen that the insertion of (A.11) and (A.12) into (A.10) generates two new relations,

$$(A.13_1)\qquad P_{f|s}(t|0) = \int_0^t S(0,t-u)\ dM_{s\to f|s}(0,\ u|0),$$

$$(A.13_2)\qquad P_{s|f}(t|0) = \int_0^t R(t0,t-u)\ d\ M_{f\to s|f}(0,\ u\ |\ 0).$$

These are part of an equations set whose structure reminds of (4.15) and (4.16). We have indeed

$$(A.14_1)\qquad P_{s|s}(t|0) = R(0,t) + \int_0^t R(0,t-u)\ dM_{s\to s|s}(0,u|0),$$

$$(A.14_2)\qquad P_{f|f}(t|0) = S(0,t) + \int_0^t S(0,t-u)\ dM_{f\to f|f}(0,u|0).$$

By definition in a renewal process all failure and restoration lives have equal p.d.f., we may then find an easy definition for the cumulative distribution functions $\mathcal{P}\{\alpha_n \leqslant t \cap (0,t)^*\} =$

$= \mathcal{P}\{\alpha_n \leqslant t\} \equiv \tilde{F}^{(n)}(0,t)$ and $\mathcal{P}\{\beta_n \leqslant t \cap (0,t^*)\} =$

$= \mathcal{P}\{\beta_n \leqslant t\} \equiv \tilde{G}^{(n)}(0,t)$ with $t \leqslant t^*$. It is

$$(A.15_1) \quad \tilde{F}^{(n)}(0,t) \equiv \mathcal{P}\{\tau'_1 + \tau'_2 + \ldots + \tau'_n \leqslant t\} =$$

$$= \int_0^t \tilde{F}^{(n-1)}(0,t-u)\, dF(0,u),$$

$$(A.15_2) \quad \tilde{G}^{(n)}(0,t) \equiv \mathcal{P}\{\tau''_1 + \tau''_2 + \ldots + \tau''_n \leqslant t\} =$$

$$\int_0^t \tilde{G}^{(n-1)}(0,t-u)\, dG(0,u),$$

where $n \geqslant 1$, $\tilde{F}^{(o)}(0,t) = \tilde{G}^{(o)}(0,t) = 1^+(t)$, and $\tilde{F}^{(1)}(0,t) \equiv F(0,t)$, $\tilde{G}^{(1)}(0,t) \equiv G(0,t)$. The analog of recurrence relations (6.8) is

$$(A.16_1) \quad F^{(n)}(0,t) = \int_0^t \tilde{G}^{(n-1)}(0,t-u)\, d\tilde{F}^{(n)}(0,u) =$$

$$= \int_0^t H^{(n-1)}(0,t-u)\, dF(0,u),$$

$$(A.16_2) \quad H^{(n)}(0,t) = \int_0^t \tilde{G}^{(n)}(0,t-u)\, d\tilde{F}^{(n)}(0,u) =$$

$$= \int_0^t \tilde{F}^{(n)}(0,t-u)\, d\tilde{G}^{(n)}(0,u) =$$

$$= \int_0^t F^{(n)}(0,u)\, dG(0,t-u)$$

with $n \geqslant 1$, $H^{(0)}(0,t) = 1^+(t)$. If (A.16) are summed up over all $n = 1,2,\ldots\infty$, through the use of (6.3) we deduce (A.11).

Next, having defined the second moment of $H^{(1)}(0,t)$,

$$\sigma^2_{\tau'\tau''} = \int_0^\infty t^2\, h^{(1)}(0,t)\, dt,$$

we can state asymptotic expressions for the expected number of failures and the expected number of renewals during (0,t) as follows [1,2]

(A.17$_1$) $$M_{s\to f|s}(0,t\,|\,0) \sim \frac{t}{m_{\tau'} + m_{\tau''}} - \frac{m_{\tau'}}{m_{\tau'} + m_{\tau''}} + \frac{\sigma^2_{\tau'\tau''}}{2(m_{\tau'} + m_{\tau''})^2},$$

(A.17$_2$) $$M_{f\to s|s}(0,t\,|\,0) = M_{s\to s|s}(0,t\,|\,0) \sim \frac{t}{m_{\tau'} + m_{\tau''}} - 1 + \frac{\sigma^2_{\tau'\tau''}}{2(m_{\tau'} + m_{\tau''})^2},$$

as $t \to \infty$.

The same definitions and methods of Sct. 6 may be used with the scope of obtaining equations for the (forward) interval reliability and interval expectance. It is found [1,3]

(A.18$_1$) $$\mathcal{P}\{\xi(t) > T\} = R(0,t+T) + \int_0^t R(0,t+T-u)\, dM_{s\to s\ s}(0,u\ 0),$$

(A.18$_2$) $$\mathcal{P}\{\zeta(t) > T\} = \int_0^t S(0,t+T-u)\, dM_{s\to f\ s}(0,u\ 0),$$

respectively. When comparing with (6.14) and (6.16) products appear to be substituted by convolution integrals.

An elegant and useful form in renewal processes has the distribution of the total on or down time. A lemma can be proved as follows [8]

(A.19) $$\mathcal{P}\{\alpha_{n+1} \leq \emptyset \cap \alpha_{n+1} + \beta_n \leq t\} - \mathcal{P}\{\beta_n \geq t-\emptyset \cap \alpha_{n+1} + \beta_n \leq t\} = \mathcal{P}\{\alpha_{n+1} \leq \emptyset \cap \beta_n < t - \emptyset\}.$$

Using (A.19) and the independence of the random variables α_n, β_n with $n = 0,1,\ldots\infty$, we have from (7.4)

(A.20) $$\mathcal{P}\{\varphi(0,t) \leq \emptyset\} = \sum_n^{0,\ldots\infty} \mathcal{P}\{\alpha_{n+1} \leq \emptyset \cap \beta_n < t-\emptyset\} - \mathcal{P}\{\alpha_{n+1} \leq \emptyset \cap \beta_{n+1} < t - \emptyset\} =$$

$$= \sum_n^{0,\ldots\infty} \mathcal{P}\{\alpha_{n+1} \leq \emptyset\} \left[\mathcal{P}\{\beta_n < t-\emptyset\} - \mathcal{P}\{\beta_{n+1} < t-\emptyset\} \right] =$$

$$= \sum_n^{0,\ldots\infty} \tilde{F}^{(n+1)}(0,\emptyset) \left[\tilde{G}^{(n)}(0,t-\emptyset) - \tilde{G}^{(n+1)}(0,t-\emptyset) \right].$$

If $\sigma_{\tau'}$ and $\sigma_{\tau''}$ are finite quantities, it can be also proved that

$$\mathcal{P}\left\{ \frac{\varphi(0,t) - m_{\tau'} t / m_{\tau'} + m_{\tau''}}{\left[(m^2_{\tau'} \; \sigma^2_{\tau''} + m^2_{\tau''} \; \sigma^2_{\tau'}) \; t / (m_{\tau'} + m_{\tau''})^3 \right]^{1/2}} \leq \emptyset \right\} \sim$$

$$\sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\emptyset} \exp\left(-\frac{u^2}{2}\right) du$$

as $t \to \infty$ [3,8] .

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