On Friedmann's subexponential lower bound for Zadeh's pivot rule*†

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Abstract

The question whether the Simplex method admits a polynomial time pivot rule remains one of the most important open questions in discrete optimization. Zadeh's pivot rule had long been a promising candidate, before Friedmann (IPCO, 2011) presented a subexponential instance, based on a close relation to policy iteration algorithms for Markov decision processes (MDPs). We investigate Friedmann's lower bound example and exhibit three flaws in the corresponding MDP: We show that (a) the initial policy for the policy iteration does not produce the required occurrence records and improving switches, (b) the specification of occurrence records is not entirely accurate, and (c) the sequence of improving switches used by Friedmann does not consistently follow Zadeh's pivot rule. In this paper, we resolve each of these issues by adapting Friedmann's construction. While the first two issues require only minor changes to the specifications of the initial policy and the occurrence records, the third issue requires a significantly more sophisticated ordering and associated tie-breaking rule that are in accordance with the LEAST-ENTERED pivot rule. Most importantly, our changes do not affect the macroscopic structure of Friedmann's MDP, and thus we are able to retain his original result.

1 Introduction

The Simplex method, originally proposed by Dantzig in 1947 (see [2]), is one of the most important algorithms to solve linear programs in practice. At its core, the method operates by maintaining a subset of basis variables while restricting non-basis variables to trivial values, and repeatedly replacing a basis variable according to a fixed *pivot rule* until the objective function value can no longer be improved. Exponential worst-case instances have been devised for many natural pivot rules (e.g., [1, 4, 6, 7]), and the question whether a polynomial time pivot rule exists remains one of the most important open problems in optimization theory.

Zadeh's LEAST-ENTERED pivot rule [9] was designed to avoid the exponential behavior on known worst-case instances for other pivot rules. The rule is *memorizing* in that it selects a variable to enter the basis that improves the objective function and has previously been selected least often among all improving variables. Indeed, for more than thirty years, Zadeh's rule defied all attempts to construct superpolynomial instances, and it seemed like a promising candidate for a polynomial pivot rule.

It was a breakthrough when Friedmann eventually presented the first superpolynomial lower bound for Zadeh's pivot rule [3]. His construction uses a connection between the Simplex Algorithm and Howard's Policy Iteration Algorithm [5] for computing optimal policies in Markov

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decision processes (MDPs). Essentially, Friedmann's construction consists of an MDP, an initial policy and an ordering of the improving switches that result in an exponential number of iterations when beginning with the initial policy and repeatedly making improving switches according to the given ordering, which obeys the LEAST-ENTERED pivot rule. The construction translates into a linear program of size $\mathcal{O}(n^2)$ where the Simplex Algorithm with Zadeh's pivot rule needs $\Omega(2^n)$ steps, which results in a superpolynomial lower bound of $2^{\Omega(\sqrt{n})}$ on the number of iterations.

Our contribution. In this paper, we expose different flaws in Friedmann's lower bound construction and present adaptations to eliminate them. We first show that the chosen initial policy does not produce the claimed occurrence records and improving switches, and we propose a modified initial policy that leads to the desired behavior. Second, we observe that the formula describing the occurrence records (that count the number of times an improving switch was made) given in [3] is inaccurate, and provide a (small) correction that does not disturb the overall argument. Note that these two modifications are necessary but relatively minor.

Finally, we exhibit a significant problem with the order in which the improving switches are applied in [3]. More precisely, we show that this order does not consistently obey Zadeh's LEAST-ENTERED pivot rule, and, in fact, that no consistent ordering exists that updates the MDP level by level in each phase according to a fixed order. This not only rules out Friedmann's ordering, but shows that a fundamentally different approach to ordering improving switches is needed. To amend this issue, we show the existence of an ordering and a tie-breaking rule compatible with the LEAST-ENTERED rule, such that applying improving switches according to the ordering still proceeds along the same macroscopic phases as intended by Friedmann. In this way, we are able to quantitatively retain Friedmann's superpolynomial lower bound on the number of iterations needed by Zadeh's LEAST-ENTERED pivot rule.

Outline. Throughout this paper, we assume some basic familiarity with the construction given in [3] and Markov decision processes in general. We review the most important aspects and notation of [3] in Section 2, and, for convenience, we provide a copy of some tables that we rely on in Appendix A. Section 3 treats issues with the initial policy and our adaptation to address them. In Section 4, we correct an inaccuracy concerning some of the occurrence records given in [3]. The main part of this paper is Section 5, where we show that the sequence of improving switches can be reordered such that the order obeys the LEAST-ENTERED rule. Finally, we summarize our findings in Section 6.

2 Preliminaries

In this section, we review the Markov decision process constructed in [3]. We introduce notation related to binary counting and explain aspects of [3] used in this paper.

2.1 Friedmann's lower bound construction

In [3], Friedmann uses the connection between the Simplex Algorithm for linear programming and the Policy Iteration Algorithm for obtaining optimal policies in Markov decision processes (MDPs). Similarly, we also restrict our discussion to policy iteration for MDPs, with the understanding that results carry over to the Simplex Algorithm. We assume knowledge of MDPs and the connection to the Simplex Algorithm and refer to [8] for more information.

We first establish some notation. Given an MDP, a player-controlled edge $e = (u, v)$ and a policy σ, we say that v is the *target* of u or u *points to* v if $\sigma(u) = v$. If $\sigma(u) \neq v$, we say that we *switch* e or *switch* u *to* v when we apply the switch (u, v) in σ. For a policy σ and an improving switch e for σ, we denote the policy obtained by applying the switch e in σ by $\sigma[e]$.

Let $n \in \mathbb{N}, n \neq 0$ be fixed. Friedmann's lower bound construction emulates an n-bit binary counter by a Markov decision process G_n. For every binary number $b = (b_n, \ldots, b_1)$ that can be represented by n bits, there is a unique policy σ_b for G_n representing b. Note we denote the least significant bit by b_1, i.e., $b = \sum_{i=1}^{n} b_i 2^{i-1}$. The Markov decision process G_n is constructed such that applying the Policy Iteration Algorithm using the LEAST-ENTERED pivot rule enumerates all policies $\sigma_0, \sigma_1, \ldots, \sigma_{2^n-1}$. Since the LEAST-ENTERED pivot rule is applied, the algorithm always chooses an improving switch that was chosen least often until now. More specifically, an *occurrence record* ϕ is maintained, and, in every step, a switch minimizing ϕ is chosen. The rule does however not determine *which* switch minimizing ϕ should be chosen if there are multiple candidates. Therefore, an explicit tie-breaking rule may be used in the construction. For an edge e and a policy σ for G_n that is calculated during the application of the Policy Iteration Algorithm, we denote the occurrence record of e at the time when σ is reached by $\phi^\sigma(e)$. We denote the set of improving switches with respect to the policy σ by I_σ.

For the remainder of this paper, we fix the following notation:

- The set of numbers that can be represented by n bits is denoted by $\mathbb{B}_n$.
- Let $b \in \mathbb{B}_n$. For $i \in \{1, \ldots, n\}$, the i-th bit of the binary representation of b is denoted by b_i. For $b \neq 0$, we denote the least significant bit of b which is equal to 1 by $\ell(b)$, that is, $\ell(b) := \min\{i \in \{1, \ldots, n\} : b_i \neq 0\}$.
- The unique policy representing $b \in \mathbb{B}_n$ constructed in [3] is denoted by σ_b.

The process G_n can be interpreted as a "fair alternating binary counter" in the following sense. Usually, when counting from 0 to $2^n - 1$ in binary, less significant bits are switched more often than more significant bits. As the LEAST-ENTERED pivot rule forces the algorithm to switch all bits equally often, the construction must ensure to operate correctly when all bits are switched equally often. This is achieved by representing every bit by *two* gadgets where only one of them *actively* represents the bit. The gadgets alternate in actively representing the bit. This enables one gadget to "catch up" with the rest of the counter while the other one represents the bit.

The lower bound construction consists of n structurally identical levels, where level i represents the i-th bit. A parameter $N \geq 7n+1$ with $N \in \mathbb{N}$ is used for defining the rewards and an additional parameter $\epsilon \in (0, N^{-(2n+11)})$ is used for defining the probabilities. The i-th level is shown in Figure 1, the coarse structure of the whole MDP in Figure 2.

A number n_v below or next to the name of a vertex v in Figure 1 denotes a reward of $(-N)^{n_v}$ associated with every edge leaving v. Other edges have a reward of 0. Let σ be a policy and v be a vertex. The *value* $\mathrm{VAL}_\sigma(v)$ of v with respect to σ is the expected accumulated reward obtained by an infinite walk starting in v. The MDP is constructed such that all values are always finite.

Each level i contains two gadgets attached to the *entry vertex* k_i. These gadgets are called *lanes*. We refer to the left lane as *lane 0* and to the right lane as *lane 1*. Lane $j \in \{0, 1\}$ of level i contains a randomization vertex A_i^j and two attached cycles with vertices $b_{i,0}^j$ and $b_{i,1}^j$. These gadgets are called *bicycles*, and we identify the bicycle containing vertex A_i^j with that vertex. For a bicycle A_i^j, the edges $(b_{i,0}^j, A_i^j), (b_{i,1}^j, A_i^j)$ are called *edges of the bicycle*. For a policy σ, the bicycle A_i^j is said to be *closed (with respect to σ)* if and only if $\sigma(b_{i,0}^j) = \sigma(b_{i,1}^j) = A_i^j$. A bicycle that is not closed is *open*.

The i-th level of G_n corresponds to the i-th bit. Which bicycle is actively representing the i-th bit depends on the setting of the $(i+1)$-th bit. When this bit is equal to 1, bicycle A_i^1 is considered active. Otherwise, bicycle A_i^0 is considered active. The i-th bit is interpreted as equal to 1 if and only if the active bicycle in level i is closed.

As initial policy, the MDP is provided the policy $\sigma^\star = \sigma_0$ representing 0. Then, a sequence of policies $\sigma_1, \sigma_2, \ldots, \sigma_{2^n-1}$ is enumerated by the Policy Iteration Algorithm using the LEAST-ENTERED pivot rule and an (implicit) tie-breaking rule. For $b \in \mathbb{B}_n, b \neq 0$, the goal is that the corresponding policy σ_b fulfills the following invariants. These invariants also apply for level n by setting $b_{n+1} := 0$ and to $b = 0$ when substituting $k_{\ell(b)}$ with t.

1. Exactly the bicycles A_i^j corresponding to bits $b_i = 1$ are closed. That is, $b_i = 1$ holds if and only if $\sigma_b(b_{i,0}^j) = \sigma_b(b_{i,1}^j) = A_i^j$ where $j = b_{i+1}$.
2. For all other bicycles A_i^j, it holds that $\sigma_b(b_{i,0}^j) = \sigma_b(b_{i,1}^j) = k_{\ell(b)}$. That is, these bicycles point to the level corresponding to the least significant set bit.
3. All entry vertices k_i point to the lane containing the active bicycle if $b_i = 1$ and to $k_{\ell(b)}$ otherwise. Formally, $\sigma_b(k_i) = c_i^j, j = b_{i+1}$ if $b_i = 1$ and $\sigma_b(k_i) = k_{\ell(b)}$ if $b_i = 0$.

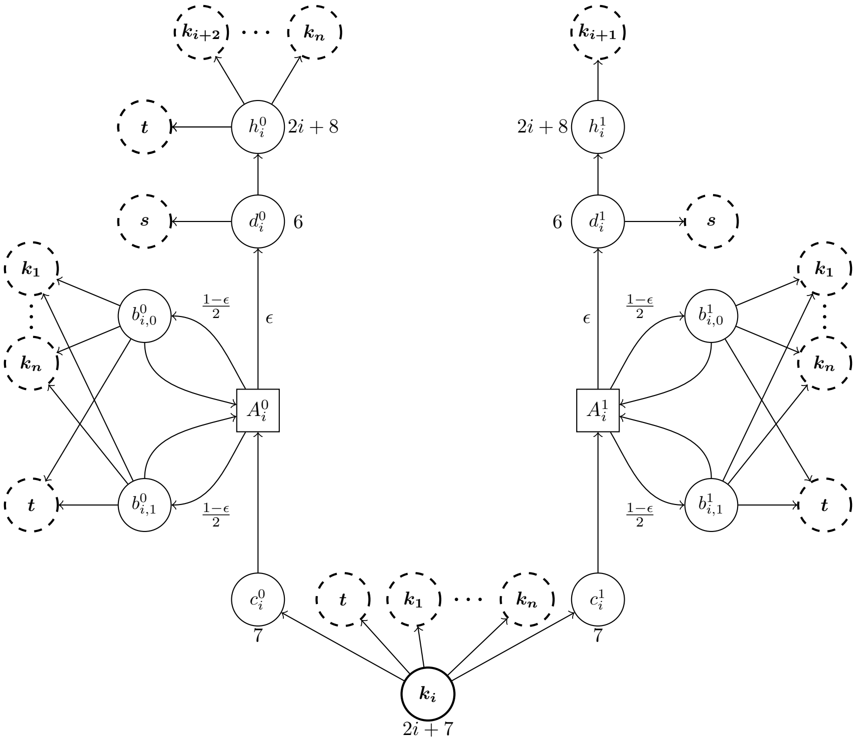

Figure 1: Level i of G_n. Circular shaped vertices are player-controlled, squares are randomization vertices. Bold vertices are global in that they can be reached from other levels. Dashed vertices do not belong to level i. Numbers on edges show the probability of taking this edge, numbers below or next to vertices show the exponent of $(-N)$ of the rewards of edges leaving this vertex. Whenever there is no number, the rewards are 0.

4. The vertex s points to the entry vertex corresponding to the least significant set bit, that is, $\sigma_b(s) = k_{\ell(b)}$.

5. All vertices h_i^0 point to the entry vertex of the first level strictly after level $i+1$ corresponding to a bit equal to 1, that is, when $l := \min\{j \in \{i+2, \dots, n\} : b_j = 1\}$, we have $\sigma_b(h_i^0) = k_l$. If no such l exists, $\sigma_b(h_i^0) = t$. Note that we do not need to specify the target vertex of h_i^1 as these vertices have an outdegree of 1.

6. The vertex d_i^j points to h_i^j if and only if $b_{i+1} = j$ and to s otherwise.

The Policy Iteration Algorithm is only allowed to switch one edge per iteration. Obviously, σ_{b+1} cannot be reached from σ_b by performing a single switch. Therefore, intermediate policies need to be introduced for the transition from σ_b to σ_{b+1}. These intermediate policies are divided into six *phases*. In each phase, a different "task" is performed. We mention here that the following description of the phases partly differs from the informal description given in [3, Pages 8,9]. We explain in detail why our description is different in Section 5. Consider the policy σ_b representing some $b \in \mathbb{B}_n$. Let $\ell' := \ell(b+1)$.

1. In phase 1, switches inside of some bicycles are performed to keep their occurrence records as balanced as possible. For every open bicycle A_i^j, at least one of the two improving switches $(b_{i,0}^j, A_i^j), (b_{i,1}^j, A_i^j)$ is applied. Some inactive bicycles are allowed to apply both of these switches in order to "catch up" with the other edges. In the active bicycle of level ℓ', we also switch both edges, as this bicycle needs to be closed.

2. In phase 2, the new least significant set bit $b_{\ell'}$ is made accessible by the rest of the MDP. Thus, $k_{\ell'}$ is switched to $c_{\ell'}^j$, where $j = (b+1)_{\ell'+1}$ denotes the lane containing the active bicycle.

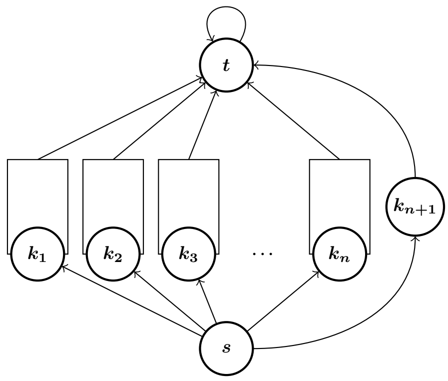

Figure 2: Coarse structure of G_n. The entry vertices are all connected to the vertices s and t. Connections between levels and from the levels to s are not shown here. An additional vertex k_{n+1} is needed for technical reasons.

3. In phase 3, we perform the "resetting process". The entry nodes of all levels i corresponding to bits with $(b+1)_i = 0$ switch to $k_{\ell'}$. The same is done for all inactive vertices $b^j_{i,l}$ contained in inactive bicycle and all vertices $b^j_{i,l}$ corresponding to levels i with $(b+1)_i = 0$. We discuss this phase in more detail in Section 5.
4. In phase 4, the vertices h^0_i are updated according to the new least significant set bit.
5. In phase 5, we connect vertex s with the new least significant set bit, i.e., we switch s to $k_{\ell'}$.
6. In phase 6, we update the vertices d^j_i such that h^0_i is the target of d^0_i if and only if $(b+1)_{i+1} = 0$ and h^1_i is the target of d^1_i if and only if $(b+1)_{i+1} = 1$.

The phases are formally defined in [3, Table 2] which we discuss in Section 2.3.

2.2 Notation related to binary counting

Let $b \in \mathbb{B}_n$. By binary counting, we refer to the process of enumerating the binary representations of all $\tilde{b} \in \{0, 1, \dots, b\}$ in their natural order. These numbers are used to determine how often and when specific edges of G_n are improving switches and will be applied.

Intuitively, we are interested in *schemes* that we observe when counting from 0 to b in binary, or, more formally, in the set of numbers that *match a scheme* with respect to the following definition.

Definition 2.1 (Scheme, matching a scheme [3]). A *scheme* is a set $S \subseteq \mathbb{N} \times \{0, 1\}$. We say that $b \in \mathbb{B}_n$ *matches the scheme* S if $b_i = q$ for all $(i, q) \in S$. We define the *match set*

$$M(b, S) \coloneqq \{\tilde{b} \in \{0, \dots, b\} : \tilde{b}_i = q \quad \forall (i, q) \in S\}$$

as the set of all numbers between 0 and b that match the scheme S.

The next definition introduces the *flip set* with respect to a number b, an index i and a scheme S. This is a subset of $M(b, S)$ that additionally fixes the i least significant bits in a specific way.

Definition 2.2 (Flip set, flip number [3]). Let $b \in \mathbb{B}_n, i \in \{1, \dots, n\}$ and S be a scheme. The set

$$F(b, i, S) \coloneqq M(b, S \cup \{(i, 1)\} \cup \{(j, 0) : j \in \{1, \dots, i-1\}\})$$

is the *flip set* corresponding to b, i and S. The *flip number* is defined as $f(b, i, S) \coloneqq |F(b, i, S)|$. For convenience, we set $F(b, i) \coloneqq F(b, i, \emptyset)$ and $f(b, i) \coloneqq f(b, i, \emptyset)$.

Finally, we define the *maximal flip number*.

Definition 2.3 (Maximal flip number [3]). Let $b \in \mathbb{B}_n, i \in \{1, \dots, n\}$ and S a scheme. The *maximal flip number* is $g(b, i, S) \coloneqq \max(\{0\} \cup \{\tilde{b} : \tilde{b} \in F(b, i, S)\})$.

We observe the following properties of the flip number and the flip set.[1]

[1]This and all other omitted proofs are deferred to Appendix B.

Proposition 2.4. *Let $b \in \mathbb{B}_n$ and $i, j \in \{1, \ldots, n\}$. Then the following hold:*

1. *Let S, S' be schemes and $S \subseteq S'$. Then $M(b, S') \subseteq M(b, S)$.*
2. *Let S, S' be schemes and $S \subseteq S'$. Then $f(b, i, S') \leq f(b, i, S)$.*
3. *It holds that $f(b, j) = f(b, j, \{(i, 0)\}) + f(b, j, \{(i, 1)\})$ and $f(b, j) = \left\lfloor \frac{b+2^{j-1}}{2^j} \right\rfloor$.*
4. *Let $i \leq j$ and S be a scheme. Then $f(b, j, S) \leq f(b, i, S)$ and thus $f(b, j) \leq f(b, i)$.*
5. *Let $i < j$. Then $F(b, j) = F(b, j, \{(i, 0)\})$ and thus $f(b, j, \{(i, 0)\}) = f(b, j)$.*

2.3 Imported tables

In this section, we briefly describe and summarize the tables introduced in [3] that we use in this work. These tables can also be found Appendix A.

The first table we use is [3, Table 2]. It formally defines when a policy σ is considered to be a phase p policy, for $p \in \{1, \ldots, 6\}$. As in [3], we say that a policy σ is a phase p policy if every vertex is mapped by σ to a choice included in the respective cell of the table. Cells that contain more than one choice indicate that policies of the respective phase are allowed to match any of the choices. As we prove later, there is an issue concerning the side conditions of phase 3. Other than correcting this issue, we rely on [3, Table 2].

The next table is [3, Table 3]. For a phase p policy σ, this table shows subsets L_σ^p and supersets U_σ^p of the set I_σ of improving switches. In general, this table does not show the complete sets of improving switches. We verified that the switches given in the sets L_σ^1 to L_σ^5 are in fact improving switches and discuss an issue related to the set L_σ^6 later. Other than correcting this issue, we rely on [3, Table 3].

The last table we use is [3, Table 4]. For $b \in \mathbb{B}_n$, this table contains the occurrence records ϕ^{σ_b} of the edges with respect to the unique policy representing the number b. This table heavily uses the notation introduced in Section 2.2. Again, we found an issue regarding the complicated conditions that we discuss in Section 4. Other than correcting this issue, we rely on [3, Table 4].

3 Initial Policy

In this section, we discuss the initial policy $\sigma^\star$ used in [3]. We show that it contradicts several aspects of [3], in particular Table 3. We also discuss how to replace $\sigma^\star$ such that the resulting issues are resolved.

3.1 Issues with Friedmann's initial policy

Before stating and discussing the issues regarding the initial policy, we correct a small but crucial typo that can be found in the beginning of [3, Section 3]. There, the following is stated: "*An edge $(u, v') \in E_0$ such that $\sigma_i(u) \neq v'$ is then said to be an improving switch if and only if either* $\mathrm{VAL}_{\sigma_i}(v') > \mathrm{VAL}_{\sigma_i}(u)$". This however seems to be incorrect and the inequality needs to be replaced by the inequality $\mathrm{VAL}_{\sigma_i}(v') > \mathrm{VAL}_{\sigma_i}(\sigma_i(u))$.

On [3, Page 11], the following is stated on the initial policy: "*As designated initial policy $\sigma^\star$, we use $\sigma^\star(d_i^j) = h_i^j$ and $\sigma^\star(\_) = t$ for all other player 0 nodes with non-singular out-degree.*" This initial policy, however, is inconsistent with the sub- and supersets of improving switches given in [3, Table 3] and [3, Lemma 4].

Issue 3.1. *The initial policy $\sigma^\star$ described above contradicts [3, Table 3].*

We prove this statement using the following lemma.

Lemma 3.2. *None of the edges $(b_{i,r}^1, A_i^1)$ for $i \in \{1, \ldots, n\}$ and $r \in \{0, 1\}$ is an improving switch with respect to $\sigma^\star$.*

Proof. Fix some $i \in \{1,\dots,n\}$ and $r \in \{0,1\}$. By definition of $\sigma^\star$, it holds that $\sigma^\star(b^1_{i,r}) = t$. Therefore, $\mathrm{VAL}_{\sigma^\star}(b^1_{i,r}) = \mathrm{VAL}_{\sigma^\star}(\sigma^\star(b^1_{i,r})) = \mathrm{VAL}_{\sigma^\star}(t) = 0$ since all edges starting in $b^1_{i,r}$ have a reward of 0. Similarly, $\mathrm{VAL}_{\sigma^\star}(b^1_{i,1-r}) = 0$. This implies that $(b^1_{i,r}, A^1_i)$ is an improving switch if and only if $\mathrm{VAL}_{\sigma^\star}(A^1_i) > 0$. But, due to $\sigma^\star(k_{i+1}) = t$,

$$\begin{aligned}
\mathrm{VAL}_{\sigma^\star}(A^1_i) &= \epsilon\,\mathrm{VAL}_{\sigma^\star}(d^1_i) + \frac{1-\epsilon}{2}\,\mathrm{VAL}_{\sigma^\star}(b^1_{i,l}) + \frac{1-\epsilon}{2}\,\mathrm{VAL}_{\sigma^\star}(b^1_{i,1-l}) \\
&= \epsilon\,\mathrm{VAL}_{\sigma^\star}(d^1_i) \\
&= \epsilon\left[(-N)^6 + \mathrm{VAL}_{\sigma^\star}(\sigma^\star(d^1_i))\right] \\
&= \epsilon\left[N^6 + \mathrm{VAL}_{\sigma^\star}(h^1_i)\right] \\
&= \epsilon\left[N^6 + (-N)^{2i+8} + \mathrm{VAL}_{\sigma^\star}(k_{i+1})\right] \\
&= \epsilon\left[N^6 + N^{2i+8} + (-N)^{2(i+1)+7} + \mathrm{VAL}_{\sigma^\star}(t)\right] \\
&= \epsilon\left[N^6 + N^{2i+8} - N^{2i+9}\right] < 0,
\end{aligned}$$

as $N \geq 7n+1 \geq 8$ and $i \geq 1$. Therefore, the edge $(b^1_{i,r}, A^1_i)$ is not an improving switch. □

Proof of Issue 3.1. By definition, $\sigma^\star$ is a phase 1 policy. Thus, according to [3, Table 3] and since $L^1_\sigma = U^1_\sigma$ holds for all phase 1 policies σ, the set of improving switches is given exactly by $I_{\sigma^\star} = \{(b^j_{i,r}, A^j_i) | \sigma^\star(b^j_{i,r}) \neq A^j_i\}$. By definition of $\sigma^\star$, we have $\sigma^\star(b^1_{i,r}) = t$ which, due to $t \neq A^1_i$, implies $\sigma^\star(b^1_{i,r}) \neq A^1_i$. Therefore, $(b^1_{i,r}, A^1_i) \in I_{\sigma^\star}$, that is, $(b^1_{i,r}, A^1_i)$ is an improving switch. But, by Lemma 3.2, $(b^1_{i,r}, A^1_i)$ is not an improving switch for any $i \in \{1,\dots,n\}$ and $r \in \{0,1\}$. This is a contradiction. □

We mention that the initial policy additionally contradicts the informal description of how a phase 1 policy is supposed to look like. In $\sigma^\star$, all vertices d^j_i point to h^j_i. This contradicts the informal description as it is intended that "*d^0_i moves higher up iff $b_{i+1} = 0$ and d^1_i moves higher up iff $b_{i+1} = 1$*" [3, Page 9].

Also, the following issue is caused by the initial policy $\sigma^\star$.

Issue 3.3. *When the Policy Iteration Algorithm is started with $\sigma^\star$, either*

1. *[3, Table 4] containing the occurrence records is incorrect for $b = 1$, or*
2. *[3, Table 3] containing the sub- and supersets of the improving switches is incorrect for $b = 1$.*

Proof of Issue 3.3. Consider the first phase 6 policy of the transition from $\sigma^\star$ to σ_1. Denote this policy by σ and fix some $i \in \{1,\dots,n\}$. Then, by [3, Table 2], we have $\sigma(s) = k_1$ and $\sigma(k_j) = k_1$ for all $j \in \{2,\dots,n\}$. Therefore since the reward of the edge (s, k_1) is equal to zero, this implies that $\mathrm{VAL}_\sigma(s) = \mathrm{VAL}_\sigma(k_1)$, and therefore

$$\begin{aligned}
\mathrm{VAL}_\sigma(\sigma(d^1_i)) &= \mathrm{VAL}_\sigma(h^1_i) \\
&= N^{2i+8} + \mathrm{VAL}_\sigma(k_{i+1}) \\
&= \underbrace{N^{2i+8} - N^{2i+9}}_{<0} + \underbrace{\mathrm{VAL}_\sigma(k_1)}_{=\mathrm{VAL}_\sigma(s)} \\
&< \mathrm{VAL}_\sigma(s),
\end{aligned}$$

hence the edge (d^1_i, s) is an improving switch for every $i \in \{1,\dots,n\}$. Now, we can either

1. apply (some or all) of these improving switches now or
2. we do not apply any of these improving switches now.

Suppose that we apply the switch (d^1_i, s) for every $i \in \{1,\dots,n\}$. By definition, phase 6 ends after these switches are applied and phase 1 of the transition from σ_1 to σ_2 begins. But, according to [3, Table 4], it should hold that $\phi^{\sigma_2}(d^1_i, s) = f(b, i+1) - 1 \cdot b_{i+1}$ for all $i \in \{1,\dots,n\}$. Because $b = 1$, we have $b_k = 0$ for all $k \in \{2,\dots,n\}$. In particular, $b_2 = 0$, implying that $\phi^{\sigma_2}(d^1_i, s) = f(b, i+1) = 0$.

This is a contradiction to the fact that we have just switched the edges (d_i^1, s). Note that this argument still holds when we only apply a subset of all of the improving switches.

Now suppose that we do not apply any of the improving switches. Then, all of the edges (d_i^1, s) remain improving switches during phase 1 of the transition from σ_1 to σ_2. This however contradicts [3, Table 3]. $\square$

3.2 Fixing the initial policy

As discussed in Issues 3.1 and 3.3, the initial policy $\sigma^\star$ needs to be changed. We propose to use the following policy instead.

Definition 3.4 (New initial policy σ^*). We define the following initial policy σ^*:

- $\sigma^*(d_i^0) := h_i^0$ for all $i \in \{1, \ldots, n\}$.
- $\sigma^*(d_i^1) := s$ for all $i \in \{1, \ldots, n\}$.
- $\sigma^*(\_) := t$ for all other player-controlled vertices with non-singular out-degree.

This policy is visualized in Figure 3. Note that this policy also represents the number 0 and [3, Lemma 1] holds for the policy σ^*. We now show that this policy resolves Issue 3.1.

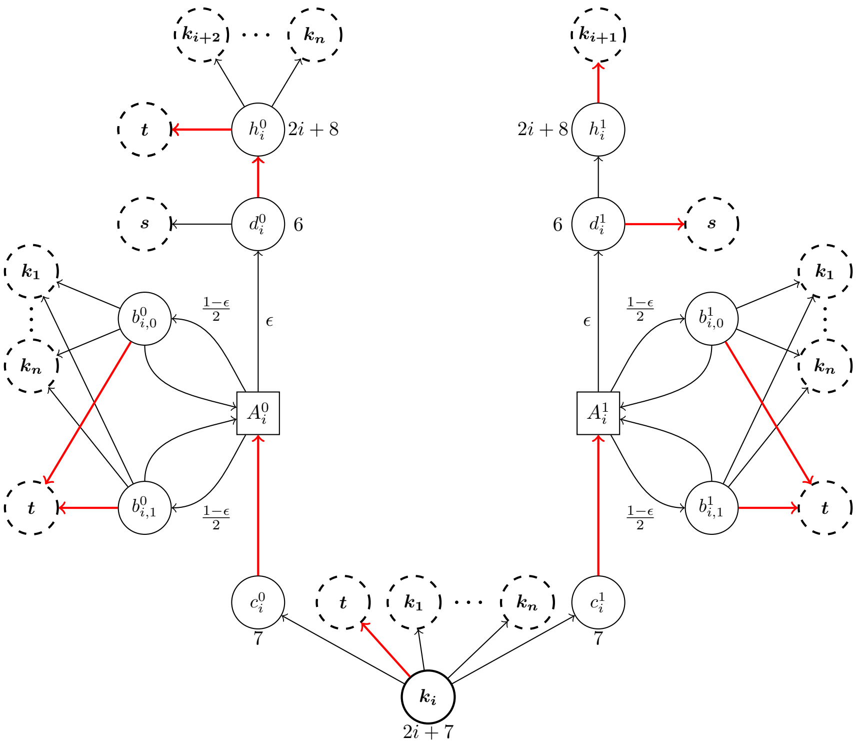

Figure 3: A level of the alternative initial policy σ^*. Thick red edges correspond to edges of σ^*.

Lemma 3.5. *For the policy σ^*, the set of improving switches is $I_{\sigma^*} = \{(b_{i,r}^j, A_i^j) | \sigma^*(b_{i,r}^j) \neq A_i^j\}$.*

Proof. Compared to $\sigma^\star$, the changes can only have an effect on the edges (d_i^1, h_i^1) and $(b_{i,r}^1, A_i^1)$. Thus it suffices to show that the edge (d_i^1, h_i^1) is not an improving switch for any $i \in \{1, \ldots, n\}$ whereas the edges $(b_{i,r}^1, A_i^1)$ are improving switches for all $i \in \{1, \ldots, n\}$ and $r \in \{0, 1\}$.

Fix some $i \in \{1, \ldots, n\}$. By the definition of σ^*, we have $\sigma^*(d_i^1) = s$ and $\sigma^*(s) = t$. Therefore, $\mathrm{VAL}_{\sigma^*}(\sigma^*(d_i^1)) = 0$. To show that (d_i^1, h_i^1) is not an improving switch, it thus suffices to show $\mathrm{VAL}_{\sigma^*}(h_i^1) < 0$. This however holds since

$$\begin{aligned}\mathrm{VAL}_{\sigma^*}(h_i^1) &= (-N)^{2i+8} + \mathrm{VAL}_{\sigma^*}(k_{i+1}) \\ &= N^{2i+8} + (-N)^{2(i+1)+7} + \mathrm{VAL}_{\sigma^*}(t) \\ &= N^{2i+8} - N^{2i+9} < 0,\end{aligned}$$

due to $N \geq 8$ and $i \geq 1$. Therefore, the edge (d_i^1, h_i^1) is not an improving switch.

Now fix some $r \in \{0,1\}$. Since it holds that $\mathrm{VAL}_{\sigma^*}(\sigma^*(b_{i,r}^1)) = \mathrm{VAL}_{\sigma^*}(\sigma^*(b_{i,1-r}^1)) = 0$, it suffices to show $\mathrm{VAL}_{\sigma^*}(A_i^1) > 0$ to prove that $(b_{i,r}^1, A_i^1)$ is an improving switch. Due to $\sigma^*(d_i^1) = s$, we have

$$\mathrm{VAL}_{\sigma^*}(A_i^1) = \epsilon\, \mathrm{VAL}_{\sigma^*}(d_i^1) = \epsilon\left[N^6 + \mathrm{VAL}_{\sigma^*}(s)\right] = \epsilon N^6 > 0,$$

so $(b_{i,r}^1, A_i^1)$ is an improving switch. □

It remains to show that our adapted initial policy σ^* also avoids Issue 3.3.

Lemma 3.6. *Starting the Policy Iteration Algorithm with the initial policy σ^* avoids Issue 3.3, that is, it does not contradict [3, Tables 3,4] for $b = 1$.*

Proof. Consider the first phase 6 policy of the transition from σ^* to σ_1. Denote this policy by σ and fix any $i \in \{1, \ldots, n\}$. Then, $\sigma(d_i^1) = s$ for all $i \in \{1, \ldots, n\}$ by the definition of σ^* and the application of improving switches. Therefore, none of the edges (d_i^1, s) is an improving switch for any $i \in \{1, \ldots, n\}$ and none of these edges can be switched. Thus, once σ_1 is reached, the occurrence record of these edges is equal to zero as they should be according to [3, Table 4]. This also implies that the edges (d_i^1, s) are no improving switches during the transition from σ_1 to σ_2, resolving the contradiction regarding [3, Table 3]. □

4 Occurrence Records

In this section, we discuss an issue related to the occurrence records of the bicycles as specified in [3, Table 4]. For $b \in \mathbb{B}_n$, this table contains the occurrence records with respect to σ_b. For a fixed $b \in \mathbb{B}_n$ and bicycle A_i^j, we use the abbreviations $g := g(b, i, \{(i+1, j)\}), z := b - g - 2^{i-1}$ and $\phi^{\sigma_b}(A_i^j) := \phi^{\sigma_b}(b_{i,0}^j, A_i^j) + \phi^{\sigma_b}(b_{i,1}^j, A_i^j)$. Using these abbreviations, the following is stated regarding the occurrence records:

$$\left|\phi^{\sigma_b}(b_{i,0}^j, A_i^j) - \phi^{\sigma_b}(b_{i,1}^j, A_i^j)\right| \leq 1 \tag{4.1}$$

$$\phi^{\sigma_b}(A_i^j) = \begin{cases} g+1 & \text{if } b_i = 1 \wedge b_{i+1} = j, \\ g+1+2z & \text{if } b_{i+1} \neq j \text{ and } z < \frac{1}{2}(b-1-g), \\ b & \text{otherwise.} \end{cases} \tag{4.2}$$

We discuss an inconsistency regarding Equation (4.2): Assuming that $\phi^{\sigma_b}(A_i^j)$ are given by Equations (4.1) and (4.2) and that the other entries of [3, Table 4] are correct causes the following contradiction.

Issue 4.1. *Let $b < 2^{n-k-1} - 1$ for some $k \in \mathbb{N}$. Then, there are edges that have a negative occurrence record according to [3, Table 4].*

Proof of Issue 4.1. Let $i \in \{n-k, \ldots, n-1\}$ and $j = 1$. By $b < 2^{n-k-1} - 1$ and $i \geq n-k$ it follows that $b < 2^i - 1$. This implies $b_i = 0$ and $b_{i+1} = 0 \neq 1 = j$. Since also $\tilde{b}_{i+1} = 0$ for all $\tilde{b} \leq b$, it follows that $g = g(b, i, \{(i+1, 1)\}) = 0$. In addition, since $b < 2^i - 1$ is equivalent to $2^i > b+1$, we get

$$2z = 2(b - 2^{i-1}) = 2b - 2^i < 2b - (b+1) = b - 1,$$

or $z < \frac{1}{2}(b-1) = \frac{1}{2}(b-1-g)$. Thus, all conditions for the second case of Equation (4.2) are fulfilled, implying

$$\begin{aligned}\phi^{\sigma_b}(b_{i,0}^j, A_i^j) + \phi^{\sigma_b}(b_{i,1}^j, A_i^j) &= g + 1 + 2z \\ &= 2z + 1 \\ &= 2(b - 2^{i-1}) + 1 \\ &< 2(2^{n-k-1} - 1 - 2^{i-1}) + 1 \\ &\leq 2(2^{n-k-1} - 1 - 2^{n-k-1}) + 1 \\ &= -1 < 0.\end{aligned}$$

Hence there must be at least one edge that has a negative occurrence record. □

We now resolve Issue 4.1. Let $b \in \mathbb{B}_n$ and A_i^j be a bicycle. We show that when applying the switches as described in [3], the occurrence records are given by the following system:

$$|\phi^{\sigma_b}(b_{i,0}^j, A_i^j) - \phi^{\sigma_b}(b_{i,1}^j, A_i^j)| \leq 1 \tag{4.3}$$

$$\phi^{\sigma_b}(A_i^j) = \begin{cases} g+1, & A_i^j \text{ is closed and active} \\ b, & A_i^j \text{ is open and active} \\ b, & A_i^j \text{ is inactive and } b < 2^{i-1} + j \cdot 2^i \\ g+1+2z, & A_i^j \text{ is inactive and } b \geq 2^{i-1} + j \cdot 2^i \end{cases} \tag{4.4}$$

This system of equations properly distinguishes between inactive bicycles that need to catch up with the counter and inactive bicycles that do not need to do so.

Informally, for $b \in \mathbb{B}_n$ the occurrence records can be described as follows:

- Every closed and active bicycle has an occurrence record corresponding to the last time it was closed.
- Every open and active bicycle has an occurrence record of b.
- Inactive bicycles are either "catching up" with other bicycles and thus have an occurrence record less than b or already finished catching up and have an occurrence record of b again.

Before proving that Equations (4.3) and (4.4) correctly describe the occurrence records, we compare them to [3, Table 4]. Equations (4.1) and (4.3) bounding the difference of the occurrence records within a bicycle are the same. Also the case of closed and active bicycles is the same since a bicycle is closed and active by definition if and only if $b_i = 1$ and $b_{i+1} = j$. Consider the second condition of Equation (4.2). This case handles inactive bicycles that do not have an occurrence record of b. This is handled by the condition $z < \frac{1}{2}(b-1-g)$, which is equivalent to $g+1+2z < b$. However, as shown in Issue 4.1, this condition does not describe inactive bicycles properly. We therefore formulate another condition, regarding the relation between b and $2^{i-1} + j \cdot 2^i$. This condition distinguishes inactive bicycles that might need to catch up with the counter because they have already been active once (if $b \geq 2^{i-1} + j \cdot 2^i$), and inactive bicycle that do not need to catch up because they have not been active before. Finally, the case of open and active bicycles, which is included in the "otherwise" case in [3, Table 4] concludes our description.

Next, we explain how the improving switches within the bicycles should be applied. This description is a reformulation of [3], specifically of the description given in the proof of [3, Lemma 5]. We remind here that we use the term *edges of bicycle* A_i^j to refer to the edges $(b_{i,0}^j, A_i^j)$ and $(b_{i,1}^j, A_i^j)$. We apply the improving switches according to the following rules for a bicycle A_i^j during phase 1 of the transition from σ_b to σ_{b+1} (rules are not stated in the order of their application):

I. If A_i^j is open and active, we switch one edge of the bicycle.

II. Let $j := b_{\ell(b+1)+1}$. In addition to rule I, the second edge of $A_{\ell(b+1)}^j$ is switched.

III. If A_i^j is inactive and $b < 2^{i-1} + j \cdot 2^i$, one edge of the bicycle is switched.

IV. If A_i^j is inactive, $b \geq 2^{i-1} + j \cdot 2^i$ and $z < \frac{1}{2}(b-1-g)$, both edges of A_i^j are switched.

V. If A_i^j is inactive, $b \geq 2^{i-1} + j \cdot 2^i$ and $z \geq \frac{1}{2}(b-1-g)$, only one edge of A_i^j is switched.

We now show that applying the improving switches according to these rules yields the occurrence records described by Equation (4.4). We additionally prove an upper bound on the occurrence records that is used heavily throughout the proof.

Theorem 4.2. *Suppose that the improving switches within the bicycles are applied as described by the rules I to V. Let $b \in \mathbb{B}_n$ and A_i^j be some bicycle. Then, Equations* (4.3) *and* (4.4) *describe the occurrence record $\phi^{\sigma_b}(A_i^j)$. In addition,*

$$\phi^{\sigma_b}(A_i^j) \leq b+1 \tag{4.5}$$

where equality holds if and only if $i = \ell(b)$ and $j = b_{\ell(b)+1}$.

To simplify the proof, we introduce the following notion. Fix $b \in \mathbb{B}_n$ and a bicycle A_i^j. The bicycle A_i^j is called *bicycle of type i with respect to σ_b* when it fulfills the i-th condition mentioned in Equation (4.4) for σ_b. We additionally establish the following abbreviations and state a lemma that is implicitly contained in the proof of Lemma 5 in [3].

- We define $g \coloneqq g(b, i, \{(i+1, j)\})$, i.e., g is the largest number smaller than or equal to b such that $g_{i+1} = j, g_i = 1$ and $g_l = 0$ for all $l < i$. We define $g' \coloneqq g(b+1, i, \{(i+1, j)\})$ analogously.
- We define $z \coloneqq b - g - 2^{i-1}$. We define $z' \coloneqq b + 1 - g' - 2^{i-1}$ analogously.
- We define $\ell \coloneqq \ell(b)$ and $\ell' \coloneqq \ell(b+1)$.

Lemma 4.3 ([3])**.** *For every $b \in \mathbb{B}_n, i \in \{1, \ldots, n\}$ with $i \neq \ell(b+1)$ and $j \in \{0,1\}$ we have $g = g'$.*

We also make use of the following lemma.

Lemma 4.4. *Let $b \in \mathbb{B}_n$ and A_i^j be some bicycle. Then, the bicycle A_i^j was closed at least once during the application of the Policy Iteration Algorithm upto policy σ_b if and only if $b \geq 2^{i-1} + j \cdot 2^i$.*

Proof. The bicycle A_i^j is closed the first time during the application of the Policy Iteration Algorithm when a number $\tilde{b} \leq b$ is reached such that $\tilde{b}_i = 1, \tilde{b}_{i+1} = j$ and $\tilde{b}_l = 0$ else. This number is exactly $2^{i-1} + j \cdot 2^i$. □

With this notation and Lemmas 4.3 and 4.4 in place, we now prove Theorem 4.2. Whenever we discuss how a bicycle should look like, we implicitly refer to the invariants introduced in Section 2.1 that describe σ_b.

Proof of Theorem 4.2. We show the statement via induction on b. Let $b = 0$. By the definition of both the original initial policy $\sigma^\star$ and the new initial policy σ^*, the target of $b_{i,l}^j$ under the corresponding policy is t for all $i \in \{1, \ldots, n\}$ and $j, l \in \{0,1\}$. Therefore, all bicycles are open and either active or inactive, regardless which of the two initial policies is considered. As $b = 0$, the inequality $b < 2^{i-1} + j \cdot 2^i$ holds for all $i \in \{1, \ldots, n\}$ and $j \in \{0,1\}$. This implies that every bicycle is either of type 2 or of type 3. Therefore, for Equation (4.4) to hold, the occurrence record of every bicycle needs to be equal to $b = 0$. But, since we consider the initial policies, no improving switch was applied yet. Therefore, $\phi^{\sigma_0}(A_i^j) = 0$ for all bicycles A_i^j. Consequently, Equation (4.4) holds. We furthermore observe that there is no least significant set bit $\ell(b)$ since $b = 0$. Hence, since $\phi^{\sigma_0}(A_i^j) = 0 < b+1$ for all bicycles A_i^j, and no bicycle is closed, Equation (4.5) holds as well.

Suppose that the statement holds for all numbers smaller or equal to $b \in \mathbb{B}_n$. We show that it also holds for $b+1$. We distinguish between the induction hypotheses with respect to Equation (4.4) and Equation (4.5) and always state to which we refer. We discuss Equation (4.3) at the end of the proof.

Fix a bicycle A_i^j for some $i \in \{1, \ldots, n\}$ and $j \in \{0,1\}$. The proof is organized as follows. We distinguish all possible cases of which "state" (*open*, *active*, …) A_i^j could be in with respect to σ_b. We then investigate of which type the bicycle is with respect to σ_b and if this type changes when transitioning to σ_{b+1}. We then state how many improving switches we need to apply according to our rules and discuss what we need to show such that Equation (4.4) remains valid for the policy σ_{b+1}.

Case 1: $\boldsymbol{A_i^j}$ **is open, active and** $\boldsymbol{i = \ell'}$**.** Then A_i^j is the active bicycle corresponding to the least significant set bit of $b+1$. By construction, it is open with respect to σ_b but needs to be closed with respect to σ_{b+1}. As $b_{\ell'+1} = (b+1)_{\ell'+1}$, the bicycle remains active. Thus, A_i^j is of type 1 with respect to σ_{b+1}. As we apply rules I and II and switch both edges of the bicycle, we thus need to show

$$\phi^{\sigma_b}(A_i^j) + 2 = g' + 1.$$

By the induction hypothesis (4.4), we have $\phi^{\sigma_b}(A_i^j) = b$ since A_i^j is a type 2 bicycle with respect to σ_b. To show Equation (4.4), it therefore suffices to show $b + 2 = g' + 1$, or, equivalently, $g' = b + 1$. This however follows since the binary representations of g' and $b+1$ both end on the subsequence $(b_{\ell'+1}, 1, 0, \dots, 0)$ of length $\ell' + 1$.

Observe that we have $\phi^{\sigma_{b+1}}(A_i^j) = (b+1) + 1$ after applying the two switches, hence Equation (4.5) remains valid as well.

Case 2: $\boldsymbol{A_i^j}$ **is open and active, but** $\boldsymbol{i \neq \ell'}$**.** We argue that A_i^j remains open and active, i.e., A_i^j is a bicycle of type 2 with respect to σ_{b+1}. By the definition of open and active, we have $b_i = 0$ and $j = b_{i+1}$. In addition, $b_1 = \dots = b_{\ell'-1} = 1$ since $\ell' = \ell(b+1)$. As all active bicycles corresponding to levels 1 to $\ell' - 1$ are closed in σ and $i \neq \ell'$, this implies $i > \ell'$. Due to only the ℓ' least significant bits (i.e., the bits b_1 to $b_{\ell'}$) being switched, A_i^j remains active with respect to $b+1$. Since the active bicycle of level ℓ' is the only bicycle that is open with respect to σ_b but closed with respect to σ_{b+1}, A_i^j remains open. Hence, since A_i^j remains open and active, it is a bicycle of type 2 with respect to σ_{b+1}. Because we only apply one improving switch in the bicycle A_i^j (rule I), we therefore need to show that

$$\phi^{\sigma_b}(A_i^j) + 1 = b + 1.$$

By the induction hypothesis (4.4) $\phi^{\sigma_b}(A_i^j) = b$, so $\phi^{\sigma_b}(A_i^j) + 1 = b + 1$. Therefore, both Equations (4.4) and (4.5) still hold.

Case 3: $\boldsymbol{A_i^j}$ **is closed, active and** $\boldsymbol{i > \ell'}$**.** We show that A_i^j is of type 1 with respect to σ_{b+1}. By the definition of closed and active, $b_i = 1$ and $b_{i+1} = j$. As only bits corresponding to indices smaller than ℓ' switch, A_i^j remains active, cf. Case 2. By $i > \ell'$, it also remains closed since only the bits b_1 to $b_{\ell'-1}$ switch from 1 to 0 and thus only bicycles corresponding to these levels are opened during phase 3. Therefore, A_i^j is a bicycle of type 1 with respect to σ_{b+1} and none of the edges $(b_{i,0}^j, A_i^j), (b_{i,1}^j, A_i^j)$ are switched. We thus need need to show

$$\phi^{\sigma_b}(A_i^j) = g' + 1.$$

By the induction hypothesis (4.4), we have $\phi^{\sigma_b}(A_i^j) = g + 1$. It therefore suffices to show that $g + 1 = g' + 1$. Since $i \neq \ell'$, this follows from Lemma 4.3. Therefore, Equation (4.4) still holds.

Equation (4.5) remains valid since $\phi^{\sigma_b}(A_i^j) \leq b$ holds by the induction hypothesis (4.5). Since $\phi^{\sigma_{b+1}}(A_i^j) = \phi^{\sigma_b}(A_i^j)$ holds by the argument above, we obtain $\phi^{\sigma_{b+1}}(A_i^j) < b + 1$.

Case 4: $\boldsymbol{A_i^j}$ **is closed, active and** $\boldsymbol{i < \ell'}$**.** We show that A_i^j is of type 4 with respect to σ_{b+1}. Because of $i < \ell'$, the bits b_i and b_{i+1} both switch. Thus, since $i < \ell'$ implies $b_i = 1$, we have $(b+1)_i = 0$. Hence A_i^j is open with respect to σ_{b+1}. Since A_i^j is active with respect to σ_b, we have $b_{i+1} = j$ and therefore, because bit b_{i+1} switches, we obtain $(b+1)_{i+1} \neq j$. Thus, A_i^j is inactive with respect to σ_{b+1}. Since A_i^j is closed, Lemma 4.4 implies $b \geq 2^{i-1} + j \cdot 2^i$. Therefore, A_i^j is a bicycle of type 4 with respect to σ_{b+1}. Because A_i^j is closed, we do not switch the edges of the bicycle and therefore need to show

$$\phi^{\sigma_b}(A_i^j) = g' + 1 + 2z'.$$

By the induction hypothesis (4.4), it holds that $\phi^{\sigma_b}(A_i^j) = g + 1$. Thus, we need to show that $g + 1 = g' + 1 + 2z'$. Since $i \neq \ell'$, Lemma 4.3 implies $g \neq g'$. It thus suffices to show that $z' = b + 1 - g' - 2^{i-1} = 0$.

From the assumptions $i < \ell'$ and that A_i^j is closed, we get $b_i = 1$. Since A_i^j is also active by assumption, it follows that $j = b_{i+1}$. This implies that $g = (b_n, \ldots, b_{i+1}, 1, 0, \ldots, 0)$. Therefore, since $i < \ell'$ implies $b = (b_n, \ldots, b_{i+1}, 1, 1, \ldots, 1)$, we get $b - g = 2^{i-1} - 1$. Because $g = g'$ holds by Lemma 4.3, this can be formulated equivalently, obtaining the equality $b + 1 - g' - 2^{i-1} = 0$. Thus Equation (4.4) remains valid.

As in Case 2, $\phi^{\sigma_{b+1}}(A_i^j) = \phi^{\sigma_b}(A_i^j)$ and since $\phi^{\sigma_b}(A_i^j) \leq b$ holds by the induction hypothesis (4.5), also Equation (4.5) follows.

Case 5: A_i^j is closed, active and $i = \ell'$. This cannot happen since both bicycles of level ℓ' are open with respect to σ_b since $b_{\ell'} = 0$.

Case 6: A_i^j is closed and inactive. This cannot happen since closed bicycles are always active (see the invariants described in Section 2.1).

Case 7: A_i^j is inactive and $b < 2^{i-1} + j \cdot 2^i$. Then, A_i^j is a bicycle of type 3. We observe that A_i^j being inactive implies that A_i^j is open. We distinguish the type of A_i^j is with respect to σ_{b+1}, since this is not clear in this case.

It cannot happen that A_i^j is closed with respect to σ_{b+1}, because the active bicycle of level ℓ' is the only bicycle which is open with respect to σ_b and closed with respect to σ_{b+1}, and A_i^j is inactive by assumption.

Suppose that A_i^j is a bicycle of type 3 with respect to σ_{b+1}. That is, it remains inactive with respect to σ_{b+1} and $b + 1 < 2^{i-1} + j \cdot 2^i$ holds. As we apply one improving switch (rule III), we thus need to show that

$$\phi^{\sigma_b}(A_i^j) + 1 = b + 1.$$

This follows immediately since $\phi^{\sigma_b}(A_i^j) = b$ by the induction hypothesis (4.4).

Suppose that A_i^j is a bicycle of type 2 with respect to σ_{b+1}. That is, it is active and open with respect to σ_{b+1}. In this case, we also need to show

$$\phi^{\sigma_b}(A_i^j) + 1 = b + 1,$$

which also follows from the induction hypotheses (4.4).

Suppose that A_i^j is a bicycle of type 4 with respect to σ_{b+1}. That is, it is inactive with respect to σ_{b+1} and $b + 1 \geq 2^{i-1} + j \cdot 2^i$. Then, since $b < 2^{i-1} + j \cdot 2^i$, it follows immediately that $b + 1 = 2^{i-1} + j \cdot 2^i$. But, by Lemma 4.4, this can only happen if the bicycle A_i^j is closed during the transition from σ_b to σ_{b+1}, contradicting the inactivity of A_i^j with respect to σ_b.

Therefore, $\phi^{\sigma_b}(A_i^j) + 1 = b + 1$ holds in all cases, and both Equation (4.4) and Equation (4.5) stay valid.

Case 8: A_i^j is inactive, $b \geq 2^{i-1} + j \cdot 2^i$ and $z < \frac{1}{2}(b - 1 - g)$. In this case, A_i^j is a bicycle of type 4 with respect to σ_b. We show that it also is a bicycle of type 4 with respect to σ_{b+1}, i.e., that A_i^j is inactive with respect to σ_{b+1} and $b + 1 \geq 2^{i-1} + j \cdot 2^i$. It then remains to show that $\phi^{\sigma_b}(A_i^j) + 2 = g' + 1 + 2z'$, or, since $\phi^{\sigma_b}(A_i^j) = g + 1 + 2z$ by the induction hypothesis (4.4), that $g + 1 + 2z + 2 = g' + 1 + 2z'$.

Observe that $b + 1 \geq 2^{i-1} + j \cdot 2^i$ immediately follows from $b \geq 2^{i-1} + j \cdot 2^i$. Towards a contradiction, assume that A_i^j is active with respect to σ_{b+1}. Since only bits with an index smaller or equal to ℓ' are switched, only inactive bicycles on levels 1 to $\ell' - 1$ can become active. As a consequence, we have $i < \ell'$.

We next show that $b - g = 2^i + 2^{i-1} - 1$ holds. First assume that $i \neq \ell' - 1$. Then, since $i < \ell' - 1$ and $b = (b_n, \ldots, b_{\ell'+1}, 0, 1, \ldots, 1)$, it follows that $b_{i+1} = 1$. Hence, by the inactivity of A_i^j with respect to σ_b, we obtain $j = 0$. Therefore,

$$g = (b_n, \ldots, b_{\ell'+1}, 0, 1, \ldots, 1, \underbrace{0}_{g_{i+1}}, \underbrace{1}_{g_i}, 0, \ldots, 0),$$

since we have $g_i = 1$ and $g_{i+1} = j = 0$ by definition of g. Consequently, $b - g = 2^i + 2^{i-1} - 1$.

Now assume that $i = \ell' - 1$. We then obtain $b_{i+1} = b_{\ell'} = 0$ and hence, by the inactivity of A_i^j, we get $j = 1$. Therefore,

$$g = (\tilde{b}_n, \dots, \tilde{b}_{\ell'+1}, 1, \underbrace{1}_{g_i = g_{\ell'-1}}, 0, \dots, 0)$$

where $(\tilde{b}_n, \dots, \tilde{b}_{\ell'+1}) = (b_n, \dots, b_{\ell'+1}) - 1$. This implies that $g + 2^i + 2^{i-1} = b + 1$ which is equivalent to $b - g = 2^i + 2^{i-1} - 1$.

Using the identities $b - g = 2^i + 2^{i-1} - 1$ and $\phi^{\sigma_b}(A_i^j) = b + 1 + 2z$ which follows from the induction hypothesis (4.4), we obtain the following estimation for $\phi^{\sigma_b}(A_i^j)$:

$$\begin{aligned}
\phi^{\sigma_b}(A_i^j) &= g + 1 + 2(b - g - 2^{i-1}) \\
&= 2b - g - 2^i + 1 \\
&= b + 2^i + 2^{i-1} - 1 - 2^i + 1 \\
&= b + 2^{i-1} > b.
\end{aligned}$$

Additionally, by assumption, $z < \frac{1}{2}(b - 1 - g)$, which implies

$$\phi^{\sigma_b}(A_i^j) = g + 1 + 2z < g + 1 + b - 1 - g = b, \tag{4.6}$$

contradicting the previous inequality. Therefore, the assumption of A_i^j being active with respect to σ_{b+1} cannot be correct, hence the bicycle must be inactive with respect to σ_{b+1} and thus be of type 4.

As discussed before, we now need to show

$$\phi^{\sigma_b}(A_i^j) + 2 = g + 1 + 2z + 2 = g' + 2 + 2z'.$$

We observe that due to the inactivity of A_i^j with respect to σ_{b+1}, we have $i \neq \ell'$ and therefore, by Lemma 4.3, also $g = g'$. Therefore,

$$\begin{aligned}
g + 1 + 2z + 2 &= g + 1 + 2b - 2g - 2^i + 2 \\
&= g' + 1 + 2(b + 1) - 2g - 2^i \\
&= g' + 1 + 2z',
\end{aligned}$$

hence Equation (4.4) still holds.

It remains to show Equation (4.5). By Equation (4.6), we have $\phi^{\sigma_b}(A_i^j) < b$, and thus, by integrality, $\phi^{\sigma_b}(A_i^j) \leq b - 1$. Thus, $\phi^{\sigma_{b+1}}(A_i^j) = \phi^{\sigma_b}(A_i^j) + 2 \leq b - 1 + 2 = b + 1$ follows since we apply two switches in A_i^j.

Case 9: A_i^j is inactive, $b \geq 2^{i-1} + j \cdot 2^i$ and $z \geq \frac{1}{2}(b - 1 - g)$. In this case, we do not distinguish the type of A_i^j with respect to σ_{b+1}. Instead, we show $g + 1 + 2z = b$. This suffices because the bicycle A_i^j cannot become closed and active (i.e., a bicycle of type 1) with respect to σ_{b+1} and, by rule V, the occurrence record of A_i^j increases by 1. Therefore, we do not need to specify the type of A_i^j if we are able to show that its occurrence record before applying the switch is equal to b.

To show that $g+1+2z = b$, we need to show $z = \frac{1}{2}(b-1-g)$. Towards a contradiction, assume that $z > \frac{1}{2}(b - 1 - g)$. Then, since A_i^j is a bicycle of type 4, by the induction hypothesis (4.4), we have $\phi^{\sigma_b}(A_i^j) = g + 1 + 2z$. Thus

$$\phi^{\sigma_b}(A_i^j) = g + 1 + 2z > g + 1 + b - 1 - g = b,$$

contradicting the induction hypothesis (4.5) requiring $\phi^{\sigma_b}(A_i^j) \leq b$. Therefore, equality holds and $\phi^{\sigma_b}(A_i^j) = g+1+(b-1-g) = b$. As we apply a single switch, we obtain $\phi^{\sigma_b}(A_i^j)+1 = b+1$, as claimed.

As we have discussed all possible cases, we successfully showed that the occurrence records given in Equation (4.4) and the estimation given in Equation (4.5) hold. Because the switches can always be applied alternatingly within a bicycle, we can ensure that Equation (4.3) holds at all times during the application of the improving switches. □

5 Improving switches of phase 3

In this section, we discuss the application of the improving switches during phase 3. There are two contradictory descriptions in [3] how to apply them. We prove that neither of the given orderings obeys the LEAST-ENTERED rule, even if the issues discussed in the previous sections are resolved. We additionally show that a natural adaptation of Friedmann's scheme still does not obey the LEAST-ENTERED rule. We then go on to prove the existence of an ordering and an associated tie-breaking rule that obey the LEAST-ENTERED rule while still producing the intended behavior of Friedmann's construction.

Throughout this section, for a fixed $b \in \mathbb{B}_n$, we use $\ell \coloneqq \ell(b)$ and $\ell' \coloneqq \ell(b+1)$ to denote the least significant set bits of b and $b+1$, respectively.

5.1 Issues with Friedmann's switching order

We start by discussing phase 3 of the transition from σ_b to σ_{b+1} for $b \in \{0, \dots, 2^n - 2\}$. In Section 2.1, we stated that in phase 3, improving switches need to be applied for every entry vertex k_i belonging to a level i with $(b+1)_i = 0$. In addition, several bicycles need to be opened, for example bicycles that correspond to bits that switch from 1 to 0. However, according to the informal description given by Friedmann [3, Pages 9–10], both the updates regarding the entry vertices and the updates regarding the bicycles should not be performed for all levels but only those with an index smaller than ℓ'. To be precise, the following is stated (where $r \in \{0, 1\}$ is arbitrary)[2]: "*In the third phase, we perform the major part of the resetting process. By resetting, we mean to unset lower bits again, which corresponds to reopening the respective bicycles. Also, we want to update all other inactive or active but not set bicycles again to move to the entry point $k_{\ell'}$. In other words, we need to update the lower entry points k_z with $z < \ell'$ to move to $k_{\ell'}$, and the bicycle nodes $b^j_{z,l}$ to move to $k_{\ell'}$. We apply these switches by first switching the entry node k_z for some $z < \ell'$ and then the respective bicycle nodes $b^j_{z,r}$.*"

We show that this description is inconsistent with several aspects of [3] and violates the LEAST-ENTERED pivot rule. Beforehand, we extract some estimations contained in the proof of [3, Lemma 3] that will be used later.

Lemma 5.1. *Let σ be a policy calculated by the Policy Iteration Algorithm during the transition from σ_b to σ_{b+1}. Denote the reward of each edge emanating from vertex v by $\langle v \rangle$. Let*

$$S_i \coloneqq \sum_{j \in \{i,\dots,n\}: b_j = 1} \left(\langle k_j \rangle + \langle c_j^0 \rangle + \langle d_j^0 \rangle + \langle h_j^0 \rangle \right) \quad \text{and} \quad T_i \coloneqq \sum_{j \in \{i,\dots,n\}: (b+1)_j = 1} \left(\langle k_j \rangle + \langle c_j^0 \rangle + \langle d_j^0 \rangle + \langle h_j^0 \rangle \right).$$

Then,

$$\begin{aligned}
\mathrm{VAL}_\sigma(s) &\in [S_1, T_1] \\
\mathrm{VAL}_\sigma(k_i) &\in [\langle k_i \rangle + S_1, T_i] \\
\mathrm{VAL}_\sigma(h_i^j) &\in [\langle h_i^j \rangle + \langle k_{i+1} \rangle + S_1, \langle h_i^j \rangle + T_{i+1}] \\
\mathrm{VAL}_\sigma(d_i^j) &\in [\langle d_i^j \rangle + S_1, \langle d_i^j \rangle + \langle h_i^j \rangle + T_{i+1}] \\
\mathrm{VAL}_\sigma(A_i^j) &\in [S_1, \langle d_i^j \rangle + \langle h_i^j \rangle + T_{i+1}] \\
\mathrm{VAL}_\sigma(b_{i,r}^j) &\in [S_1, \langle d_i^j \rangle + \langle h_i^j \rangle + T_{i+1}] \\
\mathrm{VAL}_\sigma(c_i^j) &\in [\langle c_i^j \rangle + S_1, \langle c_i^j \rangle + \langle d_i^j \rangle + \langle h_i^j \rangle + T_{i+1}].
\end{aligned}$$

In this section, we only refer to [3, Table 3] when discussing occurrence records of improving switches since we do not consider the occurrence records of edges $(b_{i,r}^j, A_i^j)$. All discussed results therefore hold independently of the previous findings in Section 4.

We begin by showing an issue regarding the informal description mentioned before.

Issue 5.2. *For every $b \in \{1, \dots, 2^{n-2} - 1\}$, the informal description of phase 3 given in [3, Pages 9–10] contradicts [3, Tables 2&4]. It additionally violates the* LEAST-ENTERED *pivot rule during the transition from σ_b to σ_{b+1} for every $b \in \{3, \dots, 2^{n-2} - 2\}$.*

[2] The notation in the quote was adapted from [3] to be in line with our paper.

Proof of Issue 5.2. Let $b \in \{1, \ldots, 2^{n-2}-1\}$. Consider the transition from σ_b to σ_{b+1}. According to [3, Table 2], for each phase 1 policy or phase 2 policy σ, it should hold that $\sigma(k_i) = k_\ell$ if $b_i = 0$ and $\sigma(k_i) = c_i^j, j = b_{i+1}$ if $b_i = 1$. But, due to $b < 2^{n-2}$, we have $\tilde{b}_n = 0$ for all $\tilde{b} \in \{0, \ldots, b\}$. In particular, $n > \ell(\tilde{b})$ for all of those $\tilde{b}$. Since phase 3 is the only phase in which the target of k_n can be changed, this implies that the target of k_n has never been changed. But for every policy σ considered so far, $\sigma(k_n) = t$ held due to $\sigma^\star(k_n) = \sigma^*(k_n) = t$. Since σ_b is a phase 1 policy by definition, this contradicts [3, Table 2], even if we change the initial policy as discussed in Section 3.2. Note that we obtain$\mathrm{VAL}_{\sigma_b}(k_n) = 0$ for all $b \in \{1, \ldots, 2^{n-2}-1\}$ by the same arguments.

As a consequence, the occurrence records of all edges (k_n, k_i) for $i \in \{1, \ldots, n-1\}$ are zero. We now discuss how this violates [3, Table 4]. Let $i \in \{1, \ldots, \lfloor \log_2(b) \rfloor + 1\}$, i.e., consider some i such that $b \geq 2^{i-1}$. According to [3, Table 4], it should then hold that $\phi^{\sigma_b}(k_n, k_i) = f(b, i, \{(n, 0)\})$. But, due to $\tilde{b}_n = 0$ for all $\tilde{b} \leq b$, we have $f(b, i, \{(n, 0)\}) = f(b, i)$. Thus, by Proposition 2.4 (3) and since $b \geq 2^{i-1}$, we have

$$f(b, i, \{(n, 0)\}) = f(b, i) = \left\lfloor \frac{b + 2^{i-1}}{2^i} \right\rfloor \geq \left\lfloor \frac{2^{i-1} + 2^{i-1}}{2^i} \right\rfloor = 1.$$

This contradicts the occurrence records of all edges (k_n, k_i) for $i \in \{1, \ldots, n-1\}$ being zero.

It remains to show that applying the improving switches as described before contradicts the LEAST-ENTERED rule. We do so by showing that the edge (k_n, k_1) is an improving switch throughout the whole transition from σ_2 to σ_3, and discuss the case of $b \in \{3, \ldots, 2^{n-2}-2\}$ afterwards. By [3, Table 4], $L_\sigma^5 = \{(s, k_{\ell'})\}$ for any phase 5 policy σ. Since only switches contained in the subsets L_σ^p are chosen as improving switches, this implies that (s, k_1) is chosen in phase 5 of the transition from σ_2 to σ_3. But, since $\ell(1) = \ell(3) = 1$, this edge has already been chosen in phase 5 of the transition from σ_0 to σ_1. Therefore, the edge has a non-zero occurrence record throughout the transition from σ_2 to σ_3. Thus, the result follows once we showed that (k_n, k_1) is an improving switch, since we already observed that it has an occurrence record of zero but is not switched.

Consider σ_b for $b = 2$. The only set bit in the binary representation of b is b_2. As observed before, we have $\sigma_2(k_n) = t$, implying $\mathrm{VAL}_{\sigma_2}(\sigma_2(k_n)) = 0$. In addition, by Lemma 5.1, for every policy σ calculated during the transition from σ_2 to σ_3, it holds that

$$\begin{aligned}
\mathrm{VAL}_{\sigma_2}(k_1) &\geq \langle k_1 \rangle + S_1 \\
&= (-N)^{2 \cdot 1 + 7} + S_1 \\
&\geq \sum_{j \in \{1, \ldots, n\} : b_j = 1} \left[(-N)^{2j+7} + (-N)^{2j+8} + (-N)^7 + (-N)^6 \right] - N^9 \\
&= (-N)^{2 \cdot 2 + 7} + (-N)^{2 \cdot 2 + 8} + (-N)^7 + (-N)^6 - N^9 \\
&= N^{12} - N^{11} - N^9 - N^7 - N^6 > 0,
\end{aligned}$$

since $N \geq 8$. Thus, (k_n, k_1) is an improving switch during the whole transition from σ_2 to σ_3.

Since $\mathrm{VAL}_{\sigma_b}(k_n) = 0$ for all $b \in \{3, \ldots, 2^{n-2}-2\}$ as discussed before, since $\ell(b) \neq n$ for all of those b, and since the values are non-decreasing, (k_n, k_1) remains an improving switch for all $b \in \{3, \ldots, 2^{n-2}-2\}$. We further observe that due to $b \geq 3$, both of the bicycles A_1^0 and A_1^1 have been closed at least once, see Lemma 4.4. This implies that all edges of these bicycles have an occurrence of at least one. Also, at least one of the edges of the inactive bicycle of level 1 is switched when transitioning from σ_b to σ_{b+1} for any $b \in \mathbb{B}_n$. Because this edge has a non-zero occurrence record whereas the edge (k_n, k_1) has an occurrence record of zero and is an improving switch, this shows that following the informal description contradicts the LEAST-ENTERED pivot rule at least once during the transition from σ_b to σ_{b+1} for every $b \in \{3, \ldots, 2^{n-1}-2\}$. □

However, in other parts of the construction, Friedmann seems to apply the improving switches differently, by not only applying them for levels with a lower index than the least significant set bit but for all levels. Especially, the side conditions specified in [3, Table 2] for defining a phase p policy rely on the fact that these switches are applied for all levels i with $(b+1)_i = 0$. According to the proof of [3, Lemma 5], the switches need to be applied in the following way[2]: "*In order to* *fulfill all side conditions for phase 3, we need to perform all switches from higher indices to smaller indices, and k_i to $k_{\ell'}$ before $b^j_{i,r}$ with $(b+1)_{i+1} \neq j$ or $(b+1)_i = 0$ to $k_{\ell'}$".*

Before showing that this variant does not obey the LEAST-ENTERED pivot rule either, we state two more lemmas. Recall that for an improving switch e and a policy σ, the policy obtained from σ by applying e is denoted by $\sigma[e]$. The first lemma characterizes the set of all improving switches that should be applied during phase 3. The second lemma shows the connection between L^3_σ and $L^3_{\sigma[e]}$ for a phase 3 policy σ and an improving switch e.

We begin by partitioning the subset L^3_σ of the set of improving switches for a phase 3 policy σ into three sets $L^{3,1}_\sigma, L^{3,2}_\sigma$ and $L^{3,3}_\sigma$. These sets are defined as follows (cf. [3, Table 3]):

- $L^{3,1}_\sigma := \{(k_i, k_{\ell'}) : \sigma(k_i) \neq k_{\ell'} \wedge (b+1)_i = 0\}$
- $L^{3,2}_\sigma := \{(b^j_{i,l}, k_{\ell'}) : \sigma(b^j_{i,l}) \neq k_{\ell'} \wedge (b+1)_i = 0\}$
- $L^{3,3}_\sigma := \{(b^j_{i,l}, k_{\ell'}) : \sigma(b^j_{i,l}) \neq k_{\ell'} \wedge (b+1)_{i+1} \neq j\}$

Note that we use a different notation than Friedmann in order to avoid using the function $\bar{\sigma}$.

Lemma 5.3. *Let $b \in \mathbb{B}_n$ and let σ be the first phase 3 policy of the transition from σ_b to σ_{b+1}. Then $L^3_\sigma = L^3_{\sigma_b}$, and $L^3_{\sigma_b}$ is the set of improving switches that should be applied during phase 3 according to [3, Table 2].*

Proof. This follows nearly immediately from the description given in [3, Page 9]. A more detailed argument can be found in Appendix B. □

Lemma 5.4. *Let σ be a phase 3 policy and let $e \in L^3_\sigma$. Then $L^3_{\sigma[e]} = L^3_\sigma \setminus \{e\}$.*

Proof. We only discuss the case $e \in L^{3,1}_\sigma$ – the cases $e \in L^{3,2}_\sigma$ and $e \in L^{3,3}_\sigma$ follow from similar arguments. Let $e \in L^{3,1}_\sigma$. Then, $e = (k_i, k_{\ell'})$ for some $i \in \{1, \ldots, n\}$ with $\sigma(k_i) \neq k_{\ell'}$ and $(b+1)_i = 0$. Hence the improving switch $(k_i, k_{\ell'})$ can be applied in σ. When the switch e is applied, we have $\sigma[e](k_i) = k_{\ell'}$ for the resulting policy $\sigma[e]$. This immediately implies $e \notin L^{3,1}_{\sigma[e]}$ and thus $e \notin L^3_{\sigma[e]}$.

Let $\tilde{e} \in L^3_\sigma$ and $\tilde{e} \neq e$. We show that $\tilde{e} \in L^3_{\sigma[e]}$. Since $\tilde{e} \in L^3_\sigma$, we have $\tilde{e} = (x, k_{\ell'})$ where either $x = k_{i'}$ or $x = b^j_{i',r}$ for some $i' \in \{1, \ldots, n\}$ and $r, j \in \{0, 1\}$. In addition, since $\tilde{e} \in L^3_\sigma$, we have $\sigma(x) \neq k_{\ell'}$. The switch $(k_i, k_{\ell'})$ is the only switch that we apply when transitioning from σ to $\sigma[e]$. Therefore, $\sigma(x) \neq k_{\ell'}$ implies $\sigma[e](x) \neq k_{\ell'}$ as the target of no vertex other than k_i changes. As furthermore the conditions $(b+1)_i = 0$ and $b_{i+1} \neq j$ remain valid, it follows that $\tilde{e} \in L^3_{\sigma[e]}$. This implies that $L^3_\sigma \subseteq L^3_{\sigma[e]} \cup \{e\}$.

Towards a contradiction, assume that there is some $\tilde{e} \in L^3_{\sigma[e]} \cup \{e\}$ but $\tilde{e} \notin L^3_\sigma$. Then, since $e \in L^3_\sigma$, we have that $e \neq \tilde{e}$. Thus, $\tilde{e} = (x, k_{\ell'})$ for some x as in the last case and $\sigma[e](x) \neq k_{\ell'}$. But since $(k_i, k_{\ell'})$ is the only switch that is applied when transitioning from σ to $\sigma[e]$, this implies that $\sigma(x) \neq k_{\ell'}$. But then, $e \in L^3_\sigma$ which is a contradiction. We therefore have $L^3_{\sigma[e]} \cup \{e\} \subseteq L^3_\sigma$ and thus $L^3_{\sigma[e]} \cup \{e\} = L^3_\sigma$. □

Corollary 5.5. *Let σ be a phase 3 policy and $e \in I_\sigma$ an improving switch for σ. Let σ' be a phase 3 policy reached after σ during the same transition. If the switch e was not applied when transitioning from σ to σ', then e is an improving switch for σ'.*

After having analyzed the set L^3_σ in more detail, we now go back to the application of the improving switches in phase 3. We prove a lemma that implies that applying the improving switches in the order described in [3, Lemma 5] contradicts the LEAST-ENTERED rule. It will also be used to show that a generalized class of orderings violates the LEAST-ENTERED pivot rule.

Lemma 5.6. *Let $i \in \{2, \ldots, n-2\}$ and $l < i$. Then, there is a number $b \in \mathbb{B}_n$ with $\ell(b+1) = l$ such that for all $j \in \{i+2, \ldots, n\}$, it holds that $\phi^{\sigma_b}(k_i, k_{\ell'}) < \phi^{\sigma_b}(k_j, k_{\ell'})$ and $(k_i, k_{\ell'}), (k_j, k_{\ell'}) \in L^3_{\sigma_b}$.*

Proof. Let $b := 2^i + 2^{l-1} - 1$ and $j \in \{i+2, \ldots, n\}$. Then, $\ell(b+1) = \ell(2^i + 2^{l-1}) = l$ since $i < l$. Furthermore, $j \geq i+2, i > l$ and $i \geq 2$ imply $b = 2^i + 2^{l-1} \leq 2^i + 2^{i-2} \leq 2^{j-2} + 2^{j-4} < 2^{j-1} - 1$.

Now consider the set $F(b,l)$ containing all $\tilde{b} \leq b$ such that $\ell(\tilde{b}) = l$, see Definition 2.2. We remind here that, by definition, $|F(b,l)| = f(b,l)$. Because $b < 2^{j-1}$, it holds that $\tilde{b}_j = 0$ for all $\tilde{b} \leq b$ and thus, $F(b,l) = F(b,l,\{(j,0)\})$. Thus, by [3, Table 4] we have

$$\phi^{\sigma_b}(k_j, k_{\ell'}) = \phi^{\sigma_b}(k_j, k_l) = f(b,l,\{(j,0)\}) = f(b,l).$$

In addition, since $b < 2^{j-1} - 1$ implies $b = 1 < 2^{j-1}$ and thus $(b+1)_j = 0$ and $\sigma_b(k_j) = k_\ell \neq k_{\ell'}$ holds due to the invariants discussed in Section 2, we have $(k_j, k_{\ell'}) \in L^3_{\sigma_b}$. However, because of $b > 2^i, i \geq 2$ and $i > l$, it holds that $\tilde{b} := 2^{i-1} + 2^{l-1} \in F(b,l)$ since $\tilde{b} \leq b$. Additionally, we have that $\tilde{b}_i = 1$. As a consequence, $\tilde{b} \notin F(b,l,\{(i,0)\})$). But this implies that $F(b,l,\{(i,0)\}) \subsetneq F(b,l)$. Since, by [3, Table 4], $\phi^{\sigma_b}(k_i, k_l) = f(b,l,\{(i,0)\})$ and, $|F(b,l\{(i,0)\})| = f(b,l,\{(i,0)\})$, this implies

$$\phi^{\sigma_b}(k_i, k_{\ell'}) = \phi^{\sigma_b}(k_i, k_l) = f(b,l,\{(i,0)\}) < f(b,l) = \phi^{\sigma_b}(k_j, k_l) = \phi^{\sigma_b}(k_j, k_{\ell'}).$$

Since $(b+1)_i = b_i = 0$ due to $i > l = \ell(b+1)$ and $\sigma_b(k_i) = k_\ell \neq k_{\ell'}$, we also have $(k_i, k_{\ell'}) \in L^3_{\sigma_b}$. $\square$

Issue 5.7. *Applying the improving switches as described in [3, Lemma 5] does not obey the* LEAST-ENTERED *pivot rule.*

Proof of Issue 5.7. According to [3, Lemma 5], the improving switches of phase 3 should be applied as follows[2]: "*[...] we need to perform all switches from higher indices to smaller indices, and k_i to $k_{\ell'}$ before $b^j_{i,l}$ with $(b+1)_{i+1} \neq j$ or $(b+1)_i = 0$ to $k_{\ell'}$*". This description is also further formalized in the side conditions of [3, Table 2].

Let $i \in \{2, \ldots, n-2\}, l < i$ and $j \in \{i+2, \ldots, n-2\}$. By Lemma 5.6, there is a number $b \in \mathbb{B}_n$ such that $l = \ell(b+1)$ and $\phi^{\sigma_b}(k_i, k_{\ell'}) < \phi^{\sigma_b}(k_j, k_{\ell'})$. In addition, $(k_i, k_{\ell'}), (k_j, k_{\ell'}) \in L^3_{\sigma_b}$. Therefore, by Lemma 5.3, the switch $(k_j, k_{\ell'})$ should be applied before the switch $(k_i, k_{\ell'})$ during the transition from σ_b to σ_{b+1} when following the description of [3].

Consider the phase 3 policy σ of this transition in which the switch $(k_j, k_{\ell'})$ should be applied. Then, since $j > i$ an we "*perform all switches from higher indices to smaller indices*", the switch $(k_i, k_{\ell'})$ was not applied yet. But, by Corollary 5.5, it is an improving switch for the current policy σ. This implies that $\phi^{\sigma_b}(k_j, k_{\ell'}) = \phi^{\sigma}(k_j, k_{\ell'})$ and additionally $\phi^{\sigma_b}(k_i, k_{\ell'}) = \phi^{\sigma}(k_i, k_{\ell'})$. Consequently, $\phi^{\sigma}(k_i, k_{\ell'}) < \phi^{\sigma}(k_j, k_{\ell'})$. Thus, since the edge $(k_i, k_{\ell'})$ is an improving switch for σ having a lower occurrence record than $(k_j, k_{\ell'})$ and σ was chosen as the policy in which $(k_j, k_{\ell'})$ should be applied, the LEAST-ENTERED rule is violated. $\square$

By stating a lemma similar to Lemma 5.6, we can even show a stronger statement. We observe that Friedmann applies the improving switches of phase 3 in the following way: During the transition from σ_b to σ_{b+1}, the improving switches are applied "one level after another" where the order of the levels depends on the least significant set bit of $b+1$, that is, $\ell(b+1)$. Our goal is now to show the following: Consider some $l \in \{1, \ldots, n-4\}$. When the improving switches of phase 3 are applied level by level according to a fixed ordering S^l during all transitions from σ_b to σ_{b+1} for which $\ell(b+1) = l$, the LEAST-ENTERED pivot rule is violated at least once.

To prove our statement we need the following lemma. Note that the occurrence records for the policies σ_b given in [3, Table 4] are independent of the ordering in which the improving switches are applied during phase 3.

Lemma 5.8. *Assume that for any transition, the switches that should be applied during phase 3 were applied in some (possibly changing) order. Let $i \in \{2, \ldots, n-2\}$ and $l < i$. Then there is a number $b \in \mathbb{B}_n$ with $\ell(b+1) = l$ such that $\phi^{\sigma_b}(k_{i+1}, k_{\ell'}) < \phi^{\sigma_b}(b^1_{i,r}, k_{\ell'})$, where $r \in \{0,1\}$ is arbitrary and $(k_{i+1}, k_{\ell'}), (b^1_{i,r}, k_{\ell'}) \in L^3_{\sigma_b}$.*

Proof. Since we assume that the same switches are applied during phase 3, the occurrence records given in [3, Table 4] remain valid. For now, consider some $b \in \mathbb{B}_n$ with $\ell(b+1) = l$. We fix its value later. By [3, Table 4] and since $\ell' = \ell(b+1) = l$,

$$\phi^{\sigma_b}(k_{i+1}, k_{\ell'}) = f(b, \ell', \{(i+1, 0)\})$$

and

$$\phi^{\sigma_b}(b^1_{i,r}, k_{\ell'}) = f(b, \ell', \{(i,0)\}) + f(b, \ell', \{(i,1), (i+1,0)\}).$$

By Proposition 2.4 (3),

$$f(b, \ell', \{(i,0)\}) = f(b, \ell', \{(i,0),(i+1,0)\}) + f(b, \ell', \{(i,0),(i+1,1)\}).$$

This implies that $\phi^{\sigma_b}(b^1_{i,r}, k_{\ell'})$ can be formulated equivalently as

$$f(b, \ell', \{(i,0),(i+1,0)\}) + f(b, \ell', \{(i,0),(i+1,1)\}) + f(b, \ell', \{(i,1),(i+1,0)\}).$$

Since $f(b, \ell', \{(i,1),(i+1,0)\}) + f(b, \ell', \{(i,0),(i+1,0)\}) = f(b, \ell', \{(i+1,0)\})$, the whole inequality can thus be formulated as

$$f(b, \ell', \{(i+1,0)\}) < f(b, \ell', \{(i+1,0)\}) + f(b, \ell', \{(i,0),(i+1,1)\}).$$

It therefore suffices to find some $b \in \mathbb{B}_n$ such that $f(b, \ell', \{(i,0),(i+1,1)\}) > 0, \ell(b+1) = l$ and $(k_{i+1}, k_{\ell'}), (b^1_{i,r}, k_{\ell'}) \in L^3_{\sigma_b}$.

We show that $b := 2^{i+1} + 2^{l-1} - 1$ fulfills this. We observe that $\ell(b+1) = \ell(2^{i+1} + 2^{l-1}) = l$ since $l < i$. In addition, since $b_{i+1} = 0$, it holds that $\sigma_b(k_{i+1}) = k_\ell \neq k_{\ell'}$. Since also $(b+1)_{i+1} = 0$, we therefore have $(k_{i+1}, k_{\ell'}) \in L^3_{\sigma_b}$. Also, since $(b+1)_{i+1} = 0 \neq 1$ and $\sigma_b(b^1_{i,r}) = k_\ell \neq k_{\ell'}$, we additionally have $(b^1_{i,\ell'}, k_{\ell'}) \in L^3_{\sigma_b}$.

Consider the number $\tilde{b} := 2^i + 2^{l-1}$. Then, $\tilde{b}_i = 0$ and $\tilde{b}_{i+1} = 1$. Since $\tilde{b} < b$, this implies $f(b, \ell', \{(i,0),(i+1,1)\}) \geq 1$. □

We now combine Lemmas 5.6 and 5.8 to prove that an entire class of orderings of the improving switches of phase 3, including Friedmann's, all violate the LEAST-ENTERED pivot rule. This class of orderings consists of all orderings such that the improving switches of phase 3 are applied "level by level", where, during the transition from σ_b to σ_{b+1}, the sequence of levels only depends on the least significant set bit of $b+1$. That is, depending on $\ell(b+1)$, an ordering $S^{\ell(b+1)}$ of the levels 1 to n is considered and when a level i_1 appears before a level i_2 within $S^{\ell(b+1)}$, all switches in level i_1 need to be applied before the improving switches of level i_2 are applied. In some sense, this shows that Friedmann's ordering needs to be changed fundamentally, and cannot be fixed by slight adaptation.

Issue 5.9. *Suppose that the improving switches of phase 3 are applied one level after another as described above. That is, the ordering of the levels in the transition from σ_b to σ_{b+1} may only depend on $\ell(b+1)$. Then, the* LEAST-ENTERED *pivot rule is violated.*

Proof. To prove Issue 5.9, we show that applying the improving switches as discussed before violates the LEAST-ENTERED rule several times by showing the following statement: Let S^i be an ordering of $\{1,\ldots,n\}$ for $i \in \{1,\ldots,n\}$. Suppose that the improving switches of phase 3 of the transition from σ_b to σ_{b+1} are applied in the order defined by $S^{\ell(b+1)}$ for all $b \in \mathbb{B}_n$. Then, for every possible least significant bit $l \in \{1,\ldots,n-4\}$, assuming that the ordering S^l obeys the LEAST-ENTERED rule results in a contradiction.

We first observe that Lemma 5.6 also holds when the improving switches are applied in some arbitrary order since we always consider the occurrence record with respect to σ_b.

Fix some $l \in \{1,\ldots,n-4\}$. Consider the ordering $S^l = (s_1,\ldots,s_n)$. For $k \in \{1,\ldots,n\}$, we denote the position of k within S^l by $k^\star$, i.e., $k^\star$ is defined such that $s_{k^\star} = k$. Assume that applying the improving switches level by level according to the ordering S^l obeys the LEAST-ENTERED rule. We show that this assumption yields both $(l+1)^\star < (n-1)^\star$ and $(n-1)^\star < (l+1)^\star$ which clearly is a contradiction.

Let $i \in \{l+1,\ldots,n-2\}$. Then, $i > l$ and therefore, by Lemma 5.8, there is a number $b \in \mathbb{B}_n$ with $\ell(b+1) = \ell' = l$ and $\phi^{\sigma_b}(k_{i+1}, k_{\ell'}) < \phi^{\sigma_b}(b^1_{i,r}, k_{\ell'})$ such that $(k_{i+1}, k_{\ell'}), (b^0_{i,r}, k_{\ell'}) \in L^3_{\sigma_b}$. Therefore, by Lemma 5.3, both switches need to be applied during the transition from σ_b to σ_{b+1}. Because of $\phi^{\sigma_b}(k_{i+1}, k_{\ell'}) < \phi^{\sigma_b}(b^1_{i,r}, k_{\ell'})$, level $i+1$ needs to appear before level i within the ordering S^l. Since Lemma 5.8 can be applied for all $i \in \{l+1,\ldots,n-2\}$, this implies that the sequence $(n-1, n-2, \ldots, l+1)$ needs to be a (not necessarily consecutive) subsequence of S^l. In particular, $(n-1)^\star < (l+1)^\star$ since we have $l+1 \neq n-1$ by assumption.

Now, let $i = l+1$ and $j \in \{i+2,\ldots,n\}$. Then, by Lemma 5.6, there is a number $b \in \mathbb{B}_n$ with $\ell(b+1) = l$ such that $\phi^{\sigma_b}(k_i, k_{\ell'}) < \phi^{\sigma_b}(k_{i+2}, k_{\ell'})$ and $(k_i, k_{\ell'}), (k_{i+2}, k_{\ell'}) \in L^3_{\sigma_b}$. Again, by

Lemma 5.3, both switches need to be applied during the transition from σ_b to σ_{b+1}. Therefore, for all $i \in \{l+1, \dots, n-2\}$, level i needs to appear before any of the levels level $j \in \{i+2, \dots, n\}$ within S^l. But this implies that the sequence $(l+1, l+3, l+4, \dots, n-1, n)$ needs to be a (not necessarily consecutive) subsequence of S^l. In particular, $(l+1)^\star < (n-1)^\star$ since $n-1 \geq l+3$ as we have $l \leq n-4$ by assumption. This however contradicts $(n-1)^\star < (l+1)^\star$.

Therefore, applying the improving switches level by level according to the ordering S^l does not obey the LEAST-ENTERED rule. $\square$

5.2 Fixing the ordering of the improving switches

In this section we prove the existence of an ordering and an associated tie-breaking rule for the application of the switches of phase 3 that obey the LEAST-ENTERED rule. We begin by giving a brief outline of this section.

Let σ be a phase 3 policy. We compare L^3_σ and U^3_σ since all improving switches that can possibly be applied during phase 3 are contained in U^3_σ (by [3, Lemma 4]). This is done via partitioning U^3_σ and considering the partition of L^3_σ used before. The comparison enables us to show that there is always a switch contained in L^3_σ minimizing the occurrence record. This justifies that "*we will only use switches from L^p_σ*" [3, Page 12] (at least for phase $p = 3$). We use structural results to show the following: All improving switches that should be applied during phase 3 according to the description in [3] can be applied (in a different order) during phase 3, without violating the LEAST-ENTERED pivot rule.

As outlined in Section 2, the transition from σ_b to σ_{b+1} is partitioned into six phases. During the third phase, the MDP is reset, that is, some bicycles are opened and the targets of some entry vertices are changed. Therefore, a phase 3 policy σ is always associated with such a transition and we implicitly consider the underlying transition from σ_b to σ_{b+1} for the corresponding $b \in \mathbb{B}_n$ when discussing a fixed phase 3 policy.

Now, fix some $b \in \{0, \dots, 2^n - 2\}$. For an edge $e = (v, w)$, we say that the edge *belongs to level* i when vertex v is part of level i of the lower bound construction.

We begin by further investigating the occurrence records of switches that should be applied during phase 3, i.e., we analyze the set L^3_σ for a phase 3 policy σ. We first show an upper bound on the occurrence record of these switches.

Lemma 5.10. *Let σ be a phase 3 policy. Then* $\max_{e \in L^3_\sigma} \phi^\sigma(e) \leq f(b, \ell')$.

Proof. As discussed in Section 5.1, the set L^3_σ can be partitioned into three subsets $L^{3,1}_\sigma, L^{3,2}_\sigma$ and $L^{3,3}_\sigma$. It thus suffices to distinguish three cases. The last two cases can be discussed together as the occurrence records of edges contained in $L^{3,2}_\sigma$ and $L^{3,3}_\sigma$ are the same, see [3, Table 4].

Case 1: $e \in L^{3,1}_\sigma$. Then, $e = (k_i, k_{\ell'})$, where $\sigma(k_i) \neq k'_\ell$ and $(b+1)_i = 0$ holds. The first of these conditions implies that the switch e was not applied yet during the transition from σ_b to σ_{b+1}. We therefore have $\phi^\sigma(k_i, k_{\ell'}) = \phi^{\sigma_b}(k_i, k_{\ell'})$. Since $\phi^{\sigma_b}(e) = f(b, \ell', \{(i,0)\})$ by [3, Table 4], this implies $\phi^\sigma(e) = f(b, \ell', \{(i,0)\})$. By Proposition 2.4 (3), we therefore have

$$\phi^\sigma(e) = f(b, \ell', \{(i,0)\}) = f(b, \ell') - f(b, \ell', \{i, 1\}) \leq f(b, \ell').$$

Case 2: $e \in L^{3,2}_\sigma$ or $e \in L^{3,3}_\sigma$. Then, $e = (b^j_{i,r}, k_{\ell'})$, where $\sigma(b^j_{i,r}) \neq k_{\ell'}$ and either $(b+1)_i = 0$ if $e \in L^{3,2}_\sigma$ or $(b+1)_{i+1} \neq j$ if $e \in L^{3,3}_\sigma$. The first condition implies that the switch e was not applied yet during the transition from σ_b to σ_{b+1}. We thus have $\phi^\sigma(b^j_{i,r}, k_{\ell'}) = \phi^{\sigma_b}(b^j_{i,r}, k_{\ell'})$. Since $\phi^{\sigma_b}(e) = f(b, \ell', \{(i,0)\}) + f(b, \ell', \{(i,1), (i+1, 1-j)\})$ by [3, Table 4], this implies

$$\phi^\sigma(e) = f(b, \ell', \{(i,0)\}) + f(b, \ell', \{(i,1), (i+1, 1-j)\}).$$

By Proposition 2.4 (2), it also holds that $f(b, \ell', \{(i,1), (i+1, 1-j)\}) \leq f(b, \ell', \{(i,1)\})$. Thus, we obtain

$$\phi^\sigma(e) \leq f(b, \ell', \{(i,0)\} + f(b, \ell', \{(i,1)\}) = f(b, \ell').$$ $\square$

Before showing that for all phase 3 policies σ, there is always an improving switch contained in L^3_σ that minimizes the occurrence record, we further discuss the superset U^3_σ. We observe that L^6_σ is contained in this set. Therefore, when analyzing U^3_σ, we need to analyze this set as well. However, there is a small error in the definition of this set that needs to be corrected.

Issue 5.11. *For every $b \in \mathbb{B}_n$ with $\ell(b+1) > 1$, there is an improving switch that should be applied in phase 6 of the transition from σ_b to σ_{b+1} but is not contained in the set L^6_σ for any phase 6 policy σ.*

Proof. Fix some $b \in \mathbb{B}_n$ such that $\ell' = \ell(b+1) > 1$. Consider the vertex $d^0_{\ell'-1}$. We show that the switch $(d^0_{\ell'-1}, s)$ needs to be applied during phase 6 of the transition from σ_b to σ_{b+1} but is not contained in L^6_σ for any phase 6 policy σ. By analyzing [3, Table 2] and the function $\bar{\sigma}$ that is used in this table, it can be shown that $b_\ell = 0$ implies $\sigma_b(d^0_{\ell'-1}) = h^0_i$. Since the ℓ'-th bit switches during the transition from σ_b to σ_{b+1}, by [3, Table 2], $\sigma_{b+1}(d^0_{\ell'-1}) = s$ needs to hold. Therefore, $(d^0_{\ell'-1}, s)$ needs to be an improving switch for some policy σ calculated during the transition from σ_b to σ_{b+1}.

Towards a contradiction, assume that there was a policy σ in which the switch $(d^0_{\ell'-1}, s)$ should be applied. Since the subsets of phase 6 policies are the only subsets that can contain this switch, σ needs to be a phase 6 policy. By [3, Lemma 4], $(d^0_{\ell'-1}, s) \in L^6_\sigma$ then holds for this policy σ. Again analyzing the function $\bar{\sigma}$, it can be shown that due to $(d^0_{\ell'-1}, s) \in L^6_\sigma$, both $\sigma(d^0_{\ell'-1}) \neq s$ and $\sigma(d^0_{\ell'-1}) = s$ need to hold. This is clearly a contradiction. As a consequence, there is no policy σ for which the switch $(d^0_{\ell'-1}, s)$ should be applied. $\square$

Issue 5.11 does not only hold for the switch $(d^0_{\ell'-1}, s)$ but in fact for all switches contained in L^6_σ for any phase 6 policy σ. This can be proven analogously to Issue 5.11 but can however be resolved easily. To be precise, L^6_σ should be defined as follows when the notation using the function $\bar{\sigma}$ is not used.

Theorem 5.12. *For any phase 6 policy σ, the subset of improving switches contained in [3, Table 3] needs to be*

$$\begin{aligned}\bar{L}^6_\sigma \coloneqq &\left\{(d^0_i, x) : \sigma(d^0_i) \neq x \wedge \sigma(d^0_i) = \begin{cases} h^0_i, & (b+1)_{i+1} = 1 \\ s, & (b+1)_{i+1} = 0 \end{cases}\right\} \cup \\ &\left\{(d^1_i, x) : \sigma(d^1_i) \neq x \wedge \sigma(d^1_i) = \begin{cases} s, & (b+1)_{i+1} = 1 \\ h^0_i, & (b+1)_{i+1} = 0 \end{cases}\right\}.\end{aligned}$$

Proof. Let σ be a phase 6 policy. We show that when we assume that the switch $(d^0_{\ell'-1}, s)$ was not applied yet, it holds that $(d^0_{\ell'-1}, s) \in \bar{L}^6_\sigma$. For all other edges contained in $\bar{L}^6_\sigma$, the statement can be shown in a similar way.

As discussed when proving Issue 5.11, $\sigma_b(d^0_{\ell'-1}) = h^0_i$ holds and $\sigma_{b+1}(d^0_{\ell'-1}) = s$ needs to hold. Since $(d^0_{\ell'-1}, s)$ was not applied yet by assumption, $\sigma(d^0_{\ell'-1}) = \sigma_b(d^0_{\ell-1'}) = h^0_i$. In particular, it holds that $\sigma(d^0_{\ell'-1}) \neq s$. But, by the definition of ℓ', we have$(b+1)_{\ell'-1+1} = (b+1)_{\ell'} = 1$. Therefore, $e \in \bar{L}^6_\sigma$. $\square$

Henceforth, we always implicitly consider $\bar{L}^6_\sigma$ as defined in Theorem 5.12 when referring to L^6_σ, that is, for any phase 6 policy σ, we redefine $L^6_\sigma \coloneqq \bar{L}^6_\sigma$.

Let $\nu^n_{i+2}(b) \coloneqq \min(\{n+1\} \cup \{j \in \{i+2, \dots, n\} : b_j = 1\}$. This term represents the next bit equal to 1 with an index of at least $i+2$. When there is no such index, $\nu^n_{i+2}(b)$ is equal to $n+1$.

As we need to analyze phase 3 in detail, we partition the superset U^3_σ contained in [3, Table 3] for a phase 3 policy σ as follows, see [3, Table 3].

$$\begin{aligned}
U^{3,1}_\sigma &\coloneqq \{(k_i, k_z) : \sigma(k_i) \notin \{k_z, k_{\ell'}\}, z \leq \ell' \wedge (b+1)_i = 0\} \\
U^{3,2}_\sigma &\coloneqq \{(b^j_{i,r}, k_z) : \sigma(b^j_{i,r}) \notin \{k_z, k_{\ell'}\}, z \leq \ell' \wedge (b+1)_i = 0\}\} \\
U^{3,3}_\sigma &\coloneqq \{(b^j_{i,r}, k_z) : \sigma(b^j_{i,r}) \notin \{k_z, k_{\ell'}\}, z \leq \ell' \wedge (b+1)_{i+1} \neq j\} \\
U^{3,4}_\sigma &\coloneqq \{(h^0_i, k_l) : l \leq \nu^n_{i+2}(b+1)\} \\
U^{3,5}_\sigma &\coloneqq \{(s, k_i) : \sigma(s) \neq k_i \wedge i < \ell'\} \\
U^{3,6}_\sigma &\coloneqq \{(d^j_i, x) : \sigma(d^j_i) \neq x \wedge i < \ell'\}
\end{aligned}$$

$$U_\sigma^{3,7} := \left\{(d_i^0, x) : \sigma(d_i^0) \neq x \wedge \sigma(d_i^0) = \begin{cases} h_i^0, & (b+1)_{i+1} = 1 \\ s, & (b+1)_{i+1} = 0 \end{cases}\right\}$$

$$U_\sigma^{3,8} := \left\{(d_i^1, x) : \sigma(d_i^1) \neq x \wedge \sigma(d_i^1) = \begin{cases} s, & (b+1)_{i+1} = 1 \\ h_i^0, & (b+1)_{i+1} = 0 \end{cases}\right\}$$

$$U_\sigma^{3,9} := \{(b_{i,l}^j, A_i^j) : \sigma(b_{i,l}^j) \neq A_i^j\}$$

Our goal is to show that whenever we have a phase 3 policy σ, we can apply a switch contained in L_σ^3 while obeying the LEAST-ENTERED rule. In Lemma 5.10, we showed an upper bound of $f(b, \ell')$ on the occurrence records of switches contained in L_σ^3 for a phase 3 policy σ. The next lemma now gives a lower bound of $f(b, \ell')$ on all switches that should be applied after phase 3. It can also be used to estimate the occurrence records of possible improving switches contained in U_σ^3. Thus, combining these two lemmas will enable us to show that the switches contained in U_σ^3 do not prevent us from applying switches in L_σ^3 due to their occurrence record.

Lemma 5.13. *Let σ be a phase 3 policy. Assume that the Policy Iteration Algorithm is started with the policy σ^* introduced in Definition 3.4. Then $\min_{e \in L_\sigma^4 \cup L_\sigma^5 \cup L_\sigma^6} \phi^\sigma(e) \geq f(b, \ell')$.*

Proof. The policy σ is calculated after the policy σ_b. Thus, $\phi^\sigma(e) \geq \phi^{\sigma_b}(e)$ holds for all edges e. It therefore suffices to show $\phi^{\sigma_b}(e) \geq f(b, \ell')$ for all $e \in L_\sigma^4 \cup L_\sigma^5 \cup L_\sigma^6$. Note that the conditions that we give here are not exactly the same as those given in [3], since we omit the additional notation $\bar{\sigma}$. They are, however, equivalent. We distinguish three cases.

Case 1: $\boldsymbol{e \in L_\sigma^4}$**.** Then, by [3, Table 3], it holds that $e = (h_i^0, k_{\nu_{i+2}^n(b+1)})$ for some $i \in \{1, \ldots, n\}$ and $\sigma(h_i^0) \notin \{k_{\nu_{i+2}^n(b+1)}, t\}$. Since $\sigma(h_i^0) \neq t$ and by the way the improving switches are applied, there needs to be a next bit equal to 1 with an index of at least $i + 2$.

Since ℓ' is the least significant bit of $b + 1$, we have $b_j = (b+1)_j$ for all $j \in \{\ell' + 1, \ldots, n\}$. Therefore, the bit equal to 1 with an index of at least $i + 2$ does not change if $i \geq \ell' - 1$. More formally, $\nu_{j+2}^n(b) = \nu_{j+2}^n(b+1)$ holds for all $j \in \{\ell' - 1, \ldots, n-2\}$. Thus, $i \leq \ell' - 2$ needs to hold since otherwise, $\sigma(h_i^0) = k_{\nu_{i+2}^n(b+1)}$, contradicting that $(h_i^0, k_{\nu_{i+2}^n(b+1)})$ is an improving switch. As also $(b+1)_j = 0$ for $j < \ell'$, it follows that $\nu_{j+2}^n(b+1) = \ell'$ for all $j \in \{1, \ldots, \ell' - 2\}$. Thus, $e = (h_i^0, k_{\ell'})$ for some $i \in \{1, \ldots, \ell' - 2\}$, and, by [3, Table 4],

$$\phi^{\sigma_b}(e) = \phi^{\sigma_b}(h_i^0, k_{\ell'}) = f(b, \ell').$$

Case 2: $\boldsymbol{e \in L_\sigma^5}$**.** Then, by [3, Table 3] and since $L_\sigma^5 = \{(s, k_{\ell'})\}$, we have $e = (s, k_{\ell'})$. Therefore, because $\phi^{\sigma_b}(s, k_{\ell'}) = f(b, \ell')$ by [3, Table 4] it holds that

$$\phi^{\sigma_b}(e) = \phi^{\sigma_b}(s, k_{\ell'}) = f(b, \ell').$$

Case 3: $\boldsymbol{e \in L_\sigma^6}$**.** By Theorem 5.12, it holds that

$$L_{\sigma_b}^6 = \{(d_{\ell'-1}^1, h_{\ell'-1}^1), (d_{\ell'-1}^0, s)\} \cup \{(d_i^0, h_i^0), (d_i^1, s) : i \in \{1, \ldots, \ell' - 2\}\}.$$

Since $L_\sigma^6 \subseteq L_{\sigma_b}^6$ can be obtained by a result similar to Lemma 5.4 it suffices to show the inequality for all $e \in L_{\sigma_b}^6$.

First, let $e = (d_{\ell'-1}^0, s)$. Then, by [3, Table 4],

$$\phi^{\sigma_b}(d_{\ell'-1}^0, s) = f(b, (\ell' - 1) + 1) + j \cdot b_{i+1} = f(b, \ell' - 1 + 1) - 0 \cdot b_{i+1} = f(b, \ell').$$

Analogously, for $e = (d_{\ell'-1}^1, h_{\ell'-1}^1)$,

$$\phi^{\sigma_b}(d_{\ell'-1}^1, h_{\ell'-1}^1) = f(b, (\ell' - 1) + 1) + (1 - j) \cdot b_{i+1} = f(b, \ell') - 0 \cdot b_{i+1} = f(b, \ell').$$

Therefore, $\phi^{\sigma_b}(e) \geq f(b, \ell')$ holds for $e \in \{(d_{\ell'-1}^0, s), (d_{\ell'-1}^1, h_{\ell'-1}^1)\}$.

Let $e = (d_i^1, s)$ for some $i \in \{1, \ldots, \ell' - 2\}$. Then, e is an improving switch if and only if the $(i+1)$-th bit switches from 1 to 0. We observe that the first transition in which (d_i^1, s) is an improving switch is therefore the transition from $\sigma_{2^{i+1}-1}$ to $\sigma_{2^{i+1}}$. As the Policy Iteration Algorithm is initialized with the policy representing the number 0, the number $b \in \mathbb{B}_n$ is represented after b many transitions. Therefore, e is an improving switch every 2^{i+1}-th transition as the $i+1$ least significant bits are all equal to 0 again once the number $b = 2^{i+1}$ is reached.

We now interpret $\phi^{\sigma_b}(e)$ as a "counter", which increases during the application of the Policy Iteration Algorithm. By what we just discussed, this counter increases every 2^{i+1} transitions and is initialized with zero. In contrast to this, the "counter" $f(b, \ell')$ increases the first time when the number $2^{\ell'-1}$ is reached. But then, after another $2^{\ell'-1}$ transitions the number $2^{\ell'}$ is reached and we have $\ell(2^{\ell'}) = \ell' + 1$. Therefore, it takes another $2^{\ell'-1}$ transitions until the counter $f(b, \ell')$ increases another time. In short, the counter $f(b, \ell')$ increases every $2^{\ell'}$ iterations, excluding the first increase which is reached after $2^{\ell'-1}$ iterations. Since $i+1 \le \ell'-1$ follows immediately from $i \le \ell'-2$, this shows that whenever the counter $f(b, \ell')$ is increased, the counter $\phi^{\sigma_b}(e)$ must have been increased at least once before or in the same iteration. Therefore, $\phi^{\sigma_b}(e) \ge f(b, \ell')$.

The statement follows for $e = (d_i^0, h_i^0)$ by the same arguments in the following way. The switch (d_i^1, s) is applied whenever the $(i+1)$-th bit is no longer equal to 1. The switch (d_i^0, h_i^0) is applied whenever the $(i+1)$-th bit becomes 0. Both of these happen whenever the $(i+1)$-th bit switches from 1 to 0 and thus, the same arguments used before can be applied. □

This lemma can now be used to show that the occurrence records of edges contained in the sets $U_\sigma^{3,4}$ to $U_\sigma^{3,9}$ are too large and that no improving switch contained in one of these sets will be applied for any phase 3 policy when following the LEAST-ENTERED rule.

Lemma 5.14. *Let σ be a phase 3 policy. For all $e \in L_\sigma^3$ and $\tilde{e} \in I_\sigma \cap \left(U_\sigma^{3,4} \cup \dots \cup U_\sigma^{3,9}\right)$, it holds that $\phi^\sigma(e) \le \phi^\sigma(\tilde{e})$.*

Proof. Let σ be a phase 3 policy and let $e \in L_\sigma^3$. Then, $\phi^\sigma(e) \le f(b, \ell')$ by Lemma 5.10. It thus suffices to show $\phi^\sigma(\tilde{e}) \ge f(b, \ell')$ for all $\tilde{e} \in I_\sigma \cap \left(U_\sigma^{3,4} \cup \dots \cup U_\sigma^{3,9}\right)$. We distinguish in which of the sets $U_{3,k}$ the switch $\tilde{e}$ is contained.

Case 1: $\tilde{e} \in U_\sigma^{3,4}$. Then $\tilde{e} = (h_i^0, k_l)$ for some $l \le \nu_{i+2}^n(b+1)$, where $\nu_{i+2}^n(b+1)$ again denotes the first bit equal to 1 with an index of at least $i+2$. If there is no such bit, $\nu_{i+2}^n(b+1)$ is equal to $n+1$. By [3, Table 4], we have $\phi^{\sigma_b}(\tilde{e}) = f(b, l)$ and since σ is reached after σ_b, we also have $\phi^\sigma(\tilde{e}) \ge \phi^{\sigma_b}(\tilde{e}) = f(b, l)$. First, assume that $l \le \ell'$. Then, by Proposition 2.4 (4), it holds that $f(b, l) \ge f(b, \ell')$, implying $\phi^\sigma(\tilde{e}) \ge f(b, \ell')$.

Now assume that $l > \ell'$. We show that this results in a contradiction. To be precise, we show that (h_i^0, k_l) is not an improving switch in this case, i.e., we show $\mathrm{VAL}_\sigma(\sigma(h_i^0)) \ge \mathrm{VAL}_\sigma(k_l)$. To simplify the notation, let $\nu := \nu_{i+2}^n(b+1)$.

First observe that $\sigma(h_i^0) \in \{t, k_{i+2}, \dots, k_n\}$, see Figure 1. Therefore $\nu \ne n+1$ needs to hold since the edge (h_i^0, k_{n+1}) does not exist. In addition, by the definition of ν and the invariants discussed in Section 2.1, $\sigma(h_i^0) = k_\nu$. We thus need to show $\mathrm{VAL}_\sigma(k_\nu) \ge \mathrm{VAL}_\sigma(k_l)$.

Since $l > \ell'$ by assumption and $\nu \ge l$ by the choice of $\tilde{e}$, also $\nu > \ell'$. Therefore, since ℓ' is the least significant set bit of $b+1$, we have $b_j = (b+1)_j$ for all $j \in \{\nu, \dots, n\}$. This implies that during phase 1, no bicycle of one of these levels was opened and the target of none of the vertices $k_\nu, \dots, k_n$ was changed during phase 2. Therefore, using the notation introduced in Lemma 5.1, it holds that $\mathrm{VAL}_\sigma(k_\nu) = S_\nu$. By the same lemma, we also get $\mathrm{VAL}_\sigma(k_l) \le T_l$. Thus, using $b_j = (b+1)_j$ for all $j > \ell'$ and $l > \ell'$, we obtain,

$$\begin{aligned}\mathrm{VAL}_\sigma(k_l) \le T_l &= \sum_{j \in \{l, \dots, n\} : (b+1)_j = 1} [N^{2j+8} - N^{2j+7} - N^7 + N^6] \\ &= \sum_{j \in \{l, \dots, n\} : b_j = 1} [N^{2j+8} - N^{2j+7} - N^7 + N^6].\end{aligned}$$

By definition, ν is the smallest index larger than or equal to $i+2$ such that the corresponding bit of $b+1$ is equal to 1. Since $\sigma(h_i^0) \in \{t, k_{i+2}, \dots, k_n\}$, also $l \ge i+2$ needs to hold, see

Figure 1. Therefore, since $l \leq \nu$, this implies that $b_l = b_{l+1} = \cdots = b_{\nu-1} = 0$ and using the previous inequality we obtain

$$\begin{aligned}
\mathrm{VAL}_\sigma(k_l) &\leq \sum_{j\in\{l,\dots,n\}:b_j=1} [N^{2j+8} - N^{2j+7} - N^7 + N^6] \\
&= \sum_{j\in\{\nu,\dots,n\}:b_j=1} [N^{2j+8} - N^{2j+7} - N^7 + N^6] \\
&= S_\nu \\
&= \mathrm{VAL}_\sigma(k_\nu).
\end{aligned}$$

This shows that the edge $\tilde{e}$ is not an improving switch. Thus, $l > \ell'$ implies $\tilde{e} \notin I_\sigma$. Therefore, $\phi^\sigma(\tilde{e}) \geq f(b,\ell')$ holds for all $\tilde{e} \in I_\sigma \cap U^{3,4}$.

<u>Case 2:</u> $\tilde{e} \in U^{3,5}_{\sigma}$. Then $\tilde{e} = (s, k_i)$ for some $i < \ell'$ and $\sigma(s) \neq k_i$. Therefore, by [3, Table 4], we have that $\phi^{\sigma_b}(s,k_i) = f(b,i)$. Since σ is reached after σ_b, we also have $\phi^\sigma(s,k_i) \geq \phi^{\sigma_b}(s,k_i)$. Since, by assumption, $i < \ell'$ and by Proposition 2.4 (4), this implies

$$\phi^\sigma(s,k_i) \geq \phi^{\sigma_b}(s,k_i) = f(b,i) \geq f(b,\ell').$$

<u>Case 3:</u> $\tilde{e} \in U^{3,6}_{\sigma}$. Then $\tilde{e} = (d_i^j, x)$ for $x \in \{s, h_i^j\}$ where $i \in \{1,\dots,n\}, j \in \{0,1\}, \sigma(d_i^j) \neq x$ and $i < \ell'$. First, assume that $x = s$. Then $\sigma(d_i^j) \neq x = s$, implying $\sigma(d_i^j) = h_i^j$. Since $i < \ell'$, it holds that $b_{i+1} = 1$ for $i \neq \ell' - 1$ and $b_{i+1} = 0$ for $i = \ell' - 1$. In addition, the target vertex of d_i^j can only be changed during phase 6 of the current transition and was thus not changed yet. This implies that we either have $d_i^j = d_i^1$ and $\sigma(d_i^1) = h_i^1$ for some $i < \ell', i \neq \ell' - 1$ or $d_i^j = d_{\ell'-1}^0$ and $\sigma(d_i^j) = \sigma(d_{\ell'-1}^0) = h_{\ell'-1}^0$ if $i = \ell' - 1$. For these switches we however already showed in the proof of Lemma 5.13 that $\phi^{\sigma_b}(\tilde{e}) \geq f(b,\ell')$ holds and thus, also $\phi^\sigma(\tilde{e}) \geq f(b,\ell')$ holds.

Now assume that $x = h_i^j$, that is, $\tilde{e} = (d_i^j, h_i^j)$. Analogously to the case $x = s$ it can then be shown that we either have $h_i^j = h_i^0$ and $\sigma(d_i^0) = h_i^0$ for $i < \ell' - 1$ or $h_i^j = h_i^1$ and $\sigma(d_i^1) = h_i^1$ for $i = \ell' - 1$. Again, these edges have already been investigated in the proof of Lemma 5.13 and the inequality $\phi^\sigma(\tilde{e}) \geq f(b,\ell')$ was shown there.

<u>Case 4:</u> $\tilde{e} \in U^{3,7}_{\sigma}$ **or** $\tilde{e} \in U^{3,8}_{\sigma}$. Since $U^{3,7}_\sigma, U^{3,8}_\sigma \subseteq L^6_\sigma$ and $\phi^\sigma(\tilde{e}) \geq f(b,\ell')$ holds for all $\tilde{e} \in L^6_\sigma$ by Lemma 5.13, the statement follows immediately.

<u>Case 5:</u> $\tilde{e} \in U^{3,9}_{\sigma}$. The set $U^{3,9}_\sigma$ contains edges that are improving switches since phase 1. We thus refer to Section 4 and the description of the application of these edges. We need to investigate the occurrence record of switches that we could have applied during phase 1 but did not apply. By the rules I to V and Theorem 4.2, we only switched one instead of two edges within a bicycle A_i^j when $\phi^{\sigma_b}(A_i^j) = b$ held at the beginning of phase 1. Since we always chose to switch the edge with the lower occurrence record in a bicycle and their occurrence records differ at most by one by Equation (4.3), this implies that for any $\tilde{e} = (b_{i,l}^j, A_i^j) \in U^{3,9}_\sigma$ with $\sigma(b_{i,l}^j) \neq A_i^j$ the equality $\phi^{\sigma_b}(b_{i,l}^j, A_i^j) = \lceil \frac{b}{2} \rceil = \lfloor \frac{b+1}{2} \rfloor$ needs to hold. Since $\lfloor \frac{b+1}{2} \rfloor = f(b,1)$ holds by Proposition 2.4 (3), $\ell' \geq 1$, and Proposition 2.4 (4), we obtain

$$\phi^\sigma(\tilde{e}) \geq \phi^{\sigma_b}(\tilde{e}) = f(b,1) \geq f(b,\ell'). \qquad \square$$

We now show that applying certain improving switches prevents other switches from being applied. To do so, we first introduce subsets of $U^{3,1}_\sigma, U^{3,2}_\sigma$ and $U^{3,3}_\sigma$. The intuitive idea behind introducing these subsets is to "slice" these sets such that for each such slice, there is an improving switch that prevents the whole slice from being applied.

Definition 5.15 (Slices). Let σ be a phase 3 policy and $i \in \{1,\dots,n\}$. Then

$$S^{3,1}_{i,\sigma} := \{(k_i, k_z) : \sigma(k_i) \notin \{k_z, k_{\ell'}\}, z \leq \ell' \wedge (b+1)_i = 0\}$$

is called *slice* of $U_\sigma^{3,1}$. For $i \in \{1, \dots, n\}$ and $j, l \in \{0, 1\}$,

$$S_{i,j,r,\sigma}^{3,2} := \{(b_{i,r}^j, k_z) : \sigma(k_i) \notin \{k_z, k_{\ell'}\}, z \leq \ell' \wedge (b+1)_i = 0\}$$

is called *slice* of $U_\sigma^{3,2}$. For $i \in \{1, \dots, n\}$ and $j, l \in \{0, 1\}$,

$$S_{i,j,r,\sigma}^{3,3} := \{(b_{i,r}^j, k_z) : \sigma(k_i) \notin \{k_z, k_{\ell'}\}, z \leq \ell' \wedge (b+1)_i \neq j\}$$

is called *slice* of $U_\sigma^{3,3}$.

Obviously, for a fixed phase 3 policy σ and each of $U_\sigma^{3,1}, U_\sigma^{3,2}$ and $U_\sigma^{3,3}$, the set of all slices as specified in Definition 5.15 partitions the corresponding set.

We now show that the switches contained in L_σ^3 prevent the improving switches contained in certain slices from being applied.

Lemma 5.16. *The following statements hold.*

1. *Let σ be the phase 3 policy in which the improving switch $(k_i, k_{\ell'})$ is applied. Let σ' be an arbitrary phase 3 policy of the same transition reached after the policy σ. Then $I_{\sigma'} \cap S_{i,\sigma'}^{3,1} = \emptyset$.*
2. *Let σ be the phase 3 policy in which the improving switch $(b_{i,l}^j, k_{\ell'})$ with $\sigma(b_{i,l}^j) \neq k_{\ell'}$ and $(b+1)_i = 0$ is applied. Let σ' be an arbitrary phase 3 policy of the same transition reached after the policy σ. Then $I_{\sigma'} \cap S_{i,j,l,\sigma'}^{3,2} = \emptyset$.*
3. *Let σ be the phase 3 policy in which the improving switch $(b_{i,l}^j, k_{\ell'})$ with $\sigma(b_{i,l}^j) \neq k_{\ell'}$ and $(b+1)_{i+1} \neq j$ is applied. Let σ' be an arbitrary phase 3 policy of the same transition reached after the policy σ. Then $I_{\sigma'} \cap S_{i,j,l,\sigma'}^{3,3} = \emptyset$.*

Proof. We show the first statement in detail and only sketch the proof of the other two statements since all of them use the same arguments.

1. Let σ' be an arbitrary phase 3 policy reached after σ. Let $\tilde{e} \in S_{i,\sigma'}^{3,1}$. We show that $\tilde{e}$ is not an improving switch with respect to σ'.

   We observe that due to the application of e in σ and since σ' is reached after σ, we have $\sigma'(k_i) = k_{\ell'}$. Since $\tilde{e} \in S_{i,\sigma'}^{3,1}$, we have $\tilde{e} = (k_i, k_z)$ for some $z \leq \ell'$ such that $\sigma'(k_i) \neq k_z$. It thus suffices to show that $\mathrm{VAL}_{\sigma'}(k_{\ell'}) \geq \mathrm{VAL}_{\sigma'}(k_z)$. Since σ' is a phase 3 policy, by Lemma 5.1,

   $$\mathrm{VAL}_{\sigma'}(k_z) \leq \sum_{j \in \{z, \dots, n\} : (b+1)_j = 1} \left[(-N)^{2j+8} + (-N)^{2j+7} + (-N)^7 + (-N)^6\right]. \tag{5.1}$$

   However, since $\sigma[e]$ is also a phase 3 policy, the active bicycle of level ℓ' was already closed (phase 1) and $k_{\ell'}$ points towards the lane containing the active bicycle (phase 2). In addition, since $\ell' = \ell(b+1)$, no bicycle corresponding to a level $j > \ell'$ was opened as $b_j = (b+1)_j$ for these indices. This implies

   $$\mathrm{VAL}_{\sigma[e]}(k_{\ell'}) = \sum_{j \in \{\ell', \dots, n\} : (b+1)_j = 1} \left[(-N)^{2j+8} + (-N)^{2j+7} + (-N)^7 + (-N)^6\right]. \tag{5.2}$$

   As the values of the vertices are non-decreasing during the application of the Policy Iteration Algorithm, we have $\mathrm{VAL}_{\sigma'}(k_{\ell'}) \geq \mathrm{VAL}_{\sigma[e]}(k_{\ell'})$. Since $(b+1)_j = 0$ for all $j < \ell'$, combining Equations (5.1) and (5.2) yields

   $$\begin{aligned} \mathrm{VAL}_{\sigma'}(k_{\ell'}) &\geq \mathrm{VAL}_{\sigma[e]}(k_{\ell'}) \\ &= \sum_{j \in \{\ell', \dots, n\} : (b+1)_j = 1} \left[(-N)^{2j+8} + (-N)^{2j+7} + (-N)^7 + (-N)^6\right] \\ &= \sum_{j \in \{1, \dots, n\} : (b+1)_j = 1} \underbrace{\left[(-N)^{2j+8} + (-N)^{2j+7} + (-N)^7 + (-N)^6\right]}_{>0 \quad \forall j \geq 1} \\ &\geq \sum_{j \in \{z, \dots, n\} : (b+1)_j = 1} \left[(-N)^{2j+8} + (-N)^{2j+7} + (-N)^7 + (-N)^6\right] \\ &\geq \mathrm{VAL}_{\sigma'}(k_z) \end{aligned}$$

   Thus, $\mathrm{VAL}_{\sigma'}(k_{\ell'}) \geq \mathrm{VAL}_{\sigma'}(k_z)$ and $\tilde{e} = (k_i, k_z)$ is not an improving switch for the policy σ'.

2. We need to show that for every phase 3 policy σ' reached after applying $e = (b_{i,r}^j, k_{\ell'})$ in σ, no switch contained in $S_{i,j,r,\sigma'}^{3,2}$ is an improving switch. Let σ' be a phase 3 policy reached after σ. Then, $\sigma[e](b_{i,r}^j) = k_{\ell'}$ and thus $\mathrm{VAL}_{\sigma[e]}(b_{i,r}^j) = \mathrm{VAL}_{\sigma[e]}(k_{\ell'})$. Since any $\tilde{e} \in S_{i,j,r,\sigma'}^{3,2}$ is of the form $\tilde{e} = (b_{i,r}^j, k_z)$ for some $z \leq \ell'$, it therefore suffices to show $\mathrm{VAL}_{\sigma'}(k_{\ell'}) \geq \mathrm{VAL}_{\sigma'}(k_z)$. This however follows by the same estimations used in the first case.

3. This is proven analogously to 2. □

This enables us to prove the following lemma which allows us to show that it is possible to always choose a switch contained in L_σ^3 when applying the LEAST-ENTERED pivot rule.

Lemma 5.17. *Let σ be a phase 3 policy. Then there is an edge $e \in L_\sigma^3 \cap \arg\min_{\tilde{e} \in I_\sigma} \phi^\sigma(\tilde{e})$.*

Proof. We first observe that $I_\sigma^3 \neq \emptyset$ for any phase 3 policy σ since the set of improving switches is empty if and only if σ is an optimal policy. Let $e \in \arg\min_{\tilde{e} \in I_\sigma} \phi^\sigma(e)$.

Since $L_\sigma^3 \subseteq I_\sigma \subseteq U_\sigma^3$ by [3, Lemma 4], either $e \in L_\sigma^3$ or $e \in U_\sigma^3 \setminus L_\sigma^3$. Assume that the second case holds, since the statement follows directly otherwise. We observe that since $U_\sigma^{3,1}, \dots, U_\sigma^{3,9}$ form a partition of U_σ^3, there is exactly one $k \in \{1, \dots, 9\}$ with $e \in U_\sigma^{3,k}$.

Assume that $k \in \{4, \dots, 9\}$. Then, by Lemma 5.14, $\phi^\sigma(e) \geq \phi^\sigma(\tilde{e})$ for all $\tilde{e} \in L_\sigma^3$ since $e \in I_\sigma$. Since e minimizes the occurrence record, this implies $\phi^\sigma(e) = \phi^\sigma(\tilde{e})$ for all $\tilde{e} \in L_\sigma^3$. This in particular implies that there is an $\tilde{e} \in L_\sigma^3$ minimizing the occurrence record, so $\tilde{e} \in \arg\min_{\tilde{e} \in I_\sigma} \phi^\sigma(\tilde{e}) \cap L_\sigma^3$.

Now assume that $k \in \{1, 2, 3\}$. We analyze these cases one after another.

<u>**Case 1:**</u> $\boldsymbol{e \in U_\sigma^{3,1}}$. Then $e = (k_i, k_z)$ for some $i \in \{1, \dots, n\}$ and some $z \in \{1, \dots, \ell'\}$ such that $\sigma(k_i) \notin \{k_z, k_{\ell'}\}$ and $(b+1)_i = 0$. Thus $e \in S_{i,\sigma}^{3,1}$. First assume that $(k_i, k_{\ell'})$ was not applied yet. Then, $\phi^\sigma(k_i, k_{\ell'}) = \phi^{\sigma_b}(k_i, k_{\ell'})$ and we additionally obtain $\phi^{\sigma_b}(k_i, k_{\ell'}) = f(b, \ell', \{(i, 0)\})$ by [3, Table 4]. Together with $z \leq \ell'$ and Proposition 2.4 (4), this implies

$$\phi^\sigma(k_i, k_{\ell'}) = \phi^{\sigma_b}(k_i, k_{\ell'}) = f(b, \ell', \{(i, 0)\}) \leq f(b, z, \{(i, 0)\}) = \phi^{\sigma_b}(e) \leq \phi^\sigma(e).$$

Since e is chosen such that it minimizes the occurrence records among all improving switches, we have $\phi^\sigma(k_i, k_{\ell'}) = \phi^\sigma(e)$. This however implies that $(k_i, k_{\ell'}) \in \arg\min_{\tilde{e} \in I_\sigma} \phi^\sigma(\tilde{e})$. Therefore, the statement follows from $(k_i, k_{\ell'}) \in L_\sigma^3$.

It remains to show that $(k_i, k_{\ell'})$ was not applied yet. Towards a contradiction, assume that it was applied before in this transition. Then there was another phase 3 policy σ' reached before σ such that $(k_i, k_{\ell'})$ was applied in σ'. But then, by Lemma 5.16, it holds that $I_\sigma \cap S_{i,\sigma}^{3,1} = \emptyset$ since the policy σ is reached after σ'. This is a contradiction since $e \in I_\sigma$ and $e \in S_{i,\sigma}^{3,1}$.

<u>**Case 2:**</u> $\boldsymbol{e \in U_\sigma^{3,2}}$. Then $e = (b_{i,r}^j, k_z)$ for some $i \in \{1, \dots, n\}$ and some $z \in \{1, \dots, \ell'\}$ such that $\sigma(b_{i,l}^j) \notin \{k_z, k_{\ell'}\}$ and $(b+1)_i = 0$. Hence, $e \in S_{i,j,r,\sigma}^{3,2}$. First assume that the improving switch $(b_{i,r}^j, k_{\ell'})$ was not applied yet. Then, since $z \leq \ell'$, by [3, Table 4] and by Proposition 2.4 (4),

$$\begin{aligned}
\phi^\sigma(b_{i,r}^j, k_{\ell'}) &= \phi^{\sigma_b}(b_{i,l}^j, k_{\ell'}) \\
&= f(b, \ell', \{(i, 0)\}) + f(b, \ell', \{(i, 1), (i+1, 1-j)\}) \\
&\leq f(b, z, \{(i, 0)\}) + f(b, z, \{(i, 1), (i+1, 1-j)\}) \\
&= \phi^{\sigma_b}(e) \\
&\leq \phi^\sigma(e).
\end{aligned}$$

Since e is chosen such that it minimizes the occurrence records among all improving switches, we have $\phi^\sigma(b_{i,r}^j, k_{\ell'}) = \phi^\sigma(e)$. This implies that $(b_{i,r}^j, k_{\ell'}) \in \arg\min_{\tilde{e} \in I_\sigma} \phi^\sigma(\tilde{e})$. Therefore, the statement follows from $(b_{i,r}^j, k_{\ell'}) \in L_\sigma^3$.

It remains to show that $(b_{i,r}^j, k_{\ell'})$ was not applied yet. However, assuming that this switch was applied before results in the same contradiction as in the last case when applying Lemma 5.16.

<u>**Case 3:**</u> $\boldsymbol{e \in U_\sigma^{3,3}}$. This follows analogously to the previous case. □

Lemma 5.17 does not immediately imply that all improving switches belonging to phase 3 that should be applied according to [3] can be applied. Although Lemma 5.17 ensures that we can chose an improving switch contained in L_σ^3 for every phase 3 policy σ, it is not clear why it cannot happen that a phase 4 policy is reached although not all switches of phase 3 were applied yet. This however can be shown when using Lemmas 5.3 and 5.4 as follows.

Theorem 5.18. *There is an ordering of the improving switches and an associated tie-breaking rule compatible with the* LEAST-ENTERED *pivot rule during phase 3 such that all improving switches contained in $L_{\sigma_b}^3$ are applied and the* LEAST-ENTERED *pivot rule is obeyed during phase 3.*

Proof. Let σ denote the first phase 3 policy of the transition from σ_b to σ_{b+1}. Then, $L_\sigma^3 = L_{\sigma_b}^3$ by Lemma 5.3. By Lemma 5.17, there is an edge $e_1 \in L_\sigma^3$ minimizing the occurrence record I_σ. By Lemma 5.4, applying this switch results in a new phase 3 policy $\sigma[e_1]$ such that $L_{\sigma[e_1]}^3 = L_\sigma^3 \setminus \{e_1\}$. Now, again by Lemma 5.17, there is an edge $e_2 \in L_{\sigma[e_1]}^3$ minimizing the occurrence record $I_{\sigma[e_1]}$.

We can now apply the same argument iteratively until we reach a phase 3 policy $\hat{\sigma}$ such that $\left|L_{\hat{\sigma}}^3\right| = 1$ while only applying switches contained in $L_{\sigma_b}^3$. Then, by construction and by Lemma 5.17, $(e_1, e_2, \dots)$ defines an ordering of the edges of $L_{\sigma_b}^3$ and an associated tie-breaking rule that always follow the LEAST-ENTERED rule. When the policy $\hat{\sigma}$ with $\left|L_{\hat{\sigma}}^3\right| = 1$ is reached, applying the remaining improving switch results in a phase 4 policy. Then, all improving switches contained in $L_{\sigma_b}^3$ were applied and the LEAST-ENTERED pivot rule was obeyed. □

Note that the ordering used in the proof of Theorem 5.18 avoids Issue 5.9: As we proved in Issue 5.9, it is not possible to apply the improving switches in level $\ell(b+1)$ and level $n-1$ consistently such that all switches of level $\ell(b+1)$ are applied before any switch of level $n+1$ is applied and vice versa. By further analyzing the proof of Issue 5.9, it can be shown that the same holds for the improving switches of other levels. Our ordering always chooses an improving switch that minimizes the occurrence record regardless of the level, and in particular does not apply improving switches level by level in an order that only depends on the least significant set bit.

6 Conclusion

In this paper we revisited the lower bound example constructed in [3] that yields a subexponential lower bound on the Simplex Algorithm using the LEAST-ENTERED pivot rule. We discussed the example in general and highlighted several issues with the construction. We proposed alterations of the construction and the application of the Policy Iteration Algorithm to resolve all of these issues. In particular, we showed that the initial policy for the policy iteration needs to be changed and provided a new initial policy (Section 3). We further showed that the description of occurence records are not entirely accurate and corrected the inaccuracy (Section 4). Most notably, we proved that the order in which Friedmann applies certain improving switches, as well as simple adaptations of this order, are inconsistent with the LEAST-ENTERED rule (Section 5), and we implicitly provided a more involved ordering and associated tie-breaking rule that overcome this issue.

Crucially, our changes retain the macroscopic properties of the construction, and, as a consequence, we are able to recover Friedmann's subexponential lower bound.

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A Imported Tables

This appendix contains tables of [3] used in this paper. They are labeled the same way as in [3] and use the exact notation used in [3]. The tables use the alternative notion $\bar{\sigma}$ for referring to the target of a vertex with respect to a policy σ that we omitted in our paper. It is defined as follows.

$\sigma(v)$	t	k_i	h_*^*	s	A_*^*	c_i^j
$\bar{\sigma}(v)$	$n+1$	i	1	0	0	$-j$

Also, as it is done in [3]we write $\bar{\sigma}(A_i^j) = 1$ if $\sigma(b_{i,0}^j) = A_i^j$ and $\sigma(b_{i,1}^j) = A_i^j$ and $\bar{\sigma}(A_i^j) = 0$ otherwise. In addition, the notation $b' = b + 1$ and $\nu_i^j := \min(\{n+1\} \cup \{j \geq i : b_j = 0\})$ is used.

The first table shows when a policy σ is considered a phase p policy.

Phase	1	2	3	4	5	6
$\bar{\sigma}(s)$	r	r	r	r	r	r'
$\bar{\sigma}(d_i^0)$	$1{-}b_{i+1}$	$1{-}b_{i+1}$	$1 - b_{i+1}$	$1 - b_{i+1}$	$1 - b_{i+1}$	$1 - b_{i+1}, 1 - b'_{i+1}$
$\bar{\sigma}(d_i^1)$	b_{i+1}	b_{i+1}	b_{i+1}	b_{i+1}	b_{i+1}	b_{i+1}, b'_{i+1}
$\bar{\sigma}(h_i^0)$	$\nu_{i+2}^n(b)$	$\nu_{i+2}^n(b)$	$\nu_{i+2}^n(b)$	$\nu_{i+2}^n(b), \nu_{i+2}^n(b')$	$\nu_{i+2}^n(b')$	$\nu_{i+2}^n(b')$
$\bar{\sigma}(b_{*,*}^*)$	$0, r$	$0, r$	$0, r, r'$	$0, r'$	$0, r'$	$0, r'$
$\bar{\sigma}(A_i^{b_{i+1}})$	b_i	$*$	$*$	$*$	$*$	$*$
$\bar{\sigma}(A_i^{b'_{i+1}})$	$*$	b'_i	b'_i	b'_i	b'_i	b'_i

Phase	1-2	3-4	5-6
$\bar{\sigma}(k_i)$	$\begin{cases} r & \text{if } b_i = 0 \\ -b_{i+1} & \text{if } b_i = 1 \end{cases}$	$\begin{cases} r, r' & \text{if } b'_i = 0 \wedge b_i = 0 \\ -b_{i+1}, r' & \text{if } b'_i = 0 \wedge b_i = 1 \\ -b'_{i+1} & \text{if } b'_i = 1 \end{cases}$	$\begin{cases} r' & \text{if } b'_i = 0 \\ -b'_{i+1} & \text{if } b_i = 1' \end{cases}$

Phase 3	Side Conditions
(a)	$\forall i.([b'_i = 0$ and $(\exists j, l. \bar{\sigma}(b_{i,l}^j) = r')]$ implies $\bar{\sigma}(k_i) = r')$
(b)	$\forall i, j.([b'_i = 0, b'_j = 0, \bar{\sigma}(k_i) = r'$ and $\bar{\sigma}(k_j) \neq r']$ implies $i > j)$

Table 2: Policy Phases where $b' = b + 1, r = \nu_1^n(b), r' = \nu_1^n(b')$ and $*$ is arbitrary

The second table shows subsets L_σ^p and supersets U_σ^p of the set of improving switches of a phase p policy. Note that this table shows the original Table contained in [3].

Ph. p	Improving Switches Subset L_σ^p	Improving Switches Superset U_σ^p
1	$\{(b_{i,l}^j, A_i^j) \mid \sigma(b_{i,l}^j \neq A_i^j\}$	L_σ^1
2	$\{(k_{r'}, c_{r'}^{b'_{r'+1}})\}$	$L_\sigma^1 \cup L_\sigma^2$
3	$\{(k_i, k_{r'}) \mid \bar{\sigma}(k_i) \neq r' \wedge b'_i = 0\} \cup$ $\{(b_{i,l}^j, k_{r'}) \mid \bar{\sigma}(b_{i,l}^j) \neq r' \wedge b'_i = 0\} \cup$ $\{(b_{i,l}^j, k_{r'}) \mid \bar{\sigma}(b_{i,l}^j) \neq r' \wedge b'_{i+1} \neq j\}$	$U_\sigma^4 \cup \{(k_i, k_z) \mid \bar{\sigma}(k_i) \notin \{z, r'\}, z \leq r' \wedge b'_i = 0\} \cup$ $\{(b_{i,l}^j, k_z) \mid \bar{\sigma}(b_{i,l}^j) \notin \{z, r'\}, z \leq r' \wedge b'_i = 0\} \cup$ $\{(b_{i,l}^j, k_z) \mid \bar{\sigma}(b_{i,l}^j) \notin \{z, r'\}, z \leq r' \wedge b'_{i+1} \neq j\}$
4	$\{(h_i^0, k_{\nu_{i+2}^n(b')}) \mid \bar{\sigma}(h_i^0) \neq \nu_{i+2}^n(b')\}$	$U_\sigma^5 \cup \{(h_i^0, k_l) \mid l \leq \nu_{i+2}^n(b')\}$
5	$\{(s, k_{r'})\}$	$U_\sigma^6 \cup \{(s, k_i) \mid \bar{\sigma}(s) \neq i \wedge i < r'\} \cup$ $\{(d_i^j, x) \mid \sigma(d_i^j) \neq x \wedge i < r'\}$
6	$\{(d_i^0, x) \mid \sigma(d_i^0) \neq x \wedge \bar{\sigma}(d_i^0) \neq b'_{i+1}\} \cup$ $\{(d_i^1, x) \mid \sigma(d_i^1) \neq x \wedge \bar{\sigma}(d_i^1)' b_{i+1})\}$	$L_\sigma^1 \cup L_\sigma^6$

Table 3: Improving Switches (where $b' = b + 1$ and $r' = \nu_1^n(b')$)

The last table shows the occurrence records of a policy σ^b representing the number $b \in \mathbb{B}_n$ according to [3]. In this table, the notation $g^* := g(b, i, \{(i+1, j)\})$ is used.

Edge e	$(*, t)$	(s, k_r)	(h^0_*, k_r)
$\phi^b(e)$	0	$f(b,r)$	$f(b,r)$

Edge e	$(b^j_{i,*}, k_r)$
$\phi^b(e)$	$f(b,r,\{(i,0)\}) + f(b,r,\{(i,1),(i+1,1-j)\})$

Edge e	(k_i, k_r)	(k_i, c^j_i)
$\phi^b(e)$	$f(b,r,\{(i,0)\})$	$f(b,i,\{(i+1,j)\})$

Edge e	(d^j_i, s)	(d^j_i, h^j_i)
$\phi^b(e)$	$f(b,i+1) - j \cdot b_{i+1}$	$f(b,i+1) - (1-j) \cdot b_{i+1}$

Complicated Conditions
$\lvert\phi^b(b^j_{i,0}, A^j_i) - \phi^b(b^j_{i,1}, A^j_i)\rvert \le 1$
$\phi^b(b^j_{i,0}, A^j_i) + \phi^b(b^j_{i,1}, A^j_i) = \begin{cases} g^* + 1 & \text{if } b_i = 1 \text{ and } b_{i+1} = j \\ g^* + 1 + 2 \cdot z & \text{if } b_{i+1} \neq j \text{ and } z := b - g^* - 2^{i-1} < \frac{1}{2}(b - 1 - g^*) \\ b & \text{otherwise} \end{cases}$

Table 4: Occurrence Records

B Omitted Proofs

This appendix contains the proofs omitted from the main part of this paper.

Proposition 2.4. *Let $b \in \mathbb{B}_n$ and $i, j \in \{1, \dots, n\}$. Then the following hold:*

1. *Let S, S' be schemes and $S \subseteq S'$. Then $M(b, S') \subseteq M(b, S)$.*
2. *Let S, S' be schemes and $S \subseteq S'$. Then $f(b, i, S') \le f(b, i, S)$.*
3. *It holds that $f(b, j) = f(b, j, \{(i, 0)\}) + f(b, j, \{(i, 1)\})$ and $f(b, j) = \lfloor \frac{b + 2^{j-1}}{2^j} \rfloor$.*
4. *Let $i \le j$ and S be a scheme. Then $f(b, j, S) \le f(b, i, S)$ and thus $f(b, j) \le f(b, i)$.*
5. *Let $i < j$. Then $F(b, j) = F(b, j, \{(i, 0)\})$ and thus $f(b, j, \{(i, 0)\}) = f(b, j)$.*

Proof. We prove the statements one after another.

1. Let S, S' be schemes such that $S \subseteq S'$. Since every number matching the scheme S' also matches the scheme S, it follows that $M(b, S') \subseteq M(b, S)$ for all numbers $b \in \mathbb{B}_n$.
2. This follows directly from (1) and by the definition of $f(b, i, S')$.
3. The first statement follows immediately since for every binary number $b \in \mathbb{B}_n$ and index $i \in \{1, \dots, n\}$, either $b_i = 0$ or $b_i = 1$.

   It remains to show that $f(b, j) = \lfloor \frac{b + 2^{j-1}}{2^j} \rfloor$ for $b \in \mathbb{B}_n$ and $j \in \{1, \dots, n\}$. We observe that 2^{j-1} is the smallest number matching $S_j = \{(j, 1), (j-1, 0), \dots, (1, 0)\}$. This implies the statement for $b < 2^{j-1}$. Now, let m_i denote the i-th number matching the scheme S_j. Then, by the previous argument, $m_1 = 2^{j-1}$. As only numbers ending on the subsequence $(1, 0, \dots, 0)$ of length j match the scheme S_j, we immediately have $m_i = (i-1) \cdot 2^j + 2^{j-1}$. Since $f(m_i, j) = i$ by definition and

   $$\left\lfloor \frac{m_i + 2^{j-1}}{2^j} \right\rfloor = \left\lfloor \frac{(i-1) \cdot 2^j + 2^{j-1} + 2^{j-1}}{2^j} \right\rfloor = \left\lfloor \frac{i \cdot 2^j}{2^j} \right\rfloor = i,$$

   we get $f(m_i, j) = \left\lfloor \frac{m_i + 2^{j-1}}{2^j} \right\rfloor$.

Now let $b \in \mathbb{B}_n$ and choose $i \in \mathbb{N}$ such that $b \in [m_i, m_{i+1})$. Then, by the definition of $f(b, j)$, we have $f(b, j) = i$. In addition, by the choice of i,

$$\left\lfloor \frac{b + 2^{j-1}}{2^j} \right\rfloor \geq \left\lfloor \frac{m_i + 2^{j-1}}{2^j} \right\rfloor = f(m_i, j) = i \tag{B.1}$$

and

$$\left\lfloor \frac{b + 2^{j-1}}{2^j} \right\rfloor < \left\lfloor \frac{m_{i+1} + 2^{j-1}}{2^j} \right\rfloor = f(m_{i+1}, j) = i + 1. \tag{B.2}$$

By integrality, Equations (B.1) and (B.2) imply that $\left\lfloor \frac{b+2^{j-1}}{2^j} \right\rfloor = i$ and thus,

$$f(b, j) = i = \left\lfloor \frac{b + 2^{j-1}}{2^j} \right\rfloor.$$

4. Let $i \leq j$ and $b \in \mathbb{B}_n$. Let $S_j := \{(j, 1), (j-1, 0), \ldots, (1, 0)\}$ and define S_i similarly. Consider any number $\tilde{b} \leq b$ matching both the schemes S_j and S. Then, since $i \leq j$ there needs to be at least one number $\hat{b} \leq \tilde{b}$ matching S_i and S. This implies $f(b, j, S) \leq f(b, i, S)$.

   The second inequality follows immediately when setting $S := \emptyset$.

5. Let $i < j$ and define $S_j := \{(j, 1), (j-1, 0), \ldots, (1, 0)\}$. Since $i < j$, we have $(i, 0) \in S_j$, immediately implying $F(b, j) = F(b, j, \{(i, 0)\})$. □

Lemma 5.3. *Let $b \in \mathbb{B}_n$ and let σ be the first phase 3 policy of the transition from σ_b to σ_{b+1}. Then $L_\sigma^3 = L_{\sigma_b}^3$, and $L_{\sigma_b}^3$ is the set of improving switches that should be applied during phase 3 according to [3, Table 2].*

Proof. Consider the invariants on given in [3, Page 9]. There it is stated that[2] "*(1) all active bicycles corresponding to set bits are closed, (2) all other bicycles are completely open, moving to the least set bit, (3) all entry points k_i move to the next active bicycle if bit i is set and to the least set bit otherwise [...]*". The set $L_\sigma^{3,1}$ contains exactly the edges corresponding to entry vertices whose corresponding bit is not set in $b+1$. That is, it contains exactly the edges that need to be applied such that aspect (3) of the description is fulfilled. In addition, the set $L_\sigma^{3,2}$ contains the edges of bicycles that need to be open since the corresponding bit of $b+1$ is equal to zero. Similarly, the set $L_\sigma^{3,3}$ contains the edges of inactive bicycles. Since the edges of all of these bicycles need to switch to the vertex $k_{\ell'}$ in order to obey the second aspect of the description given above, the set L_σ^3 completely describes the set of improving switches that should be applied during phase 3. □