A note on the paper "Demonstrating Johnson's algorithm via resource constrained scheduling"

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Abstract

In this paper we demonstrate that the relation between two jobs defined by $min\{a_i, b_j\} \leq min\{b_i, a_j\}$, used in Johnson's theorem, is not transitive. However, both the theorem and Johnson's algorithm are correct.

Keywords: Flow shop, scheduling, makespan, Johnson's theorem, transitivity

1 Introduction

Cheng and Lin (2017), in Section 2, talk about the Johnson's algorithm (Johnson 1954) and in the third paragraph they say:

"*Derivations of the optimal schedule lead to the following rule to schedule jobs (Johnson, 1954):*

If $min\{a_i, b_j\} \leq min\{b_i, a_j\}$ *then job i precedes job j in some optimal schedule.* (1)

Rule (1) is transitive, i.e. if $min\{a_i, b_j\} \leq min\{b_i, a_j\}$ *and* $min\{a_j, b_k\} \leq min\{b_j, a_k\}$,

then $min\{a_i, b_k\} \leq min\{b_i, a_k\}$"

Despite the authors state that Rule (1) is transitive, it is easy to show, in a counterexample, that it is not. This fact was already noticed by (Companys 2003) and (Baker and Trietsch 2009) who detected that Johnson's rule is not transitive in some of the cases when there are jobs that have the same processing time in stage 1 and 2.

2 Counterexample

Consider the example shown in Table 1. Three jobs have to be processed in a two-machine flow shop, which consists of machine M_a, in stage 1, and machine M_b, in stage two. Let a_j and b_j be the processing times of job j on machine M_a and machine M_b, respectively.

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jobs	1	2	3
a_i	26	20	31
b_i	20	20	30

Table1. Three Jobs 2-machine problem

It is easy to see that:

$\min\{a_1, b_2\} \leq \min\{b_1, a_2\}$ and $\min\{a_2, b_3\} \leq \min\{b_2, a_3\}$ but $\min\{a_1, b_3\} > \min\{b_1, a_3\}$. Therefore, we have demonstrated that Rule (1) is not transitive.

3 Johnson's theorem revisited

Johnson used four cases to demonstrate the transitivity of rule 1. But, he saw, in case 4, that "*when $b_2 \leq a_1, a_2, b_1$ and $a_2 \leq a_3, b_3, b_2$ then $a_2 = b_2$ and we have item 2 indifferent to item 1 and item 3. In this case, item 1 may or may not precede item 3 but there is no contradiction to transitivity as long as we order item 1 and item 3 first and then put item 2 anywhere*", which evidence the no transitivity in some cases.

Hence, to demonstrate the theorem in all cases it is necessary to extend rule (1) in the following way:

If $\min\{a_i, b_j\} < \min\{b_i, a_j\}$ or ($\min\{a_i, b_j\} = \min\{b_i, a_j\}$ and $a_i - b_i \leq a_j - b_j$), then job i precedes job j in some optimal schedule.

If we apply the extended rule (1) to case 4, job 1 dominates job 2 if $a_1 – b_1 \leq a_2 – b_2 = 0$ and job 2 dominates job 3 if $0 = a_2 – b_2 \leq a_3 – b_3$.

Therefore, $\min\{a_1, b_3\} \leq \min\{a_3, b_1\}$ and $a_1 – b_1 < a_3 – b_3$ then job 1 dominates job 3 and the extended rule is transitive. A similar reasoning can be applied when $a_1 = b_1$ or/and $a_3 = b_3$.

Now, thanks to the transitivity of the extended rule we can apply a sorting algorithm to obtain an optimal solution to any set of jobs.

As an example, the bubble sort algorithm is applied to the counterexample shown in section 2. The initial sequence is 1– 2 – 3. Then, job 1 is compared to job 2.

As $\min\{26,20\} = 20 = \min\{20,20\}$ but 26 – 20 = 6 and 20 – 20 = 0, job 2 dominates job 1, i.e. job 2 has to be processed before job 1. Therefore, the new sequence is 2 – 1 – 3.

Next, similarly, job 1 is compared to job 3. It can be seen that job 3 dominates job 1. Then the new sequence is 2 – 3 – 1. The procedure starts again by comparing job 2 to job 3. Job 2 dominates job 3. Therefore, the sequence 2 – 3 – 1 is maintained.

Next, although it is not necessary, job 3 is compared against job 1 and it is seen that job 3 dominates job 1. Hence, the final sequence is 2 – 3 – 1, which is an optimal sequence.

4 Discussion

The extended rule is less general than Johnson's rule but the former guarantees the transitivity. However, the use of this rule leads to eliminate some optimal solutions than can be found by using Johnson's algorithm. Although, it is easy to show, that Johnson's algorithm is not able to enumerate all the optimal solutions either.

This fact is only relevant if a second criterion has to be considered since, in this case, it is better to be able to evaluate several optimal solutions, according the first criterion, in order to choose the one which minimizes the second one, as it is done in (Rajendran 1992) to minimize the total flowtime subject to obtaining the optimal makespan for two-stage flow shop problem.

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