Locating domination in bipartite graphs and their complements

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Abstract

A set S of vertices of a graph G is *distinguishing* if the sets of neighbors in S for every pair of vertices not in S are distinct. A *locating-dominating set* of G is a dominating distinguishing set. The *location-domination number* of G, $\lambda(G)$, is the minimum cardinality of a locating-dominating set. In this work we study relationships between $\lambda(G)$ and $\lambda(\overline{G})$ for bipartite graphs. The main result is the characterization of all connected bipartite graphs G satisfying $\lambda(\overline{G}) = \lambda(G) + 1$. To this aim, we define an edge-labeled graph G^S associated with a distinguishing set S that turns out to be very helpful.

Keywords: domination; location; distinguishing set; locating domination; complement graph; bipartite graph.

AMS subject classification: 05C12, 05C35, 05C69.

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1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The distance between two vertices v and w is denoted by $d_G(v, w)$. The *neighborhood* of a vertex $u \in V$ is $N_G(u) = \{v : uv \in E\}$. We write $N(u)$ or $d(v, w)$ if the graph G is clear from the context. For any $S \subseteq V$, $N(S) = \cup_{u\in S} N(u)$. A set $S \subseteq V$ is *dominating* if $V = S \cup N(S)$ (see [7]). For further notation and terminology, we refer the reader to [4].

A set $S \subseteq V$ is *distinguishing* if $N(u) \cap S \neq N(v) \cap S$ for every pair of different vertices $u, v \in V \setminus S$. In general, if $N(u) \cap S \neq N(v) \cap S$, we say that S *distinguishes* the pair u and v. A *locating-dominating set*, *LD-set* for short, is a distinguishing set that is also dominating. Observe that there is at most one vertex not dominated by a distinguishing set. The *location-domination number* of G, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an *LD-code* [12, 13]. Certainly, every LD-set of a non-connected graph G is the union of LD-sets of its connected components and the location-domination number is the sum of the location-domination number of its connected components. Both, LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1, 2, 3, 5, 6, 8, 9, 10]. A complete and regularly updated list of papers on locating-dominating codes is to be found in [11].

The *complement* of G, denoted by $\overline{G}$, has the same set of vertices of G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. This work is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$ for connected bipartite graphs.

It follows immediately from the definitions that a set $S \subseteq V$ is distinguishing in G if and only if it is distinguishing in $\overline{G}$. A straightforward consequence of this fact are the following results.

Proposition 1 ([9])**.** *Let $S \subseteq V$ be an LD-set of a graph $G = (V, E)$. Then, S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$;*

Proposition 2 ([8])**.** *Let $S \subseteq V$ be an LD-set of a graph $G = (V, E)$. Then, the following properties hold.*

(a) There is at most one vertex $u \in V \setminus S$ such that $N(u) \cap S = S$, and in the case it exists, $S \cup \{u\}$ is an LD-set of $\overline{G}$.

(b) S is an LD-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ such that $N(u) \cap S = S$.

Theorem 1 ([8])**.** *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

According to the preceding inequality, $\lambda(\overline{G}) \in \{\lambda(G)-1, \lambda(G), \lambda(G)+1\}$ for every graph G, all cases being feasible for some connected graph G. We intend to determine graphs such that $\lambda(\overline{G}) > \lambda(G)$, that is, we want to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$. This problem was completely solved in [9] for the family of block-cactus.

In this work, we carry out a similar study for bipartite graphs. For this purpose, we first introduce in Section 2 the graph associated with a distinguishing set. This graph turns out to be very helpful to derive some properties related to LD-sets and the location-domination number of G, and will be used to get the main results in Section 3.

In Table 1, the location-domination number of some families of bipartite graphs are displayed, along with the location-domination number of its complement graphs. Concretely, we consider the path P_n of order $n \geq 4$; the cycle C_n of (even) order $n \geq 4$; the star $K_{1,n-1}$ of order $n \geq 4$, obtained by joining a new vertex to $n-1$ isolated vertices; the complete bipartite graph $K_{r,n-r}$ of order $n \geq 4$, with $2 \leq r \leq n-r$ and stable sets of order r and $n-r$, respectively; and finally, the bi-star $K_2(r,s)$ of order $n \geq 6$ with $3 \leq r \leq s = n-r$, obtained by joining the central vertices of two stars $K_{1,r-1}$ and $K_{1,s-1}$ respectively.

Proposition 3 ([9])**.** *Let G be a graph of order $n \geq 4$. If G is a graph belonging to one of the following classes: $P_n, C_n, K_{1,n-1}, K_{r,n-r}, K_2(r,s)$, then the values of $\lambda(G)$ and $\lambda(\overline{G})$ are known and they are displayed in Table 1.*

G	P_n	P_n	C_n	C_n
n	$4 \leq n \leq 6$	$n \geq 7$	$4 \leq n \leq 6$	$n \geq 7$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$
$\lambda(\overline{G})$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$

G	$K_{1,n-1}$	$K_{r,n-r}$	$K_2(r,s)$
n	$n \geq 4$	$2 \leq r \leq n-r$	$3 \leq r \leq s$
$\lambda(G)$	$n-1$	$n-2$	$n-2$
$\lambda(\overline{G})$	$n-1$	$n-2$	$n-3$

Table 1: The values of $\lambda(G)$ and $\lambda(\overline{G})$ for some families of bipartite graphs.

Notice that in all cases considered in Proposition 3, we have $\lambda(\overline{G}) \leq \lambda(G)$. Moreover, for every pair of integers (r,s), with $3 \leq r \leq s$ we have examples of bipartite graphs with stable sets of order r and s respectively, such that $\lambda(\overline{G}) = \lambda(G)$ and such that $\lambda(\overline{G}) = \lambda(G) - 1$.

2 The graph associated with a distinguishing set

Let S be a distinguishing set of a graph G. We introduce in this section a labeled graph associated with S and study some general properties. Since LD-sets are distinguishing sets that are also dominating, this graph allows us to derive some properties related to LD-sets and the location-domination number of G.

Definition 1. Let S be a distinguishing set of cardinality k of a graph $G = (V, E)$ of order n. The so-called *S-associated graph*, denoted by G^S, is the edge-labeled graph defined as follows.

i) $V(G^S) = V \setminus S$;

ii) If $x, y \in V(G^S)$, then $xy \in E(G^S)$ if and only if the sets of neighbors of x and y in S differ in exactly one vertex $u(x, y) \in S$;

iii) The label $\ell(xy)$ of edge $xy \in E(G^S)$ is the only vertex $u(x, y) \in S$ described in the preceding item.

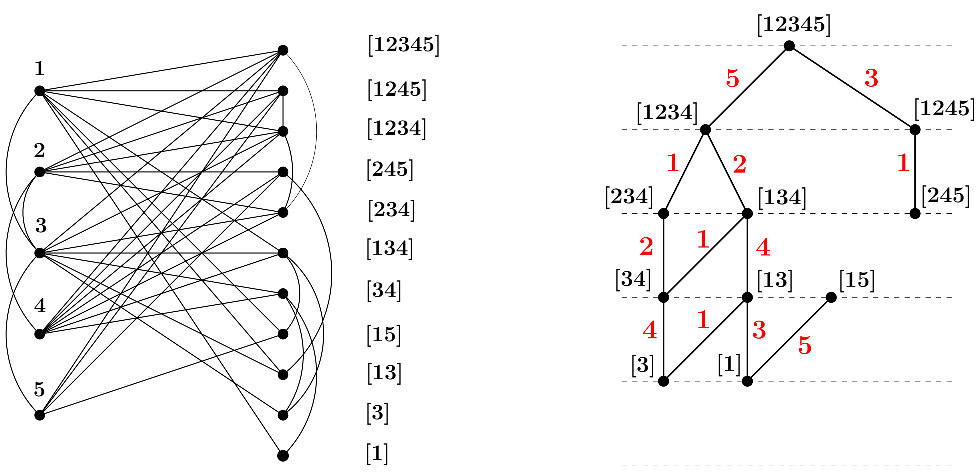

Figure 1: A graph G (left) and the graph G^S associated with the distinguishing set $S = \{1, 2, 3, 4, 5\}$ (right). The neighbors in S of each vertex are those enclosed in brackets.

Notice that if $xy \in E(G^S$, $\ell(xy) = u \in S$ and $|N(x) \cap S| > |N(y) \cap S|$, then $N(x) \cap S = (N(u) \cap S) \cup \{u\}$. Therefore, we can represent the graph G^S with the vertices lying on $|S| + 1 = k + 1$ levels, from bottom (level 0) to top (level k), in such a way that vertices with exactly j neighbors in S are at level j. For any $j \in \{0, 1, \ldots, k\}$ there are at most $\binom{k}{j}$ vertices at level j. So, there is at most one vertex at level k and, if it is so, this vertex is adjacent to all vertices of S. There is at most one vertex at level 0 and, if it is so, this vertex has no neighbors in S. The vertices at level 1 are those with exactly one neighbor in S. See Figure 1 for an example of an LD-set-associated graph.

Next, we state some basic properties of the graph associated with a distinguishing set that will be used later.

Proposition 4. *Let S be a distinguishing set of $G = (V, E)$, $x, y \in V \setminus S$ and $u \in S$. Then,*

(1) S is a distinguishing set of $\overline{G}$.

(2) The associated graphs G^S and $\overline{G}^S$ are equal.

(3) The representation by levels of $\overline{G}^S$ is obtained by reversing bottom-top the representation of G^S.

(4) $xy \in E(G^S)$ and $\ell(xy) = u$ if and only if x and y have the same neighborhood in $S \setminus \{u\}$ and (thus) they are not distinguished by $S \setminus \{u\}$.

(5) If $xy \in E(G^S)$ and $\ell(xy) = u$, then $S \setminus \{u\}$ is not a distinguishing set.

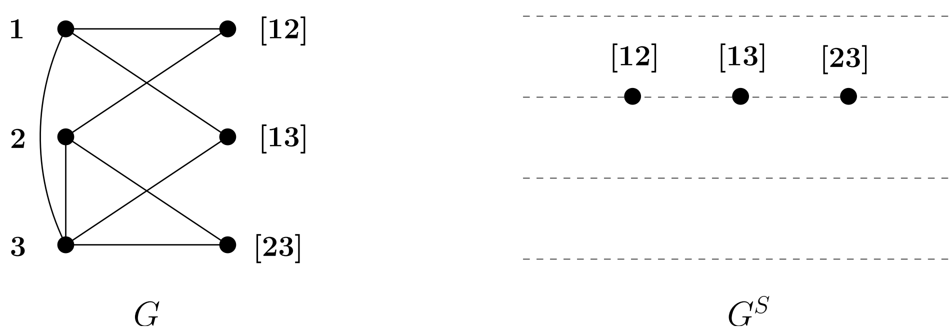

Figure 2: $S = \{1, 2, 3\}$ is distinguishing, $S' = \{1, 2\}$ is not distinguishing and G^S has no edges.

The converse of Proposition 4 (5) is not necessarily true. For example, consider the graph G of order 6 displayed in Figure 2. By construction, $S = \{1, 2, 3\}$ is a distinguishing set. However, $S' = S \setminus \{3\} = \{1, 2\}$ is not a distinguishing set, because $N(3) \cap S' = N([12]) \cap S' = \{1, 2\}$, and the S-associated graph G^S has no edge with label 3 (in fact, G^S has no edges since the neighborhood in S of all vertices not in S have the same size).

As a straight consequence of Proposition 4 (5), the following result is derived.

Corollary 1. *Let S be a distinguishing set of G and let $S' \subseteq S$. Consider the subgraph $H_{S'}$ of G^S induced by the edges with a label from S'. Then, all the vertices belonging to the same connected component in $H_{S'}$ have the same neighborhood in $S \setminus S'$, concretely, it is the neighborhood in S of a vertex from the connected component lying on the lowest level.*

For example, consider the graph shown in Figure 1. If $S' = \{1,2\}$, then vertices of the same connected component in $H^{S'}$ have the same neighborhood in $S \setminus S'$. Concretely, the neighborhood of vertices [1234], [234], [134] and [34] in $S \setminus \{1,2\}$ is $\{3,4\}$; the neighborhood of vertices [13] and [3] in $S \setminus \{1,2\}$ is $\{3\}$; and the neighborhood of vertices [1245] and [245] in $S \setminus \{1,2\}$ is $\{4,5\}$ (see Figure 3).

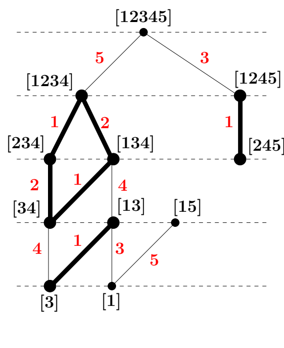

Figure 3: If $S' = \{1,2\}$, then $H^{S'} \cong C_4 + 2K_2$ has three components. Vertices of the same component in $H^{S'}$ have the same neighborhood in $S \setminus S'$.

Proposition 5. *Let S be a a distinguishing set of cardinality k of a connected graph $G = (V,E)$ of order n. Let G^S be its associated graph. Then, the following conditions hold.*

1. $|V(G^S)| = n - k$.
2. *G^S is bipartite.*
3. *Incident edges of G^S have different labels.*
4. *Every cycle of G^S contains an even number of edges labeled v, for all $v \in S$.*
5. *Let ρ be a walk with no repeated edges in G^S. If ρ contains an even number of edges labeled v for every $v \in S$, then ρ is a closed walk.*
6. *If $\rho = x_i x_{i+1} \dots x_{i+h}$ is a path satisfying that vertex x_{i+h} lies at level $i+h$, for any $h \in \{0,1,\dots,h\}$, then*
   (a) *the edges of ρ have different labels;*
   (b) *for all $j \in \{i+1, i+2, \dots, i+h\}$, $N(x_j) \cap S$ contains the vertex $\ell(x_k x_{k+1})$, for any $k \in \{i, i+1, \dots, j-1\}$.*

Proof. 1. It is a direct consequence from the definition of G^S.

2. Take the sets $V_1 = \{x \in V(G^S) : |N(x) \cap S| \text{ is odd}\}$ and $V_2 = \{x \in V(G^S) : |N(x) \cap S| \text{ is even}\}$. Then, $V(G^S) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Since $||N(x) \cap S| - |N(y) \cap S|| = 1$ for any $xy \in E(G^S)$, it is clear that the vertices x, y are not in the same subset V_i, $i = 1, 2$.

3. Suppose that edges $e_1 = xy$ and $e_2 = yz$ have the same label $l(e_1) = l(e_2) = v$. This means that $N(x) \cap S$ and $N(y) \cap S$ differ only in vertex v, and $N(y) \cap S$ and $N(z) \cap S$ differ only in vertex v. It is only possible if $N(x) \cap S = N(z) \cap S$, implying that $x = z$.

4. Let ρ be a cycle such that $E(\rho) = \{x_0x_1, x_1x_2, \dots x_hx_0\}$. The set of neighbors in S of two consecutive vertices differ exactly in one vertex. If we begin with $N(x_0) \cap S$, then each time we add (remove) the vertex of the label of the corresponding edge, we have to remove (add) it later in order to obtain finally the same neighborhood, $N(x_0) \cap S$. Therefore, ρ contains an even number of edges with label v.

5. Consider the vertices $x_0, x_1, x_2, x_3, ..., x_{2k}$ of ρ. In this case, $N(x_{2k}) \cap S$ is obtained from $N(x_0) \cap S$ by either adding or removing the labels of all the edges of the walk. As every label appears an even number of times, for each element $v \in S$ we can match its appearances in pairs, and each pair means that we add and remove (or remove and add) it from the neighborhood in S. Therefore, $N(x_{2k}) \cap S = N(x_0) \cap S$, and hence $x_0 = x_{2k}$.

6. It straightly follows from the fact that $N(x_j) \cap S = N(x_{j-1} \cap S) \cup \{\ell(x_{j-1}x_j)\}$, for any $j \in \{i+1, \dots, i+h\}$. □

In the study of distinguishing sets and LD-sets using its associated graph, a family of graphs is particularly useful, the *cactus graph* family. A *block* of a graph is a maximal connected subgraph with no cut vertices. A connected graph G is a *cactus* if all its blocks are either cycles or edges. Cactus are characterized as those connected graphs with no edge shared by two cycles.

Lemma 1. *Let S be a distinguishing set of a graph G and $S' \subseteq S$. Consider a subgraph H of G^S induced by a set of edges containing exactly two edges with label u, for each $u \in S' \subseteq S$. Then, all the connected components of H are cactus.*

Proof. We prove that there is no edge lying on two different cycles of H. Suppose, on the contrary, that there is an edge e_1 contained in two different cycles C_1 and C_2 of H. Note that C_1 and C_2 are cycles of G^S, since $S' \subseteq S$. Hence, if the label of e_1 is $u \in S' \subseteq S$, then by Proposition 5 both cycles C_1 and C_2 contain the other edge e_2 of H with label u. Suppose that $e_1 = x_1y_1$ and $e_2 = x_2y_2$ and assume w.l.o.g. that there exist $x_1 - x_2$ and $y_1 - y_2$ paths in C_1 not containing edges e_1, e_2. Let P_1 and P_1' denote respectively those paths (see Figure 4 a).

We have two possibilities for C_2: (i) there are $x_1 - x_2$ and $y_1 - y_2$ paths in C_2 not containing neither e_1 nor e_2. Let P_2 denote the $x_1 - x_2$ path in C_2 in that case (see Figure 4 b); (ii) there are $x_1 - y_2$ and $y_1 - x_2$ paths in C_2 not containing neither e_1 nor e_2 (see Figure 4 c).

In case (ii), the closed walk formed with the path P_1, e_1 and the $y_1 - x_2$ path in C_2 would contain a cycle with exactly an edge labeled with u, which is a contradiction (see Figure 4 d).

In case (i), at least one the following cases hold: either the $x_1 - x_2$ paths in C_1 and in C_2, P_1 and P_2, are different, or the $y_1 - y_2$ paths in C_1 and in C_2 are different (otherwise, $C_1 = C_2$).

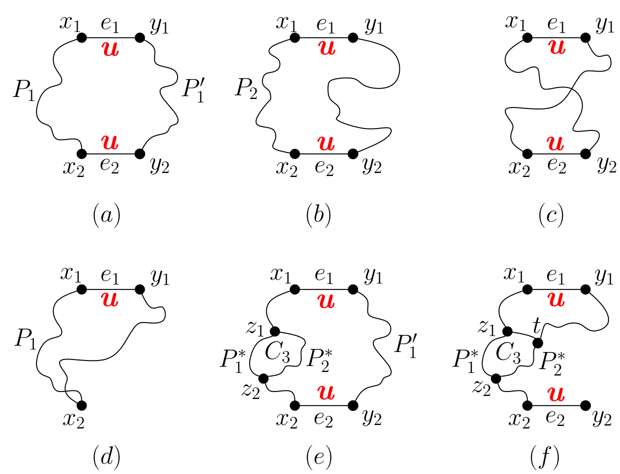

Figure 4: All connected components of the subgraph H are cactus.

Assume that P_1 and P_2 are different. Let z_1 be the last vertex shared by P_1 and P_2 advancing from x_1 and let z_2 be the first vertex shared by P_1 and P_2 advancing from z_1 in P_2. Notice that $z_1 \neq z_2$. Take the cycle C_3 formed with the $z_1 - z_2$ paths in P_1 and P_2. Let P_1^* and P_2^* be respectively the $z_1 - z_2$ subpaths of P_1 and P_2 (see Figure 4 e). We claim that the internal vertices of P_2^* do not lie in P_1'. Otherwise, consider the first vertex t of P_1' lying also in P_2^*. The cycle beginning in x_1, formed by the edge e_1, the $y_1 - t$ path contained in P_1', the $t - z_1$ path contained in P_2^* and the $z_1 - x_1$ path contained in P_1 has exactly one appearance of an edge with label u, which is a contradiction (see Figure 4 f). By Proposition 5, the labels of edges belonging to P_1^* appear exactly two times in cycle C_3, but they also appear exactly two times in cycle C_1. But this is only possible if they appear exactly two times in P_1^*, since H contains exactly to edges with the same label. By Proposition 5, P_1^* must be a closed path, which is a contradiction. □

Next, we establish some properties relating parameters of bipartite graphs having cactus as connected components.

Lemma 2. *Let H be a bipartite graph of order at least 4 such that all its connected components are cactus. Let $cc(H)$ denote the number of connected components of H. Then, $|V(H)| \geq \frac{3}{4}|E(H)| + cc(H) \geq \frac{3}{4}|E(H)| + 1$.*

Proof. Let $\mathrm{cc}(H)$ be the number of connected components and $\mathrm{cy}(H)$ the number of cycles of H. Since H is a planar graph with $\mathrm{cy}(H)+1$ faces and $\mathrm{cc}(H)$ connected components, the equality follows from the generalization of Euler's Formula:

$$(\mathrm{cy}(H) + 1) + |V(H)| = |E(H)| + (\mathrm{cc}(H) + 1).$$

Let $\mathrm{ex}(H) = |E(H)| - 4\,\mathrm{cy}(H)$. Then,

$$\begin{aligned}|V(H)| = |E(H)| - \mathrm{cy}(H) + \mathrm{cc}(H) &= |E(H)| - \frac{1}{4}(|E(H) - \mathrm{ex}(H)) + \mathrm{cc}(H) \\ &= \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H).\end{aligned}$$

But $\mathrm{cc}(H) \geq 1$, and $\mathrm{ex}(H) \geq 0$ because all cycles of a bipartite graph have at least 4 edges. Therefore,

$$|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H) \geq \frac{3}{4}|E(H)| + \mathrm{cc}(H) \geq \frac{3}{4}|E(H)| + 1. \qquad \square$$

Corollary 2. *Let S be a distinguishing set of a graph G. Consider a subgraph $H_{S'}$ of G^S induced by a set of edges containing exactly two edges with label u for each $u \in S' \subseteq S$. Let $r' = |S'|$. Then, $|V(H_{S'})| \geq \frac{3}{2}r' + 1$.*

3 The bipartite case

This section is devoted to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$ when we restrict ourselves to bipartite graphs. In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n = r + s \geq 4$, such that $V = U \cup W$, being U,W their stable sets and $1 \leq |U| = r \leq s = |W|$.

In the study of LD-sets, vertices with the same neighborhood play an important role, since at least one of them must be in an LD-set. We say that two vertices u and v are *twins* if either $N(u) = N(v)$ or $N(u) \cup \{u\} = N(v) \cup \{v\}$.

Lemma 3. *Let S be an LD-code of G. Then, $\lambda(\overline{G}) \leq \lambda(G)$ if any of the following conditions hold.*

i) $S \cap U \neq \emptyset$ *and* $S \cap W \neq \emptyset$.

ii) $r < s$ *and* $S = W$.

iii) $2^r \leq s$.

Proof. If S satisfies item i), then the LD-code of G is a distinguishing set of $\overline{G}$ and it is dominating in $\overline{G}$ because there is no vertex in G with neighbors in both stable sets. Thus, $\lambda(\overline{G}) \leq \lambda(G)$.

Next, assume that $r < s$ and $S = W$. In this case, $\lambda(G) = |W| > |U|$ and thus U is not an LD-set, but is it a dominating set since G is connected. Therefore, there exists a pair of vertices $w_1, w_2 \in W$ such that $N(w_1) = N(w_2)$. Hence, $W - \{w_1\}$ is an LD-set of $G - w_1$. Let $u \in S$ be a vertex adjacent to w_1 (it exists since G is connected), and notice that $(W \setminus \{w_1\}) \cup \{u\}$ is an LD-code of G with vertices in both stable sets, which, by the preceding item, means that $\lambda(\overline{G}) \leq \lambda(G)$.

Finally, if $2^r \leq s$ then $S \neq U$, which means that S satisfies either item i) or item ii). □

Proposition 6. *If G has order at least 3 and $1 \leq r \leq 2$, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proof. If $r = 1$, then G is the star $K_{1,n-1}$ and $\lambda(\overline{G}) = \lambda(G) = n - 1$.

Suppose that $r = 2$. We distinguish cases (see Figure 5).

- If $s \geq 2^2 = 4$ then, by Lemma 3, $\lambda(\overline{G}) \leq \lambda(G)$.
- If $s = 2$, then G is either P_4 and $\lambda(\overline{P_4}) = \lambda(P_4) = 2$, or G is C_4 and $\lambda(\overline{C_4}) = \lambda(C_4) = 2$.
- If $s = 3$, then G is either P_5, $K_{2,3}$, $K_2(1,2)$, or the banner P, and $\lambda(\overline{P_5}) = \lambda(P_5) = 2$, $\lambda(\overline{K_{2,3}}) = \lambda(K_{2,3}) = 3$, $2 = \lambda(\overline{K_2(1,2)}) < \lambda(K_2(1,3)) = 3$, and $2 = \lambda(\overline{\mathrm{P}}) < \lambda(\mathrm{P}) = 3$. □

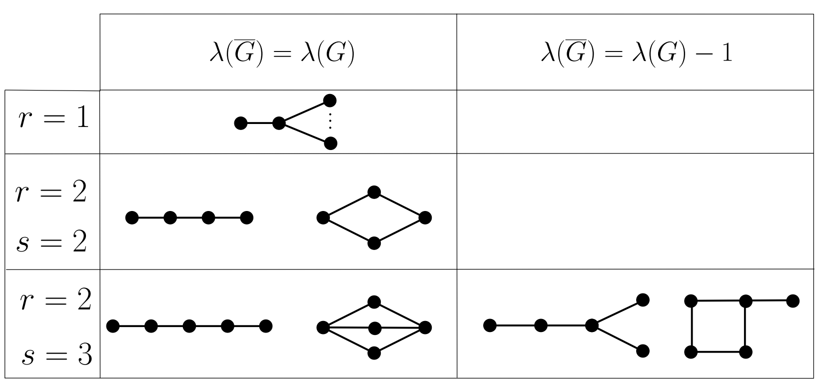

Figure 5: Some bipartite graphs with $1 \leq r \leq 2$.

From now on, we assume that $r \geq 3$.

Proposition 7. *If $r = s$, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proof. If G has an LD-code with vertices at both stable sets, then $\lambda(\overline{G}) \leq \lambda(G)$ by Lemma 3. In any other case, G has at most two LD-codes, U and W.

If both U and W are LD-codes, then we distinguish the following cases.

- If there is no vertex $u \in U$ such that $N(u) = W$, then W is an LD-set of $\overline{G}$, and consequently, $\lambda(\overline{G}) \leq \lambda(G)$.
- Analogously, if there is no vertex $w \in W$ such that $N(w) = U$, then we derive $\lambda(\overline{G}) \leq \lambda(G)$.
- If there exist vertices $u \in U$ and $w \in W$ such that $N(u) = W$ and $N(w) = U$, then $(U - \{u\}) \cup \{w\}$ would be an LD-set of $\overline{G}$, and thus $\lambda(\overline{G}) \leq \lambda(G)$.

Next, assume that U is an LD-code and W is not an LD-code of G. If there is no vertex $w \in W$ such that $N(w) = U$, then U is an LD-set of $\overline{G}$, and so $\lambda(\overline{G}) \leq \lambda(G)$. Finally, suppose that there is a vertex $w \in W$ such that $N(w) = U$. Note that W is not a distinguishing set of G (otherwise, it would be an LD-code because W is a dominating set of size r). Therefore, there exist vertices $x, y \in U$ such that $N(x) = N(y)$. In such a case, $(U \setminus \{x\}) \cup \{w\}$ is an LD-set of $\overline{G}$, and thus $\lambda(\overline{G}) \leq \lambda(G)$. □

From Lemma 3 and Proposition 7 we derive the following result.

Corollary 3. *If $\lambda(\overline{G}) = \lambda(G) + 1$, then $r < s \leq 2^r - 1$ and U is the only LD-code of G.*

Next theorem characterizes connected bipartite graphs satisfying the equation $\lambda(\overline{G}) = \lambda(G) + 1$ in terms of the graph associated with a distinguishing set.

Theorem 2. *Let $3 \leq r < s$. Then, $\lambda(\overline{G}) = \lambda(G) + 1$ if and only if the following conditions hold.*

i) W has no twins.

ii) There exists a vertex $w \in W$ such that $N(w) = U$.

iii) For every vertex $u \in U$, the graph G^U has at least two edges with label u.

Proof. $\Leftarrow$) Condition i) implies that U is an LD-set of G and, hence, $\lambda(G) \leq r$. Let S be an LD-code of $\overline{G}$. We next prove that S has at least $r + 1$ vertices. Note that $S \neq U$, since U is not a dominating set in $\overline{G}$. Consider the graph G^U associated with U. Let $H_{U \setminus S}$ be the subgraph of G^U induced by the set of edges with a label in $U \setminus S \neq \emptyset$. By Corollary 1, the vertices of the same connected component in $H_{U\setminus S}$ have the same neighborhood in $U\cap S$. Besides, W induces a complete graph in $\overline{G}$. Hence, $S\cap W$ must contain at least all but one vertices of the same connected component of $H_{U\setminus S}$, otherwise $\overline{G}$ would contain vertices with the same neighborhood in S. Therefore, $S\cap W$ has at least $|V(H_{U\setminus S})|-cc(H_{U\setminus S})$ vertices, where $cc(H_{U\setminus S})$ is the number of connected components of $H_{U\setminus S}$. Lemmas 1 and 2 imply that $|V(H_{U\setminus S})|-cc(H_{U\setminus S})\geq \frac{3}{4}|E(H_{U\setminus S})|$, and condition iii) implies that $|E(H_{U\setminus S})|\geq 2|U\setminus S|$. Therefore,

$$\begin{aligned}|S| &= |S\cap U|+|S\cap W|\geq |S\cap U|+|V(H_{U\setminus S})|-cc(H_{U\setminus S})\\ &\geq |S\cap U|+\frac{3}{4}|E(H_{U\setminus S})|\geq |S\cap U|+\frac{3}{2}|U\setminus S|\\ &= |U|+\frac{1}{2}|U\setminus S|>|U|=r\,.\end{aligned}$$

$\Rightarrow$) By Corollary 3, U is the only LD-code of G and hence, U is not an LD-set of $\overline{G}$. Therefore, W has no twins and $N(w)=U$ for some $w\in W$. It only remains to prove that condition iii) holds. Suppose on the contrary that there is at most one edge in G^U with label u for some $u\in U$. We consider two cases.

If there is no edge with label u, then by Proposition 4, $U\setminus\{u\}$ distinguishes all pairs of vertices of W in $\overline{G}$. Let $S=(U\setminus\{u\})\cup\{w\}$. We claim that S is an LD-set of $\overline{G}$. Indeed, S is a dominating set in $\overline{G}$, because u is adjacent to any vertex in $U\setminus\{u\}$ and vertices in $W\setminus\{w\}$ are adjacent to w. It only remains to prove that S distinguishes the pairs of vertices of the form u and v, when $v\in W\setminus\{w\}$. But $w\in N_{\overline{G}}(v)\cap S$ and $w\notin N_{\overline{G}}(u)\cap S$. Thus, S is an LD-set of $\overline{G}$, implying that $\lambda(\overline{G})\leq|S|=|U|=\lambda(G)$, a contradiction.

If there is exactly one edge xy with label u, then only one of the vertices x or y is adjacent to u in G. Assume that $ux\in E(G)$. Recall that $x,y\in W$. By Proposition 4, $U\setminus\{u\}$ distinguishes all pairs of vertices of W, except the pair x and y, in $\overline{G}$. Let $S=(U\setminus\{u\})\cup\{x\}$. We claim that S is an LD-set of $\overline{G}$. Indeed, S is a dominating set in $\overline{G}$, because u is adjacent to any vertex in $U\setminus\{u\}$ and vertices in $W\setminus\{x\}$ are adjacent to x. It only remains to prove that S distinguishes the pairs of vertices of the form u and v, when $v\in W\setminus\{x\}$. But $x\in N_{\overline{G}}(v)\cap S$ and $x\notin N_{\overline{G}}(u)\cap S$. Thus, S is an LD-set of $\overline{G}$, implying that $\lambda(\overline{G})\leq|S|=|U|=\lambda(G)$, a contradiction. □

Observe that condition *iii)* of Theorem 2 is equivalent to the existence of al least two pairs of twins in $G-u$, for every vertex $u\in U$. Therefore, it can be stated as follows.

Theorem 3. *Let* $3\leq r<s$. *Then,* $\lambda(\overline{G})=\lambda(G)+1$ *if and only if the following conditions hold:*

i) W *has no twins;*

ii) There exists a vertex $w\in W$ *such that* $N(w)=U$*;*

iii) For every vertex $u \in U$, the graph $G - u$ has at least two pairs of twins in W.

We already know that it is not possible to have $\lambda(\overline{G}) = \lambda(G) + 1$ when $s > 2^r$. However, the condition $s \leq 2^r - 1$ is not sufficient to ensure the existence of bipartite graphs satisfying $\lambda(\overline{G}) = \lambda(G) + 1$. We next show that there are graphs satisfying this equation if and only if $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.

Proposition 8. *If $r \geq 3$ and $\lambda(\overline{G}) = \lambda(G) + 1$, then $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.*

Proof. By Corollary 3, we have that $s \leq 2^r - 1$. Consider a subgraph H of G^U with exactly two edges with label u for every $u \in S$. Now inequeality $\frac{3r}{2} + 1 \leq s$ immediately follows from Proposition 2 and Corollary 2. □

Proposition 9. *For every pair (r, s), $r, s \in \mathbb{N}$, such that $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, there exists a bipartite graph $G(r, s)$ such that $\lambda(\overline{G}) = \lambda(G) + 1$.*

Proof. Let $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$. Take the bipartite graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ such that $V = U \cup W$, $U = [r] = \{1, 2, \ldots, r\}$, and $W \subseteq \mathcal{P}([r]) \setminus \{\emptyset\}$ is defined as follows (see Figure 3). For $r = 2k$ even:

$$W = \Big\{[r]\Big\} \cup \Big\{[r] \setminus \{i\} : i \in [r]\Big\} \cup \Big\{[r] \setminus \{2i-1, 2i\} : 1 \leq i \leq k\Big\}$$

and for $r = 2k + 1$ odd:

$$W = \Big\{[r]\Big\} \cup \Big\{[r] \setminus \{i\} : i \in [r]\Big\} \cup \Big\{[r] \setminus \{2i-1, 2i\} : 1 \leq i \leq k\Big\} \cup \Big\{[r] \setminus \{r-1, r\}\Big\}.$$

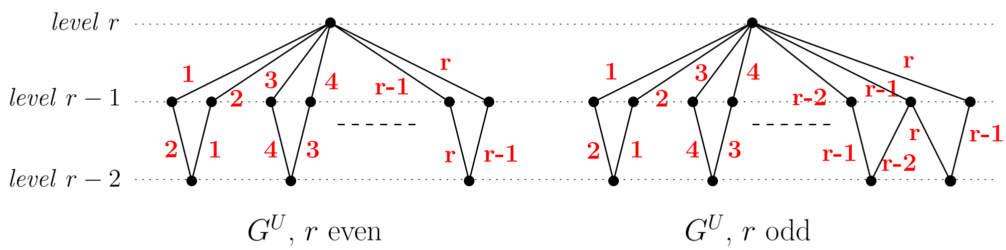

Figure 6: The labeled graph G^U, for $G = G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ and $U = \{1, \ldots, r\}$.

By construction, W has no twins, there is a vertex w such that $N(w) = U$ and the U-associated graph, G^U has at least two edges with label u for every $u \in U$. Hence, $\lambda(\overline{G}) = \lambda(G) + 1$ by Theorem 2.

For $s > \lceil \frac{3r}{2} + 1 \rceil$, we can add up to $2^r - 1 - r$ vertices to the set W of the graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ taking into account that the neighborhoods in S of vertices in W must be different and non-empty. □

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