

Master of Science in Advanced Mathematics and Mathematical Engineering

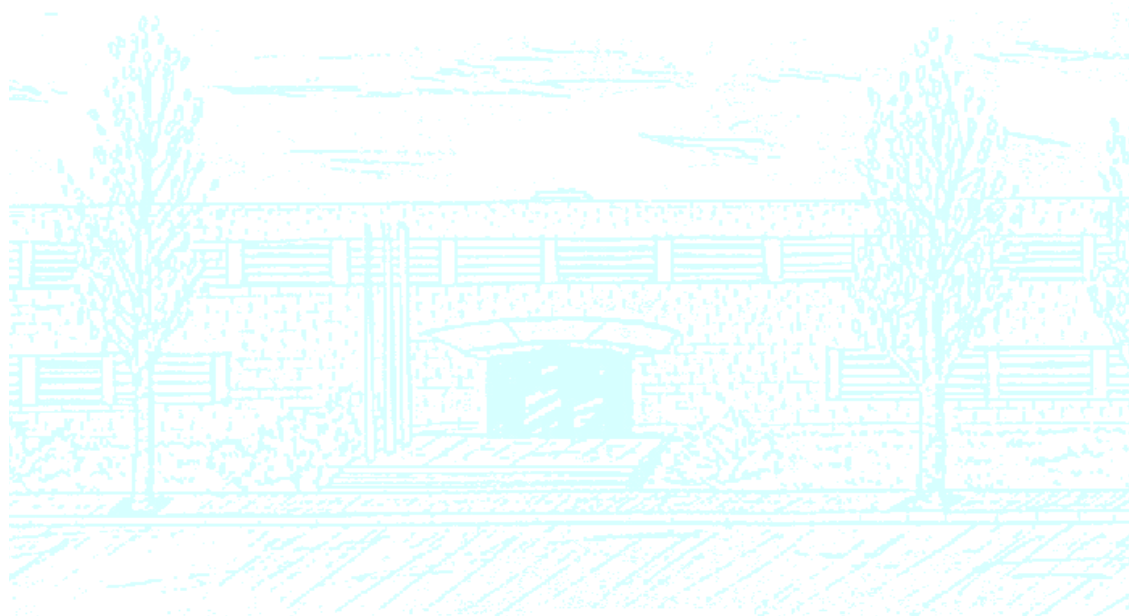
Title: Morita equivalence and decomposition spaces

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MASTER'S THESIS



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Facultat de Matemàtiques i Estadística

Universitat Politècnica de Catalunya
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Master in Advanced Mathematics and Mathematical Engineering
Master's thesis

Morita Equivalence and Decomposition Spaces

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Abstract

Inspired by prior results by Stanley [19] and Leroux [16] showing what information can be recovered from an isomorphism of incidence algebras, we investigate the very same idea applied to decomposition spaces. We review the work of Stanley and Leroux and provide sufficient background on the homotopy theory of groupoids to be able to define decomposition spaces, equivalences of them as linear functors and solve the isomorphism problem for both the groupoid-level coalgebra and for the numerical incidence algebra.

Keywords

Incidence algebra, isomorphism problem, groupoids, homotopy linear algebra, simplicial groupoids, decomposition spaces.

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1. Introduction

The field of combinatorics is full of examples of elegant proofs based on both algebraic and analytic methods. In the late 20th century, the probabilistic method appeared as well. Theories like generating functions and the symbolic method provide concise and rigorous solutions to purely combinatorial problems and can be applied in a systematic manner.

However, it is often said that algebraic proofs do not always provide the combinatorial insight that a direct proof in terms of bijections would. With the development of category theory in the last decades, there has appeared a new solution to this problem: combinatorial species and groupoids for instance provide a way to encode isomorphisms while still being able to do some algebra on them. For instance, we are going to show how one can perform linear algebra with groupoid coefficients based on the work of Gálvez-Carrillo, Kock and Tonks in [9]. Later we use these techniques to define the *incidence coalgebra* at the groupoid level as in [8], without collapsing isomorphisms and symmetries to rational coefficients.

One of these external sources of combinatorial information is the construction known as the *incidence algebra*. It is possible to define a k -algebra that reflects most of the combinatorial structure of some object, be it a poset, a monoid, a category or, more generally, a *decomposition space* [8]. Many properties can be recovered from inversion in this algebra or product formulas. One particularly well-known example is Möbius inversion. In elementary terms, for two functions $f, g : \mathbb{N}_{\geq 1} \rightarrow \mathbb{N}_{\geq 1}$ such that

$$g(n) = \sum_{d|n} f(d), \quad n \geq 1,$$

then

$$f(n) = \sum_{d|n} \mu(d)g\left(\frac{n}{d}\right), \quad \mu(d) = \begin{cases} 1 & \text{if } d \text{ is square-free with an even number of factors} \\ -1 & \text{if } d \text{ is square-free with an odd number of factors} \\ 0 & \text{otherwise} \end{cases}$$

This can be restated and generalized in terms of the incidence algebra of the divisibility poset: The analog statement states that the zeta function $\zeta(n) = 1$ is invertible as an element of the incidence algebra of the divisibility poset and that its inverse is the Möbius function, μ . This generalization step then provides special cases of this inversion formula for any suitable category [3] or decomposition space [7].

An interesting question to ask is how much information from the combinatorial object is actually reflected in the incidence algebra. A way to make this statement precise is to study whether an isomorphism of incidence algebras induces an isomorphism between the objects they were constructed from. The proof then hopefully provides an explicit bijection between components of the original structure. This kind of question is generally regarded as *Morita theory*. Originally as studied by Kiiti Morita, it consisted in characterizing rings that had equivalent module categories, but nowadays the concept is closer to the general idea of defining an equivalence relation based on an equivalence of associated structures. His idea has been successfully applied to multiple areas like ring theory, algebraic geometry and homotopy theory. In our case we consider posets, Möbius categories or decomposition spaces with isomorphic incidence algebras to be Morita-equivalent.

This particular problem regarding incidence algebras has been studied and solved in a couple of cases. Stanley proved in [19] that incidence algebras constructed from posets preserve the entire order relation, and an isomorphism of two incidence algebras induces an isomorphism between the two posets they were

constructed from. Leroux proves a similar statement for Möbius categories in [16], but the conclusion is not as strong as one could expect. We extend their work to the theory of *Segal spaces* and decomposition spaces and find appropriate conditions to be able to prove a similar result.

This work is organized in four main sections. In the section 2 we introduce the classical definition of the incidence algebra for posets with results of Stanley and examples. Then we explain the generalization by Leroux in section 3 and show a simple counter example showing that the result can not be strengthened much further. After that, we provide basic notions from the homotopy theory of groupoids and some category-theoretic constructions, including *homotopy linear algebra*. Finally, we use the tools from section 4 in section 5 to generalize the work of Leroux for the framework of decomposition spaces. There, we discuss two versions of the isomorphism problem: one at the groupoid level, and one which is closer to the classical variant in terms of a \mathbb{Q} -algebra obtained by computing the cardinality of the one at the groupoid level.

2. The incidence algebra associated to a poset

In this section we introduce the most basic setting to work with incidence algebras and the result that we aim to generalize for decomposition spaces. We are going to provide some basic results and intuition about incidence algebras in order to have some familiarity with them when working on more general constructions.

First we include the basic definitions and the key result by Stanley [19], where he completely solves the isomorphism problem for posets. We do not provide some proofs as we are going to prove more general versions of them in the next sections. In some cases however, it is illustrative to provide a sketch of the proof, as in the main theorem of this section. Then we prove the (much easier) converse implication and look some examples.

Recall that a partially ordered set (poset) is said to be locally finite if all its intervals

$$[x, y] = \{z \in P : x \leq z \leq y\}$$

are finite.

Then, given a locally finite poset P and a field k , one can construct the free k -vector space on its set of intervals $\langle \text{int } P \rangle_k$ and then define the operation $\Delta : \langle \text{int } P \rangle_k \rightarrow \langle \text{int } P \rangle_k \otimes \langle \text{int } P \rangle_k$ given by

$$\begin{aligned} \Delta([x, y]) &= \sum_{x \leq z \leq y} [x, z] \otimes [z, y] \\ &= [x, x] \otimes [x, y] + [x, y] \otimes [y, y] + \sum_{x < z < y} [x, z] \otimes [z, y]. \end{aligned}$$

Together with

$$\begin{aligned} \varepsilon : \langle \text{int } P \rangle_k &\rightarrow k \\ [x, y] &\mapsto \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

it defines a *coalgebra* structure on $\langle \text{int } P \rangle_k$ that we call the *incidence coalgebra* of P . Verifying the coassociativity is a simple exercise of applying the definition

$$\begin{aligned} (\text{id} \otimes \Delta)(\Delta([x, y])) &= (\text{id} \otimes \Delta) \left(\sum_{x \leq z \leq y} [x, z] \otimes [z, y] \right) \\ &= \sum_{x \leq z \leq y} [x, z] \otimes \left(\sum_{z \leq w \leq y} [z, w] \otimes [w, y] \right) \\ &= \sum_{x \leq z \leq w \leq y} [x, z] \otimes [z, w] \otimes [w, y] \\ &= (\Delta \otimes \text{id})(\Delta([x, y])) \end{aligned}$$

Counitality is even simpler:

$$\begin{aligned} (\text{id} \otimes \varepsilon)(\Delta([x, y])) &= (\text{id} \otimes \varepsilon) \left(\sum_{x \leq z \leq y} [x, z] \otimes [z, y] \right) \\ &= \sum_{x \leq z=y} [x, z] \otimes \varepsilon([z, y]) \\ &= [x, y] = (\varepsilon \otimes \text{id})(\Delta([x, y])). \end{aligned}$$

Note that the hypothesis that P is locally finite is important here. Otherwise we could not ensure that the sum in the definition of Δ is finite.

Although this coalgebra structure perfectly expresses the intent to study decompositions in P , coalgebras are nearly not as common as rings or k -algebras, so their treatment may feel a bit unintuitive or "backwards". For this reason, we switch to algebras instead: there is a standard procedure to define a k -algebra from any k -coalgebra, the resulting algebra is usually called the *convolution algebra*.

Given a k -coalgebra (C, Δ, ε) , one can obtain its convolution algebra as follows. Consider the dual vector space $A = \text{Hom}_k(C, k)$ of C and define the *convolution product* of $\phi, \psi \in A$ as

$$(\phi * \psi)(x) = (\phi \otimes \psi)(\Delta(x)),$$

with unit $\varepsilon \in A$. Diagrammatically, $\phi * \psi$ is given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\phi \otimes \psi} k \otimes k \xrightarrow{\cong} k.$$

Moreover, any morphism of coalgebras $f : C \rightarrow C'$ defines a morphism of algebras $f^* : \text{Hom}_k(C', k) \rightarrow \text{Hom}_k(C, k)$ by precomposition with f . Then one can verify that it preserves the multiplication: given $\phi', \psi' \in A'$, one has a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\Delta} & C \otimes C & & \\ \downarrow f & & \downarrow f \otimes f & & \\ C' & \xrightarrow{\Delta} & C' \otimes C' & \xrightarrow{\phi' \otimes \psi'} & k \otimes k \xrightarrow{\cong} k. \end{array}$$

The proof for unitality is similar but simpler. One can readily see that this operation is a contravariant functor from the category of k -coalgebras and coalgebra morphisms to the category of k -algebras and algebra morphisms $\mathbf{Coalg}_k^{\text{op}} \rightarrow \mathbf{Alg}_k$.

Applying this technique to the convolution algebra of a poset we obtain the *incidence algebra* $I(P)$ of P . Explicitly, it can be described as maps $\phi, \psi : \text{int } P \rightarrow k$ (recall that there is a natural bijection between set maps $\text{int } P \rightarrow k$ and linear maps $\langle \text{int } P \rangle_k \rightarrow k$) with multiplication

$$(\phi * \psi)([x, y]) = \sum_{x \leq z \leq y} \phi([x, z])\psi([z, y]).$$

A particularly illustrating subset of these maps is given by the dual basis of $\text{int } P$. For each $[x, y] \in \text{int } P$, one has the *characteristic function* $\chi_{[x, y]} : \text{int } P \rightarrow k$ which is 1 at $[x, y]$ and 0 at any other interval. Then, one can interpret the multiplication of these functions as a multiplication or composition of intervals

$$\begin{aligned} (\chi_{[x, z]} * \chi_{[z, y]})([s, t]) &= \begin{cases} 1 & \text{if } [s, t] = [x, y] \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{[x, y]}([s, t]). \end{aligned}$$

In the section 3 we will see how this precisely defines a composition in a suitable category, and how it allows to generalize this theory.

We also denote the characteristic functions at *degenerate* intervals as $\chi_x = \chi_{[x,x]}$. These constitute a surprisingly important submonoid of $I(P)$, where multiplication simply becomes pointwise multiplication in k :

$$(\chi_x * \chi_y)([s, t]) = \chi_x([s, t])\chi_y([s, t]) = \begin{cases} 1 & \text{if } s = t = x = y \\ 0 & \text{otherwise} \end{cases}$$

This is, either $\chi_x * \chi_y = \chi_x = \chi_y$ or $\chi_x * \chi_y = 0$.

The reason why these maps are useful is that they can be identified as the dual bases of the free vector space spanned by P itself. We consider this vector space, $\text{Hom}_{\text{Set}}(P, k)$, to be another k -algebra with pointwise multiplication

$$(\phi * \psi)(x) = \phi(x)\psi(x), \quad \phi, \psi \in \text{Hom}_{\text{Set}}(P, k)$$

and unit the constant map 1. Then it can be readily seen that the map

$$e_P : \text{Hom}_{\text{Set}}(P, k) \rightarrow I(P) \\ \phi \mapsto \bar{\phi},$$

where $\bar{\phi} : \text{int } P \rightarrow k$ is the extension of ϕ with 0's for nondegenerate intervals

$$\bar{\phi}([x, y]) = \begin{cases} \phi(x) & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

This clearly maps characteristic functions $\chi_x : P \rightarrow k$ to what we purposely denote in the same way, namely $\chi_{[x,x]} : \text{int } P \rightarrow k$.

Next, if we define

$$J(P) = \{\phi \in I(P) : \phi([x, x]) = 0 \forall x \in P\},$$

it can be easily proved to be a two-sided ideal of $I(P)$. Then we have a close relationship between $J(P)$ and the collection of characteristic functions χ_x .

Proposition 2.1. *The composition of the extension map e_P with the projection $I(P) \rightarrow I(P)/J(P)$ is an isomorphism of k -algebras.*

$$\text{Hom}_{\text{Set}}(P, k) \cong I(P)/J(P)$$

Moreover, its inverse is the passage to the quotient of the restriction map to degenerate intervals, r_P .

Proof. To see that it is injective, let $\phi \in \text{Hom}_{\text{Set}}(P, k)$ with $e_P(\phi) \in J(P)$. Then $\phi(x) = 0$ for all $x \in P$, so $\phi = 0$. For surjectivity, let $[\phi] \in I(P)/J(P)$ and observe that $[\phi] = [e_P(\phi|_P)]$, where we identify P again with the set of degenerate intervals. \square

We need to remind some algebraic terminology now. An *idempotent* element in a ring R (or algebra) is an element $x \in R$ such that $x^2 = x$. Two elements $x, y \in R$ are said to be *orthogonal* if $xy = yx = 0$ and $x \in R$ is *primitive* if $x = y + z$ for orthogonal idempotents y and z implies that $y = 0$ or $z = 0$. Now one can easily verify the following proposition:

Lemma 2.2. *The set of characteristic functions in $\text{Hom}_{\text{Set}}(P, k)$ is the unique maximal set of primitive orthogonal idempotents of $\text{Hom}_{\text{Set}}(P, k)$.*

Moreover, the key in Stanley's proof of the isomorphism problem lies in a purely ring-theoretic lemma.

Lemma 2.3 ([19]). *Let R be an associative ring. Suppose e, f, e', f' are idempotents in R such that $e' - e$ and $f' - f$ belong to a two-sided ideal J satisfying $\bigcap_{n \geq 1} J^n = 0$. Then $e \cdot R \cdot f = 0$ if and only if $e' \cdot R \cdot f' = 0$.*

Finally, one can solve the main isomorphism problem by combining all the results we have seen so far.

Theorem 2.4 ([19]). *Let P, Q be locally finite posets and $\Phi : I(P) \cong I(Q)$ an isomorphism of algebras. Then $P \cong Q$.*

Proof (Sketch). Firstly, observe that $x \leq y \iff \chi_x * I(P) * \chi_y \neq 0$. With this in mind, the family $\{\chi_x\}$ can be partially ordered as $\chi_x \leq \chi_y \iff \chi_x * I(P) * \chi_y \neq 0$, which gives a poset P' that is isomorphic to P . The next step is to prove that the construction of P' does not depend on the order relation in P by replacing $\{\chi_x\}_x$ by any maximal family of primitive orthogonal idempotents of $I(P)$.

For any other such family of primitive orthogonal idempotents $\{\psi_\alpha\}_{\alpha \in A}$, one can combine Proposition 2.1 and Lemma 2.2 to conclude that there is a bijection $\tau : P \rightarrow A$ such that $\chi_x - \psi_{\tau x} \in J(P)$. This is, $\psi_\alpha|_P = \chi_x$.

Then, by 2.3 together with the bijection τ , the partial order defined on $\{\psi_\alpha\}_\alpha$ as

$$\psi_\alpha \leq \psi_\beta \iff \psi_\alpha * I(P) * \psi_\beta \neq 0$$

is isomorphic to P' .

Finally, let $\Phi : I(P) \rightarrow I(Q)$ be an isomorphism of k -algebras, $\{\chi_x\}_x$ the family of idempotents of $I(P)$, P' the partial order induced by $\{\chi_x\}_x$ and Q' the partial order induced on $I(Q)$ by its corresponding family of idempotents. It is clear that Φ maps $\{\chi_x\}_x$ to $\{\Phi(\chi_x)\}_{x \in P}$, a maximal family of primitive orthogonal idempotents in $I(Q)$, which is a partial order isomorphic to Q' . In fact, the restriction of Φ to P' is also an isomorphism of partial orders:

$$\begin{aligned} \chi_x \leq \chi_y &\iff \chi_x * I(P) * \chi_y \neq 0 \\ &\iff \Phi(\chi_x * I(P) * \chi_y) = \Phi(\chi_x) * I(Q) * \Phi(\chi_y) \neq 0 \\ &\iff \Phi(\chi_x) \leq \Phi(\chi_y). \end{aligned}$$

Therefore, we get $P \cong P' \cong \Phi(P') \cong Q' \cong Q$. □

2.1 Functoriality

We now study the easier converse implication. To do this, we first detect which class of monotone maps between posets always induce algebra morphisms. These turn out to be the *local isomorphisms*.

Definition 2.5. Let P and Q be posets. A monotone map $f : P \rightarrow Q$ is a *local isomorphism* if f restricts to an isomorphism $[x, y] \rightarrow [f(x), f(y)]$ for each $x, y \in P$.

Observe that for a local isomorphism f , it is always true that $[f(x), f(y)] = f([x, y])$. The inclusion \supseteq is true in general, and \subseteq is due to the surjectivity of $f|_{[x, y]} : [x, y] \rightarrow [f(x), f(y)]$. Another immediate consequence is the following¹:

¹Recall that $x < y$ if and only if $x < y$ and there is no z with $x < z < y$.

Proposition 2.6. *A local isomorphism reflects the covering relation. In particular, it reflects the length of unrefinable chains.*

Proof. Let $f : P \rightarrow Q$ be a local isomorphism and P and Q locally finite posets. Observe that

$$x \triangleleft y \iff |[x, y]| = 2, \quad |[f(x), f(y)]| = 2 \iff f(x) \triangleleft f(y).$$

Since f provides a bijection $[x, y] \rightarrow [f(x), f(y)]$, we have $x \triangleleft y \iff f(x) \triangleleft f(y)$. \square

Example 2.7. The inclusion of an interval $[x, y] \subseteq P$ is always a local isomorphism. Indeed, if $[z, w] \subseteq [x, y]$, then the image of the inclusion is $[z, w]$ itself.

Notice that the special fact about local isomorphisms is that they preserve the decomposition structure of an interval. The following proposition exploits this fact in order to prove the functoriality of the incidence coalgebra (we write $\mathbf{LPos}_{\text{li,iso}}$ for the category of locally finite posets and local isomorphisms between them).

Lemma 2.8. *The rule $C : P \mapsto \langle \text{int } P \rangle_k$ defines a functor $\mathbf{LPos}_{\text{li,iso}} \rightarrow \mathbf{Coalg}_k$.*

Proof. For a monotone map $P \rightarrow Q$, define $\hat{f} : \langle \text{int } P \rangle_k \rightarrow \langle \text{int } Q \rangle_k$ as $\hat{f}([x, y]) = [f(x), f(y)]$, extended by linearity. It is obvious that this action preserves compositions and identities, so we just need to prove that these are coalgebra homomorphisms. Firstly, it preserves the counit ε

$$\begin{aligned} \varepsilon(\hat{f}([x, y])) &= \varepsilon([f(x), f(y)]) = \begin{cases} 1 & \text{if } f(x) = f(y) \\ 0 & \text{if } f(x) \neq f(y) \end{cases} \\ &= \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \end{aligned} \quad (f : [x, y] \xrightarrow{\cong} [f(x), f(y)])$$

and $(\hat{f} \otimes \hat{f})\Delta = \Delta\hat{f}$

$$\begin{aligned} (\hat{f} \otimes \hat{f})(\Delta([x, y])) &= (\hat{f} \otimes \hat{f})\left(\sum_{t \in [x, y]} [x, t] \otimes [t, y]\right) \\ &= \sum_{t \in [x, y]} \hat{f}([x, t]) \otimes \hat{f}([t, y]) \\ &= \sum_{t \in [x, y]} [f(x), f(t)] \otimes [f(t), f(y)] \\ &= \sum_{z \in f([x, y])} [f(x), z] \otimes [z, f(y)] \quad (f : [x, y] \xrightarrow{\cong} f([x, y])) \\ &= \sum_{z \in [f(x), f(y)]} [f(x), z] \otimes [z, f(y)] \quad (f([x, y]) = [f(x), f(y)]) \\ &= \Delta([f(x), f(y)]) = \Delta(\hat{f}[x, y]). \end{aligned}$$

\square

Now, we have seen above that the convolution algebra defines a (contravariant) functor. Thus, by composing we have that the incidence algebra is a functor

$$\mathbf{LPos}_{l.iso}^{op} \xrightarrow{C^{op}} \mathbf{Coalg}_k^{op} \xrightarrow{\text{Hom}_{\text{Set}}(-, k)} \mathbf{Alg}_k,$$

Functoriality then provides an immediate consequence: if $P \cong Q$, then $I(P) \cong I(Q)$. Together with Theorem 2.4, one obtains that $P \cong Q$ if and only if $I(P) \cong I(Q)$.

2.2 Example: subgroup lattices

To conclude this section, we show some examples with subgroup lattices of well-known groups. For G a group, we write $S(G)$ for the lattice of subgroups of G ordered by inclusion. In order to simplify the notation, let **grp** be the full subcategory of **Grp** (the category of all groups) with finite groups only.

Let G, H be two groups and $f : G \rightarrow H$ a group morphism. Then, f induces a monotonic function, $f(-) : S(G) \rightarrow S(H)$ that maps a subgroup to its image via f . This gives rise to a functor

$$\begin{aligned} S : \mathbf{Grp} &\rightarrow \mathbf{Pos} \\ G &\mapsto S(G) \\ f &\mapsto f(-) \end{aligned}$$

Proposition 2.9. *The functor S restricts to a functor $S : \mathbf{grp} \rightarrow \mathbf{LPos}$ from the category of finite groups to the category of locally finite posets and monotone maps. In fact, the following are equivalent for any group G :*

1. G is finite.
2. $S(G)$ is locally finite.
3. $S(G)$ is finite.

Proof. Clearly, (1) \Rightarrow (2) and (1) \Rightarrow (3). We also have that (2) \Leftrightarrow (3) because $S(G)$ is equal to the interval $[1, G]$. To check that (3) \Rightarrow (1), suppose G infinite and consider two cases:

- If all elements have finite order, $\{\langle x \rangle\}_{x \in G}$ is an infinite collection of subgroups, so $S(G)$ must be infinite.
- If there is an element $x \in G$ with infinite order, then $\{\langle x^n \rangle\}_{n \geq 0}$ is also an infinite collection of subgroups. □

Example 2.10. In this example we compute the incidence algebra of the subgroup lattice of the cyclic group $C_4 = \mathbb{Z}_4$. Note that C_4 has exactly two subgroups: the trivial one and $H = \langle 2 \rangle \cong C_2$. Therefore, the set of intervals of the poset $P = S(C_4) = \{0 < H < C_4\}$ is

$$\text{int}(S(C_4)) = \{[0, 0], [0, H], [0, C_4], [H, H], [H, C_4], [C_4, C_4]\}.$$

First, we compute an example of a comultiplication in the incidence coalgebra of P :

$$\Delta([0, C_4]) = [0, 0] \otimes [0, C_4] + [0, H] \otimes [H, C_4] + [0, C_4] \otimes [C_4, C_4].$$

Next, we have that the incidence algebra of this poset over some field k is $\text{Hom}(\text{int}(P), k)$, with multiplication given by

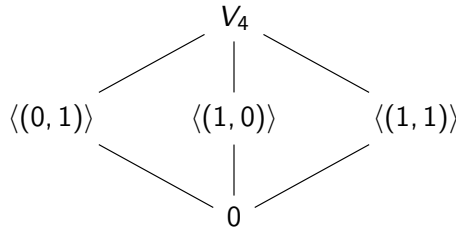
$$(\phi * \psi)([G_1, G_2]) = \sum_{G_1 \hookrightarrow G_3 \hookrightarrow G_2} \phi([G_1, G_3])\psi([G_3, G_2]).$$

In addition, notice that $I(S(C_4))$ has finite dimension 6 and its canonical basis consists of

- Characteristic functions at degenerate intervals: $\chi_0, \chi_H, \chi_{C_4}$.
- Characteristic functions at nondegenerate intervals: $\chi_{[0,H]}$, $\chi_{[H,C_4]}$ and χ_{0,C_4} .

As we have seen, we can understand most of the incidence algebra by looking at these elements.

Example 2.11. We now repeat a similar process with the Klein group $V_4 = C_2 \times C_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have the following subgroup lattice, P :



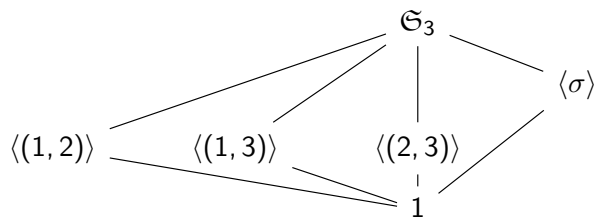
Similarly to the previous case, the incidence algebra of P is generated by all the intervals of the form $[G_1, G_2]$, with $G_1 \hookrightarrow G_2 \hookrightarrow V_4$. The only difference in this case is that the whole poset interval $[0, V_4]$ is not a chain.

Now the basis consists of 5 characteristic functions at degenerate intervals and 6 at nondegenerate ones. Moreover, we have a new trait that did not appear in the example with C_4 as it was a chain:

$$\chi_{[0, \langle(0,1)\rangle]} * \chi_{[\langle(0,1)\rangle, V_4]} = \chi_{[0, \langle(1,0)\rangle]} * \chi_{[\langle(1,0)\rangle, V_4]} = \chi_{[0, \langle(1,1)\rangle]} * \chi_{[\langle(1,1)\rangle, V_4]} = \chi_{[0, V_4]}$$

This is, if two chains have the same endpoints, then their corresponding product of characteristic functions coincide and equals the interval defined by the two endpoints.

Example 2.12. Next, we look at the incidence algebra of the subgroup lattice of the symmetric group \mathfrak{S}_3 . We know that $\mathfrak{S}_3 = \langle(1, 2), (1, 3), (2, 3), \sigma = (1, 2, 3)\rangle$. Then, we have the following subgroup lattice:



Observe that this lattice is extremely similar to V_4 's: instead of 3 intermediate subgroups we have 4. Then, the same results hold almost verbatim with the only difference being that we now have one more chain that reaches the top element.

3. Möbius categories

In this section we are going to generalize these results from the previous section about posets to Möbius categories as an intermediate step before delving into decomposition spaces. Doing so will expose some issues that necessarily show up when studying the isomorphism problem for decomposition spaces. Most of the material in this section is originally due to Content, Lemay and Leroux ([3], [16]).

First, we need to explain what the concepts that we used to define incidence algebras for posets become in this setting. Let P be a poset and regard it as a category \mathcal{C} (so its objects are the elements of P and there is one arrow $x \rightarrow y$ if and only if $x \leq y$). There is a bijection between $\text{int } P$ and $\text{Mor } \mathcal{C}$, the set of morphisms of \mathcal{C} :

$$\begin{aligned} \text{int } P &\rightarrow \text{Mor } \mathcal{C} \\ [x, y] &\mapsto x \xrightarrow{\exists!} y \end{aligned}$$

Moreover, identities (which are unique at each object in any category) correspond to degenerate intervals $[x, x]$. This already indicates that the incidence algebra of a category should be defined in terms of its set of morphisms.

Next, we define the analogue of locally finite posets for categories. If a category \mathcal{C} is a poset and $f : x \rightarrow y$ is the arrow in \mathcal{C} corresponding to an interval $[x, y]$, there is a bijection

$$\begin{aligned} (f) &\rightarrow [x, y] \\ (f', f'') &\mapsto \text{cod } f' = \text{dom } f'', \end{aligned}$$

where

$$(f) = \{(f', f'') : f'f'' = f\}$$

is the set of factorizations of f . Therefore, we must require these sets to be finite.

Definition 3.1. A *decomposition-finite* category is a small category \mathcal{C} where (f) is finite for any $f \in \text{Mor } \mathcal{C}$.

We can now proceed to define the incidence algebra of a decomposition-finite category.

Definition 3.2. Similarly as for locally finite posets, the *incidence algebra* over a field k of a decomposition-finite category \mathcal{C} is defined as the set of maps from $\text{Mor } \mathcal{C}$ to the field k

$$I(\mathcal{C}) = \text{Hom}_{\text{Set}}(\text{Mor } \mathcal{C}, k)$$

with multiplication of $\phi, \psi : \text{Mor } \mathcal{C} \rightarrow k$ given by

$$(\phi * \psi)(f) = \sum_{f=f'f''} \phi(f') \cdot \psi(f'')$$

and unit

$$\delta(f) = \begin{cases} 1 & \text{if } f = \text{id}_a \text{ for some } a \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

Notice that this definition is essentially what we did for posets, although in this case we skipped the construction of the coalgebra (the incidence algebra being its linear dual) for brevity. The actual isomorphism

is the classical one from the construction of the free vector space, which is a coalgebra isomorphism as well due to how the convolution algebra is defined:

$$I(\mathcal{C}) = \text{Hom}_{\text{Set}}(\text{Mor } \mathcal{C}, k) \cong \text{Hom}_{\text{Vect}_k}(\langle \text{Mor } \mathcal{C} \rangle, k) = \langle \text{Mor } \mathcal{C} \rangle_k^*$$

Associativity is a direct consequence of the associativity of composition in \mathcal{C}

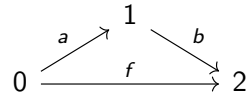
$$\begin{aligned} (\phi * (\psi * \theta))(f) &= \sum_{f=f_1 f_2} \phi(f_1)(\psi * \theta)(f_2) \\ &= \sum_{f=f_1 f_2} \sum_{f_2=f_{21} f_{22}} \phi(f_1)\psi(f_{21})\theta(f_{22}) \\ &= \sum_{f=f_1 f_2 f_3} \phi(f_1)\psi(f_2)\theta(f_3) \\ &= ((\phi * \psi) * \theta)(f) \end{aligned}$$

and unitality from elementary facts about composition with identities

$$(\delta * \phi)(f) = \sum_{f=f_1 f_2} \delta(f_1)\phi(f_2) = \delta(\text{id}_{\text{cod } f})\phi(f) = \phi(f) = \phi(f)\delta(\text{id}_{\text{dom } f}) = (\phi * \delta)(f).$$

As before, for $f \in \text{Mor } \mathcal{C}$ we denote the characteristic function which is 1 at f and 0 otherwise by $\chi_f : \text{Mor } \mathcal{C} \rightarrow k$. For a subset U of $\text{Mor } \mathcal{C}$, we define $\chi_U = \{\chi_f : f \in U\}$. We will also write any restriction of χ_f as χ_f and deduce its domain from the context. The following example illustrates these operations and similarities with the case of posets.

Example 3.3. Consider the category



where $f = ba$. Including identities, it consists of 6 arrows, so $I(\mathcal{C}) \cong k^6$ as vector spaces, where the basis of $I(\mathcal{C})$ is given by the characteristic functions $\chi_0, \chi_1, \chi_2, \chi_a, \chi_b, \chi_f : \text{Mor } \mathcal{C} \rightarrow k$

$$\chi_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$$

Then the multiplication is just

$$\chi_g * \chi_h = \begin{cases} \chi_{gh} & \text{if } g \circ h \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Observe that this category is just the reinterpretation of the poset $\{0 < 1 < 2\}$ and that their incidence algebras coincide²: both have dimension 6 and it is easily verified that the correspondence explained above between arrows and intervals induces the desired isomorphism by linear extension.

²Here multiplication factors are reversed due to composition syntax. Some authors simply give an alternative definition, but they are all functionally equivalent in practice.

Recall that a group (or a ring, field or k -algebra) structure on a set A is *topological* if A is a topological space and the operations of A are continuous. If k is a topological field, then $I(\mathcal{C})$ naturally becomes a topological k -algebra with the product topology regarding $\text{Hom}_{\text{Set}}(\text{Mor } \mathcal{C}, k)$ as the product $k^{\text{Mor } \mathcal{C}}$. This is yet another difference from the simpler case with posets, where the main proof did not require any topology (explicitly).

Our work with the topology of $I(\mathcal{C})$ will be based on convergent *nets* (sometimes called *generalized sequences*). A net in $I(\mathcal{C})$ is a map $A \rightarrow I(\mathcal{C})$ for A a directed set (i.e. any two elements in A have a common upper bound). These provide direct generalizations of results about sequences in metric spaces to topological spaces. A classical reference on nets in topology can be found in Kelley's *General Topology* [11].

Proposition 3.4. *For k a topological field, a net (ϕ_α) in $I(\mathcal{C})$ converges to ϕ if and only if $(\phi_\alpha(f))$ converges to $\phi(f)$ in k for all $f \in \text{Mor } \mathcal{C}$. Moreover, $I(\mathcal{C})$ is a topological k -algebra and it is Hausdorff whenever k is.*

Proof. We only need to show that scalar multiplication, internal multiplication and addition are continuous, the rest of the facts follow directly from completely general results explained in [11, Chapter 2]. The continuity of these operations is proved by exploiting the characterization of limits and the continuity of the operations in k . The proof for each operation is almost identical, for instance addition is continuous because

$$\lim_{\alpha} \phi_{\alpha} + \lim_{\beta} \psi_{\beta} = \lim_{(\alpha, \beta)} (\phi_{\alpha} + \psi_{\beta}) \iff \lim_{\alpha} \phi_{\alpha}(f) + \lim_{\beta} \psi_{\beta}(f) = \lim_{(\alpha, \beta)} (\phi_{\alpha} + \psi_{\beta})(f) \quad \forall f \in \text{Mor } \mathcal{C}. \quad \square$$

We will regard $I(\mathcal{C})$ as a Hausdorff topological k -algebra with respect to this topology for the rest of this section. If k is not inherently a Hausdorff topological field, then we can assume the discrete topology for k . The following result is a well-known fact about topological rings (see [20, §1.4]), but it is a simple example of the kind of argument that we are going to use throughout this section.

Proposition 3.5. *Let I be a left (right) ideal of a (not necessarily commutative) topological ring R . The topological closure \bar{I} of I is again a left (right) ideal of R .*

Proof. Any $r, s \in \bar{I}$ can be rewritten as limits of convergent nets (r_{α}) and (s_{β}) of I . Then by continuity of addition and iterating the sum of limits we get that

$$r + s = \lim_{\alpha} r_{\alpha} + \lim_{\beta} s_{\beta} = \lim_{\alpha} \lim_{\beta} (r_{\alpha} + s_{\beta}) = \lim_{(\alpha, \beta)} (r_{\alpha} + s_{\beta}),$$

hence $r + s \in \bar{I}$ because each $r_{\alpha} + s_{\beta}$ belongs to I . Similarly, for any $t \in R$,

$$t \lim_{\alpha} r_{\alpha} = \lim_{\alpha} t r_{\alpha} \in \bar{I}$$

again because each $t r_{\alpha}$ belongs to I . □

Proposition 3.6. *Let $(\lambda_f)_{f \in U}$ a collection of scalars in k indexed by $U \subseteq \text{Mor } \mathcal{C}$. Consider the set*

$$\mathcal{P}_{\text{fin}}(U) = \{S \subseteq U : S \text{ is finite}\},$$

which is a lattice when ordered by inclusion and, in particular, a directed set. Then the net $(\sum_{f \in S} \lambda_f \chi_f)_{S \in \mathcal{P}_{\text{fin}}(U)}$ converges to

$$\begin{aligned} \phi : \text{Mor } \mathcal{C} &\rightarrow k \\ f &\mapsto \begin{cases} \lambda_f & \text{if } f \in U \\ 0 & \text{if } f \notin U. \end{cases} \end{aligned}$$

This is, $\phi = \lim_{S \subseteq_{\text{fin}} U} \sum_{f \in S} \lambda_f \chi_f$.

Proof. For any $g \in \text{Mor } \mathcal{C}$, consider two cases:

- If $g \notin U$, then for any finite subset S of U

$$\left(\sum_{f \in S} \lambda_f \chi_f \right)(g) = 0,$$

so $(\sum_{f \in S} \lambda_f \chi_f)(g) \rightarrow 0$ in k because it is a constant net.

- If $g \in U$, then for any finite subset S of U greater than (i.e. containing) $\{g\}$,

$$\left(\sum_{f \in S} \lambda_f \chi_f \right)(g) = \lambda_g \chi_g(g) = \lambda_g.$$

This implies that the (cofinal) subnet indexed by $\mathcal{P}_{\text{fin}}(U)$ is the constant net λ_g , so $(\sum_{f \in S} \lambda_f \chi_f)(g) \rightarrow \lambda_g$ in k .

This means that $(\sum_{f \in S} \lambda_f \chi_f)_S \rightarrow \phi$ in $I(\mathcal{C})$. □

In light of this result, we may express any $\phi \in I(\mathcal{C})$ as a (possibly infinite) sum

$$\phi = \sum_{f \in U} \phi(f) \chi_f,$$

where U is either the support of ϕ or simply $\text{Mor } \mathcal{C}$, just like an infinite linear combination or the sum of a numerical series.

Next we have one of the core definitions of this theory, again due to Leroux et al. in [3] and [16]. The length of a morphism essentially measures how decomposable an arrow is. For instance, the composition of three indecomposable arrows has length at least 3.

Definition 3.7. Let \mathcal{C} be a category and f a morphism in \mathcal{C} . The *length* of f (if it exists) is

$$\ell(f) = \sup\{n \in \mathbb{N} : \exists f_1, \dots, f_n \neq \text{id s.t. } f = f_1 \cdots f_n\},$$

We also define

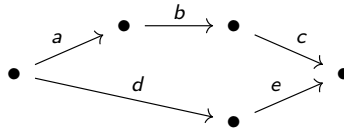
$$\mathcal{C}_n = \{f \in \text{Mor } \mathcal{C} : \ell(f) = n\}, \quad \mathcal{C}_n(A, B) = \text{Hom}_{\mathcal{C}}(A, B) \cap \mathcal{C}_n, \quad n \geq 1.$$

For the rest of this section, we will often write A for id_A .

Based on this concept of length one defines what turns out to be the appropriate class of categories to work with incidence algebras and Möbius inversion.

Definition 3.8. A *Möbius category* is a decomposition-finite category in which $\ell(f)$ exists for any morphism f .

Observe that Möbius categories never contain non-identity idempotents or (left or right) invertible morphisms. In particular, they are *skeletal* (i.e. isomorphism classes are trivial). Thus, in a Möbius category, $\ell(f) = 0$ precisely when f is an identity, so $\mathcal{C}_0 = \{\text{id}_A : A \in \mathcal{C}\}$ corresponds to the set of objects of \mathcal{C} . It is also easily seen that $\ell(f) + \ell(g) \leq \ell(fg)$ for any two morphisms f and g of \mathcal{C} . The equality does not hold in general though, as the example illustrates:



If $cba = ed$ and all displayed arrows have length 1, then $\ell(ed) = 3 > \ell(e) + \ell(d) = 2$.

A particularly important theorem introduced in [3, Theorem 1.1] provides an inversion formula for elements of the incidence algebra, a generalization of Möbius inversion.

Theorem 3.9 (Möbius inversion). *A Möbius category is a decomposition-finite category in which any $\phi \in I(\mathcal{C})$ is invertible if and only if $\phi(\text{id}_A) \neq 0 \ \forall A \in \mathcal{C}$.*

Perhaps even more importantly, the proof of this theorem provides an explicit (recursive) formula for inversion in the incidence algebra:

$$\phi^{-1}(f) = \begin{cases} \phi(f)^{-1} & \text{if } \ell(f) = 0 \\ -\phi(\text{dom } f)^{-1} \sum_{\substack{f=gh \\ \ell(h) > 0}} \phi^{-1}(g)\phi(h) & \text{if } \ell(f) > 0. \end{cases}$$

3.1 Functoriality of the incidence algebra

In this section we are going to take a look at which functors induce algebra morphisms in a similar fashion as local isomorphisms did for posets. This should allow us to prove the easy part of the isomorphism problem: do equivalent Möbius categories have isomorphic incidence algebras?

Definition 3.10. We say that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *local isomorphism* if $F \times F : (f) \rightarrow (F(f))$ is a bijection for each f in \mathcal{C} .

Given a local isomorphism functor $F : \mathcal{C} \rightarrow \mathcal{D}$, it induces a linear map $F^* : I(\mathcal{D}) \rightarrow I(\mathcal{C})$ defined as $F^* : \phi \mapsto \phi \circ F$.

Proposition 3.11. *Local isomorphisms are length-preserving.*

Proof. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a local isomorphism and \mathcal{C}, \mathcal{D} Möbius categories. First, we show that $F(f) = \text{id}$ implies $f = \text{id}$. Recall that $(\text{id}, \text{id}) \in (\text{id})$ and $(\text{id}, f), (f, \text{id}) \in (f)$. Since F is a local isomorphism, $(\text{id}, f) \mapsto (\text{id}, F(f)) = (\text{id}, \text{id})$ and $(f, \text{id}) \mapsto (F(f), \text{id}) = (\text{id}, \text{id})$ implies that $(f, \text{id}) = (\text{id}, f)$, so $f = \text{id}$.

Next, we prove that $\ell(F(f)) = 1$ implies $\ell(f) = 1$. Observe that any non-degenerate decomposition $f = f_1 f_2$ would be mapped to a non-degenerate decomposition $F(f) = F(f_1)F(f_2)$, so $\ell(f) \leq 1$. By functoriality $\ell(f) = 0$ would imply $\ell(F(f)) = 0$, hence $\ell(f) = 1$.

Now, let f be an arrow of length $n \geq 1$ and let $f = f_1 \cdots f_n$ be a decomposition of f into arrows of length 1. Then $F(f) = F(f_1) \cdots F(f_n)$ with each $F(f_i)$ of length 1, so $\ell(F(f)) \geq \ell(f)$. Finally, we prove that $\ell(f) = \ell(F(f))$ by induction on $\ell(F(f))$. The base case is already proved by combining all the previous facts, so assume that $\ell(F(f)) = n > 1$ and that $\ell(g) = \ell(F(g))$ for any g with $\ell(F(g)) < n$. Let $F(f) = h_1 h_2$ with $\ell(h_1) = 1$ and $\ell(h_2) = n - 1$. Then $(h_1, h_2) \in F(f)$, so there exists $(f_1, f_2) \in (f)$ with $(F(f_1), F(f_2)) = (h_1, h_2)$. By induction hypothesis, $\ell(f_i) = \ell(F(f_i)) = \ell(h_i)$, hence

$$\ell(f) = \ell(f_1 f_2) \geq \ell(f_1) + \ell(f_2) = \ell(h_1) + \ell(h_2) = \ell(F(f)). \quad \square$$

Proposition 3.12. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a local isomorphism between Möbius categories, then $F^* : I(\mathcal{D}) \rightarrow I(\mathcal{C})$ is a morphism of algebras.*

Proof. Firstly, $F^*(\delta) = \delta$ by Proposition 3.11:

$$\begin{aligned} (\delta \circ F)(f) &= \delta(F(f)) = \begin{cases} 1 & \text{if } \ell(F(f)) = 0 \\ 0 & \text{if } \ell(F(f)) \neq 0 \end{cases} \\ &= \begin{cases} 1 & \text{if } \ell(f) = 0 \\ 0 & \text{if } \ell(f) \neq 0 \end{cases} \\ &= \delta(f). \end{aligned}$$

Finally, the preservation of the convolution product is proved in an analogous way as for posets.

$$\begin{aligned} (F^*(\phi) * F^*(\psi))(f) &= \sum_{f' f'' = f} \phi(F(f')) \psi(F(f'')) \\ &= \sum_{h' h'' = F(f)} \phi(h') \psi(h'') \\ &= (\phi * \psi)(F(f)) = F^*(\phi * \psi)(f). \end{aligned} \quad \square$$

Observe that the construction of the incidence algebra can be described in terms of the simpler functors Mor^3 and $\text{Hom}(-, k)$. If we let $\mathbf{Möb}$ denote the category of Möbius categories and local isomorphisms, $U : \mathbf{Alg}_k \rightarrow \mathbf{Vect}_k$ the forgetful functor and $I : \mathbf{Möb} \rightarrow \mathbf{Alg}_k^{\text{op}}$ is the incidence algebra functor (precomposition is always functorial and we just proved that it is well-defined)

$$\begin{array}{ccccc} \mathbf{Möb} & \xrightarrow{\text{Mor}} & \mathbf{Set} & \xrightarrow{\text{Hom}(-, k)} & \mathbf{Vect}_k^{\text{op}} \\ & & & & \uparrow U^{\text{op}} \\ & & & & \mathbf{Alg}_k^{\text{op}} \\ & \searrow I & & & \end{array}$$

³The action of Mor on morphisms simply maps a functor F to the disjoint union all the maps $\text{Hom}(a, b) \rightarrow \text{Hom}(F(a), F(b))$

In particular, note that Mor maps faithful functors to injective maps (Hom sets are disjoint by definition) and $\text{Hom}(-, k)$ is easily verified to send injective maps to surjections. Therefore a faithful local isomorphism $\mathcal{C} \rightarrow \mathcal{D}$ induces a surjection $I(\mathcal{D}) \twoheadrightarrow I(\mathcal{C})$ (so $I(\mathcal{C})$ is isomorphic to a quotient of $I(\mathcal{D})$) because U is the identity on morphisms.

If we consider equivalences $\mathcal{C} \rightarrow \mathcal{D}$, these are simply isomorphisms because Möbius categories are skeletal. Clearly an isomorphism is also a local isomorphism, hence the equivalence $\mathcal{C} \rightarrow \mathcal{D}$ induces an isomorphism of algebras $I(\mathcal{D}) \rightarrow I(\mathcal{C})$ by functoriality.

3.2 The isomorphism problem

The idea behind this solution to the isomorphism problem is to reflect enough combinatorial information in the ring structure of $I(\mathcal{C})$, which is necessarily preserved by k -algebra isomorphisms. Using Theorem 3.9 it is possible encode some of these in algebraic terms. The key fact is the relation between length and the powers of *Jacobson radical*.

Definition 3.13. For $n \geq 0$, define

$$J_n(\mathcal{C}) = \{\phi \in I(\mathcal{C}) : \forall f, \ell(f) < n \implies \phi(f) = 0\}.$$

We will use the shorthand notation $J_n = J_n(\mathcal{C})$ if \mathcal{C} is clear from the context.

One of the characterizations of the Jacobson radical of a ring R that can be found in most noncommutative algebra books is (for incidence algebras we write $J(\mathcal{C}) = J(I(\mathcal{C}))$ or even just J as long as it is not ambiguous)

$$J(R) = \{a \in R : \forall b, c \in R, 1 - bac \text{ is invertible}\}.$$

A priori, these two appear to be unrelated, but the following proposition shows that they are extremely similar.

Proposition 3.14. *Let \mathcal{C} be a Möbius category and $J^n(\mathcal{C})$ the n th power of Jacobson radical of $I(\mathcal{C})$. Then*

1. $J_1 = J$ [16, Proposition 0.1],
2. $J_n = \overline{J^n}$ (the topological closure in $I(\mathcal{C})$) and
3. $J_n/J_{n+1} \cong \text{Hom}(\mathcal{C}_n, k)$ as k -vector spaces via the restriction to \mathcal{C}_n of maps $\text{Mor } \mathcal{C} \rightarrow k$. Thus, $\text{Hom}(\mathcal{C}_0, k)$ becomes a k -algebra.

Proof. We have that $\phi \in J(I(\mathcal{C}))$ if and only if for all $\eta, \psi \in I(\mathcal{C})$ we have that $\delta - \eta * \phi * \psi$ is invertible. By 3.9, this is equivalent to

$$0 \neq (\delta - \eta * \phi * \psi)(A) = 1 - \eta(A)\phi(A)\psi(A) \quad \forall A \in \mathcal{C}_0$$

Summing up,

$$\phi \in J(I(\mathcal{C})) \iff \forall \eta, \psi, A, \eta(A)\phi(A)\psi(A) \neq 1.$$

By choosing η, ψ and A appropriately one may always achieve $\eta(A)\phi(A)\psi(A) = 1$ for some A unless $\phi(A) = 0$ for all A , so this must be the case.

For (2), let $\phi_1 * \cdots * \phi_n$ be a generator of J^n such that $\phi_i \in J = J_1$ and $f \in \text{Mor } \mathcal{C}$ with $\ell(f) < n$. Then

$$\phi(f) = (\phi_1 * \cdots * \phi_n)(f) = \sum_{f_1 \cdots f_n = f} \phi_1(f_1) \cdots \phi_n(f_n).$$

Since $\ell(f) < n$, any decomposition $f_1 \cdots f_n = f$ will have some $f_i \in \mathcal{C}_0$, hence $\phi_i(f_i) = 0$, $\phi(f) = 0$ and $\phi \in J_n$. To see that J_n is closed, let (ϕ_α) be a net of J_n that converges to ϕ in $I(\mathcal{C})$. Then $\phi_\alpha(f)$ converges to $\phi(f)$, but all $\phi_\alpha(f)$ are 0 whenever $\ell(f) < n$, hence $\phi(f) = 0$ whenever $\ell(f) < n$ so J_n is closed. This proves that $J_n \supseteq \overline{J^n}$.

For the converse inclusion, let $\phi \in J_n$, $n > 0$ and rewrite it as

$$\phi = \sum_{f \in U} \phi(f) \chi_f, \quad U = \{f : \ell(f) \geq n\}$$

as in Proposition 3.6. Furthermore, if $\ell(f) \geq n$, then there is a non-degenerate decomposition $f = f_1 \cdots f_n$ and hence $\chi_f = \chi_{f_1} * \cdots * \chi_{f_n}$ with each $\chi_{f_i} \in J_1$. This implies that if $\ell(f) = n$, then $\chi_f \in J^n$. Since each partial sum $\sum_{f \in S} \phi(f) \chi_f$ in the limit

$$\lim_{S \subseteq_{\text{fin}} U} \sum_{f \in S} \phi(f) \chi_f$$

belongs to J^n , we conclude that $\phi \in \overline{J^n}$. The case for $n = 0$ needs special treatment but it is trivial since $J_0 = I(\mathcal{C})$.

Finally, we consider the quotient J_n/J_{n+1} . Define a linear map

$$\begin{aligned} J_n &\rightarrow \text{Hom}(\mathcal{C}_n, k) \\ \phi &\mapsto \phi|_{\mathcal{C}_n}, \end{aligned}$$

which is clearly surjective by definition of J_n and its kernel is precisely J_{n+1} (again, by definition of J_{n+1}). Therefore $J_n/J_{n+1} \cong \text{Hom}(\mathcal{C}_n, k)$. \square

Corollary 3.15. *If \mathcal{C} has finitely many morphisms, then $J^n(\mathcal{C}) = J_n(\mathcal{C})$.*

Proof. Since finite products of discrete spaces are discrete, all subsets of the incidence algebra are closed hence $J^n(\mathcal{C}) = \overline{J^n(\mathcal{C})} = J_n(\mathcal{C})$. \square

Note that the composite $I(\mathcal{C}) \rightarrow I(\mathcal{C})/J(\mathcal{C}) \rightarrow \text{Hom}(\mathcal{C}_0, k)$ ($r_{\mathcal{C}}$ from now on) has a section: it is the extension of a map $\mathcal{C}_0 \rightarrow k$ with zeros outside of \mathcal{C}_0 , which we denote it by $e_{\mathcal{C}}$. This fact is obvious because restricting a map that has been extended with zeros yields the original map.

Proposition 3.14 has a few important consequences. Firstly, observe that the induced multiplication on $\text{Hom}(\mathcal{C}_0, k)$ is simply pointwise multiplication and unit the constant map 1, so $\text{Hom}(\mathcal{C}_0, k)$ is just the algebra given by a direct product of $|\mathcal{C}_0|$ copies of k . We also have the following technical lemmas which are derived from this proposition and will prove useful later to recover a bijection between the objects of two categories from an isomorphism of incidence algebras.

The first one shows how the set of characteristic functions on vertices can be precisely determined by purely algebraic properties regarding the algebra structure. For this, recall an idempotent of a ring R is an element $e \in R$ with $e^2 = e$, two elements are orthogonal if their product is zero and a primitive idempotent is one that can not be expressed as the sum of two nonzero orthogonal idempotents.

Lemma 3.16. *There is a unique maximal set of primitive orthogonal idempotents in $\text{Hom}(\mathcal{C}_0, k)$ and it is $\chi_{\mathcal{C}_0}$.*

Proof. First, we prove that any set of primitive orthogonal idempotents is contained in $\chi_{\mathcal{C}_0}$. Let ϕ be a (nonzero) primitive idempotent in $\text{Hom}(\mathcal{C}_0, k)$ and suppose that ϕ does not belong to $\chi_{\mathcal{C}_0}$. The elements of $\chi_{\mathcal{C}_0}$ can be characterized as maps $\mathcal{C}_0 \rightarrow k$ which are 1 at a single element of \mathcal{C}_0 and 0 everywhere else. Then, ϕ must be 1 at least at two identities (a nonzero idempotent in a field can only be 1), let id_A be one of them. Now define

$$\phi_1 = \chi_A \quad \phi_2 = \phi - \chi_A.$$

These two maps are (nonzero) idempotents, orthogonal and add up to ϕ . This contradicts the assumption that ϕ was primitive, hence $\phi \in \chi_{\mathcal{C}_0}$.

Finally, we just need to prove that $\chi_{\mathcal{C}_0}$ is a set of primitive orthogonal idempotents.

- They are orthogonal and idempotent

$$(\chi_A * \chi_B)(\text{id}_C) = \chi_A(\text{id}_C)\chi_B(\text{id}_C) = \begin{cases} 1 & \text{if } A = B = C, \\ 0 & \text{otherwise.} \end{cases}$$

- To check that they are primitive, suppose $\chi_A = \phi + \psi$ for some orthogonal idempotents ϕ and ψ . Since ϕ and ψ are idempotents, the only values they may yield are either 0 or 1 and they are never 1 simultaneously by orthogonality. Therefore $\phi = \chi_A$ and $\psi = 0$ (or viceversa) because $1 = \phi(\text{id}_A) + \psi(\text{id}_A)$ and 0 everywhere else.

□

Lemma 3.17 ([16, Corollary 1.3]). *Let \mathcal{C} and \mathcal{D} be Möbius categories and $\Psi : I(\mathcal{C}) \rightarrow I(\mathcal{D})$ be a k -algebra morphism. Then $\Psi(J(\mathcal{C})) \subseteq J(\mathcal{D})$.*

Proof. Suppose that there is $\phi \in J(\mathcal{C})$ such that $\Psi(\phi) \notin J(\mathcal{D})$. This is, $\Psi(\phi)(\text{id}_A) = \lambda \neq 0$ for some A and λ . However, $\delta - \frac{1}{\lambda}\phi$ is invertible in $I(\mathcal{C})$ (because $\phi \in J(\mathcal{C})$) whereas $\Psi(\delta - \frac{1}{\lambda}\phi) = \delta - \frac{1}{\lambda}\Psi(\phi)$ is not invertible in $I(\mathcal{D})$ (it is zero at id_A). Since invertible elements are mapped to invertible elements, this is a contradiction. □

Given any morphism $\Psi : I(\mathcal{C}) \rightarrow I(\mathcal{D})$, we now have two unique morphisms of algebras such that the diagram

$$\begin{array}{ccccc} I(\mathcal{C}) & \twoheadrightarrow & I(\mathcal{C})/J(\mathcal{C}) & \xrightarrow{\cong} & \text{Hom}(\mathcal{C}_0, k) \\ \downarrow \Psi & & \downarrow & & \downarrow \Psi_0 \\ I(\mathcal{D}) & \twoheadrightarrow & I(\mathcal{D})/J(\mathcal{D}) & \xrightarrow{\cong} & \text{Hom}(\mathcal{D}_0, k) \end{array}$$

commutes and Ψ_0 is an isomorphism if Ψ is. The following proposition shows that Ψ_0 provides the bijection between $\text{Ob } \mathcal{C}$ and $\text{Ob } \mathcal{D}$.

Proposition 3.18. *Let \mathcal{C} and \mathcal{D} be Möbius categories and $\Psi : I(\mathcal{C}) \rightarrow I(\mathcal{D})$ an isomorphism of k -algebras. Then, using the same notation as above, it induces a bijection of sets $\tau_\Psi : \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{D}$ such that there is a commutative diagram of bijections*

$$\begin{array}{ccccc}
 \text{Hom}(\mathcal{C}_0, k) & \longleftarrow & \chi_{\mathcal{C}_0} & \xleftarrow{\cong} & \text{Ob } \mathcal{C} \\
 \cong \downarrow \Psi_0 & & \cong \downarrow \Psi_0|_{\chi_{\mathcal{C}_0}} & & \cong \downarrow \tau_\Psi \\
 \text{Hom}(\mathcal{D}_0, k) & \longleftarrow & \chi_{\mathcal{D}_0} & \xleftarrow{\cong} & \text{Ob } \mathcal{D}
 \end{array}$$

Proof. It is routine to verify that such isomorphisms preserve maximal sets of orthogonal primitive idempotents, so $\Psi_0(\chi_{\mathcal{C}_0}) = \chi_{\mathcal{D}_0}$ by Lemma 3.16 and we can restrict Ψ_0 as follows:

$$\begin{array}{ccc}
 \text{Hom}(\mathcal{C}_0, k) & \longleftarrow & \chi_{\mathcal{C}_0} \\
 \cong \downarrow \Psi_0 & & \cong \downarrow \Psi_0|_{\chi_{\mathcal{C}_0}} \\
 \text{Hom}(\mathcal{D}_0, k) & \longleftarrow & \chi_{\mathcal{D}_0}
 \end{array}$$

Finally, τ_Ψ is obtained from $\Psi_0|_{\chi_{\mathcal{C}_0}}$ and the correspondence $\chi_A \leftrightarrow A$. □

The last step before proving the final theorem is to show that this bijection is the restriction of some suitable isomorphism $I(\mathcal{C}) \rightarrow I(\mathcal{D})$, which can then be used to obtain coherent bijections between \mathcal{C}_n and \mathcal{D}_n .

For any invertible $\psi \in I(\mathcal{C})$, let γ_ψ denote the inner automorphism of $I(\mathcal{C})$ given by conjugation with ψ :

$$\gamma_\psi(\phi) = \psi * \phi * \psi^{-1}.$$

Proposition 3.19 ([16, Proposition 1.4]). *Let \mathcal{C} and \mathcal{D} be Möbius categories and $\Psi : I(\mathcal{C}) \rightarrow I(\mathcal{D})$ an isomorphism of k -algebras. Then there is an invertible element $\psi \in I(\mathcal{D})$ such that the composite*

$$I(\mathcal{C}) \xrightarrow{\Psi} I(\mathcal{D}) \xrightarrow{\gamma_\psi} I(\mathcal{D})$$

extends the bijection $\Psi_0|_{\chi_{\mathcal{C}_0}}$ to $I(\mathcal{C})$

$$\begin{array}{ccccc}
 I(\mathcal{C}) & \longleftarrow & \chi_{\mathcal{C}_0} & \xleftarrow{e_{\mathcal{C}}} & \chi_{\mathcal{C}_0} \\
 \cong \downarrow \gamma_\psi \Psi & & \cong \downarrow (\gamma_\psi \Psi)|_{\chi_{\mathcal{C}_0}} & & \cong \downarrow \Psi_0|_{\chi_{\mathcal{C}_0}} \\
 I(\mathcal{D}) & \longleftarrow & \chi_{\mathcal{D}_0} & \xleftarrow{e_{\mathcal{D}}} & \chi_{\mathcal{D}_0}
 \end{array}$$

and $r_{\mathcal{D}}(\psi) = 1$ (so $r_{\mathcal{D}}\gamma_\psi = r_{\mathcal{C}}$).

Proof. For each $D \in \mathcal{D}$, let $\psi_D = (\Psi e_{\mathcal{C}} \Psi_0^{-1} r_{\mathcal{D}})(\chi_D)$. In more detail, letting $\tau = \tau_\Psi$,

$$\chi_D \xrightarrow{r_{\mathcal{D}}} \chi_D \xrightarrow{\Psi_0^{-1}} \chi_{\tau^{-1}D} \xrightarrow{e_{\mathcal{C}}} \chi_{\tau^{-1}D} \xrightarrow{\Psi} \psi_D.$$

Firstly, note that $r_{\mathcal{D}}(\psi_D) = \chi_D$ because

$$r_{\mathcal{D}}\Psi e_{\mathcal{C}}\Psi_0^{-1}r_{\mathcal{D}} = \Psi_0 r_{\mathcal{C}} e_{\mathcal{C}} \Psi_0^{-1} r_{\mathcal{D}} = r_{\mathcal{D}}.$$

Next define $\psi \in I(\mathcal{D})$ as $\psi(f) = \psi_{\text{cod } f}(f)$. Then $r_{\mathcal{D}}(\psi) = 1$

$$r_{\mathcal{D}}(\psi)(\text{id}_D) = \psi(\text{id}_D) = \psi_D(\text{id}_D) = r_{\mathcal{D}}(\psi_D)(\text{id}_D) = \chi_D(\text{id}_D) = 1$$

and ψ is invertible as its action on identities is determined by $r_{\mathcal{D}}(\psi) = 1$, which is nonzero at all identities.

Moreover

$$(\chi_D * \psi)(f) = \begin{cases} \psi_D(f) & \text{if } \text{cod } f = D \\ 0 & \text{if } \text{cod } f \neq D \end{cases}$$

and

$$\begin{aligned} (\psi * \psi_D)(f) &= \sum_{f=f'f''} \psi_{\text{cod } f'}(f')\psi_D(f'') \\ &= (\psi_{\text{cod } f} * \psi_D)(f) \\ &= \Psi(\chi_{\tau^{-1} \text{cod } f} * \chi_{\tau^{-1} D})(f) \\ &= \begin{cases} \psi_D(f) & \text{if } \text{cod } f = D \\ 0 & \text{if } \text{cod } f \neq D, \end{cases} \end{aligned}$$

for any $f \in \text{Mor } \mathcal{D}$, so $\chi_D = \psi * \psi_D * \psi^{-1}$. At this point, substituting the definition of ψ_D states that $(\gamma_{\psi} \Psi e_{\mathcal{C}})(\chi_{\mathcal{C}}) = (e_{\mathcal{D}} \Psi_0)(\chi_{\mathcal{C}})$. \square

For \mathcal{C} a small category, any set of morphisms of \mathcal{C} describes a (multi)digraph with vertices the objects of \mathcal{C} and edges the set of morphisms with source and target their domain and codomain respectively. In particular, we consider the *length n graph* $(\mathcal{C}, \mathcal{C}_n)$ of a Möbius category: the graph with vertices objects of \mathcal{C} and edges arrows of length n . Then an isomorphism of incidence algebras (as *topological algebras*) yields an isomorphism of length n graphs.

Theorem 3.20 ([16, Theorem 2.2]). *Let \mathcal{C} and \mathcal{D} be Möbius categories with countable \mathcal{C}_n and \mathcal{D}_n -sets⁴ for some $n \geq 1$ and $I(\mathcal{C}) \cong I(\mathcal{D})$ as topological k -algebras. Then $(\mathcal{C}, \mathcal{C}_n) \cong (\mathcal{D}, \mathcal{D}_n)$ as graphs.*

Proof. Since inner automorphisms are continuous, by Proposition 3.19 we can assume that there is an isomorphism $\Psi : I(\mathcal{C}) \rightarrow I(\mathcal{D})$ of topological algebras restricting to a bijection $\Psi|_{\chi_{\mathcal{C}_0}} : \chi_{\mathcal{C}_0} \rightarrow \chi_{\mathcal{D}_0}$. This provides a bijection of vertices, namely τ .

We must show that it induces a bijection $\mathcal{C}_n(A, B) \cong \mathcal{D}_n(\tau A, \tau B)$ for each $A, B \in \mathcal{C}$. Together with Proposition 3.14 and Lemma 3.17, the fact that Ψ is a continuous closed isomorphism gives

$$\Psi(J_n(\mathcal{C})) = \Psi(\overline{J^n(\mathcal{C})}) = \overline{\Psi(J^n(\mathcal{C}))} = \overline{J^n(\mathcal{D})} = J_n(\mathcal{D}),$$

so Ψ restricts to an isomorphism of vector spaces

$$\chi_A * J_n(\mathcal{C}) * \chi_B \xrightarrow{\cong} \Psi(\chi_A * J_n(\mathcal{C}) * \chi_B) = \chi_{\tau A} * J_n(\mathcal{D}) * \chi_{\tau B}$$

that induces

$$\frac{\chi_A * J_n(\mathcal{C}) * \chi_B}{\chi_A * J_{n+1}(\mathcal{C}) * \chi_B} \xrightarrow{\cong} \frac{\chi_{\tau A} * J_n(\mathcal{D}) * \chi_{\tau B}}{\chi_{\tau A} * J_{n+1}(\mathcal{D}) * \chi_{\tau B}}.$$

⁴This is, $\mathcal{C}_n(A, B)$ is countable for any $A, B \in \mathcal{C}$ and similarly for \mathcal{D} .

As in Prop 3.14, one can see that extension and restriction to $\mathcal{C}_n(A, B)$ and $\mathcal{D}_n(\tau A, \tau B)$ respectively become linear isomorphisms in the quotient

$$\mathrm{Hom}(\mathcal{C}_n(A, B), k) \longrightarrow \frac{\chi_A * J_n(\mathcal{C}) * \chi_B}{\chi_A * J_{n+1}(\mathcal{C}) * \chi_B}, \quad \frac{\chi_{\tau A} * J_n(\mathcal{D}) * \chi_{\tau B}}{\chi_{\tau A} * J_{n+1}(\mathcal{D}) * \chi_{\tau B}} \longrightarrow \mathrm{Hom}(\mathcal{D}_n(\tau A, \tau B), k),$$

hence we have

$$\mathrm{Hom}(\mathcal{C}_n(A, B), k) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{D}_n(\tau A, \tau B), k).$$

If $\mathcal{C}_n(A, B)$ or $\mathcal{D}_n(\tau A, \tau B)$ is finite then both are vector spaces of equal finite dimension d , hence $|\mathcal{C}_n(A, B)| = d = |\mathcal{D}_n(\tau A, \tau B)|$. Otherwise the hypothesis of countability implies that

$$|\mathcal{C}_n(A, B)| = \aleph_0 = |\mathcal{D}_n(\tau A, \tau B)|.$$

□

3.3 Continuity and finiteness conditions

Observe that Theorem 3.20 requires the isomorphism of incidence algebras to be a homeomorphism as well. In this section we prove sufficient conditions under which an isomorphism of incidence algebras (as k -algebras) is continuous.

One of these conditions is that the category should be finitely generated. This is, it should have (locally) a finite number of indecomposable arrows and infinite chains of composable arrows are forbidden.

Definition 3.21. A category \mathcal{C} is *finitely generated* if the sets

1. $\mathcal{C}_1(A, B)$
2. $[A, B] = \{C \in \mathcal{C} : \mathrm{Hom}(A, C) \neq \emptyset, \mathrm{Hom}(C, B) \neq \emptyset\}$

are finite for all $A, B \in \mathcal{C}$.

These conditions imply seemingly stronger finiteness consequences and further relate J_n with J^n . We need the following lemma to prove that finite generation is sufficient for an isomorphism to be continuous.

Lemma 3.22 ([16, Proposition 2.5]). *Let \mathcal{C} be a finitely generated Möbius category. Then each set $\mathcal{C}_n(A, B)$ is finite and $\chi_B * J_n * \chi_A = \chi_B * J^n * \chi_A$ for $n \geq 1$.*

Proof. Given $n \geq 1$ and $A, B \in \mathcal{C}$, choose a decomposition for each arrow in $\mathcal{C}_n(A, B)$ into an arrow of length 1 and an arrow of length $n - 1$. This clearly defines an injective map

$$\mathcal{C}_n(A, B) \longrightarrow \prod_{C \in [A, B]} \mathcal{C}_1(C, B) \times \mathcal{C}_{n-1}(A, C) \tag{1}$$

To prove that $\mathcal{C}_n(A, B)$ is finite, proceed by induction. The base case is $\mathcal{C}_1(A, B)$, which is known to be finite by hypothesis. For the induction step suppose that all $\mathcal{C}_{n-1}(A, B)$ are finite for all A, B . Then

$$|\mathcal{C}_n(A, B)| \leq \left| \prod_{C \in [A, B]} \mathcal{C}_1(C, B) \times \mathcal{C}_{n-1}(A, C) \right| = \sum_{C \in [A, B]} |\mathcal{C}_1(C, B)| \cdot |\mathcal{C}_{n-1}(A, C)| < \infty$$

by the finiteness of $[A, B]$ and injectivity of (1).

We prove the second part by induction as well. For $n = 1$ we already have $J_1 = J$, so $\chi_B * J_1 * \chi_A = \chi_B * J * \chi_A$. Next suppose that $\chi_B * J_{n-1} * \chi_A = \chi_B * J^{n-1} * \chi_A$ for all $A, B \in \mathcal{C}$. The map (1) defines two maps, one that chooses the length 1 arrow (x) and another one that chooses the length $n - 1$ one (y).

$$\begin{aligned} x : \mathcal{C}_n(A, B) &\longrightarrow \coprod_{C \in [A, B]} \mathcal{C}_1(C, B) \\ y : \mathcal{C}_n(A, B) &\longrightarrow \coprod_{C \in [A, B]} \mathcal{C}_{n-1}(A, C). \end{aligned}$$

This is, for $f \in \mathcal{C}_n(A, B)$ there exists $C \in [A, B]$ such that $f = x(f)y(f)$ with $x(f) : C \rightarrow B$ of length 1 and $y(f) : A \rightarrow C$ of length $n - 1$. Define

$$S_C = \{x(f) : f \in \mathcal{C}_n(A, B), x(f) \in \mathcal{C}_1(C, B)\}, \quad C \in [A, B]$$

and

$$V_g = \{h \in \mathcal{C}_{n-1}(A, C) : x(gh) = g, y(gh) = h\} \quad g \in S_C, C \in [A, B].$$

Let $\phi \in \chi_B * J_n * \chi_A$, we must show that $\phi \in \chi_B * J^n * \chi_A$. For each $C \in [A, B]$ and $g \in S_C$, define

$$\psi_g(h) = \begin{cases} \phi(gh) & \text{if } h \in V_g \\ 0 & \text{if } h \notin V_g. \end{cases}$$

Observe that $\psi_g(h) \in J_{n-1}$: if $\ell(h) < n - 1$, then $h \notin \mathcal{C}_{n-1} \supseteq V_g$, so $\psi_g(h) = 0$. It is also true that $\psi_g = \chi_C * \psi_g * \chi_A$ as it is already zero for maps with domain and codomain different from A and C respectively. Therefore

$$\psi_g \in \chi_C * J_{n-1} * \chi_A = \chi_C * J^{n-1} * \chi_A,$$

so $\psi_g = \chi_C * \psi'_g * \chi_A$ for some $\psi'_g \in J^{n-1}$ by the induction hypothesis. Now, ϕ can be rewritten in terms of the ψ_g as

$$\phi = \sum_{C \in [A, B]} \sum_{g \in S_C} \chi_g * \psi_g$$

because, when evaluating at some f , all summands vanish except for the one corresponding to $x(f)$ and then ψ_g is evaluated at $y(f)$. Therefore

$$\begin{aligned} \phi &= \sum_{C \in [A, B]} \sum_{g \in S_C} \chi_g * \psi_g \\ &= \sum_{C \in [A, B]} \sum_{g \in S_C} \chi_g * \chi_C * \psi'_g * \chi_A \\ &= \sum_{C \in [A, B]} \sum_{g \in S_C} \chi_g * \psi'_g * \chi_A \\ &= \chi_B * \left(\sum_{C \in [A, B]} \sum_{g \in S_C} \chi_g * \psi'_g \right) * \chi_A \end{aligned}$$

where each $\chi_g * \psi'_g \in J^n$, so $\phi \in \chi_B * J^n * \chi_A$.

Finally, the converse inclusion is true in general: $J^n \subseteq \overline{J^n} = J_n$. □

Corollary 3.23. *If a finitely generated Möbius category has a finite set of objects, then \mathcal{C}_n is finite and $J_n = J^n$ for all $n \geq 1$.*

We prove a slightly more general version of the above statement: instead of an isomorphism, we simply require either a finite number of objects or a morphism of k -algebras that restricts to a bijection of objects.

Theorem 3.24 ([16, Theorem 2.6]). *Let \mathcal{C} and \mathcal{D} be Möbius categories, \mathcal{C} finitely generated, and $\Psi : I(\mathcal{C}) \rightarrow I(\mathcal{D})$ a morphism of k -algebras. If \mathcal{C} has a finite set of objects or Ψ restricts to a bijection $\chi_{\mathcal{C}_0} \rightarrow \chi_{\mathcal{D}_0}$, then it is continuous.*

Proof. Let $(\phi_\alpha)_{\alpha \in A}$ be a net in $I(\mathcal{C})$ with $\lim_\alpha \phi_\alpha = 0$. This is, for all $f \in \text{Mor } \mathcal{C}$ there exists α_f such that $\phi_\alpha(f) = 0$ for any $\alpha \geq \alpha_f$. We need to prove that $\lim_\alpha \Psi(\phi_\alpha) = 0$.

There are two cases to consider:

1. \mathcal{C} has a finite number of objects. Consider the sets

$$U_n = \bigcup_{k=1}^{n-1} \mathcal{C}_k = \{g \in \text{Mor } \mathcal{C} : \ell(g) < n\}, \quad n \geq 1.$$

Since they are finite, one can obtain an upper bound α_n of $\{\alpha_g : g \in U_n\}$ by directedness of A for each n . Now $\phi_\alpha(g) = 0$ for any $\alpha \geq \alpha_n$ and $g \in U_n$, so $\phi_\alpha \in J_n(\mathcal{C})$ for $\alpha \geq \alpha_n$. Applying Ψ ,

$$\Psi(\phi_\alpha) \in \Psi(J_n(\mathcal{C})) = \Psi(J^n(\mathcal{C})) = \Psi(J(\mathcal{C}))^n \subseteq J^n(\mathcal{D}) \subseteq J_n(\mathcal{D}), \quad \alpha \geq \alpha_n, \quad n \geq 1.$$

For any $f \in \text{Mor } \mathcal{D}$, choose $n = \ell(f) + 1$. Then $\Psi(\phi_\alpha)(f) = 0$ for any $\alpha \geq \alpha_n$, so $\lim_\alpha \Psi(\phi_\alpha) = 0$.

2. Ψ restricts to a bijection of $\chi_{\mathcal{C}_0} \rightarrow \chi_{\mathcal{D}_0}$. Similarly as before, define

$$U_n(A, B) = \bigcup_{k=1}^{n-1} \mathcal{C}_k(A, B) = \{g \in \text{Hom}(A, B) : \ell(g) < n\}, \quad A, B \in \mathcal{C}, \quad n \geq 1.$$

They are all finite as well, so choose an upper bound $\alpha_n(A, B)$ of each $\{\alpha_g : g \in U_n(A, B)\}$. Then $\phi_\alpha(g) = 0$ for any $\alpha \geq \alpha_n(A, B)$ and $g \in U_n(A, B)$, hence $\chi_B * \phi_\alpha * \chi_A \in \chi_B * J_n(\mathcal{C}) * \chi_A$. Applying Ψ again,

$$\begin{aligned} \chi_{\tau B} * \Psi(\phi_\alpha) * \chi_{\tau A} &\in \chi_{\tau B} * \Psi(J_n(\mathcal{C})) * \chi_{\tau A} \\ &= \chi_{\tau B} * \Psi(J^n(\mathcal{C})) * \chi_{\tau A} \\ &= \chi_{\tau B} * \Psi(J(\mathcal{C}))^n * \chi_{\tau A} \\ &\subseteq \chi_{\tau B} * J(\mathcal{D})^n * \chi_{\tau A} \\ &\subseteq \chi_{\tau B} * J_n(\mathcal{D}) * \chi_{\tau A}, \quad \alpha \geq \alpha_n(A, B), \quad A, B \in \mathcal{C}, \quad n \geq 1. \end{aligned}$$

Finally, for $f : D \rightarrow E$ in $\text{Mor } \mathcal{D}$, choose $n = \ell(f) + 1$. Then

$$\Psi(\phi_\alpha)(f) = (\chi_E * \Psi(\phi_\alpha) * \chi_D)(f) = 0$$

for any $\alpha \geq \alpha_n(\tau^{-1}D, \tau^{-1}E)$. This is, $\lim_\alpha \Psi(\phi_\alpha) = 0$. □

To sum up, we combine this last theorem with 3.20 to obtain the main result in this section, a similar result to Theorem 2.4 for finitely generated Möbius categories without any references to the topology of $I(\mathcal{C})$.

Theorem 3.25 ([16, Corollary 2.7]). *Let \mathcal{C} and \mathcal{D} be finitely generated Möbius categories with isomorphic incidence algebras. Then $(\mathcal{C}, \mathcal{C}_n)$ and $(\mathcal{D}, \mathcal{D}_n)$ are isomorphic as graphs for all $n \geq 1$.*

Proof. By Proposition 3.19, we can assume that the isomorphism $I(\mathcal{C}) \rightarrow I(\mathcal{D})$ restricts to an isomorphism $\chi_{\mathcal{C}_0} \rightarrow \chi_{\mathcal{D}_0}$. Then Theorem 3.24 applied to both Ψ and Ψ^{-1} proves that they are continuous, hence $I(\mathcal{C})$ and $I(\mathcal{D})$ are homeomorphically isomorphic. Finally, Theorem 3.20 provides the isomorphism $(\mathcal{C}, \mathcal{C}_n) \cong (\mathcal{D}, \mathcal{D}_n)$. \square

The natural question to ask is whether this result can be improved to an isomorphism. The answer is that it is not possible in general, as the following example shows.

Example 3.26 ([16, Example 1.6]). Consider the two categories \mathcal{C} and \mathcal{D} generated by the graphs



and relations

$$\begin{cases} ad = ae \\ bd = cd \end{cases} \quad \begin{cases} xu = xv \\ xu = yu \end{cases}$$

respectively. To prove that they are not isomorphic, suppose that there is an isomorphism $\mathcal{C} \rightarrow \mathcal{D}$. This functor must preserve the shape of the category while maintaining coherency with the relations. Such an isomorphism is then entirely defined by two bijections $\{d, e\} \rightarrow \{u, v\}$ and $\{a, b, c\} \rightarrow \{x, y, z\}$. If $d \mapsto v$, then $bd \mapsto F(b)v$ and $bd = cd \mapsto F(c)v$. Since v is an epimorphism in \mathcal{D} , $F(b) = F(c)$, a contradiction because F is bijective. Conversely, if $d \mapsto u$, there are two new cases to consider. If $a \mapsto x$, then $ad \mapsto xu$ and $bd \mapsto yu = xv$, so F would not be injective. Otherwise $ad \mapsto F(a)u$ and $ae \mapsto F(a)v$, implying that F is ill-defined.

Now, the isomorphism between their incidence algebras is given by

$$\begin{aligned} \chi_a &\mapsto \chi_x & \chi_d &\mapsto \chi_u \\ \chi_b &\mapsto \chi_y + \chi_z & \chi_e &\mapsto \chi_v \\ \chi_c &\mapsto \chi_z + \chi_x \end{aligned}$$

and $\chi_n \mapsto \chi_n$ for $n = 0, 1, 2$. The rest are determined forcing it to be an algebra morphism:

$$\begin{aligned} \chi_{bd} &\mapsto \chi_{yu} + \chi_{zu} & \chi_{cd} &\mapsto \chi_{zu} + \chi_{xu} = \chi_{zu} + \chi_{yu} \\ \chi_{ae} &\mapsto \chi_{xv} & \chi_{ad} &\mapsto \chi_{xu} = \chi_{xv} \\ \chi_{be} &\mapsto \chi_{yv} + \chi_{zv} \\ \chi_{ce} &\mapsto \chi_{zv} + \chi_{xv} \end{aligned}$$

This shows that relations are preserved and the basis is mapped to linearly independent vectors. Since both algebras have the same dimension as vector spaces (both categories have the same number of arrows), then it is an isomorphism.

This example is far from contrived, and the point is that the isomorphism of algebras may map the standard basis of characteristic functions to a completely different one thus losing combinatorial information. As we have seen in the previous theorems, the preservation of the product forces any isomorphisms to preserve the number of arrows between any two objects.

4. Homotopy theory of groupoids

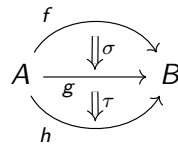
The passage from Möbius categories to decomposition spaces requires some background on groupoids and homotopy theory. Since the theory is mostly developed around ∞ -groupoids (Lurie [17] being one of the main sources), most bibliography is in that setting as well. Thankfully, propositions can often be specialized directly and, although their proofs tend to remain correct, they can be reworked using simpler tools and reasoning for 1-groupoids.

The downside to restricting ourselves to 1-groupoids is that we must keep track in each case whether we are dealing with 1-categories, 2-categories, bicategories, and so on. We are also going to provide a brief introduction to 2-category theory and enriched category theory in order to be able to deal with some concepts that appear here.

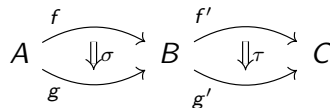
4.1 Preliminary (2-)category-theoretic notions

Before continuing with homotopy theory, we should recall a few notions from basic category theory and 2-category theory. Some good references for this material are the well-known [14] by S. MacLane and [1] by S. Awodey for 1-category theory and [13] by S. Lack or [15] by T. Leinster for the 2-categorical notions. This section is not (nor does it pretend to be) a complete, detailed explanation of 2-category theory, but it serves as a quick overview of some concepts that we are going to be using throughout the rest of this work.

Firstly, there are two classes of compositions of natural transformations. Given two natural transformations $\sigma : f \rightarrow g$ and $\tau : g \rightarrow h$, their vertical composition is a natural transformation $F \rightarrow H$ and it is defined componentwise: $(\tau \circ \sigma)_a = \tau_a \circ \sigma_a$.



The lesser known composition operation is horizontal composition. For functors $f, g : A \rightarrow B$ and $f', g' : B \rightarrow C$ given two natural transformations $\sigma : f \rightarrow g$ and $\tau : f' \rightarrow g'$ one can define $f'\sigma : f'f \rightarrow f'g$ as $(f'\sigma)_a = f'(\sigma_a)$ and $\tau f : f'f \rightarrow g'f$ as $(\tau f)_a = \tau_{f(a)}$. Combining both gives what is written $\tau \cdot \sigma : f'f \rightarrow g'f'$ (or simply $\tau\sigma$) and it is defined as $g'\sigma \circ \tau f = \tau f \circ f'\sigma$.



In order to manipulate these compositions, we are implicitly going to make use of what is called the *interchange law*. For functors $f, g, h : A \rightarrow B$ and $f', g', h' : B \rightarrow C$ and natural transformations $f \xrightarrow{\sigma} g \xrightarrow{\tau} h$, $f' \xrightarrow{\sigma'} g' \xrightarrow{\tau'}$, we have

$$\tau' \tau \circ \sigma' \sigma = (\tau' \circ \sigma') \cdot (\tau \circ \sigma).$$

Another useful construction is the *comma category*. Roughly, it is a category whose objects are arrows and arrows are commutative squares. As we get into 2-category theory (specially with groupoids), it will become useful as it can be understood as lifting the ambient dimension by turning arrows into objects and commutative squares into morphisms.

Definition 4.1. Given functors $f : B \rightarrow C$ and $g : D \rightarrow C$, the *comma category* is the category $(f \downarrow g)$ whose objects are triples (b, d, ϕ) , where $b \in B$, $d \in D$ and $\phi : f(b) \rightarrow g(d)$. A morphism $(b, d, \phi) \rightarrow (b', d', \phi')$ is a pair of arrows $\beta : b \rightarrow b'$ and $\delta : d \rightarrow d'$ in B and D respectively such that the diagram

$$\begin{array}{ccc} f(b) & \xrightarrow{f(\beta)} & f(b') \\ \downarrow \phi & & \downarrow \phi' \\ g(d) & \xrightarrow{g(\delta)} & g(d') \end{array}$$

Later on, especially in section 4.3, we are also going to talk about *monoidal categories*. A *monoidal category* is simply a category together with a monoid-like structure. Precisely, it includes an object $i \in C$ and a bifunctor $\otimes : C \times C \rightarrow C$ and natural isomorphisms

$$i \otimes a \cong a \cong a \otimes i \quad (a \otimes b) \otimes c \cong a \otimes (b \otimes c)$$

subject to some coherence conditions (see [15]). A typical example is **Set** together with its terminal object 1 and the cartesian product. In addition, a monoidal category is said to be *symmetric* if there is a specified natural isomorphism $a \otimes b \cong b \otimes a$.

Now we move on to 2-categories. The first example of 2-category that one encounters is usually **Cat**, the category of small categories: it has small categories as objects, and between two categories C, D one has the functor category $\text{Fun}(C, D)$ whose objects are functors $C \rightarrow D$ and morphisms natural transformations between them.

Definition 4.2. A (*strict*) 2-category is a category whose Hom sets are categories and composition is functorial. For a 2-category \mathcal{C} and $C, D \in \mathcal{C}$, the objects of $\text{Hom}_{\mathcal{C}}(C, D)$ are called *1-cells* and its morphisms are *2-cells*. If composition is only unital and associative up to (specified) natural isomorphisms satisfying some coherence conditions (see [15]), it is said to be a *bicategory* or *weak 2-category* instead.

Note that this definition already provides us with enough tools to define *equivalences* as a generalization of isomorphisms: given two objects $C, D \in \mathcal{C}$ in a 2-category, they are said to be equivalent if there exist 1-cells $f : C \rightarrow D$ and $g : D \rightarrow C$ and invertible 2-cells $\text{id} \rightarrow fg$ and $\text{id} \rightarrow gf$. If $\mathcal{C} = \mathbf{Cat}$ then this becomes the definition of an equivalence of categories.

A particularly relevant 2-category for our study is the category **Grpd** of small groupoids. Recall that a *groupoid* is a category in which all arrows are invertible. It is clearly a full subcategory of **Cat**, and we say that a 2-category is **Grpd**-enriched if all its Hom-categories are groupoids. In that case, we denote these Hom-categories as $\text{Map}_E(x, y)$ and call them *mapping spaces* or *mapping groupoids*. In the case of **Grpd**, its mapping spaces have objects functors and morphisms natural transformations between them.

The next step after defining 2-categories is to define its appropriate notion of morphism. In this case there are two perfectly valid (more, actually) such notions: just like 2-categories and bicategories, 2-functors are a stricter notion than pseudofunctors, but both have their uses. In our case however, we just need 2-functors.

Definition 4.3. A (strict) 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between 2-categories \mathcal{C} and \mathcal{D} is a rule that assigns an object $F(C) \in \mathcal{D}$ to each object $C \in \mathcal{C}$. Moreover, for each $C, C' \in \mathcal{C}$ one has a functor

$$F_{C,C'} : \text{Hom}_{\mathcal{C}}(C, C') \rightarrow \text{Hom}_{\mathcal{D}}(F(C), F(C'))$$

such that $F(\text{id}_C) = \text{id}_{F(C)}$ and $F(f) \circ F(g) = F(fg)$. If these two equalities are not strict but rather (specified) natural isomorphisms satisfying some additional coherence properties (regarding unitality and associativity of composition, see [15]), one says that it is a *pseudofunctor* or *weak 2-functor* instead.

Note that pseudofunctors can be defined between any two bicategories, but 2-functors are more appropriate between 2-categories than between bicategories. Moreover, bifunctors also arise naturally in regular category theory. For instance, limits and colimits in **Cat** are only defined up to isomorphism and so is their action on functors. These situations are usually handled in a cleaner manner by using pseudofunctors. Next, we define 2-natural transformations.

Definition 4.4. Given (possibly weak) 2-functors between (possibly weak) 2-categories $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a 2-natural transformation σ is a 1-cell $\sigma_C : F(C) \rightarrow G(C)$ for each $C \in \mathcal{C}$ together with natural isomorphisms $(\sigma_f)_f$ for every $C, C' \in \mathcal{C}$

$$\begin{array}{ccc} F(C) & \xrightarrow{\sigma_C} & G(C) \\ F(f) \downarrow & \nearrow_{\sigma_f \cong} & \downarrow G(f) \\ F(C') & \xrightarrow{\sigma_{C'}} & G(C') \end{array}$$

which are compatible with other coherence isomorphisms for the pseudofunctors (see [15]). If all the σ_f are identities, then it is a *strict 2-natural transformation*.

In this case we also have morphisms between natural transformations and they are called modifications.

Definition 4.5. A modification $m : \sigma \rightarrow \tau$ between two pseudonatural transformations $\sigma, \tau : F \rightarrow G$, of functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ is a collection of 2-cells $m_A : \sigma_C \rightarrow \tau_C$ indexed by $C \in \mathcal{C}$ that commute with the 2-cell components of σ and τ .

As for plain 1-categories, we can consider the 2-category of 2-functors $\text{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$ between any two 2-categories. Its objects are 2-functors, 1-cells are 2-natural transformations and 2-cells are modifications. In addition, one also has $\text{PsFun}(\mathcal{C}, \mathcal{D})$, the 2-category of pseudofunctors, pseudonatural transformations and modifications. If \mathcal{C} and \mathcal{D} are just 1-categories, then both coincide and can be regarded as 2-categories by considering their Hom-sets as discrete categories (i.e. categories whose only morphisms are just identities) and then they all match the regular 1-functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$.

Now that we have all the concepts set up, we review what is an equivalence $\mathcal{C} \simeq \mathcal{D}$ of 2-categories. It consists of two 2-functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ together with 2-natural equivalences $\sigma : \text{Id} \rightarrow FG$ and $\tau : GF \rightarrow \text{Id}$. This is, there exist modifications $m : \text{id} \rightarrow \sigma \circ \tau$ and $m' : \tau \circ \sigma \rightarrow \text{id}$ which are isomorphisms. In simpler terms, it can be seen that a 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is part of an equivalence if and only if it is essentially surjective on objects (i.e. any object in the codomain is *equivalent* to some object in the image) and $F_{C,C'}$ is an equivalence on each Hom-category $\text{Hom}_{\mathcal{C}}(C, C')$. By replacing strict items with their weak variants one obtains the definition of a biequivalence, i.e. an equivalence of bicategories.

A commonly mentioned remark usually attributed to S. Lack is that every naturally occurring bicategory is equivalent to a 2-category. One particular example that we are going to meet in this work is the bicategory of spans in **Grpd**, which is going to be equivalent to the category of groupoid slices and linear functors defined later. Other important coherence theorems include MacLane's coherence theorem for monoidal categories (roughly, it is safe to consider that the natural isomorphisms in the definition of a monoidal category are identities) and standard procedures that given any pseudofunctor $C \rightarrow \mathbf{Cat}$ produce an equivalent 2-functor $C \rightarrow \mathbf{Cat}$ (and similarly for **Grpd**, see [12] for the general statement).

Finally, observe that the strategy followed in the definition of these 2-categorical notions can be carried over to define n -categories, n -functors, and so on. Most concepts from regular category theory translate well in this theory (like adjoints or limits), so we use them freely. In this work we are only going to go up to 3-categories, at very specific places. For us it suffices to know that a 2-bicategory is the same as a bicategory, but lifting all concepts one level: the required isomorphisms become equivalences, and Hom-categories become 2-categories. Then an equivalence of 3-categories (or 2-bicategories) is an essentially surjective assignment of objects and compatible equivalences of Hom-2-categories.

4.2 Groupoids and homotopy theory

From now on and unless stated explicitly, we restrict ourselves to small groupoids. We will also denote groupoids by capital letters and groupoid maps (functors) by lowercase letters. The basic definition for this section is that of an *homotopy* of groupoids. As we will see throughout this section, natural transformations are surprisingly similar to homotopies between continuous maps.

Definition 4.6. For two groupoid maps $f, g : X \rightarrow Y$, a *homotopy between f and g* is a natural transformation $\alpha : f \rightarrow g$ (which is necessarily an isomorphism). We say that f is *homotopic* to g (denoted $f \simeq g$) if there exists a homotopy $f \rightarrow g$. Two groupoids are *homotopy equivalent* if they are equivalent as categories.

Let us recall the topological analogues of these concepts. A path $x \rightarrow x'$ between two points x, x' of a topological space X is a continuous function $\sigma : [0, 1] \rightarrow X$ such that $\sigma(0) = x$ and $\sigma(1) = x'$. On the other hand, a path $x \rightarrow x'$ in a groupoid is just an (invertible) arrow in X or, equivalently, an object of the arrow category X^2 (the category of functors $\mathbf{2} \rightarrow X$, where $\mathbf{2}$ is the category with two objects and one arrow between them). Similarly, we can think of natural transformations between $f, g : X \rightarrow Y$ as objects of the functor category $(Y^X)^2$, which correspond by adjunction to functors $X \times \mathbf{2} \rightarrow Y$, just like homotopies between continuous functions are represented by continuous functions $X \times [0, 1] \rightarrow Y$. In addition, note that the notion of *mapping space* is also borrowed from topology.

These facts are of course not a coincidence. There is a canonical model structure on **Grpd** that shows that most homotopy theory can be developed in **Grpd** (again, usually $\infty\mathbf{Grpd}$) just like for regular topological spaces. Therefore, it is going to be convenient to imagine a groupoid as the fundamental group(oid) of some topological space. The following definitions adhere to this philosophy.

Definition 4.7. Let X be a groupoid.

1. It is *connected* if it is nonempty and $\text{Hom}_X(x, y) \neq \emptyset$ for all $x, y \in X$.
2. A *component* of X is a full subgroupoid of X with set of objects an isomorphism class of X . For $x \in X$, we write $[x]$ or $X_{[x]}$ for the component of X containing x . The set of components of X will

be denoted by $\pi_0 X$.

3. For $x \in X$, we define $\pi_1(X, x) = \text{Aut}_X(x) = \text{Hom}_X(x, x)$.
4. A groupoid is *discrete* if it is homotopy equivalent to a set (i.e. a category without non-identity morphisms). If it is also connected, then it is called *contractible*.

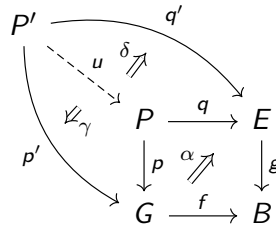
We haven't introduced any new concepts yet, this definition simply translates even more topological definitions to groupoids. For instance, observe that a connected groupoid is a groupoid which is path-connected in the sense that any two points can be connected by some path. A contractible groupoid is just a groupoid which is homotopy equivalent to the groupoid $\mathbf{1}$ with one object and one (identity) arrow. Then a discrete groupoid is a groupoid whose connected components are contractible, like a totally disconnected topological space.

Definition 4.8. A groupoid X is *locally finite* if $\pi_1(X, x)$ is finite for all $x \in X$ and (*homotopy*) *finite* if $\pi_0 X$ is also finite. We write **grpd** for the full subcategory of **Grpd** whose objects are finite groupoids.

By moving from sets to groupoids, we have to adopt 2-categorical versions of classical concepts like pullbacks and fibres. These account for 2-cells (homotopies in our case) in their universal properties and generally replace arrow equalities with isomorphisms.

Definition 4.9. Given two maps $G \xrightarrow{f} B \xleftarrow{g} E$, their *homotopy pullback* is a groupoid P , two maps $p : P \rightarrow G$ and $q : P \rightarrow E$ and a homotopy $\alpha : fp \rightarrow gq$ satisfying:

1. For any other homotopy commutative diagram $\beta : fp' \rightarrow gq'$ there exists a map $u : P \rightarrow P'$ and homotopies $\gamma : pu \rightarrow p'$, $\delta : qu \rightarrow q'$ that factor β through $\alpha \cdot u : fpu \rightarrow gqu$:



Explicitly,

$$\beta = g\delta \circ \alpha u \circ f\gamma^{-1}.$$

2. For any other map $v : P' \rightarrow P$ and homotopies $\gamma' : pv \rightarrow p'$ and $\delta' : qv \rightarrow q'$ satisfying the previous condition, there exists a unique homotopy $\theta : v \rightarrow u$ such that $\gamma' = \gamma \circ p\theta$ and $\delta' = \delta \circ q\theta$.

Here, P , p and q are determined up to homotopy. These squares are said to be *homotopy cartesian* or *homotopy pullbacks*. Moreover, the map p is called is the *homotopy pullback of g along f* and written $p = f^*(g)$.

Recall that pullbacks in **Set** are given by the fibre product

$$G \times_B E = \{(x, y) \in G \times E : f(x) = g(y)\}.$$

When dealing with groupoids, we switch to the homotopical version of this concept, the *homotopy fibre product*.

Definition 4.10. The *homotopy fibre product* of maps $f : G \rightarrow B$ and $g : E \rightarrow B$ is the comma category $(f \downarrow g)$ (which is a groupoid in our case, see [14, §II.6]) and will be denoted by $G \times_B E$.

Explicitly, the objects of $G \times_B E$ consist of triples (x, y, ϕ) where $\phi : f(x) \rightarrow g(y)$ for $x \in G, y \in E$. An arrow $(x, y, \phi) \rightarrow (x', y', \phi')$ is a pair of arrows $\gamma : x \rightarrow x'$ and $\varepsilon : y \rightarrow y'$ in G and E respectively such that the square

$$\begin{array}{ccc} f(x) & \xrightarrow{f(\gamma)} & f(x') \\ \downarrow \phi & & \downarrow \phi' \\ g(y) & \xrightarrow{g(\varepsilon)} & g(y') \end{array}$$

commutes in B .

For each such product, define two projections

$$\begin{array}{ll} p : G \times_B E \rightarrow G & q : G \times_B E \rightarrow E \\ (x, y, \phi) \mapsto x & (x, y, \phi) \mapsto y \\ (\gamma, \varepsilon) \mapsto \gamma & (\gamma, \varepsilon) \mapsto \varepsilon. \end{array}$$

Then the square

$$\begin{array}{ccc} G \times_B E & \xrightarrow{q} & E \\ \downarrow p & & \downarrow g \\ G & \xrightarrow{f} & B \end{array}$$

is easily seen to be commutative up to homotopy via $\alpha : fp \rightarrow gq$, where $\alpha_{(x,y,\phi)} = \phi : f(x) \rightarrow g(y)$. This is, in fact, the homotopy pullback of $E \rightarrow G \leftarrow B$.

Lemma 4.11. *The homotopy fibre product together with the two projections described above is a homotopy pullback.*

Proof (Sketch). Given A and the maps $A \rightarrow G$ and $A \rightarrow E$,

define

$$\begin{array}{lll} u : & A & \rightarrow & G \times_B E \\ & a & \mapsto & (p'(a), q'(a), \beta_a) \\ & (\alpha : a \rightarrow a') & \mapsto & (p'(\alpha), q'(\alpha)) \end{array}$$

and $\delta = \text{id}$, $\gamma = \text{id}$. All the required properties are then easily verified. \square

Example 4.12. For any groupoids G and H , the homotopy fibre product $G \times_1 H$ is $G \times H$. The diagram is the regular one from the cartesian product:

$$\begin{array}{ccc} G \times H & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow \\ H & \longrightarrow & 1 \end{array}$$

Even if apparently trivial, the next lemma provides both a simple example and useful criteria to prove that a given square is a homotopy pullback.

Lemma 4.13. *If the vertical maps in a homotopy commutative square*

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ f \downarrow & \nearrow \alpha & \downarrow f' \\ B & \xrightarrow{b} & B' \end{array}$$

are equivalences, then it is a homotopy pullback square. In particular, homotopy pullbacks of equivalences along any maps is are equivalences.

Proof. We proceed by definition. Since f and f' are equivalences, we may assume that both are part of adjoint equivalences. Then, there exist g and g' together with homotopies $\eta : \text{id} \rightarrow gf$ and $\varepsilon : fg \rightarrow \text{id}$ for f and η', ε' for f' that we can assume to be the units and counits of the adjunctions $f \dashv g$ and $f' \dashv g'$. Firstly, define the homotopy $\bar{\alpha} = g'b\varepsilon \circ g'\alpha g \circ \eta'ag : ag \rightarrow g'b$.

Let $B \xleftarrow{p} C \xrightarrow{q} A'$ be another span equipped with an homotopy $\beta : bp \rightarrow f'q$. Then consider the map $gp : C \rightarrow A$ together with the homotopies

$$\gamma = \varepsilon p : fgp \rightarrow p$$

and

$$\delta = \eta'^{-1}q \circ g'\beta \circ \bar{\alpha}p : agp \rightarrow q.$$

Using that $\varepsilon'f \circ f\eta' = \text{id}$ and the definition of $\bar{\alpha}$, one can readily show that $\beta = f'\delta \circ gp\alpha \circ b\gamma^{-1}$. Moreover, for any other $u : C \rightarrow A$ with corresponding homotopies $\gamma' : fu \rightarrow p$ and $\delta' : au \rightarrow q$, define $\theta = g\gamma' \circ \eta u : u \rightarrow gp$.

$$\begin{array}{ccccc} C & & & & \\ & \searrow u & \nearrow \delta' & & \\ & & A & \xrightarrow{a} & A' \\ & \searrow \gamma' & \downarrow f & \nearrow \alpha & \downarrow f' \\ & & B & \xrightarrow{b} & B' \\ & \searrow p & \downarrow g & \nearrow \bar{\alpha} & \downarrow f' \\ & & A & \xrightarrow{a} & A' \\ & \searrow gp & & & \end{array}$$

Clearly, θ and γ' determine each other under the condition that $\gamma' = \gamma \circ f\theta$ (f is faithful), so θ is unique in this sense. Finally, $\delta' = \delta \circ a\theta$ is proved again by substituting δ and applying the corresponding triangular identity. \square

The prism lemma is the main source of pullback squares. The proof is completely analogous to the one for the classical pullback pasting lemma of strict pullbacks ([1, Lemma 5.10], [1, Corollary 5.11]), the only difference being that one needs to ensure that homotopies are coherent with each other.

Lemma 4.14 (Prism lemma, [17, Lemma 4.4.2.1]). *Given a homotopy commutative diagram*

$$\begin{array}{ccc}
 E & \longrightarrow & E' \\
 \downarrow & \searrow & \downarrow \\
 & G & \longrightarrow & G' \\
 \downarrow & \swarrow & \downarrow & \swarrow \\
 H & \longrightarrow & H'
 \end{array}$$

where the bottom face $(GG'HH')$ is a homotopy pullback, the top face $(EE'GG')$ is a homotopy pullback if and only if the back one $(EE'HH')$ is.

Particularizing this lemma to (strictly) commutative triangular faces yields the homotopical version of the pullback pasting lemma, a name that we will use to refer to Lemma 4.14 especially when its triangular faces are strictly commutative. This allows us to show an example of a quite general square that can be easily shown to be homotopy pullback thanks to this lemma.

Corollary 4.15. *The following diagram is a homotopy pullback for any G, G', H and $f : G \rightarrow G'$:*

$$\begin{array}{ccc}
 G \times H & \xrightarrow{f \times \text{id}} & G' \times H \\
 \downarrow p_G \lrcorner & & \downarrow p_{G'} \\
 G & \xrightarrow{f} & G'
 \end{array}$$

Here $p_G : G \times H \rightarrow G$ and $p_{G'} : G' \times H \rightarrow G'$ are the canonical projections.

Proof. Observe that both the outer square and the right hand square are simply cartesian product pullback squares, hence homotopy pullbacks:

$$\begin{array}{ccccc}
 G \times H & \xrightarrow{f \times \text{id}} & G' \times H & \xrightarrow{p_H} & H \\
 \downarrow p_G \lrcorner & & \downarrow p_{G'} \lrcorner & & \downarrow \\
 G & \xrightarrow{f} & G' & \longrightarrow & 1
 \end{array}$$

The prism lemma ensures that the left hand square is a homotopy pullback. \square

Now, the main selling point of homotopy pullbacks is that they are *homotopy invariant*. This is, one can replace any object in the original diagram with an homotopy equivalent one and still obtain the same pullback.

Proposition 4.16 (Functoriality and invariance of the pullback). *Given a map of diagrams*

$$\begin{array}{ccc}
 G \times_B E & \longrightarrow & E \\
 \swarrow & \dashv & \swarrow \\
 G & \longrightarrow & B \\
 \downarrow & \dashv & \downarrow \\
 G' \times_{B'} E' & \longrightarrow & E' \\
 \swarrow & \dashv & \swarrow \\
 G' & \longrightarrow & B'
 \end{array}$$

there exists a unique map (up to homotopy) $u : G \times_B E \rightarrow G' \times_{B'} E'$ such that the cube is commutative. If all vertical arrows are equivalences, then so is u and all faces become pullbacks.

Proof. The diagram shows two maps $G \times_B E \rightarrow E'$ and $G \times_B E \rightarrow G'$ commuting with the bottom face, so by the universal property of $G' \times_{B'} E'$ there exists $u : G \times_B E \rightarrow G' \times_{B'} E'$ making the whole diagram commute.

Whenever the rest of the vertical arrows are equivalences, we have that all previously existing faces of the cube are pullbacks by Lemma 4.13. Then the pullback pasting lemma applied to the top and front faces ensures that the internal diagonal is a homotopy pullback. Hence the back face is a homotopy pullback because we have a prism with faces the internal diagonal, bottom and back faces. Now recall that the homotopy pullback of an equivalence along any map is again an equivalence, hence u is an equivalence because it is the pullback of $E \rightarrow E'$ along $G' \times_{B'} E' \rightarrow E'$. Finally, the left face is a homotopy pullback because both of its vertical arrows are equivalences. \square

Once we have defined homotopy pullbacks, fibres are naturally the next definition. Recall that in **Set**, a fibre is just the preimage of an element $b \in B$ by some map $A \rightarrow B$, which can be expressed as a pullback of the map along the map $1 \rightarrow B$ that chooses b .

Definition 4.17. Given a map of groupoids $p : E \rightarrow B$, the *homotopy fibre of p over $b \in B$* is the homotopy pullback of p along the *name map*

$$\begin{array}{ccc}
 \ulcorner b \urcorner & : & 1 \rightarrow B \\
 * & \mapsto & b.
 \end{array}$$

Denoting $\text{hfib}(p, b) = 1 \times_B E$, we have the homotopy cartesian square

$$\begin{array}{ccc}
 \text{hfib}(p, b) & \longrightarrow & E \\
 \downarrow & \lrcorner & \downarrow p \\
 1 & \xrightarrow{\ulcorner b \urcorner} & B
 \end{array}$$

If the choice of p is not ambiguous, we also write E_b for $\text{hfib}(f, b)$. In particular, the homotopy fibre $\text{hfib}(\ulcorner b \urcorner, b)$ is called the *loop groupoid*, $\Omega_b B$.

Let us take a look at $\Omega_b B$. Its objects are $(*, *, \phi : b \rightarrow b)$ for each $\phi \in \text{Aut}_B(b)$, and its morphisms can only be identities $(\text{id}_*, \text{id}_*)$ or the empty set. Thus, $\Omega_b B$ is discrete and its set of objects is effectively $\text{Aut}_B(b)$. Put simply, $\Omega_b B$ is the set of automorphisms at b .

Similarly, the homotopy fibre of a map $p : E \rightarrow B$ at $b \in B$ has objects $(*, e, \phi : p(e) \rightarrow b)$ that we identify with (e, ϕ) as writing the object $* \in 1$ is redundant. A morphism $(e, \phi) \rightarrow (e', \phi')$ in the homotopy fibre is a pair $(\text{id}_*, \varepsilon : e \rightarrow e')$ such that $\phi = \phi' \varepsilon$. For the same reason as before, we do not account for id_* and consider $\varepsilon : e \rightarrow e'$ only. It is going to be convenient to think of morphisms as commutative triangles like

$$\begin{array}{ccc} p(e) & \xrightarrow{p(\varepsilon)} & p(e') \\ & \searrow \phi & \swarrow \phi' \\ & & b \end{array}$$

where we account for ε before applying p .

To further clarify the difference between homotopy fibres and regular set-theoretic preimages, remember that the preimage of $\ulcorner b \urcorner : 1 \rightarrow B$ at b is simply 1 , whereas its homotopy fibre at b is $\Omega_b B$. More generally, if E is discrete then the homotopy fibre is discrete and its objects are all pairs (e, ϕ) with $\phi : p(e) \rightarrow b$. In particular cases however, some similarities become clear.

Remark 4.18. If $p : E \rightarrow B$ is a map of groupoids and B is discrete (e.g. a set), then $\text{hfib}(p, b)$ is easily seen to be homotopy equivalent to what is called the *essential preimage* of p at b . This is, the full subgroupoid of E that is mapped to the connected component of b .

Then, it makes sense to consider the notation for connected components as a particular case of the homotopy fibre construction

$$\begin{array}{ccc} B_{[b]} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \ulcorner [b] \urcorner \\ B & \longrightarrow & \pi_0 B \end{array}$$

and even extend it to denote essential preimage of a map $E \rightarrow B$

$$\begin{array}{ccccc} E_{[b]} & \longrightarrow & B_{[b]} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \ulcorner [b] \urcorner \\ E & \longrightarrow & B & \longrightarrow & \pi_0 B \end{array}$$

It is important to note that all of these constructions are functorial. It is readily verified that both constructing a name map $\ulcorner - \urcorner : B \rightarrow B^1$ and the formation of the comma category $(- \downarrow -) : (B^G)^{\text{op}} \times B^E \rightarrow \mathbf{Grpd}$ are functorial, so the functoriality of the fibre then follows from the fact that every groupoid is isomorphic to its opposite via the assignment $f^{\text{op}} \mapsto f^{-1}$.

One more definition that needs adjustment for homotopy theory is that of slice categories. Usually, a slice category $\mathcal{C}_{/I}$ is the category whose objects are arrows $X \rightarrow I$ of \mathcal{C} and arrows are commuting triangles (see [14] or [1] for instance).

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \swarrow \\ & & I \end{array}$$

It is just a particular case of a comma category [14, §II] as one has $\mathcal{C}_{/I} = (\text{Id}_{\mathcal{C}} \downarrow \ulcorner I \urcorner)$, where $\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is the identity functor and $\ulcorner I \urcorner : 1 \rightarrow \mathcal{C}$ is the name map.

Again, the homotopical version replaces equality of arrows with homotopies.

Definition 4.19. For a **Grpd**-enriched category \mathcal{C} , the *homotopy slice category* $\mathcal{C}_{/I}$ (\mathcal{C} over $I \in \mathcal{C}$) is the (again **Grpd**-enriched) category whose objects are morphisms $X \rightarrow I$ in \mathcal{C} and a morphism $(X \xrightarrow{f} I) \rightarrow (X' \xrightarrow{f'} I)$ is a morphism $g : X \rightarrow X'$ together with an isomorphism $\alpha : f \rightarrow f'g$ in the groupoid $\text{Map}_{\mathcal{C}}(X, I)$.

A 2-cell between two morphisms $(g, \alpha), (g', \alpha') : f \rightarrow g$ in $\mathcal{C}_{/I}$ is a homotopy $\beta : g \rightarrow g'$ which is coherent with α and α' . We use the shorthand notation $\text{Map}_{/I}(f, g)$ for the mapping space between two objects $f, g \in \mathcal{C}_{/I}$ whenever \mathcal{C} (usually **Grpd**) can be deduced from the context.

Observe that an equivalence $f \simeq g$ in $\mathbf{Grpd}_{/I}$ with $f : X \rightarrow I$ and $f' : X' \rightarrow I$ is a homotopy equivalence $g : X \rightarrow X'$ together with a homotopy $f \rightarrow f'g$. In particular, if $X = X'$ and the maps $X \rightarrow X', X' \rightarrow X$ are identities, the 2-cells $f \rightarrow f'\text{id}$ and $f' \rightarrow f\text{id}$ are nothing but a homotopy $f \rightarrow f'$ and its inverse. Therefore, equivalences in $\mathbf{Grpd}_{/I}$ generalize homotopies to equivalences between maps of homotopy equivalent domains.

From now on and for brevity, we will refer to the homotopy variants we have defined like commutative diagrams up to homotopy, homotopy pullbacks, homotopy slices, homotopy fibres and so on without the homotopy qualifier.

4.3 Homotopy linear algebra

Recall that some basic operations on finite sets (products and disjoint unions) hold strong similarities with the natural numbers. Then one can start developing this algebra as in the symbolic method, where it is used to describe complex objects from simpler ones in order to obtain the generating function of the complex object rather easily.

This idea can also be applied to **Grpd** with products and coproducts as well, but we are more interested their homotopical versions. While products remain the same, homotopy colimits or homotopy sums generalize coproducts in a way that behaves correctly with homotopy fibres and the whole homotopy setting in general. Then we will be able to perform some linear algebra on groupoid maps (regarding them as generalized vectors with groupoid coordinates).

We are especially interested in this framework because later on it will become an important tool as an intermediate step to both define the incidence algebra of a decomposition space and to prove some key results. This section is primarily based on [9], where it is developed in full generality for ∞ -groupoids.

The first new concept is that of the *Grothendieck construction*. Given a family of groupoids indexed by a groupoid, to take its homotopy *homotopy colimit* roughly consists in taking the disjoint union of all of them and then adding morphisms based on the indexing (see [10, §A.1.1.7, §B1.3.1]).

Definition 4.20. The *homotopy colimit* or *homotopy sum* of a functor $F : B \rightarrow \mathbf{Grpd}$ is the groupoid $\int F$

(usually denoted $\int^{b \in B} F(b)$) with objects

$$\text{Ob } \int F = \coprod_{b \in B} \text{Ob } F(b) = \{(b, e) : b \in B, e \in F(b)\}$$

and morphisms

$$\begin{aligned} \text{Hom}_{\int F}((b, e), (b', e')) &= \coprod_{\beta: b \rightarrow b'} \text{Hom}_{F(b')} (F(\beta)(e), e') \\ &= \{(\beta, \varepsilon) : \beta : b \rightarrow b', \varepsilon : F(\beta)(e) \rightarrow e'\}. \end{aligned}$$

The identities of $\int F$ are simply $(\text{id}_b, \text{id}_e)$ and the composition of (β', ε') and (β, ε) where $\varepsilon : F(\beta)(e) \rightarrow e'$ and $\varepsilon' : F(\beta')(e') \rightarrow e''$ is $(\beta' \circ \beta, \varepsilon' \circ F(\beta')(\varepsilon))$.

$$F(\beta' \circ \beta)(e) = F(\beta')(F(\beta)(e)) \xrightarrow{F(\beta')(\varepsilon)} F(\beta')(e') \xrightarrow{\varepsilon'} e''$$

This groupoid is naturally equipped with a projection

$$\begin{aligned} \int F &\rightarrow B \\ (b, e) &\mapsto b \\ (\beta, \varepsilon) &\mapsto \beta \end{aligned}$$

This projection is the *Grothendieck construction* of F .

A particularly important example is the case in which B is discrete. Then it is readily seen that

$$\int^{b \in B} F(b) = \coprod_{b \in B} F(b)$$

and the projection is the obvious one induced by the coproduct $\coprod_{b \in B} F(b) \rightarrow B$. Thus, we may think of the Grothendieck construction as a homotopy generalization of the coproduct where symmetries between indices are taken into account.

Just like coproducts, this construction is functorial as a mapping $\mathbf{Grpd}^B \rightarrow \mathbf{Grpd}_{/B}$ and can be iterated. Explicitly, given a functor $I \times J \rightarrow \mathbf{Grpd}$, we have equivalences

$$\int^{i \in I} \int^{j \in J} F(i, j) \simeq \int^{j \in J} \int^{i \in I} F(i, j) \simeq \int^{(i, j) \in I \times J} F(i, j).$$

that commute with their projections to $I \times J$. Functoriality then gives that homotopy sums are invariant under homotopy equivalences. This is, given an equivalence $F \simeq G$, one has an equivalence $\int F \simeq \int G$ which again induces an equivalence in $\mathbf{Grpd}_{/B}$ for the respective Grothendieck constructions.

We already mentioned that homotopy sums behave well in conjunction with homotopy fibres. For sets there is a duality between maps $p : S \rightarrow I$ (as objects in $\mathbf{Set}_{/I}$) and collections of sets $\{S_i\}_{i \in I}$ (functors $I \rightarrow \mathbf{Set}$). Roughly, the duality is given by the assignments

$$\begin{aligned} \mathbf{Set}^I &\rightarrow \mathbf{Set}_{/I} & \mathbf{Set}_{/I} &\rightarrow \mathbf{Set}^I \\ \{S_i\}_i &\mapsto p = \coprod_{i \in I} (S_i \rightarrow \{i\}) & p &\mapsto \{S_i = p^{-1}(i)\}_{i \in I} \end{aligned}$$

For instance, given a map $p : S \rightarrow \mathbb{N}$, one would obtain $S_n = p^{-1}(n)$ and then p is recovered as

$$\begin{array}{ccccccc} p^{-1}(0) & p^{-1}(1) & & (p^{-1}(0) \amalg p^{-1}(1) \amalg \dots) & & S & \\ \downarrow & + \downarrow & + \dots = & \downarrow & \cong & \downarrow p & \\ \{0\} & \{1\} & & \mathbb{N} & & \mathbb{N} & \end{array}$$

where each map $p^{-1}(i) \rightarrow \mathbb{N}$ is the restriction of p to $p^{-1}(i)$, i.e., the constant map i .

This duality allows us to think of maps $S \rightarrow I$ as indexed collections of sets, where sets play the role of scalars. Then a map $S \rightarrow \mathbf{2}$ becomes a 2-dimensional “vector” with **Set** coefficients because it can be seen as a pair of sets.

The following theorem shows that homotopy fibres and the Grothendieck construction are naturally inverses of each other, just like preimages and disjoint unions of sets. It can be proved in full generality from results in [17, §2.2.1] or in the special case of 1-categories and 1-groupoids as in [10, §B1.2].

Theorem 4.21 (Fundamental equivalence). *The assignments*

$$\begin{array}{ccc} \text{hfib} : \mathbf{Grpd}_{/B} & \rightarrow & \mathbf{Grpd}^B & \int : \mathbf{Grpd}^B & \rightarrow & \mathbf{Grpd}_{/B} \\ p : E \rightarrow B & \mapsto & \text{hfib}(p, -) & F & \mapsto & \int F \rightarrow B \end{array}$$

constitute an equivalence of the 2-categories. Moreover, the diagrams

$$\begin{array}{ccc} \mathbf{Grpd}_{/B} & \xrightarrow{\text{hfib}} & \mathbf{Grpd}^B & & \mathbf{Grpd}^B & \xrightarrow{\int} & \mathbf{Grpd}_{/B} \\ \downarrow f^* & & \downarrow (-\circ f) & & \downarrow (-\circ f) & & \downarrow f^* \\ \mathbf{Grpd}_{/B'} & \xrightarrow{\text{hfib}} & \mathbf{Grpd}^{B'} & & \mathbf{Grpd}^{B'} & \xrightarrow{\int} & \mathbf{Grpd}_{/B'} \end{array}$$

commute (up to 2-equivalence) for any $f : B' \rightarrow B$. More precisely, hfib and \int are 3-natural in B .

Given that homotopy sums can be iterated, it is now easy to define homotopy sums of both collections of functors and collections of maps.

Definition 4.22. The homotopy sum of $\mathcal{F} : I \rightarrow \mathbf{Grpd}^B$ is calculated componentwise

$$\left(\int \mathcal{F} \right)(b) := \int^{i \in I} \mathcal{F}(i)(b)$$

The fundamental equivalence now provides a definition for the homotopy sum of an indexed family of $\mathbf{Grpd}_{/B}$. Since any map $g \in \mathbf{Grpd}_{/B}$ is the Grothendieck construction $\text{hfib}(g, -)$

$$g \simeq \int^{b \in B} \text{hfib}(g, b) \downarrow B,$$

we define the *homotopy sum of an indexed family of maps* $F : I \rightarrow \mathbf{Grpd}_{/B}$ as the usual projection

$$\int^{i \in I} g_i := \int^{b \in B} \int^{i \in I} \text{hfib}(g_i, b) \downarrow B \in \mathbf{Grpd}_{/B},$$

where g_i denotes $F(i)$.

If $g_i : E_i \rightarrow B$, observe that in the definition of $\int^i g_i$ we have

$$\int^{b \in B} \int^{i \in I} \text{hfib}(g_i, b) \simeq \int^{i \in I} \int^{b \in B} \text{hfib}(g_i, b) \simeq \int^{i \in I} E_i,$$

so we can consider that $\int^i g_i : \int^i E_i \rightarrow B$. It is also easy to verify that $\text{hfib}(\int^i g_i, b) \simeq \int^i \text{hfib}(g_i, b)$, so fibres of homotopy sums of maps can be calculated easily. Recall that we have $\text{hfib}(\int^i g_i, i) \simeq E_i$ too, given directly by the fundamental equivalence. Moreover, homotopy sums of families indexed over some $I \times J$ can also be iterated:

$$\int^{(i,j) \in I \times J} g_{ij} \simeq \int^{i \in I} \int^{j \in J} g_{ij}.$$

We now elaborate on a common strategy to compute homotopy sums. Given a family g_i of maps $E_i \rightarrow B$, consider the map $g : \int^{(i,b)} \text{hfib}(g_i, b) \rightarrow I \times B$. Then the following squares are cartesian and g and the family g_i determine each other up to homotopy: g determines the g_i 's by pullback along $\ulcorner i \urcorner \times \text{id}$ and the g_i determine g via the Grothendieck construction.

$$\begin{array}{ccccc} E_i & \xrightarrow{g_i} & 1 \times B & \longrightarrow & 1 \\ \downarrow & \lrcorner & \ulcorner i \urcorner \times \text{id} \downarrow & \lrcorner & \ulcorner i \urcorner \downarrow \\ E & \xrightarrow{g} & I \times B & \longrightarrow & I \end{array}$$

This means that we can compute $\int^i g_i$ by finding an appropriate $g : E \rightarrow I \times B$ which fits in these pullback diagrams and then composing with the projection $I \times B \rightarrow B$.

Definition 4.23. Given $S \in \mathbf{Grpd}$ and $g \in \mathbf{Grpd}/_B$ with $g : E \rightarrow B$, we define the *scalar multiplication* $S \otimes g \in \mathbf{Grpd}/_B$ as

$$S \times E \xrightarrow{t_S \times g} 1 \times B \xrightarrow{\cong} B,$$

where $t_S : S \rightarrow 1$ is the unique map $S \rightarrow 1$.

It is routine to verify that homotopy sums commute with constant factors. Explicitly

$$\int^{i \in I} f \times g_i \simeq f \times \int^{i \in I} g_i$$

and, in particular,

$$\int^{i \in I} S \otimes g_i \simeq S \otimes \int^{i \in I} g_i.$$

For illustrative purposes, let us now examine a *linear combination* where objects of $\mathbf{Grpd}/_B$ are vectors and scalars are groupoids (objects of \mathbf{Grpd})

$$\int^{i \in I} S_i \otimes g_i \in \mathbf{Grpd}/_B.$$

Here the S_i are an indexed collection of groupoids $I \rightarrow \mathbf{Grpd}$ and the g_i are also indexed by the groupoid I as $I \rightarrow \mathbf{Grpd}/_B$. Thus, this homotopy sum is the sum of the family $S_i \otimes g_i : S_i \times E_i \rightarrow B$, where E_i is the domain of g_i .

As before, we can obtain two maps $f : S \rightarrow I$ and $g : E \rightarrow I \times B$ via the Grothendieck construction that determine the families S_i and g_i up to equivalence:

$$f \simeq \int^{i \in I} S_i \quad g \simeq \int^{(i,b) \in I \times B} \text{hfib}(g_i, b)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ I & & I \times B \end{array}$$

We can now form the fibre product $S \times_I E$ of the maps f and $p_1 g$ where $p_1 : I \times B \rightarrow I$ is the obvious projection. Moreover, the projections of the fibres $S_i \rightarrow S$ and $E_i \rightarrow E$ induce a map $S_i \times E_i \rightarrow S \times_I E$. Then we have a homotopy commutative diagram

$$\begin{array}{ccccc} & & S_i & \longrightarrow & 1 \\ & \nearrow & \downarrow & \lrcorner & \nearrow \\ S_i \times E_i & \longrightarrow & E_i & & \\ \downarrow & \searrow & \downarrow & \dashrightarrow & \downarrow \\ & & S & \longrightarrow & I \\ \downarrow & \nearrow & \downarrow & \nearrow & \\ S \times_I E & \longrightarrow & E & & \end{array}$$

where the dotted maps are just the compositions through S_i and 1 respectively. By previous observations, the back, top, bottom and right faces are pullbacks. Applying the pullback pasting lemma with the top and back faces ensures that the internal diagonal square is also a pullback. By the prism lemma, since the bottom face is a pullback, then the front face is also a pullback.

Thus the family $S_i \otimes g_i$ is the pullback along $\ulcorner i \urcorner \times \text{id}$ of the map $S \times_I E \rightarrow E \rightarrow I \times B$,

$$\begin{array}{ccccccc} S_i \times E_i & \xrightarrow{t_{S_i} \times \text{id}} & 1 \times E_i \cong E_i & \xrightarrow{g_i} & B \cong 1 \times B & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \ulcorner i \urcorner \times \text{id} \downarrow & \lrcorner & \ulcorner i \urcorner \downarrow \\ S \times_I E & \xrightarrow{q} & E & \xrightarrow{g} & I \times B & \longrightarrow & I \end{array}$$

so

$$\int^{i \in I} S_i \otimes g_i \simeq \begin{array}{c} S \times_I E \\ \downarrow q \\ E \\ \downarrow p_2 g \\ B \end{array}$$

or, more explicitly, the homotopy sum of the family $S_i \otimes g_i : S_i \times E_i \rightarrow B$ is equivalent to

$$\int^i S_i \times_I \int^i E_i \xrightarrow{q} \int^i E_i \xrightarrow{\int^i g_i} B.$$

This shows both a reinterpretation of the fibre product as a scalar product of two vectors (when $B = 1$) and simplified way to verify some linear combinations. A particularly important case of this construction is the following lemma, where we recover the previous idea of recovering the set map $p : S \rightarrow \mathbb{N}$ as the sum of maps $p^{-1}(i) \rightarrow \{i\}$.

Lemma 4.24. *Any $f \in \mathbf{Grpd}_{/B}$, $f : E \rightarrow B$ can be expressed as*

$$f \simeq \int^{e \in E} \ulcorner f(e) \urcorner \simeq \int^{b \in B} \text{hfib}(f, b) \otimes \ulcorner b \urcorner.$$

Proof. Notice that all squares in the following diagram are pullbacks:

$$\begin{array}{ccccc} 1 & \xrightarrow{\ulcorner f(e) \urcorner} & B \cong 1 \times B & \longrightarrow & 1 \\ \ulcorner e \urcorner \downarrow & \lrcorner & \ulcorner e \urcorner \times \text{id} \downarrow & \lrcorner & \ulcorner e \urcorner \downarrow \\ E & \xrightarrow{(id, f)} & E \times B & \longrightarrow & E \end{array}$$

Then the bottom left arrow (id, f) composed with $p_2 : E \times B \rightarrow B$, is equivalent to the given homotopy sum.

The second equivalence is proved similarly:

$$\begin{array}{ccccc} E_b & \longrightarrow & 1 & \xrightarrow{\ulcorner b \urcorner} & B \cong 1 \times B & \longrightarrow & 1 \\ \downarrow & \lrcorner & \ulcorner b \urcorner \downarrow & \lrcorner & \text{id} \times \ulcorner b \urcorner \downarrow & \lrcorner & \downarrow \ulcorner b \urcorner \\ E & \xrightarrow{f} & B & \xrightarrow{(id, id)} & B \times B & \longrightarrow & B, \end{array}$$

□

This lemma ensures that any map $E \rightarrow B$ can be expressed as the linear combination $\int^b E_b \otimes \ulcorner b \urcorner$. Thus, we can now think of *name maps* $\ulcorner b \urcorner$ as a basis of $\mathbf{Grpd}_{/B}$ (or, at least, a system of generators). This will prove especially useful later to derive explicit formulas for linear functors between slices $\mathbf{Grpd}_{/B} \rightarrow \mathbf{Grpd}_{/B'}$.

Let $f : B \rightarrow B'$ be a map of groupoids. Then f induces two functors: $f_!$ is just postcomposition with f and f^* takes the pullback of a map along f .

$$\begin{array}{ccc} f_! : \mathbf{Grpd}_{/B} & \rightarrow & \mathbf{Grpd}_{/B'} & & f^* : \mathbf{Grpd}_{/B'} & \rightarrow & \mathbf{Grpd}_{/B} \\ g : X \rightarrow B & \mapsto & fg : X \rightarrow B' & & g : X \rightarrow B' & \mapsto & f^*(g) : X \times_{B'} B \rightarrow B \end{array}$$

These two functors are (homotopy) adjoint for any f .

Proposition 4.25. *For any map $f : B \rightarrow B'$, there is a natural equivalence of the mapping groupoids*

$$\text{Map}_{/B'}(f_!(g), g') \simeq \text{Map}_{/B}(g, f^*(g')).$$

Proof. Let $g : A \rightarrow B$ and $g' : C \rightarrow B'$. Then we have a homotopy cartesian square

$$\begin{array}{c}
 A \\
 \searrow g \\
 \begin{array}{ccccc}
 & B \times_{B'} C & \xrightarrow{p} & C & \\
 & \downarrow f^*(g') & \lrcorner & \downarrow g' & \\
 & B & \xrightarrow{f} & B' &
 \end{array}
 \end{array}$$

A map $f_!(g) \rightarrow g'$ is a map $A \rightarrow C$ such that the new outer square commutes, hence there exists a unique map (up to homotopy) $h : A \rightarrow B \times_{B'} C$ together with a homotopy $f^*(g')h \rightarrow g$, constituting a map $g \rightarrow f^*(g')$. A morphism between two maps $f_!(g) \rightarrow g'$ then becomes the unique homotopy between the corresponding two equivalent maps $A \rightarrow B \times_{B'} C$ by the statement of the universal property about 2-cells.

Given a map $g \rightarrow f^*(g')$, we obtain a map $A \rightarrow C$ by composing with p . That provides a map $f_!(g) \rightarrow g'$. This assignment is clearly functorial and it is straightforward to verify that it is the homotopy inverse of the firstly defined map again by the uniqueness requirements of the 2-cells provided by the universal property. \square

Recall that the prism lemma provides a homotopical version of the regular pullback pasting lemma:

$$\begin{array}{ccc}
 \bullet & \longrightarrow & \bullet \\
 \downarrow & \lrcorner & \downarrow \\
 H & \xrightarrow{q} & E \\
 \downarrow p & \lrcorner & \downarrow g \\
 G & \xrightarrow{f} & B
 \end{array}$$

One of its consequences is that the composition of the left arrows is homotopic to the pullback of the long right hand arrow along the bottom one by uniqueness of the pullback. The Beck-Chevalley lemma is a more precise statement of this fact in terms of an equivalence of pullback and postcomposition (lowershriek) functors: $f^*g_! \simeq p_!q^*$.

Lemma 4.26 (Beck-Chevalley, [5]). *For any pullback square*

$$\begin{array}{ccc}
 H & \xrightarrow{q} & E \\
 \downarrow p & \lrcorner & \downarrow g \\
 G & \xrightarrow{f} & B
 \end{array}$$

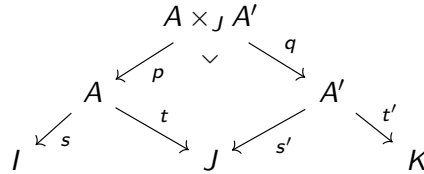
there is a natural equivalence

$$\begin{array}{ccc}
 \mathbf{Grpd}_{/H} & \xrightarrow{q_!} & \mathbf{Grpd}_{/E} \\
 p^* \uparrow & \Downarrow & \uparrow g^* \\
 \mathbf{Grpd}_{/G} & \xrightarrow{f_!} & \mathbf{Grpd}_{/B}
 \end{array}$$

Definition 4.27. A *linear functor* is a 2-functor $F : \mathbf{Grpd}_{/I} \rightarrow \mathbf{Grpd}_{/J}$ which is equivalent to $p_! \circ q^*$ for some span $I \xleftarrow{q} A \xrightarrow{p} J$:

$$\mathbf{Grpd}_{/I} \xrightarrow{q^*} \mathbf{Grpd}_{/A} \xrightarrow{p_!} \mathbf{Grpd}_{/J}.$$

It is easy to verify that identities are linear and that composition of linear functors is again linear: for two spans $I \xleftarrow{s} A \xrightarrow{t} J$ and $J \xleftarrow{s'} A' \xrightarrow{t'} K$, consider the pullback



Then the Beck-Chevally lemma ensures that

$$(t')_!(s')^* t_! s^* \simeq (t')_! q_! p^* s^* \simeq (t'q)_!(sp)^*,$$

so there is a category of homotopy slices of \mathbf{Grpd} and linear functors. Although some results remain valid for arbitrary base objects, we restrict them to locally finite groupoids for simplicity.

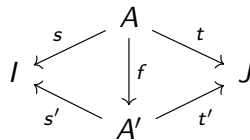
Definition 4.28. We denote the category of homotopy slices of \mathbf{Grpd} and linear functors by \mathbf{LIN} . It is a symmetric monoidal 3-category with identity object $\mathbf{Grpd}_{/1} \cong \mathbf{Grpd}$ and tensor product

$$\mathbf{Grpd}_{/I} \otimes \mathbf{Grpd}_{/J} = \mathbf{Grpd}_{/I \times J}.$$

The Hom-2-category between two slices is the full sub-2-category of the 2-functor 2-category $\text{Fun}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/J})$ with objects linear functors only. We denote it by $\mathbf{LIN}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/J})$.

Remark 4.29. In the setting of ∞ -groupoids, linear functors defined in terms of (homotopy) colimit-preserving functors [9]⁵. Since regular groupoids are a particular case of ∞ -groupoids, we now have another characterization of linear functors.

Given that the composition of spans is only defined up to homotopy, it is natural to expect \mathbf{LIN} to be a strictification of a category of spans. This is indeed the case even in a more general setting for polynomial diagrams and polynomial functors. It is explained in greater detail by Gambino and Kock in [4]. We specialize one of the key results to linear functors: we say that a map of spans from $I \xleftarrow{s} A \xrightarrow{t} J$ to $I \xleftarrow{s'} A' \xrightarrow{t'} J$ is a homotopy commutative diagram



Explicitly, the data consists of a map $f : A \rightarrow A'$ together with two homotopies $s \rightarrow s'f$ and $t \rightarrow t'f$.

Then the construction of a linear functor from a span becomes functorial: if $\varepsilon_f : f_! f^* \rightarrow \text{id}$ denotes the counit of the adjunction from Proposition 4.25, the aforementioned map of spans induces a natural transformation

⁵What we denote as \mathbf{LIN} is expressed as $\underline{\mathbf{LIN}}$ in [9].

$$\begin{array}{ccccc}
\mathbf{Grpd}/I & \xrightarrow{s^*} & \mathbf{Grpd}/A & \xrightarrow{t_!} & \mathbf{Grpd}/J \\
& \searrow (s')^* & \downarrow \simeq & \downarrow \simeq & \nearrow (t')_! \\
& & \mathbf{Grpd}/A' & \xrightarrow{t_!} & \mathbf{Grpd}/A' \\
& & \downarrow \varepsilon_f & & \\
& & \mathbf{Grpd}/A' & \xrightarrow{t_!} & \mathbf{Grpd}/A'
\end{array}$$

Using these definitions we can now make the connection between spans and linear functors significantly more precise.

Proposition 4.30 ([4, Theorem 2.17]). *If $\mathbf{Span}(I, J)$ denotes the category of spans of groupoids from I to J then the construction of a linear functor out of a span gives an equivalence of 2-categories*

$$\mathbf{Span}(I, J) \simeq \mathbf{LIN}(\mathbf{Grpd}/I, \mathbf{Grpd}/J).$$

Moreover, together with the assignment $I \mapsto \mathbf{Grpd}/I$, it becomes an equivalence of the 2-bicategory \mathbf{Span} of (locally finite) groupoids and spans between them and \mathbf{LIN} (which is a 2-bicategory in particular).

Now, given that each linear functor is induced by a *unique* span, we may freely call such a span *the underlying span* of the linear functor.

Since a span $I \leftarrow A \rightarrow J$ is essentially a map $g : A \rightarrow I \times J$, we can then look at this map as a collection of maps $g_i : A_i \rightarrow J$ as before or as a collection of scalars A_{ij} (the fibres of g) which coincide with the fibres $(A_i)_j$ of g_i :

$$\begin{array}{ccccc}
(A_i)_j & \longrightarrow & 1 \times 1 & & \\
\downarrow & \lrcorner & \text{id} \times \lrcorner j \lrcorner & \downarrow & \\
A_i & \xrightarrow{g_i} & 1 \times J & \xrightarrow{p_1} & 1 \\
\downarrow & \lrcorner & \lrcorner i \lrcorner \times \text{id} & \lrcorner & \lrcorner i \lrcorner \\
A & \xrightarrow{g} & I \times J & \xrightarrow{p_1} & I
\end{array}$$

In addition, we have already proved that an element of \mathbf{Grpd}/I can be expressed as a linear combination of its fibres with basis elements, so we can compute the action of the linear functor $F : \mathbf{Grpd}/I \rightarrow \mathbf{Grpd}/J$ induced by g :

$$\begin{aligned}
F\left(\int^{i \in I} E_i \otimes \lrcorner i \lrcorner\right) &\simeq \int^{i \in I} E_i \otimes \int^{j \in J} (A_i)_j \otimes \lrcorner j \lrcorner \\
&\simeq \int^{j \in J} \left(\int^{i \in I} A_{ij} \times E_i\right) \otimes \lrcorner j \lrcorner.
\end{aligned}$$

This is extremely similar to the formula for matrix multiplication by a vector with coordinates E_i . With a similar procedure, one can also verify that composition of linear functors corresponds to matrix multiplication: for $A = (A_{ij})$ and $B = (B_{jk})$ two linear functors, the coefficients of the composition $B \circ A$ are

$$(B \circ A)_{ik} \simeq \int^{j \in J} B_{jk} \times A_{ij}.$$

Then it is not surprising that linear functors preserve linear combinations, since a linear combination $\int^i S_i \otimes g_i$ with $g_i : A_i \rightarrow J$ is simply the application of the linear functor $A = ((A_i)_j)$ to the vector $\int^i S_i \otimes \ulcorner i \urcorner$. In fact, we have the following proposition:

Proposition 4.31 ([9, 2.10]). *The functor*

$$\begin{array}{ccc} \mathbf{Grpd}_{/I \times J} & \rightarrow & \mathbf{LIN}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/J}) \\ (A \rightarrow I \times J) & \mapsto & (I \leftarrow A \rightarrow J) \end{array}$$

is an equivalence of 2-categories.

This means that **LIN** has internal hom functors. Furthermore, both addition and multiplication by scalars work componentwise almost by definition (in many aspects, $\mathbb{M}_{n \times m}(k)$ is isomorphic to $k^{m \times n}$ as $\mathbf{LIN}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/J})$ is equivalent to $\mathbf{Grpd}_{/I \times J}$).

A similar result from linear algebra is that map is linear (i.e. it preserves linear combinations) if and only if it admits a matrix representation. One implication is already set (we have just seen that spans correspond to matrices), we now prove the converse.

Proposition 4.32. *Let $F : \mathbf{Grpd}_{/I} \rightarrow \mathbf{Grpd}_{/J}$ be a homotopy sum-preserving functor. Then it is linear.*

Proof. Firstly, note that any functor preserving homotopy sums necessarily preserves linear combinations, as any groupoid is the homotopy sum of the fibres (Lemma 4.24) of its identity map. Then define A_i to be the domain of $F(\ulcorner i \urcorner) \in \mathbf{Grpd}_{/J}$ for each $i \in I$ and define $g = \int^{i \in I} F(\ulcorner i \urcorner) : A \rightarrow I \times J$ via the Grothendieck construction of $F(\ulcorner i \urcorner)$:

$$\begin{array}{ccccc} A_i & \xrightarrow{F(\ulcorner i \urcorner)} & J & \longrightarrow & \mathbf{1} \\ \downarrow \lrcorner & & \downarrow \ulcorner i \urcorner \times \text{id} & & \downarrow \ulcorner i \urcorner \\ A & \xrightarrow{g} & I \times J & \xrightarrow{p_1} & I \end{array}$$

We claim that the linear functor with underlying span $g : A \rightarrow I \times J$ is equivalent to F . Since F preserves linear combinations, it suffices to prove it for the name maps $\ulcorner i \urcorner$. By the above diagram,

$$(p_2 g)_!((p_1 g)^*(\ulcorner i \urcorner)) \simeq (p_2 \circ (\ulcorner i \urcorner \times \text{id}))_!(F(\ulcorner i \urcorner)) \simeq F(\ulcorner i \urcorner).$$

The check for its action on morphisms is the same expression. □

Now, we consider the dual space of a slice.

Corollary 4.33. *By the previous proposition and the fundamental equivalence, the assignment*

$$\begin{array}{ccccc} \mathbf{LIN}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/1}) & \rightarrow & \mathbf{Grpd}_{/I} & \rightarrow & \mathbf{Grpd}^I \\ (I \leftarrow A \rightarrow \mathbf{1}) & \mapsto & (A \rightarrow I) & \mapsto & \text{hfib}(I \rightarrow A, -) \end{array}$$

is an equivalence.

Although $\mathbf{Grpd}_{/I}$ and \mathbf{Grpd}^I are always equivalent (unlike dual spaces in regular linear algebra), we will see that \mathbf{Grpd}^I is the most natural choice for the dual space of $\mathbf{Grpd}_{/I}$. First, we can easily obtain the “dual basis” of $\ulcorner b \urcorner \in \mathbf{Grpd}_{/B}$ by computing its fibres. Recall that $\text{hfib}(\ulcorner b \urcorner, b')$ is the discrete groupoid $\text{Hom}_B(b, b')$, so the dual of $\ulcorner b \urcorner$ is $h^b = \text{Hom}(b, -)$. Again, just like for vector spaces, we have the obvious pairing

$$\begin{aligned} \langle -, - \rangle : \mathbf{Grpd}_{/I} \times \mathbf{Grpd}^I &\rightarrow \mathbf{Grpd} \cong \mathbf{Grpd}_{/1} \\ (\ulcorner i \urcorner, h^j) &\mapsto \text{Hom}(j, i) \cong \begin{cases} \Omega_i B & \text{if } i \cong j, \\ \emptyset & \text{if } i \not\cong j \end{cases} \end{aligned}$$

that arises from application of h^j regarded as an object of $\mathbf{LIN}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/1})$.

4.4 Finite dimensional homotopy linear algebra

In this section we show how all the linear algebra in \mathbf{Grpd} translates into classical linear algebra over a field (we limit ourselves to \mathbb{Q}) when some finiteness conditions are met. If homotopy linear algebra becomes unmanageable at some point, this technique allows to seamlessly translate most concepts to the classical setting. More importantly, this is the key step to recover the classical incidence algebra of decomposition spaces from the intermediate step in homotopy theory.

Definition 4.34. The cardinality of a finite groupoid X is

$$|X| = \sum_{[x] \in \pi_0 X} \frac{1}{|\pi_1(X, x)|} \in \mathbb{Q}.$$

Observe that for the particular case of connected groupoids we have that

$$|X| = \frac{1}{|\pi_1(X, x)|}, \quad \text{for any } x \in X.$$

For discrete groupoids

$$|X| = \sum_{x \in \pi_0 X} \frac{1}{|\pi_1(X, x)|} = \sum_{x \in \pi_0 X} 1 = |\text{Ob } X|,$$

and, conversely, $|X| = \frac{1}{|\Omega_x X|}$ for any connected groupoid X and $x \in X$. It is also important to notice that cardinality is invariant under homotopy equivalences (since both automorphism groups and connected components are).

Especially when taking the cardinality of a homotopy sum it will be convenient to use the following notation. For any function $f : \pi_0 X \rightarrow V$ with X a finite groupoid and V a \mathbb{Q} -vector space, we denote

$$\int^{x \in X} f(x) = \sum_{[x] \in \pi_0 X} \frac{1}{|\Omega_x X|} f([x]) \in V.$$

Recall that any category is equivalent to its skeleton. For groupoids, one has that every groupoid is equivalent to a disjoint union of its automorphism groups (one for each connected component). This is the idea behind the proof of the analogue of 4.24.

Lemma 4.35 ([9, Lemma 3.5]). *For any map $E \rightarrow B$, one has that*

$$|E| = \int^{b \in B} |E_b| \in \mathbb{Q}.$$

whenever both sides exists.

It is important to note that this ensures that cardinality commutes with homotopy sums by the fundamental equivalence:

$$\left| \int^{b \in B} E_b \right| = \int^{b'} \left| \text{hfib} \left(\int^{b \in B} E_b, b' \right) \right| = \int^{b' \in B} |E_{b'}|$$

The first equality is from Lemma 4.35 and then the fundamental equivalence provides $\text{hfib}(\int^{b \in B} E_b, b') \simeq E_{b'}$.

In order to translate more concepts from the previous section, we are going to need some finiteness conditions. One of them is that sums in matrix multiplications should be finite.

Definition 4.36. A map of groupoids is (*homotopy*) *finite* if all its homotopy fibres are finite. In particular, if all its fibres are empty or contractible, then it is a *homotopy monomorphism*.

Proposition 4.37. *Pullbacks of homotopy monomorphisms are homotopy monomorphisms and pullbacks of homotopy finite maps are again homotopy finite.*

Proof. Let $p : E \rightarrow B$, $f : C \rightarrow B$. The naturality of the fibre functor (by the fundamental equivalence) ensures that $\text{hfib}(f^*(p), -) \simeq \text{hfib}(p, -) \circ f$, so $f^*(p)$ is homotopy finite (or a homotopy monomorphism) if p is. \square

Definition 4.38. A span $I \xleftarrow{r} A \rightarrow J$ is *finite* if r is. A linear functor is *finite* if the underlying span is.

The following proposition is an immediate consequence of Proposition 4.37:

Proposition 4.39 ([9, Proposition 4.3]). *Any finite linear functor $\mathbf{Grpd}_{/I} \rightarrow \mathbf{Grpd}_{/J}$ arising from a span $I \leftarrow A \rightarrow J$ with locally finite I and J restricts to a linear functor $\mathbf{grpd}_{/I} \rightarrow \mathbf{grpd}_{/J}$.*

Then we have the finite groupoid counterpart of **LIN** and linear functors, $\underline{\mathbf{lin}}$. We also include an ad hoc definition of its dual $\overleftarrow{\mathbf{lin}}$ for convenience, the full construction is explained in detail in [9, §6].

Definition 4.40. The category of slices $\mathbf{grpd}_{/I}$ with I locally finite and finite linear functors between them will be denoted by $\underline{\mathbf{lin}}$. Dually, we define $\overleftarrow{\mathbf{lin}}$ as $\underline{\mathbf{lin}}^{\text{op}}$ with objects replaced by functor categories via the fundamental equivalence. We also call morphisms $F : \mathbf{grpd}^I \rightarrow \mathbf{grpd}^J$ in $\underline{\mathbf{lin}}$ *linear* and $F^* = F^{\text{op}} : \mathbf{grpd}_{/J} \rightarrow \mathbf{grpd}_{/I}$ is its *dual*.

We will next define a vector space from its homotopy analogue, a homotopy slice. Again, we require locally finite bases and finite domains.

Definition 4.41. For a slice $\mathbf{grpd}_{/I}$ (with I locally finite), consider the free \mathbb{Q} -vector space $\mathbb{Q}_{\pi_0 I}$ generated by $\delta_i := [i]$ for each $[i] \in \pi_0 I$ and denote $\|\mathbf{Grpd}_{/I}\| = \mathbb{Q}_{\pi_0 I}$. Then for any linear functor F defined by a finite span $I \leftarrow A \rightarrow J$ there is a linear map

$$\begin{aligned} \|F\| : \mathbb{Q}_{\pi_0 I} &\rightarrow \mathbb{Q}_{\pi_0 J} \\ \delta_i &\mapsto \int^{j \in J} |A_{ij}| \delta_j = \sum_{[j] \in \pi_0 J} \frac{|A_{ij}|}{|\Omega_j J|} \delta_j \end{aligned}$$

Since the composition of linear functors is just matrix multiplication and cardinality preserves homotopy sums (see Lemma 4.35) and products, this defines a *global cardinality functor*

$$\|-\| : \underline{\mathbf{lin}} \rightarrow \mathbf{Vect}_{\mathbb{Q}}.$$

The global cardinality functor is monoidal: it maps $\mathbf{grpd}_{/1}$ to \mathbb{Q} and we also have that the cardinality of $\mathbf{grpd}_{/I} \otimes \mathbf{grpd}_{/J}$ is

$$\|\mathbf{grpd}_{/I \times J}\| = \mathbb{Q}_{\pi_0(I \times J)} \cong \mathbb{Q}_{\pi_0 I \times \pi_0 J} \cong \mathbb{Q}_{\pi_0 I} \otimes_{\mathbb{Q}} \mathbb{Q}_{\pi_0 J}.$$

Definition 4.42. The *local cardinality* of an object $(f : E \rightarrow I) \in \mathbf{grpd}_{/I}$ is the row matrix of the linear functor $1 \leftarrow E \xrightarrow{f} I$. Explicitly,

$$|f| = \int^{i \in I} |E_{1i}| \delta_i = \sum_{[i] \in \pi_0 I} \frac{|E_{1i}|}{|\Omega_i I|} \delta_i.$$

If we take $f = \ulcorner i \urcorner : 1 \rightarrow I$, then $|f|$ is precisely δ_i , the global cardinality of the linear functor $\ulcorner i \urcorner$ induced by $1 \leftarrow 1 \rightarrow I$. One may also consider multiplying the matrix obtained by taking the cardinality of each fibre A_{ij} of the underlying span $I \leftarrow A \rightarrow J$ of a linear functor $F : \mathbf{grpd}_{/I} \rightarrow \mathbf{grpd}_{/J}$ times the local cardinality of an object $p \in \mathbf{grpd}_{/I}$. With minor algebraic manipulation, it follows from the definition that $|F(p)| = (|A_{ij}|) \cdot |p|$. By a similar argument it is easily seen that $(|A_{ij}|) \cdot |p| = \|F\|(|p|)$ as well, so $|F(p)| = \|F\|(|p|)$.

Since we have just seen that both concepts of cardinality are highly compatible, we are going to use the same name (cardinality) and notation $(|-\|)$ in both cases.

Finally, we provide analogues of all these definitions for the corresponding dual spaces.

Definition 4.43. The cardinality of a linear functor $F : \mathbf{grpd}^I \rightarrow \mathbf{grpd}^J$ is the linear dual of the cardinality of the dual linear functor $F^* : \mathbf{grpd}_{/J} \rightarrow \mathbf{grpd}_{/I}$:

$$|F| = |F^*|^* : \mathbb{Q}^{\pi_0 I} \rightarrow \mathbb{Q}^{\pi_0 J},$$

where $|\mathbf{grpd}^I| = \mathbb{Q}^{\pi_0 I} = \mathbf{Hom}_{\mathbf{Vect}_{\mathbb{Q}}}(\mathbb{Q}_{\pi_0 I}, \mathbb{Q}) = \mathbb{Q}_{\pi_0 I}^*$. The linear duals of the basis elements δ_i will be denoted by δ^i .

Similarly, the cardinality of a functor $g \in \mathbf{grpd}^I$ is defined to be the cardinality of the corresponding linear functor $\mathbf{grpd}_{/I} \rightarrow \mathbf{grpd}_{/1}$ according to 4.33.

From the definition we can easily deduce that

$$|h^i|(\delta_j) = \sum_{[*] \in \pi_0 1} \frac{|1_{j1}|}{|\Omega_* 1|} = \begin{cases} |\Omega_i I| & \text{if } i \cong j \\ 0 & \text{if } i \not\cong j. \end{cases} \implies |h^i| = |\Omega_i I| \delta^i$$

and, more importantly, cardinality commutes with the pairing $\mathbf{grpd}_{/I} \times \mathbf{grpd}^I \rightarrow \mathbf{grpd}_{/1}$ and its linear counterpart $\mathbb{Q}_{\pi_0 I} \otimes \mathbb{Q}^{\pi_0 I} \rightarrow \mathbb{Q}$.

$$\langle \ulcorner i \urcorner, |h^j| \rangle = \langle \delta_i, |\Omega_j I| \delta^j \rangle = |\Omega_j I| \delta^j(\delta_i) = |\langle \ulcorner i \urcorner, h^j \rangle|.$$

5. Decomposition spaces

In this section we are finally going to work with general decomposition spaces. Since they are just simplicial groupoids satisfying some additional conditions, we need to provide a few basic definitions and results about simplicial objects (simplicial sets and simplicial groupoids specially) before proceeding to decomposition spaces.

Simplicial sets are extremely similar to simplicial complexes and the same intuition about segments, triangles and tetrahedrons applies most of the time. The passage to simplicial groupoids does not add as much conceptual complexity as one would expect, most of the time it simply requires replacing equalities with isomorphisms and restating some conditions in terms of pullbacks.

Definition 5.1. The *simplex category* $\mathbf{\Delta}$ is the category with objects finite nonempty ordinals

$$[n] = \{0 < 1 < \dots < n\}$$

and morphisms monotone maps.

Injective monotone maps $\partial^i : [n-1] \rightarrow [n]$ with image $[n] \setminus \{i\}$ are called *coface maps* and the surjective ones $\sigma^i : [n+1] \rightarrow [n]$ with $\sigma(i) = \sigma(i+1)$ are *codegeneracy maps*.

Any identities satisfied by coface and codegeneracy maps are called *cosimplicial identities*.

It is important to notice that any monotone map $[m] \rightarrow [n]$ can be expressed as a composition of cofaces and codegeneracies (see [18, Chapter I] for more details). Thus, $\mathbf{\Delta}$ is generated by coface maps and codegeneracy maps modulo cosimplicial identities.

Definition 5.2. A morphism $f : [n] \rightarrow [m]$ in $\mathbf{\Delta}$ is *free* if it is distance-preserving. This is, $f(k) + 1 = f(k+1)$. A *generic* map is an endpoint-preserving morphism $g : [n] \rightarrow [m]$ in $\mathbf{\Delta}$: $g(0) = 0$ and $g(n) = m$.

One can check that free maps are precisely those that are generated by outer coface maps $\partial^\perp = \partial^0$, $\partial^\top = \partial^n$ and generic maps are generated by codegeneracies and inner cofaces ∂^i , $0 < i < n$.

The fact that all morphisms in $\mathbf{\Delta}$ are generated by cofaces and codegeneracies provides a more straightforward way to prove that a sequence of objects (where the object X_n is the image of $[n]$) and morphisms (the images of cofaces and codegeneracies) defines a simplicial object. Although we choose to define them in terms of $\mathbf{\Delta}$, in practice we are going to use the explicit combinatorial characterization given below.

Definition 5.3. A simplicial object in a category \mathcal{C} is a functor $X : \mathbf{\Delta}^{\text{op}} \rightarrow \mathcal{C}$. The object $X([n])$ will be denoted X_n and the morphisms $X(\partial^i) = d_i : X_n \rightarrow X_{n-1}$ and $X(\sigma^i) = s_i : X_n \rightarrow X_{n+1}$ are *face* and *degeneracy maps* respectively for each n and i . The images of the cosimplicial identities are the *simplicial identities*.

Explicitly, a simplicial object can be given as a sequence of objects X_n , $n \geq 0$, a collection of face maps $d_i : X_n \rightarrow X_{n-1}$ and a collection of degeneracy maps $s_i : X_n \rightarrow X_{n+1}$ satisfying the simplicial identities

$$d_i s_i = d_{i+1} s_i = \text{id}, \quad d_i d_j = d_{j-1} d_i, \quad s_j s_i = s_i s_{j-1}, \quad d_{j+1} s_i = s_i d_j, \quad d_i s_j = s_{j-1} d_i,$$

for any $n \geq 0$ and $0 \leq i < j \leq n$.

Observe that these identities reflect the intuition from simplicial complexes: consider the case where $\mathcal{C} = \mathbf{Set}$. If we call the elements of X_0 vertices, the elements of X_1 edges and, in general, the elements of X_n n -simplices (with 2-simplices being triangles), we observe that the degeneracy $s_0 : X_0 \rightarrow X_1$ corresponds to regarding a vertex as a (degenerate) edge, while $d_0 : X_1 \rightarrow X_0$ (also written d_\perp) removes the first vertex and returns the remaining one. In general, we say that inner face maps $d_i : X_n \rightarrow X_{n-1}$ with $0 < i < n$ return the inner faces for $n > 2$ or the *the long edge* for $n = 2$. Conversely, the edges obtained by repeatedly applying the outer face maps d_\perp and d_\top are generally called *principal edges*.

Moreover, notice that these identities show that any composition of inner face maps $X_n \rightarrow \dots \rightarrow X_1$ is equal, so we write d_1^{n-1} for this unique generic map even though each d_1 denotes a different morphism $d_1 : X_k \rightarrow X_{k-1}$. We also say that d_1^{n-1} gives the long edge.

Next we define maps between simplicial objects. Again, we provide two equivalent definitions: an explicit, combinatorial one in terms of each object X_n and maps $X_n \rightarrow X_{n-1}$ and $X_{n-1} \rightarrow X_n$ or in terms of functors $\mathbf{\Delta}^{\text{op}} \rightarrow \mathcal{C}$.

Definition 5.4. A *simplicial map* $F : X \rightarrow Y$ of simplicial objects is a natural transformation $X \rightarrow Y$. A simplicial map of simplicial groupoids is *cartesian* with respect to a map $g : [m] \rightarrow [n]$ in $\mathbf{\Delta}$ if the (strictly) commutative square

$$\begin{array}{ccc} X_n & \xrightarrow{F_n} & Y_n \\ X(g) \downarrow & \lrcorner & \downarrow Y(g) \\ X_m & \xrightarrow{F_m} & Y_m \end{array}$$

is homotopy cartesian.

Alternatively, a simplicial map $F : X \rightarrow Y$ is a sequence of maps $F_n : X_n \rightarrow Y_n$ that commute with all faces and all degeneracies:

$$\begin{array}{ccccccc} X_0 & \xleftarrow{d_{i_0}} & X_1 & \xleftarrow{d_{i_1}} & X_2 & \xleftarrow{d_{i_2}} & \dots \\ F_0 \downarrow & & F_1 \downarrow & & F_2 \downarrow & & \\ Y_0 & \xleftarrow{d_{i_0}} & Y_1 & \xleftarrow{d_{i_1}} & Y_2 & \xleftarrow{d_{i_2}} & \dots \end{array} \qquad \begin{array}{ccccccc} X_0 & \xrightarrow{s_{i_0}} & X_1 & \xrightarrow{s_{i_1}} & X_2 & \xrightarrow{s_{i_2}} & \dots \\ F_0 \downarrow & & F_1 \downarrow & & F_2 \downarrow & & \\ Y_0 & \xrightarrow{s_{i_0}} & Y_1 & \xrightarrow{s_{i_1}} & Y_2 & \xrightarrow{s_{i_2}} & \dots \end{array}$$

Finally, we explain two important constructions that show how simplicial sets generalize categories. Recall that functors between posets (regarded as categories) are exactly monotone maps, so we can consider $\mathbf{\Delta}$ to be a (full) subcategory of \mathbf{Cat} . This provides a straightforward definition for the category of simplicial objects in a given category \mathcal{C} : the functor category⁶ $\text{Fun}(\mathbf{\Delta}^{\text{op}}, \mathcal{C})$. We can then give a one-line definition of the *nerve* of a category.

Definition 5.5. The (*strict*) *nerve* of a small category \mathcal{C} is the simplicial set

$$N\mathcal{C} = \text{Hom}_{\mathbf{Cat}}(-, \mathcal{C}) : \mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Set}.$$

⁶Note that this refers to functors in the classical sense, even if \mathcal{C} is a 2-category like \mathbf{Grpd} .

A closer look at this definition reveals that an n -simplex of NC is just a sequence

$$a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n$$

of n composable arrows of \mathcal{C} . Then d_0 removes the first arrow, d_n removes the last arrow and d_i with $0 < i < n$ composes the arrows $a_{i-1} \rightarrow a_i \rightarrow a_{i+1}$. An especially important face map is $d_1 : X_2 \rightarrow X_1$, which is exactly the composition operation of \mathcal{C} . On the other hand, a degeneracy s_i simply replaces an object a_i with the identity $a_i \rightarrow a_i$. In particular, observe that $(NC)_1 = \text{Mor } \mathcal{C}$ and $(NC)_0$ is the set of objects of \mathcal{C} .

This is, in fact, how a category can be regarded as an ∞ -category. It is well-known in the literature that this functor $N : \mathbf{Cat} \rightarrow \text{Fun}(\Delta^{\text{op}}, \mathbf{Cat})$ is full and faithful, thus embedding \mathbf{Cat} into $\infty\mathbf{Cat} \hookrightarrow \mathbf{sSet} = \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})$. Since we are mostly working with groupoids, there is a related definition that includes isomorphisms between equivalent simplices, the *fat nerve*.

Definition 5.6. The *fat nerve* of a small category \mathcal{C} is the simplicial groupoid

$$NC = \text{Fun}(-, \mathcal{C})^{\text{iso}} : \Delta^{\text{op}} \rightarrow \mathbf{Grpd}.$$

where $\text{Ob}(NC)_n = (NC)_n$ and morphisms are natural isomorphisms.

The fat nerve is similar to the strict one, but it also includes isomorphisms of n -simplices, i.e., commutative diagrams

$$\begin{array}{ccccccc} a_0 & \longrightarrow & a_1 & \longrightarrow & \cdots & \longrightarrow & a_n \\ \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong \\ b_0 & \longrightarrow & b_1 & \longrightarrow & \cdots & \longrightarrow & b_n \end{array}$$

In this case, $(NC)_1 = (\mathcal{C}^2)^{\text{iso}}$ is the arrow groupoid (the maximal subgroupoid of the arrow category) and $(NC)_0 = \mathcal{C}^{\text{iso}}$ is the maximal subgroupoid of \mathcal{C} .

Although it is not possible to define composition in an arbitrary simplicial groupoid, the fat nerve preserves additional structure from the category that allows us to compose two n -simplices to obtain an $(n+1)$ -simplex. This property is what defines Segal spaces.

Definition 5.7. A simplicial groupoid $X : \Delta^{\text{op}} \rightarrow \mathbf{Grpd}$ is a *Segal space* if all squares

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_{\perp}} & X_n \\ \downarrow d_{\top} & & \downarrow d_{\top} \\ X_n & \xrightarrow{d_{\perp}} & X_{n-1} \end{array}$$

are homotopy pullbacks.

A particularly important case of these homotopy pullbacks is when $n = 1$, which states that $X_2 \simeq X_1 \times_{X_0} X_1$ and, by induction, $X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$. This is, a simplex in X_n is the “composition” of n simplices of X_1 . Given $a, b \in X_1$ with $d_{\perp} a \cong d_{\top} b$, we write $a \cdot b$ for any 2-simplex σ with faces $d_{\perp} \sigma \cong b$ and $d_{\top} \sigma \cong a$.

The fat nerve of a category provides a particularly important class of examples of Segal spaces. Objects of the fibre product $(\mathbf{NC})_n \times_{(\mathbf{NC})_{n-1}} (\mathbf{NC})_n$ are pairs of n -simplices σ_1, σ_2 together with isomorphisms $d_{\perp}\sigma_1 \rightarrow d_{\top}\sigma_2$

$$\begin{array}{ccccccc} \sigma_1 : & a_0 & \longrightarrow & a_1 & \longrightarrow & \cdots & \longrightarrow & a_n \\ & & & \downarrow \cong & & & & \downarrow \cong \\ \sigma_2 : & & & b_1 & \longrightarrow & \cdots & \longrightarrow & b_n & \longrightarrow & b_{n+1} \end{array}$$

that can be contracted to an $n + 1$ simplex

$$a_0 \rightarrow a_1 \cong b_1 \rightarrow \cdots \rightarrow b_n \rightarrow b_{n+1}.$$

In this case, given $a, b \in (\mathbf{NC})_1$ the condition $d_{\perp}a \cong d_{\top}b$ above expresses composability up to isomorphism and $a \cdot b \cong (\bullet \xrightarrow{a} \bullet \cong \bullet \xrightarrow{b} \bullet)$.

All the definitions of incidence (co)algebras that we have given only use decompositions of elements rather than composition. But, as we have already seen, the Segal condition expresses composability of the objects in X_1 to recover objects in X_2 . Decomposition spaces generalize Segal spaces by requiring decomposition only, and these are exactly all that is required in order to define the incidence coalgebra on X_1 .

Definition 5.8. A simplicial groupoid $X : \mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Grpd}$ is a *decomposition space* if it maps any pushout square

$$\begin{array}{ccc} [n] & \xrightarrow{f} & [m] \\ \downarrow g & & \downarrow g' \\ [q] & \xrightarrow{f'} & [p] \end{array}$$

with free f, f' and generic g, g' to a homotopy pullback

$$\begin{array}{ccc} X_p & \longrightarrow & X_q \\ \downarrow \lrcorner & & \downarrow \\ X_m & \longrightarrow & X_n. \end{array}$$

This condition ensures we always have the pullbacks

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{s_{k+1}} & X_{n+2} & \xleftarrow{d_{k+2}} & X_{n+3} \\ \downarrow d_{\perp} \lrcorner & & \downarrow d_{\perp} & & \downarrow d_{\perp} \\ X_n & \xrightarrow{s_k} & X_{n+1} & \xleftarrow{d_{k+1}} & X_{n+2} \end{array} \qquad \begin{array}{ccccc} X_{n+1} & \xrightarrow{s_k} & X_{n+2} & \xleftarrow{d_{k+1}} & X_{n+3} \\ \downarrow d_{\top} \lrcorner & & \downarrow d_{\top} & & \downarrow d_{\top} \\ X_n & \xrightarrow{s_k} & X_{n+1} & \xleftarrow{d_{k+1}} & X_{n+2} \end{array}$$

for any $n \geq 0$ and $0 \leq k \leq n$. In fact, it is enough to verify a few of these to prove that a simplicial groupoid is a decomposition space. The proof of the following proposition can be found in [6].

Proposition 5.9 ([6, Proposition 3.3]). *A simplicial groupoid $X : \Delta^{\text{op}} \rightarrow \mathbf{Grpd}$ is a decomposition space if and only if the squares*

$$\begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ d_{\perp} \downarrow & \lrcorner & \downarrow d_{\perp} \\ X_0 & \xrightarrow{s_0} & X_1 \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ d_{\top} \downarrow & \lrcorner & \downarrow d_{\top} \\ X_0 & \xrightarrow{s_0} & X_1 \end{array}$$

are homotopy pullbacks and for each $n \geq 2$ there is some i , $0 < i < n$ such that the squares

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_{i+1}} & X_n \\ d_{\perp} \downarrow & \lrcorner & \downarrow d_{\perp} \\ X_n & \xrightarrow{d_i} & X_{n-1} \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ d_{\top} \downarrow & \lrcorner & \downarrow d_{\top} \\ X_n & \xrightarrow{d_i} & X_{n-1} \end{array}$$

are also homotopy pullbacks.

Using this characterization, one can easily show that decomposition spaces really generalize Segal spaces.

Corollary 5.10 ([6, Proposition 3.5]). *Any Segal space is a decomposition space.*

Proof. First, we show that the left square in

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_n} & X_n & \xrightarrow{d_{\top}} & X_{n-1} \\ d_{\perp} \downarrow & & d_{\perp} \downarrow & \lrcorner & \downarrow d_{\perp} \\ X_n & \xrightarrow{d_{n-1}} & X_{n-1} & \xrightarrow{d_{\top}} & X_{n-2} \end{array}$$

is a pullback for $n \geq 2$. The right hand one is a pullback because it is precisely the Segal condition. Moreover, the horizontal composites are $d_n d_{\top} = d_{\top} d_{\top}$ and $d_{n-1} d_{\top} = d_{\top} d_{\top}$ by the simplicial identities, so the outer square is just the pasting of two Segal squares

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_{\top}} & X_n & \xrightarrow{d_{\top}} & X_{n-1} \\ d_{\perp} \downarrow & \lrcorner & d_{\perp} \downarrow & \lrcorner & \downarrow d_{\perp} \\ X_n & \xrightarrow{d_{\top}} & X_{n-1} & \xrightarrow{d_{\top}} & X_{n-2}, \end{array}$$

hence a pullback as well. This completes the first part of the proof.

Next, we need to show that the left square in

$$\begin{array}{ccccc} X_1 & \xrightarrow{s_1} & X_2 & \xrightarrow{d_{\top}} & X_1 \\ d_{\perp} \downarrow & & d_{\perp} \downarrow & \lrcorner & \downarrow d_{\perp} \\ X_0 & \xrightarrow{s_0} & X_1 & \xrightarrow{d_{\top}} & X_0 \end{array}$$

is also a pullback. Again, the right hand square is the Segal condition, hence a pullback, and the horizontal composites are identities, so the outer square is a pullback. This proves that the left square is a pullback.

The proof for vertical d_{\perp} arrows is analogous. □

Before continuing, it is going to be illustrative to show an example of a decomposition space which is not Segal. In particular it does not arise as the nerve of a category.

Example 5.11 ([8, Example 1.1.5]). Consider any full groupoid G of the (large) groupoid of finite graphs and graph isomorphisms between them. We allow these to have multiple edges and loops. Take $X_0 = 1$ and $X_1 = G$. For $n \geq 2$, we define X_n to be the groupoid with objects of graphs in G together with an ordered partition (V_1, \dots, V_n) of its vertex set into n subsets. Isomorphisms of X_n are partition-index-preserving graph isomorphisms (i.e. such that they map vertices in the i -th component of the partition to the i -component of the partition of the target graph).

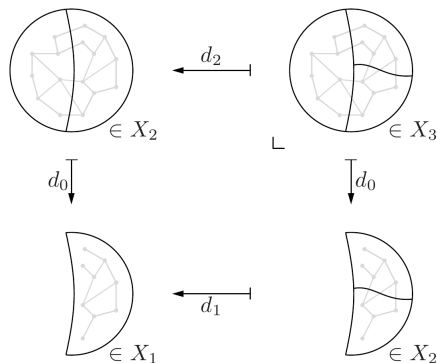
This structure can be equipped with face and degeneracy maps defined as

- Outer face maps d_{\perp} and d_{\top} delete the first (resp. last) set of the partition and return the corresponding subgraph with the specified set of edges removed.
- The inner face map d_i performs the union of the i -th partition set and the $(i + 1)$ -th.
- The degeneracy map s_i inserts an empty partition set at the i th position.

These operations clearly satisfy the simplicial identities, and the pullback condition from the decomposition space axioms

$$\begin{array}{ccc} X_2 & \xleftarrow{d_2} & X_3 \\ \downarrow d_0 & \lrcorner & \downarrow d_0 \\ X_1 & \xleftarrow{d_1} & X_2 \end{array}$$

states that a three-part partition (G, V_1, V_2, V_3) can be recovered from $(G, V_1, V_2 \cup V_3)$ and $(G \setminus V_1, V_2, V_3)$.



The reason why this decomposition space is not a Segal groupoid is that it is not possible to *uniquely* reconstruct a graph G from two disjoint subgraphs G_1, G_2 (with $V(G) = V(G_1) \cup V(G_2)$) as all the information about edges in G between G_1 and G_2 is lost.

5.1 The incidence (co)algebra of a decomposition space

Using tools from homotopy linear algebra, we can define the incidence coalgebra of a decomposition space X . Recall that a coalgebra (in the classical sense) is a k -vector space V equipped with two linear maps

$$\varepsilon : V \rightarrow k \quad \text{and} \quad \Delta : V \rightarrow V \otimes V$$

satisfying some coassociative and counital laws. Equivalently, it is a comonoid object in the monoidal category of k -vector spaces with the tensor product. In the homotopical setting we have replaced vector spaces by slices and linear maps by linear functors, so the incidence coalgebra of X is going to be some slice $\mathbf{Grpd}_{/B}$ together with corresponding linear functors

$$\varepsilon : \mathbf{Grpd}_{/B} \rightarrow \mathbf{Grpd}_{/1} \quad \text{and} \quad \Delta : \mathbf{Grpd}_{/B} \rightarrow \mathbf{Grpd}_{/B} \otimes \mathbf{Grpd}_{/B}.$$

The requirements in the definition of decomposition space then become precisely what is needed to prove that these maps are coassociative and counital (up to equivalence).

Let $B = X_1$ (recall that for the fat nerve of a category, $(\mathbf{NC})_1$ is the groupoid of all arrows). Consider the linear functors ε and Δ induced by the spans

$$X_1 \xleftarrow{s_0} X_1 \longrightarrow 1 \quad \text{and} \quad X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2, d_0)} X_1 \times X_1$$

respectively. This structure is the *incidence coalgebra* of X .

Theorem 5.12 ([6, §7]). *For a decomposition space X , the slice $\mathbf{Grpd}_{/X_1}$ is a strong monoid object in the symmetric monoidal category \mathbf{LIN} together with the linear functors ε and Δ described above.*

Proof. We just prove coassociativity, counitality is analogous and simpler. We need to show that the square

$$\begin{array}{ccc} \mathbf{Grpd}_{/X_1} & \xrightarrow{\Delta} & \mathbf{Grpd}_{/X_1 \times X_1} \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\ \mathbf{Grpd}_{/X_1 \times X_1} & \xrightarrow{\text{id} \otimes \Delta} & \mathbf{Grpd}_{/X_1 \times X_1 \times X_1} \end{array}$$

commutes up to equivalence. To see this, we expand it with the corresponding spans, which can be then be connected via X_3

$$\begin{array}{ccccc} X_1 & \xleftarrow{d_1} & X_2 & \xrightarrow{(d_2, d_0)} & X_1 \times X_1 \\ d_1 \uparrow & & d_1 \uparrow & \lrcorner & d_1 \times \text{id} \uparrow \\ X_2 & \xleftarrow{d_2} & X_3 & \xrightarrow{(d_3, d_0 d_0)} & X_2 \times X_1 \\ (d_2, d_0) \downarrow & & \downarrow (d_2 d_2, d_0) & & \downarrow (d_2, d_0) \times \text{id} \\ X_1 \times X_1 & \xleftarrow{\text{id} \times d_1} & X_1 \times X_2 & \xrightarrow{\text{id} \times (d_2, d_0)} & X_1 \times X_1 \times X_1 \end{array}$$

The top right square is a pullback because the outer square in the diagram

$$\begin{array}{ccccc}
 X_2 & \xrightarrow{(d_2, d_0)} & X_1 \times X_1 & \xrightarrow{p_1} & X_1 \\
 d_1 \uparrow & & d_1 \times \text{id} \uparrow & \lrcorner & \uparrow d_1 \\
 X_3 & \xrightarrow{(d_3, d_0 d_0)} & X_2 \times X_1 & \xrightarrow{p_1} & X_2
 \end{array}$$

is (X is a decomposition space), and the rightmost one is a pullback in general. The argument for the bottom left square is symmetric.

By the Beck-Chevalley lemma, we now have that $(d_1 \times \text{id})^* \circ (d_2, d_0)! \simeq (d_3, d_0 d_0)! \circ d_1^*$, so

$$\begin{aligned}
 (\Delta \otimes \text{id}) \circ \Delta &\simeq ((d_2, d_0) \times \text{id})! \circ (d_1 \times \text{id})^* \circ (d_2, d_0)! \circ d_1^* \\
 &\simeq ((d_2, d_0) \times \text{id})! \circ (d_3, d_0 d_0)! \circ d_1^* \circ d_1^* \\
 &\simeq ((d_2 d_3, d_0 d_3, d_0 d_0) \times \text{id})! \circ (d_1 d_1)^*
 \end{aligned}$$

This is, $(\Delta \otimes \text{id}) \circ \Delta$ is the linear functor induced by the diagonal span $X_1 \leftarrow X_3 \rightarrow X_1 \times X_1 \times X_1$. Similarly one proves that $\Delta \circ (\text{id} \otimes \Delta)$ is also determined by this span, so $(\Delta \otimes \text{id}) \circ \Delta \simeq \Delta \circ (\text{id} \otimes \Delta)$. \square

Recall that names $\ulcorner x \urcorner : 1 \rightarrow X_1$ play the role of basis elements in \mathbf{Grpd}/X_1 , so these should provide a simpler description of Δ and ε analogous to the one for locally finite posets and Möbius categories, where

$$\Delta([x, y]) = \sum_{z \in [x, y]} [x, z] \otimes [z, y] \quad \text{and} \quad \Delta(f) = \sum_{f=gh} g \otimes h$$

respectively. By Lemma 4.24 we have

$$\begin{aligned}
 \Delta(\ulcorner x \urcorner) &\simeq \int^{(a,b) \in X_1 \times X_1} ((X_2)_x)_{a,b} \otimes \ulcorner a, b \urcorner \\
 &\simeq \int^{(a,b) \in X_1 \times X_1} (X_2)_{x,a,b} \otimes \ulcorner a \urcorner \otimes \ulcorner b \urcorner,
 \end{aligned}$$

where $(X_2)_{x,a,b} = \text{hfib}((d_1, d_2, d_0), (x, a, b))$. In particular, if X is the strict nerve of a small category, the fibre is either a singleton or empty

$$(X_2)_{x,a,b} \simeq \begin{cases} \{(x, a, b)\} & \text{if } x = a \cdot b \\ \emptyset & \text{if } x \neq a \cdot b \end{cases}$$

and the comultiplication is the projection

$$\coprod_{x=a \cdot b} \mathbf{1} \rightarrow X_1 \times X_1 = \text{Mor } \mathcal{C} \times \text{Mor } \mathcal{C}$$

which, written in a more suggestive notation, is

$$\Delta(\ulcorner x \urcorner) \simeq \sum_{x=a \cdot b} \ulcorner a \urcorner \otimes \ulcorner b \urcorner.$$

More generally, if X is the fat nerve of a category (or any Segal space), one can easily verify (see [8]) that

$$(X_2)_{x,a,b} \simeq \begin{cases} \Omega_{d_1 a} X_0 \times \Omega_x X_1 & \text{if } x \cong a \cdot b, \\ \emptyset & \text{if } x \not\cong a \cdot b, \end{cases}$$

so

$$\Delta(\ulcorner x \urcorner) \simeq \Omega_x X_1 \otimes \int^{(a,b) \in X_1 \times X_1}_{f \cong a \cdot b} \Omega_{d_1 a} X_0 \otimes \ulcorner a \urcorner \otimes \ulcorner b \urcorner$$

The computation is even simpler for ε . Given $\ulcorner x \urcorner \in \mathbf{Grpd}/X_1$, we have

$$\varepsilon(\ulcorner x \urcorner) = (X_0)_x \otimes \ulcorner 1 \urcorner : (X_0)_x \rightarrow 1,$$

with $(X_0)_x = \text{hfib}(s_0, x)$. In particular, observe that

$$(X_0)_x = \begin{cases} 1 & \text{if } x \text{ is an identity} \\ \emptyset & \text{otherwise} \end{cases}$$

when $X = \mathcal{NC}$.

Naturally, the next step is to define the *incidence algebra* in terms of the coalgebra. As before, we simply consider the linear dual. In this case, it is the category

$$\mathbf{LIN}(\mathbf{Grpd}/X_1, \mathbf{Grpd}/1).$$

The convolution product $F * G$ of two linear functors $F, G : \mathbf{Grpd}/X_1 \rightarrow \mathbf{Grpd}/1$ is the composite

$$\mathbf{Grpd}/X_1 \xrightarrow{\Delta} \mathbf{Grpd}/X_1 \otimes \mathbf{Grpd}/X_1 \xrightarrow{F \otimes G} \mathbf{Grpd}/1 \otimes \mathbf{Grpd}/1 \xrightarrow{\cong} \mathbf{Grpd}/1,$$

and the unit just $\varepsilon : \mathbf{Grpd}/X_1 \rightarrow \mathbf{Grpd}/1$. Associativity and unitality follow directly from coassociativity and counitality. Rewritten in terms of the equivalence 4.33, the convolution product is simply

$$(F * G)(x) \simeq \int^{(a,b) \in X_1} (X_2)_{x,a,b} \times F(a) \times G(b),$$

and the unit becomes $\delta(x) \simeq (X_0)_x$. Since \mathbf{Grpd}^{X_1} is clearly more convenient to work in than $\mathbf{LIN}(\mathbf{Grpd}/X_1, \mathbf{Grpd}/1)$, we are going to consider \mathbf{Grpd}^{X_1} as the incidence algebra of X .

Finally, we define the characteristic functions $\chi_a : X_1 \rightarrow \mathbf{Grpd}$ for $a \in X_1$. The definition is identical to the one for Möbius categories:

$$\chi_a : X_1 \rightarrow \mathbf{Grpd} \\ x \mapsto \begin{cases} 1 & \text{if } x \cong a \\ \emptyset & \text{otherwise} \end{cases}$$

Expressed as a linear functor, it is the one induced by the span

$$X_1 \xleftarrow{\ulcorner a \urcorner} 1 \rightarrow 1,$$

so its coefficients are $\Omega_a X_1$ at indices isomorphic to a and 0 elsewhere.

Interestingly, one can recover the zeta functor ζ as $\zeta(x) \simeq \int^{a \in X_1} \chi_a$ and, in some cases (namely *complete* decomposition spaces, more on them in section 5.4), $\delta(x) \simeq \int^{v \in X_0} \chi_{s_0 v}$ as well.

5.2 CULF functors and equivalences of decomposition spaces

As always, the next step after defining a new structure or concept is usually giving the appropriate notion of homomorphism between two of them. In our case these are a particular kind of simplicial maps just like decomposition spaces are a particular kind of simplicial groupoids. Later we are going to see why this is the appropriate notion.

Definition 5.13. A simplicial map between two simplicial groupoids $F : X \rightarrow Y$ is *conservative* if the square

$$\begin{array}{ccc} X_n & \xrightarrow{s_i} & X_{n+1} \\ F_n \downarrow & \lrcorner & \downarrow F_n \\ Y_n & \xrightarrow{s_i} & Y_{n+1} \end{array}$$

is a homotopy pullback for all $0 \leq i \leq n$. Similarly, we say that it has *Unique Liftings of Factorizations* (ULF) if the square

$$\begin{array}{ccc} X_{n+1} & \xleftarrow{d_{i+1}} & X_{n+2} \\ F_{n+1} \downarrow & \lrcorner & \downarrow F_{n+2} \\ Y_{n+1} & \xleftarrow{d_{i+1}} & Y_{n+2} \end{array}$$

is a homotopy pullback for $0 \leq i \leq n$. A functor which is both conservative and ULF is called *CULF*.

Note that this definition does not mention decomposition spaces, it is defined in terms of any simplicial groupoid. The improvement brought by decomposition spaces is that it becomes much easier to check whether a map is CULF:

Proposition 5.14 ([6, Proposition 4.2]). *If X is a decomposition space, any ULF functor $F : X \rightarrow Y$ is conservative.*

Proposition 5.15 ([6, Lemma 4.3]). *A simplicial map $F : X \rightarrow Y$ between decomposition spaces is CULF if and only if it is cartesian on the generic map $\partial^1 : [1] \rightarrow [2]$:*

$$\begin{array}{ccc} X_1 & \xleftarrow{d_1} & X_2 \\ F_1 \downarrow & \lrcorner & \downarrow F_2 \\ Y_1 & \xleftarrow{d_1} & Y_2 \end{array}$$

Notice that a CULF functor is a generalization of what we called a *local isomorphism* in sections 2 and 3. For the nerve of a category, the set (f) is simply the fibre of the composition map $c : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$, and a local isomorphism is a functor that induces an isomorphism $(f) \rightarrow (F(f))$. For a CULF map $F : X \rightarrow Y$ and $x \in X_1$ we have the following diagram by the pullback pasting lemma 4.14:

$$\begin{array}{ccc}
1 & \longleftarrow & (X_2)_x \\
\downarrow \lrcorner x \lrcorner & & \downarrow \\
X_1 & \xleftarrow{d_1} & X_2 \\
\downarrow F_1 & & \downarrow F_2 \\
Y_1 & \xleftarrow{d_1} & Y_2
\end{array}$$

This is, the fibres $(X_2)_x$ and $(Y_2)_{F_1(x)}$ coincide. For the particular case of the strict nerve of a category this is precisely a bijection $(x) \rightarrow (F_1(x))$ which is induced by the fact that the square $c \circ (F_1 \times F_1) = F_1 \circ c$ is a pullback in **Set**:

$$\begin{aligned}
X_2 &= \{(x_1, x_2) \in X_1 \times X_1 : \text{dom } x_1 = \text{cod } x_2\} \\
&\cong \{(x, (y_1, y_2)) \in X_1 \times Y_2 : F(x) = y_1 y_2\}
\end{aligned}$$

Now, the reason why this is the appropriate notion of homomorphism is that CULF functors always induce coalgebra homomorphisms, making the incidence coalgebra construction functorial.

Proposition 5.16 ([6, Lemma 8.2]). *If $F : X \rightarrow Y$ is a CULF functor, then $F_{1!} : \mathbf{Grpd}_{X_1} \rightarrow \mathbf{Grpd}_{Y_1}$ preserves ε and Δ .*

Proof. Since we have two homotopy pullback squares

$$\begin{array}{ccc}
X_1 & \xleftarrow{d_1} & X_2 \\
F_1 \downarrow & \lrcorner & \downarrow F_2 \\
Y_1 & \xleftarrow{d_1} & Y_2
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_1 & \xleftarrow{s_0} & X_0 \\
F_1 \downarrow & \lrcorner & \downarrow F_0 \\
Y_1 & \xleftarrow{s_0} & Y_0
\end{array}$$

we can apply the Beck-Chevalley lemma (4.26) and obtain

$$\begin{aligned}
\Delta \circ F_{1!} &\simeq (d_2, d_0)_! \circ d_1^* \circ F_{1!} & \varepsilon \circ F_{1!} &\simeq 1_! \circ s_0^* \circ F_{1!} \\
&\simeq (d_2, d_0)_! \circ F_{2!} \circ d_1^* & &\simeq 1_! \circ F_{0!} \circ s_0^* \\
&\simeq (F_1 \times F_1)_! \circ (d_2, d_0)_! \circ d_1^* & \text{and} & \simeq 1_! \circ s_0^* \\
&\simeq (F_1 \times F_1)_! \circ \Delta & & \simeq \varepsilon,
\end{aligned}$$

where $1_! : \mathbf{Grpd}_{/I} \rightarrow \mathbf{Grpd}_{/1}$ denotes postcomposition with the terminal map. □

In this work we care about isomorphisms particularly. Since we are in a 2-categorical context, the appropriate notion of *sameness* is that of equivalence of 2-categories. Given that a simplicial groupoid is a functor $\Delta^{\text{op}} \rightarrow \mathbf{Grpd}$, an equivalence of two of these must be a strict 2-natural transformation which in turn is again an equivalence on each component. Recall that by Lemma 4.13 any commutative square whose vertical maps are equivalences is automatically a homotopy pullback square, so any simplicial map which is an equivalence at each n must be CULF. Moreover, by the (2-)functoriality of the lowershriek $f \mapsto f_!$ this assignment maps any equivalences $F_1 G_1 \simeq \text{Id}$ and $G_1 F_1 \simeq \text{Id}$ to equivalences $F_{1!} G_{1!} \simeq \text{Id}$ and $G_{1!} F_{1!} \simeq \text{Id}$. Thus, we have the immediate result that equivalent decomposition spaces yield equivalent incidence coalgebras.

Theorem 5.17. *If $F : X \rightarrow Y$ is an homotopy equivalence at each degree, then it is CULF and $F_{1!} : \mathbf{Grpd}/_{X_1} \rightarrow \mathbf{Grpd}/_{Y_1}$ is an equivalence of 2-categories that preserves ε and Δ .*

In particular, an equivalence of categories $\mathcal{C} \rightarrow \mathcal{D}$ induces levelwise equivalences $NC \rightarrow ND$ and $\mathbf{NC} \rightarrow \mathbf{ND}$, so $\mathbf{Grpd}/_{(NC)_1} \simeq \mathbf{Grpd}/_{(ND)_1}$ and $\mathbf{Grpd}/_{(NC)_1} \simeq \mathbf{Grpd}/_{(ND)_1}$. Since the nerve functor is full and faithful, this implication actually holds in both directions. An immediate consequence of this fact is that solving the isomorphism problem for decomposition spaces particularizes easily for Möbius categories.

5.3 Cardinality of the incidence (co)algebra

As we announced previously, the incidence coalgebra on $\mathbf{Grpd}/_{X_1}$ induces a coalgebra structure on $\mathbb{Q}_{\pi_0 X_1}$ by taking cardinalities as in 4.4. This section explains how this is done in detail and how it relates to the incidence algebra of a poset or a Möbius category.

In order to be able to compute cardinalities, we first need to ensure some finiteness conditions. The following definition is directly motivated by the requirements of Proposition 4.39 applied to Δ and ε .

Definition 5.18. A decomposition space X is *locally finite* if X_1 is a locally finite groupoid and $s_0 : X_0 \rightarrow X_1$ and $d_1 : X_2 \rightarrow X_1$ are finite maps.

Thus, for a locally finite decomposition space X we can restrict Δ and ε to finite groupoids

$$\Delta : \mathbf{grpd}/_{X_1} \rightarrow \mathbf{grpd}/_{X_1} \otimes \mathbf{grpd}/_{X_1} \quad \varepsilon : \mathbf{grpd}/_{X_1} \rightarrow \mathbf{grpd}/_1.$$

From the formulas above, one can recover the classical incidence coalgebra by taking cardinalities. The comultiplication becomes

$$\begin{aligned} |\Delta| : \mathbb{Q}_{\pi_0 X_1} &\rightarrow \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \\ \delta_x &\mapsto \int_{a,b \in X_1} |(X_2)_{x,a,b}| \delta_a \otimes \delta_b = \sum_{[a],[b] \in \pi_0 X_1} c_{a,b}^x \delta_a \otimes \delta_b, \end{aligned}$$

where the *section coefficients* $c_{a,b}^x$ denote $\frac{|(X_2)_{x,a,b}|}{|\Omega_a X_1| |\Omega_b X_1|}$. The counit is simply

$$\begin{aligned} |\varepsilon| : \mathbb{Q}_{\pi_0 X_1} &\rightarrow \mathbb{Q} \\ \delta_x &\mapsto |(X_0)_x|. \end{aligned}$$

As one would expect, when X is the strict nerve of a Möbius category $|\Delta|$ and $|\varepsilon|$ are the operations defined in section 3

$$f \mapsto \sum_{f=a \cdot b} a \otimes b \quad \text{and} \quad f \mapsto \begin{cases} 1 & \text{if } f \in \mathcal{C}_0 \\ 0 & \text{otherwise.} \end{cases}$$

For the special case in which X a Segal space one can prove [8, Proposition 1.2.6] that

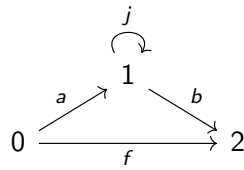
$$c_{a,b}^x = \begin{cases} \frac{|\Omega_{s_0 d_0 a} X_1| |\Omega_x X_1|}{|\Omega_a X_1| |\Omega_b X_1|} & \text{if } x \cong a \cdot b \\ 0 & \text{if } x \not\cong a \cdot b. \end{cases}$$

There is not much to say about $|\varepsilon|$ other than the fact that the result for NC remains true if we simply require s_0 to be a (homotopy) monomorphism. This fact will become more relevant later when studying complete decomposition spaces.

As a final remark, note that the linear duals of $|\Delta|$ and $|\varepsilon|$ correspond to the cardinality of the structure maps ($*$ and δ) of the incidence algebra. We denote this numerical incidence algebra by $I(X)$.

We complete this section with the explicit calculation of the numerical incidence algebra that perfectly illustrates the isomorphism problem for (Segal) decomposition spaces.

Example 5.19. Consider the fat nerve X of the *hanger category*



where $f = ba$, $a = ja$, $b = bj$ and $jj = id_1$.

It has $\pi_0 X_0 = \{[0], [1], [2]\}$ and an additional loop at $[1]$ given by j , $\text{Aut}_{X_0}(1) = \{id_1, j\}$. The groupoid X_1 of edges has connected components $\{[id_0], [id_1], [id_2], [a], [b], [f]\}$ and the only components with loops are $[id_1]$, $[a]$ and $[b]$, all three of them with exactly two automorphisms: one is the identity and the other one is a combination of identities and j . For instance, the automorphisms at $a \in X_1$ are

$$\begin{array}{ccc} 0 & \xrightarrow{a} & 1 \\ \parallel & & \parallel \\ 0 & \xrightarrow{a} & 1 \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{a} & 1 \\ \parallel & & \downarrow j \\ 0 & \xrightarrow{a} & 1 \end{array}$$

The groupoid X_2 is similar to X_1 : any connected components involving j can be replaced with one involving 1 instead, and any such component has two automorphisms: the identity and a diagram with some vertical j .

With this information and some manual work (which can be easily automated for these small scenarios) one can easily obtain the coefficients for the comultiplication

$$\begin{aligned} \Delta(\delta_i) &= \delta_i \otimes \delta_i \quad i = 0, 1, 2 \\ \Delta(\delta_a) &= \delta_0 \otimes \delta_a + \delta_a \otimes \delta_1 \\ \Delta(\delta_b) &= \delta_1 \otimes \delta_b + \delta_b \otimes \delta_2 \\ \Delta(\delta_f) &= \delta_0 \otimes \delta_f + \delta_f \otimes \delta_2 + \frac{1}{2} \delta_a \otimes \delta_b \end{aligned}$$

The fractional coefficient has appeared due to the loops introduced by j . If we let $\sigma \in X_2$ denote the 2-simplex $0 \xrightarrow{a} 1 \xrightarrow{b} 2$, then one can easily verify that

- $|\text{Aut}_{X_1}(a)| = |\text{Aut}_{X_1}(b)| = 2$
- The fibre $(X_2)_{f,a,b}$ is a discrete groupoid with two connected components. These are represented by

the identity map $(d_1\sigma, d_2\sigma, d_0\sigma) \rightarrow (f, a, b)$ and

$$\left(\begin{array}{ccc} 0 & \xrightarrow{d_1\sigma} & 2 \\ \parallel & & \parallel \\ 0 & \xrightarrow{f} & 2 \end{array}, \begin{array}{ccc} 0 & \xrightarrow{d_2\sigma} & 1 \\ \parallel & & \parallel \\ 0 & \xrightarrow{a} & 2 \end{array}, \begin{array}{ccc} 0 & \xrightarrow{d_0\sigma} & 1 \\ \downarrow j & & \parallel \\ 0 & \xrightarrow{b} & 2 \end{array} \right) : (d_1\sigma, d_2\sigma, d_0\sigma) \rightarrow (f, a, b)$$

The hanger category is clearly not equivalent to the category of the poset $[2] = \{0 < 1 < 2\}$ shown in Example 3.3. In fact, it is not even a Möbius category because $\ell(\text{id}_1) = \infty$ as one can always decompose $\text{id}_1 = jj$. Nevertheless, if we define $\Psi : I(X) \rightarrow I([2])$ by mapping $\delta_a \mapsto 2\delta_{0 \leq 1}$ and every other δ_x to its corresponding counterpart in $I([2])$, it is readily seen to be an isomorphism of coalgebras:

$$\begin{aligned} (\Psi \otimes \Psi)(\Delta(\delta_f)) &= \Psi(\delta_0) \otimes \Psi(\delta_f) + \Psi(\delta_f) \otimes \Psi(\delta_2) + \frac{1}{2}\Psi(\delta_a) \otimes \Psi(\delta_b) \\ &= \delta_{0 \leq 0} \otimes \delta_{0 \leq 2} + \delta_{0 \leq 2} \otimes \delta_{2 \leq 2} + \frac{1}{2}2\delta_{0 \leq 1} \otimes \delta_{1 \leq 2} \\ &= \Delta(\delta_{0 \leq 2}) \\ &= \Delta(\Psi(\delta_f)). \end{aligned}$$

5.4 Completeness and Möbius decomposition spaces

Before dealing with the isomorphism problem, we need more information about the comultiplication. For Möbius categories we had some useful formulas like $\chi_f * \chi_g = \chi_{fg}$ or $\chi_{\text{cod } f} * \chi_f = \chi_f$. These are not obvious in this case, and it turns out that some additional hypotheses are required in order to obtain similar results. Most of the content in this section is based on [7], although most proofs have been reworked according to our simpler setting.

To do this, we need to translate a few concepts like length conditions from Möbius categories. First, let us fix some notation. For a simplicial groupoid X , let \vec{X}_n denote the full subgroupoid of X_n of simplices with nondegenerate principal edges (this is, $d_+^{k-1}d_+^{n-k}\sigma \not\cong s_0v$ for any $1 \leq k \leq n$ and $v \in X_0$) and $\vec{X}_0 = X_0$ by convention. In addition, for each $n \geq 1$, let Φ_n be the linear functor given by the span

$$X_1 \xleftarrow{d_1^{n-1}} \vec{X}_n \longrightarrow 1$$

and $\Phi_0 = \varepsilon$ for $n = 0$:

$$X_1 \xleftarrow{s_0} X_0 \rightarrow 1.$$

Unless stated otherwise, we will regard Φ_n as an element of \mathbf{Grpd}^{X_1} under the equivalence in 4.33. This is, $\Phi_n(x) \simeq \text{hfib}(d_1^{n-1}|_{\vec{X}_n}, x)$.

The role of these functors is to measure the amount of decompositions of a simplex $x \in X_1$ into n nondegenerate simplices. In the rest of this section we are going to restrict ourselves to a class of decomposition spaces that is particularly well-behaved for vertices and degeneracies.

Definition 5.20. A decomposition space is *complete* if s_0 is a monomorphism.

The point of this definition becomes much clearer when combined with the following general fact about homotopy monomorphisms.

Proposition 5.21. *A map of groupoids $f : E \rightarrow B$ is a homotopy monomorphism if and only if it is full and faithful.*

Proof. To prove that it is a bijection on morphisms, let $\beta : f(e) \rightarrow f(e')$ be an isomorphism in B . Then regard it as an object of the fibre $E_{f(e')}$. Since the fibre is contractible, there exists a unique isomorphism between β and $\text{id}_{f(e')} : f(e') \rightarrow f(e')$

$$\begin{array}{ccc} f(e) & \xrightarrow{f(\varepsilon)} & f(e') \\ & \searrow \beta & \swarrow \cong \\ & & f(e') \end{array}$$

This is, there is a unique $\varepsilon : e \rightarrow e'$ with $f(\varepsilon) = \beta$. The converse can be easily deduced from observations in section 4: if f is full then the fibre is connected, and if f is faithful the fibre is discrete. Combining both gives that the fibre is empty or contractible. \square

The first consequence of completeness is that the inclusion $X_0 + \vec{X}_1 \rightarrow X_1$ is in fact an equivalence. For this reason, we identify X_0 with a full subgroupoid of X_1 when dealing with complete spaces (for instance we write $v \in X_1$ for any $v \in X_0$). This result allows us to extend it to X_n because in a decomposition space all degeneracies are pullbacks of s_0 along compositions of outer face maps (from the definition and the pullback pasting lemma), so we obtain the following immediate result by recalling Proposition 4.37.

Proposition 5.22 ([7, Lemma 2.3]). *All degeneracies in a complete decomposition space are monomorphisms.*

This implies that X_n is the disjoint union of the full subgroupoids of X_n where a fixed combination of degenerate and nondegenerate principal edges appears.

$$X_n \simeq \sum_{k=0}^n \binom{n}{k} \vec{X}_k$$

For instance,

$$X_2 \simeq \vec{X}_2 + s_0(\vec{X}_1) + s_1(\vec{X}_1) + s_0s_0(X_0)$$

Using this decomposition into disjoint subgroupoids we can recover a familiar expression for the multiplication of the incidence coalgebra:

$$\Delta(\ulcorner X \urcorner) \simeq \int^{(a,v) \in \vec{X}_1 \times X_0} (X_2)_{xav} \otimes \ulcorner a \urcorner \otimes \ulcorner v \urcorner + \int^{(v,b) \in X_0 \times \vec{X}_1} (X_2)_{xvb} \otimes \ulcorner v \urcorner \otimes \ulcorner b \urcorner \quad (2)$$

$$+ \int^{(a,b) \in \vec{X}_1 \times \vec{X}_1} (X_2)_{xab} \otimes \ulcorner a \urcorner \otimes \ulcorner b \urcorner + \int^{(v,w) \in X_0 \times X_0} (X_2)_{xvw} \otimes \ulcorner v \urcorner \otimes \ulcorner w \urcorner \quad (3)$$

The first two terms correspond to the trivial decompositions $f = \text{id}f = f\text{id}$ for Möbius categories, the third term is about nontrivial ones and lastly the case when $f = \text{id}$. When taking cardinalities, these summands become even simpler by turning homotopy sums into regular (weighted) sums.

Proposition 5.23. *Given $x, a \in X_1$ and $v \in X_0$, we have*

$$(X_2)_{xav} \simeq \begin{cases} \Omega_x X_1 \times \Omega_{d_0 x} X_1 & \text{if } a \cong x \text{ and } v \cong d_0 x \\ 0 & \text{otherwise} \end{cases}$$

and similarly for $(X_2)_{xva}$.

Proof. Considering that in the diagram

$$\begin{array}{ccc} (X_2)_{xav} & \longrightarrow & 1 \\ \downarrow & & \downarrow \lceil (x, a, v) \rceil \\ X_1 & \xrightarrow{(\text{diag}, d_0)} & X_1 \times X_1 \times X_0 \longrightarrow X_0 \\ \downarrow s_1 & \lrcorner & \downarrow \text{id} \times s_0 \\ X_2 & \xrightarrow{(d_1, d_2, d_0)} & X_1 \times X_1 \times X_1 \longrightarrow X_1 \end{array}$$

the bottom left square is a pullback (since the outer rectangle is cartesian by the decomposition space axioms), it suffices to prove the upper square. Clearly, $(X_1)_{xav} = 0$ if $x \not\cong a$ or $d_0 x \not\cong v$. For the other case assume without loss of generality that $x = a$ and $d_0 x = v$ and consider the canonical map

$$\begin{array}{ccc} u \quad \Omega_x X_1 \times \Omega_{d_0 x} X_0 & \rightarrow & (X_1)_{x, x, d_0 x} \\ (\phi_1, \phi_2) & \mapsto & ((x, x, d_0 x), (\text{id}, \phi_1, \phi_2) : (x, x, d_0 x) \rightarrow (x, x, d_0 x)) \end{array}$$

We show that it is essentially surjective. For any other object $(\psi_1, \psi_2, \psi_3) : (y, y, d_0 y) \rightarrow (x, x, d_0 x)$ of the fibre, there is an object $(\text{id}, \psi_2 \psi_1^{-1}, \psi_3 \psi_1^{-1}) = u(\psi_2 \psi_1^{-1}, \psi_3 \psi_1^{-1})$ in the image which is isomorphic to it via $\psi_1^{-1} : x \rightarrow y$.

Finally, it is faithful because the domain is discrete. It is full because any isomorphism $\varphi : (\text{id}, \phi_1, \phi_2) \rightarrow (\text{id}, \phi'_1, \phi'_2)$ must be the identity by inspecting the first component: $\text{id} = \text{id}\varphi$.

$$\begin{array}{ccc} (x, x, d_0 x) & \xrightarrow{(\text{diag} \varphi, d_0 \varphi) = (\varphi, \varphi, d_0 \varphi)} & (x, x, d_0 x) \\ & \searrow (\text{id}, \phi_1, \phi_2) & \swarrow (\text{id}, \phi'_1, \phi'_2) \\ & (x, x, d_0 x) & \end{array}$$

Then we get $\Omega_x X_1 \times \Omega_{d_0 x} X_0 \simeq (X_1)_{x, x, d_0 x}$. Note that $\Omega_{d_0 x} X_0 \cong \Omega_{d_0 x} X_1$ because s_0 is full and faithful (a homotopy monomorphism). \square

This implies that the first summands of the comultiplication of δ_x , $x \in \vec{X}_1$, become⁷

$$\begin{aligned} \int^{(a, v) \in \vec{X}_1 \times X_0} |(X_2)_{xav}| \delta_a \otimes \delta_v &= \frac{|(X_2)_{x, x, d_0 x}|}{|\Omega_x X_1| |\Omega_{d_0 x} X_1|} \delta_x \otimes \delta_{d_0 x} = \delta_x \otimes \delta_{d_0 x} \\ \int^{(v, b) \in X_0 \times \vec{X}_1} |(X_2)_{xvb}| \delta_v \otimes \delta_b &= \frac{|(X_2)_{x, d_2 x, x}|}{|\Omega_x X_1| |\Omega_{d_2 x} X_1|} \delta_{d_2 x} \otimes \delta_x = \delta_{d_2 x} \otimes \delta_x \end{aligned}$$

⁷Here $|\Omega_{d_0 x} X_0| = |\Omega_{d_0 x} X_1|$ because s_0 is a monomorphism.

Next, we obtain a few more properties by describing \vec{X}_n in terms of the pullback

$$\begin{array}{ccc} \vec{X}_n & \hookrightarrow & X_n \\ (d_{\perp}^{k-1} d_{\top}^{n-k})_k \downarrow & \lrcorner & \downarrow (d_{\perp}^{k-1} d_{\top}^{n-k})_k \\ \vec{X}_1^n & \hookrightarrow & X_1^n \end{array}$$

It is routine to verify that the canonical map $\vec{X}_n \rightarrow \vec{X}_1^n \times_{X_1^n} X_n$ mapping $\sigma \mapsto (\sigma, (d_{\perp}^{k-1} d_{\top}^{n-k} \sigma)_k, \text{id})$ is essentially surjective, full and faithful, hence the diagram is a pullback square.

This observation can be used to prove that, given a 2-simplex whose principal edges correspond to the long edges of some n - and m -simplices, its principal edges can be replaced with the corresponding n - and m -simplices obtaining a $(n+m)$ -simplex. Clearly, the result still holds for an n -simplex with principal edges corresponding to m_i -simplices, but the proof is analogous and the notation is much simpler in the binary case.

Proposition 5.24 ([7, Lemma 3.5]). *There is a pullback square*

$$\begin{array}{ccccc} \vec{X}_{n+m} & \hookrightarrow & X_{n+m} & \xrightarrow{d_2^{m-1} d_1^{n-1}} & X_2 \\ (d_{\top}^m, d_{\perp}^n) \downarrow & \lrcorner & (d_{\top}^m, d_{\perp}^n) \downarrow & \lrcorner & \downarrow (d_{\top}, d_{\perp}) \\ \vec{X}_n \times \vec{X}_m & \hookrightarrow & X_n \times X_m & \xrightarrow{d_1^{n-1} \times d_1^{m-1}} & X_1 \times X_1 \end{array}$$

Proof. In the diagram

$$\begin{array}{ccccccc} \vec{X}_{n+m} & \hookrightarrow & X_{n+m} & \longrightarrow & X_{1+m} & \longrightarrow & X_2 \\ \downarrow & & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \vec{X}_n \times \vec{X}_m & \hookrightarrow & X_n \times X_m & \longrightarrow & X_1 \times X_m & \longrightarrow & X_1 \times X_1 \\ \downarrow & \lrcorner & \downarrow & & & & \\ \vec{X}_1^{n+m} & \hookrightarrow & X_1^{n+m} & & & & \end{array}$$

the bottom square is a product of pullback squares and the outer left rectangle is also cartesian, so the top left square is. The two rightmost squares are pullbacks by repeated application the decomposition space axioms. \square

The length of a simplex is then a straightforward definition in terms of Φ_n and \vec{X}_n .

Definition 5.25. The length of a 1-simplex $x \in X_1$ is

$$\ell(x) = \sup\{n \geq 0 : \Phi_n(x) \neq \emptyset\}.$$

and $\ell(x) = \infty$ if $\Phi_n(x) \neq \emptyset$ for all $n \geq 0$. For $\sigma \in X_n$, the length of σ is defined as the length of its long edge.

Note that $\Phi_n(x) \neq \emptyset$ is equivalent to the existence of an n -simplex with nondegenerate principal edges whose long edge is x . Moreover, observe that ℓ is invariant under isomorphism by the functoriality of Φ_n and defines a functor $X_n \rightarrow \mathbb{N}$ (regarding \mathbb{N} as a discrete groupoid) that we can use to partition X_n into disjoint groupoids by length

$$\begin{array}{ccc} X_{n,k} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \lrcorner k \\ X_n & \xrightarrow{\ell} & \mathbb{N} \end{array}$$

where $X_n = \sum_k X_{n,k}$ and $X_{n,k}$ is the full subgroupoid of X_n whose simplices have length k . We now define what is going to be the analog definition of Möbius category for decomposition spaces.

Definition 5.26. A complete decomposition space is of *locally finite length* if all simplices of X_1 have finite length.

Definition 5.27. A complete decomposition space is *Möbius* if it is both locally finite and of locally finite length.

These hypotheses already allow us to prove some basic (and intuitive) results that were more or less obvious for Möbius categories. Firstly, recall that $\ell(f) + \ell(g) \leq \ell(fg)$.

Proposition 5.28 ([7, §6.20]). *If X is of locally finite length and $\sigma \in X_n$, then $\ell(\sigma) \geq \sum_{k=1}^n \ell(d_{\perp}^{k-1} d_{\top}^{n-k} \sigma)$.*

Proof. Let $m_k = \ell(d_{\perp}^{k-1} d_{\top}^{n-k} \sigma)$ be the length of the k th principal edge, we need to prove that $\Phi_{m_1+\dots+m_n}(d_1 \sigma) \neq \emptyset$. Let $\sigma_k \in \vec{X}_{m_k}$ with $d_1^{m_k-1} \sigma_k \cong d_{\perp}^{k-1} d_{\top}^{n-k} \sigma$. Then we have a simplex $\bar{\sigma} \in \vec{X}_{m_1+\dots+m_n}$ with long edge $d_1^{n-1} \sigma$ by Proposition 5.24. \square

A key fact about Möbius categories is that morphisms were not invertible. More generally, the composition of any two non-identity morphisms never yielded an identity. This one is going to take several to prove, the first one being that at least one of them must not be an identity.

Proposition 5.29 ([7, Corollary 6.9]). *If X is any decomposition space, a simplex $\sigma \in X_n$ is degenerate if and only if $d_{\perp} \sigma$ or $d_{\top} \sigma$ is.*

Proof. First, assume that $d_{\top} \sigma = s_k \tau$. Then the pullback condition

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{s_k} & X_n \\ d_{\top} \downarrow & \lrcorner & \downarrow d_{\top} \\ X_{n-2} & \xrightarrow{s_k} & X_{n-1} \end{array}$$

ensures that $\sigma = s_k \bar{\sigma}$ for some $\bar{\sigma} \in X_{n-1}$. The case $d_{\perp} \sigma = s_k \tau$ is analogous. Conversely, if $\tau = s_k \sigma$, then $d_0 s_k \sigma = s_{k-1} d_0 \sigma$. \square

Corollary 5.30. *In a decomposition space, an n -simplex is degenerate if and only if some of its principal edges are.*

Proof. Iterate Proposition 5.29. □

Another intuitive fact is that vertices have length zero. Again, the proof is not as simple as one would expect but the fact still holds.

Proposition 5.31 ([7, Lemma 6.6]). *If X is of locally finite length and $x \in X_1$, $\ell(x) = 0$ if and only if $x \cong s_0 v$ for some $v \in X_0$.*

Proof. One implication is trivial: if $\ell(x) = 0$ then $\Phi_1(x) = 0$ and $x \notin \vec{X}_1$, so $x \in X_0$.

We prove the converse by contradiction. Let $v \in X_0$ and assume that $\ell(s_0 v) = n > 0$. Then, there exists $\sigma \in \vec{X}_n$ with long edge $s_0 v$. By repeated application of the decomposition space axioms, we have the following pullback diagrams (note that $d_\top d_2^{n-1} = d_\top^n$):

$$\begin{array}{ccccc}
 X_{n+n} & \xrightarrow{d_1^{n-1}} & X_{1+n} & \xrightarrow{d_\perp} & X_n \\
 \downarrow d_\top^n & \lrcorner & \downarrow d_2^{n-1} & \lrcorner & \downarrow d_1^{n-1} \\
 & & X_2 & \xrightarrow{d_\perp} & X_1 \\
 & & \downarrow d_\top & & \\
 X_n & \xrightarrow{d_1^{n-1}} & X_1 & &
 \end{array}$$

Now consider $s_0 s_0 v \in X_2$ and $\sigma \in \vec{X}_n$. Clearly, $d_\perp s_0 s_0 v = s_0 v = d_1^{n-1} \sigma$, hence there exists $v\sigma \in X_{n+1}$ with $d_2^{n-1}(v\sigma) \cong s_0 s_0 v$ and $d_\perp(v\sigma) \cong \sigma$. Next, observe that $d_\perp^n(v\sigma) \cong s_0 v = d_1^{n-1} \sigma$, so there exists $\sigma\sigma \in X_{n+n}$ with $d_\top^n(\sigma\sigma) \cong \sigma$ and $d_\perp^n(\sigma\sigma) = d_\perp d_1^{n-1}(\sigma\sigma) \cong \sigma$. Since σ has all principal edges nondegenerate, so does $\sigma\sigma$, hence $\sigma\sigma \in \vec{X}_{n+n}$ with $d_1^{2n-1}(\sigma\sigma) = d_1(s_0 s_0 v) = s_0 v$ contradicting $\ell(s_0 v) = n < n + n$. □

We can finally prove the statement about the lack of isomorphisms in our analogy with Möbius categories. Note that the hypotheses reflect the fact that all arrows in a Möbius category must have finite length, which is the key of the proof in that particular case.

Proposition 5.32 ([7, Corollary 6.10]). *If X is of locally finite length, an n -simplex $\sigma \in X_n$ has a degenerate long edge if and only if all its principal edges are degenerate or, equivalently, $\sigma \cong s_0^n v$.*

Proof. If the long edge is degenerate, then $\ell(\sigma) = 0$ by definition and Proposition 5.31. By Proposition 5.28,

$$0 = \ell(\sigma) = \ell(d_1^{n-1} \sigma) \geq \sum_k \ell(d_\perp^{k-1} d_\top^{n-k} \sigma).$$

Therefore $\ell(d_\perp^{k-1} d_\top^{n-k} \sigma) = 0$ and each principal edge is degenerate.

We prove the second part by induction on n . The base case $n = 1$ is trivial, so suppose that it holds any $(n - 1)$ -simplex. Given that all the principal edges of $\sigma \in X_n$ are degenerate, $d_\top \sigma$ is completely degenerate by the induction hypothesis, so let $v \in X_0$ such that $s_0^{n-1} v \cong d_\top \sigma$. Since the square

$$\begin{array}{ccc}
 X_1 & \xrightarrow{s_0^{n-1}} & X_n \\
 d_{\top} \downarrow & \lrcorner & \downarrow d_{\top} \\
 X_0 & \xrightarrow{s_0^n} & X_{n-1}
 \end{array}$$

is a pullback, there exists $x \in X_1$ such that $\sigma \cong s_0^{n-1}x$ and $v \cong d_{\top}x$. However, the last principal edge of σ , $d_{\perp}^{n-1}\sigma \cong d_{\perp}^{n-1}s_0^{n-1}x = x$ is also degenerate, so $x \cong s_0w$ for some w . We conclude that $\sigma \cong s_0^n w$. Finally, the fact that the long edge is degenerate is a direct consequence of simplicial identities. \square

Combining all these results, we can finally obtain a simple formula for the numerical comultiplication. After taking cardinalities in (2) and applying our previous observations, we obtain the familiar expressions

$$|\Delta|(\delta_v) = \frac{(X_2)_{vw}}{|\Omega_v X_1| |\Omega_v X_1|} \delta_v \otimes \delta_v = \delta_v \otimes \delta_v$$

for $v \in X_0$ and

$$\begin{aligned}
 |\Delta|(\delta_x) &= \delta_x \otimes \delta_{d_{\perp}x} + \delta_{d_{\top}x} \otimes \delta_x + \int^{(a,b) \in \vec{X}_1 \times \vec{X}_1} |(X_2)_{xab}| \delta_a \otimes \delta_b \\
 &= \delta_x \otimes \delta_{d_{\perp}x} + \delta_{d_{\top}x} \otimes \delta_x + \sum_{\substack{[a] \in \pi_0 X_1 \\ 0 < \ell(a) < n}} \sum_{\substack{[b] \in \pi_0 X_1 \\ 0 < \ell(b) < n}} c_{ab}^x \delta_a \otimes \delta_b
 \end{aligned}$$

for $x \in \vec{X}_1$, $\ell(x) = n > 0$. This alone is enough to prove the inversion formula in the incidence algebra as in Theorem 3.9.

Finally, observe that by definition in a Möbius decomposition space the sequence $\Phi_1(x), \Phi_2(x), \dots$ is eventually 0 and each $\Phi_n(x)$ is finite. Then it follows that the map

$$\sum_{n \geq 1} d_1^{n-1} : \prod_{n \geq 1} \vec{X}_n \rightarrow X_1$$

is finite. For the strict nerve of a small category \mathcal{C} , this means that for any arrow in \mathcal{C} there is a finite amount of decompositions into indecomposable arrows.

5.5 The isomorphism problem for decomposition spaces

The intermediate step at slice coalgebras in the passage from decomposition spaces to numerical incidence algebras gives us an opportunity to approach the isomorphism problem differently. Recall that the entire process for a Möbius category \mathcal{C} can be described in multiple steps:

$$\mathcal{C} \mapsto \mathbf{NC} \mapsto C(\mathbf{NC}) \mapsto I(\mathbf{NC})$$

First, one generates the (fat) nerve of \mathcal{C} . It is well known that this operation is full and faithful, so we can safely say that the nerves of non-isomorphic categories are never going to be isomorphic nor equivalent.

On the other hand, we still know nothing about how much information does the slice coalgebra preserve. By analogy with vector spaces, one could think of it as the vector space spanned by the 1-simplices of the

decomposition space together with an operation to provide all possible decompositions of a simplex. It is no longer clear whether this suffices to reconstruct the entire decomposition space or not, so we will first study this new version of the isomorphism problem.

Next, we have the analogue of the classical notion with the passage to the incidence algebra. For instance, two different decomposition spaces could have isomorphic numerical incidence algebras (as we saw in the counterexamples in section 3) but different slice coalgebras. Now it is clearer that taking cardinalities loses important chunks of information: firstly, all the group structure and the topology of a groupoid is compressed as a rational number and, secondly, the vector space is spanned by the connected components, again lacking any knowledge about automorphism groups.

5.5.1 Equivalent slice coalgebras

Since the context of this section is entirely contained at the groupoid level, we need to work with linear functors and spans. To do so, we are going to make heavy use of the equivalence between **LIN** and **Span**, especially of the one regarding the hom-2-categories **Span**(I, J).

Lemma 5.33. *Let $s : I \rightarrow J$ be an equivalence. If s^{-1} denotes its homotopy inverse, then $s^* \simeq (s^{-1})_!$ as linear functors.*

Proof. Consider the following morphism (homotopy commutative diagram) between the underlying spans of s^* and $(s^{-1})_!$ respectively

$$\begin{array}{ccc}
 & I & \\
 s \swarrow & & \searrow \\
 J & & I \\
 \parallel & \simeq \downarrow s & \parallel \\
 & J & \\
 & \uparrow s^{-1} &
 \end{array}$$

Since the middle morphism s is an equivalence, it induces an equivalence $s^* \simeq (s^{-1})_!$ by Proposition 4.30. \square

Proposition 5.34. *Let $\Psi : \mathbf{Grpd}_{/I} \rightarrow \mathbf{Grpd}_{/J}$ be an equivalence induced by a span $I \xleftarrow{s} A \xrightarrow{t} J$. Then s and t are equivalences. Moreover $\Psi \simeq ts^{-1}$, where s^{-1} denotes the homotopy inverse of s .*

Proof. Let $\Psi' : \mathbf{Grpd}_{/J} \rightarrow \mathbf{Grpd}_{/I}$ be the inverse of Ψ , i.e., $\Psi'\Psi \simeq \text{Id}$ and $\Psi\Psi' \simeq \text{Id}$. Note that by being part of an equivalence with a linear functor (Ψ), Ψ' must preserve homotopy sums:

$$\Psi' \left(\int^{k \in K} g_k \right) \simeq \Psi' \left(\int^{k \in K} \Psi(\Psi'(g_k)) \right) \simeq \Psi' \left(\Psi \left(\int^{k \in K} \Psi'(g_k) \right) \right) \simeq \int^{k \in K} \Psi'(g_k),$$

so by Proposition 4.32 Ψ' is linear. If $I \xleftarrow{s'} A' \xrightarrow{t'} J$ is the span that defines Ψ' , then the composite $\Psi'\Psi$ is given by $I \xleftarrow{sp} A \times_J A' \xrightarrow{t'q} J$, where

$$\begin{array}{ccccc}
 I & \xleftarrow{s} & A & \xleftarrow{p} & A \times_J A' \\
 & & \downarrow t & & \downarrow q \\
 & & J & \xleftarrow{s'} & A' \\
 & & & & \downarrow t' \\
 & & & & I
 \end{array}$$

Since this composition is equivalent to the identity, by Proposition 4.30 their corresponding spans are equivalent, so there exists an equivalence $f : A \times_J A' \rightarrow I$ and a (homotopy) commutative diagram

$$\begin{array}{ccc}
 & A \times_J A' & \\
 sp \swarrow & & \searrow t'q \\
 I & & I \\
 \parallel & \simeq \downarrow f & \parallel \\
 & I &
 \end{array}$$

This is, $sp \simeq t'q \simeq t$, so they are equivalences. This implies that s and t' are essentially surjective and the symmetric argument proves the same for s' and t . Moreover, one can easily verify (using the pullback pasting lemma, 4.14) that pullbacks of essentially surjective maps are essentially surjective, so p and q are essentially surjective as well. Now, s is also full because sp is an equivalence and p is essentially surjective, hence the fibre A_i is always nonempty and connected. Repeating the same argument for s' and applying the pullback pasting lemma again we obtain that $\Omega_* A_i$ (where $*$ is any object in the unique connected component of A_i) is connected because s' is full, hence $A_i \simeq 1$.

$$\begin{array}{ccccc}
 & & 1 & \longleftarrow & \Omega_* A_i \\
 & & \downarrow \Gamma_*^{-1} & & \downarrow \Gamma_* \\
 1 & \longleftarrow & A_i & \longleftarrow & 1 \\
 \downarrow \Gamma_i^{-1} & & \downarrow \Gamma_i & & \downarrow \Gamma_i \\
 I & \xleftarrow{s} & A & \xleftarrow{p} & A \times_J A' \\
 & & \downarrow t & & \downarrow q \\
 & & J & \xleftarrow{s'} & A'
 \end{array}$$

Since all the fibres of s are nonempty and contractible, it is an equivalence. The same argument applies for s' , so it is an equivalence as well. Let s^{-1} and $(s')^{-1}$ be their homotopy inverses. Then we have that $\Psi \simeq (ts^{-1})_!$ and $\Psi' \simeq (t'(s')^{-1})_!$ by Lemma 5.33. Now, their composition is equivalent to the identity, so we have two more commutative diagrams

$$\begin{array}{ccc}
 I & \xrightarrow{t'(s')^{-1}ts^{-1}} & I \\
 \parallel & & \parallel \\
 I & & I
 \end{array}
 \qquad
 \begin{array}{ccc}
 J & \xrightarrow{ts^{-1}t'(s')^{-1}} & J \\
 \parallel & & \parallel \\
 J & & J
 \end{array}$$

that ensure that ts^{-1} and $t'(s')^{-1}$ are homotopy inverses of each other. Both ts^{-1} and s^{-1} are homotopy equivalences, so t must be an homotopy equivalence as well. \square

What Proposition 5.34 is essentially stating is that any equivalence given by linear functors can not be too far from a permutation matrix. The idea behind this proposition is that groupoid coefficients can never be negative and usually there are no inverses for the cartesian product, so the usual matrix inversion formulas all fail for non-permutation matrices.

In light of this result it is reasonable to expect an equivalence of coalgebras to induce an equivalence of decomposition spaces. More precisely, we have the following theorem, to whose proof we dedicate the rest of this subsection.

Theorem 5.35. Let X and Y be decomposition spaces and $\Psi : \mathbf{Grpd}/_{X_1} \rightarrow \mathbf{Grpd}/_{Y_1}$ an equivalence in **LIN** commuting (up to equivalence) with the corresponding ε and Δ :

$$\begin{array}{ccc} \mathbf{Grpd}/_{X_1} & & \mathbf{Grpd}/_{X_1} \xrightarrow{\Delta} \mathbf{Grpd}/_{X_1} \otimes \mathbf{Grpd}/_{X_1} \\ \downarrow \Psi & \searrow \varepsilon & \downarrow \Psi \qquad \qquad \downarrow \Psi \otimes \Psi \\ \mathbf{Grpd}/_{Y_1} & \xrightarrow{\varepsilon} \mathbf{Grpd}/_1 & \mathbf{Grpd}/_{Y_1} \xrightarrow{\Delta} \mathbf{Grpd}/_{Y_1} \otimes \mathbf{Grpd}/_{Y_1} \end{array}$$

Then, regarding X and Y as pseudofunctors $\mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Grpd}$, there exists an equivalence $X \rightarrow Y$ in the 2-category $\text{PsFun}(\mathbf{\Delta}^{\text{op}}, \mathbf{Grpd})$. Moreover, this map is CULF and the induced functor $\mathbf{Grpd}/_{X_1} \rightarrow \mathbf{Grpd}/_{Y_1}$ is precisely Ψ .

Given two decomposition spaces X and Y , suppose that $\Psi : \mathbf{Grpd}/_{X_1} \rightarrow \mathbf{Grpd}/_{Y_1}$ is an equivalence of coalgebras. By Proposition 5.34 we have that $\Psi \simeq \Psi_1!$ for some equivalence $\Psi_1 : X_1 \rightarrow Y_1$. Moreover, since it is an equivalence of coalgebras we have $\varepsilon_Y \simeq \varepsilon_X \circ \Psi_1!$ and $(\Psi_1 \otimes \Psi_1)! \circ \Delta_X \simeq \Delta_Y \circ \Psi_1!$. From the first equivalence we obtain

$$\varepsilon_Y \simeq 1_! s_0^* \simeq 1_! s_0^* \Psi_1! \simeq 1_! (\Psi_1^{-1} s_0)$$

Again, by Proposition 4.30 we recover an equivalence of spans

$$\begin{array}{ccc} & X_0 & \\ & \swarrow s_0 & \searrow \\ X_1 & & 1 \\ & \swarrow \Psi_1^{-1} s_0 & \searrow \\ & Y_0 & \end{array} \quad \begin{array}{c} \simeq \\ \downarrow \Psi_0 \end{array}$$

that becomes the square

$$\begin{array}{ccc} X_0 & \xrightarrow{s_0} & X_1 \\ \Psi_0 \downarrow \simeq \lrcorner & & \simeq \downarrow \Psi_1 \\ Y_0 & \xrightarrow{s_0} & Y_1, \end{array}$$

which is a pullback by 4.13.

On the other hand, the condition $(\Psi_1 \otimes \Psi_1)! \circ \Delta_X \simeq \Delta_Y \circ \Psi_1!$ gives

$$(\Psi_1 \otimes \Psi_1)! (d_{\top}, d_{\perp})! d_1^* \simeq (\Psi_1 \otimes \Psi_1)! \circ \Delta_X \simeq \Delta_Y \circ \Psi_1! \simeq (d_{\top}, d_{\perp})! (\Psi_1^{-1} d_1)^*.$$

Applying Proposition 4.30 once again

$$\begin{array}{ccccc} & X_2 & & & \\ & \swarrow d_1 & \searrow (d_{\top}, d_{\perp}) & & \\ X_1 & & X_1 \times X_1 & \xrightarrow{\Psi_1 \otimes \Psi_1} & Y_1 \times Y_1 \\ & \swarrow \Psi_1^{-1} & \searrow \Psi_2 & & \\ & Y_1 & & & Y_2 \\ & & \swarrow d_1 & \searrow (d_{\top}, d_{\perp}) & \end{array}$$

hence

$$\begin{array}{ccc}
 X_2 & \xrightarrow{d_1} & X_1 \\
 \psi_2 \downarrow \simeq \lrcorner & & \simeq \downarrow \psi_1 \\
 Y_2 & \xrightarrow{d_1} & Y_1
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 X_2 & \xrightarrow{(d_\top, d_\perp)} & X_1 \times X_1 \\
 \psi_2 \downarrow \simeq \lrcorner & & \simeq \downarrow \psi_1 \otimes \psi_1 \\
 Y_2 & \xrightarrow{(d_\top, d_\perp)} & Y_1 \times Y_1
 \end{array}$$

by a similar argument as before.

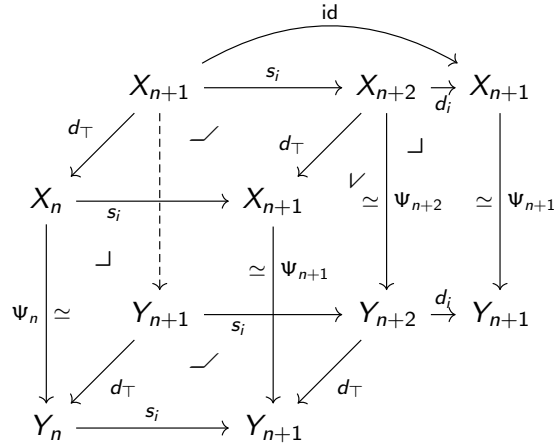
Now we prove the existence of Ψ_n inductively for $n \geq 3$. Suppose that there are equivalences $\Psi_k : X_k \rightarrow Y_k$ for each k , $2 \leq k \leq n+1$ such that the squares

$$\begin{array}{ccc}
 X_k & \xrightarrow{d_i} & X_{k-1} \\
 \psi_k \downarrow \simeq \lrcorner & & \simeq \downarrow \psi_{k-1} \\
 Y_k & \xrightarrow{d_i} & Y_{k-1}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 X_k & \xrightarrow{(d_\top, d_\perp)} & X_{k-1} \times X_{k-1} \\
 \psi_k \downarrow \simeq \lrcorner & & \simeq \downarrow \psi_{k-1} \otimes \psi_{k-1} \\
 Y_k & \xrightarrow{(d_\top, d_\perp)} & Y_{k-1} \times Y_{k-1}
 \end{array}$$

are cartesian for any i , $1 \leq i < k$. For $n+2$ and $1 \leq i \leq n$, we have the commutative cubes (missing one edge) by the induction hypothesis

In both cases there exist unique (up to homotopy) dashed maps which are equivalences and turn all faces into pullbacks by Proposition 4.16. Now, we must prove that all dashed arrows (for each cube and for each i) are equivalent. Firstly note that varying i in a fixed cube does not change its corresponding dashed arrow because it is the pullback on the right face. Finally, the dashed arrow in the cube is equivalent to the one on the right because both are equivalent to $d_2^*(\Psi_{n+1})$.

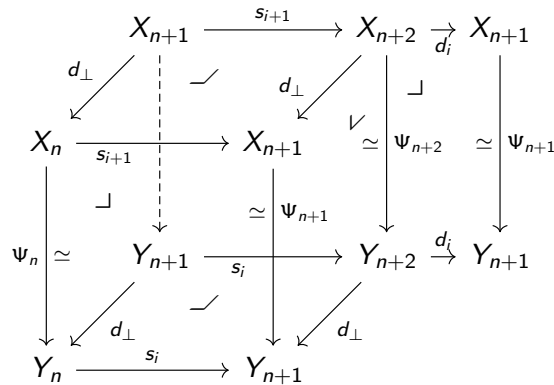
There are two points left to prove: that Ψ_1 commutes with outer face maps and Ψ_0 and that each Ψ_n commutes with degeneracies and Ψ_{n+1} . The first point will follow directly from the proof of the second, and we proceed by induction to prove it. We already proved the base case ($n = 0$) from the preservation of the counit: $\Psi_1 s_0 \simeq s_0 \Psi_0$. Now, for $n+1$ we have the following diagram for any i , $0 \leq i \leq n+1$:



By a completely analogous argument as for face maps, we obtain a new map $X_{n+1} \rightarrow Y_{n+1}$ turning both new faces in the cube into pullbacks. Now, since the composites of the horizontal arrows in the background diagram are the identity, we obtain that the dashed arrow is simply $\text{id}^*(\psi_{n+1}) \simeq \psi_{n+1}$. Finally, letting $n = 0$ we recover the pullback

$$\begin{array}{ccc}
 X_0 & \xleftarrow{d_\perp} & X_1 \\
 \psi_0 \downarrow \simeq & & \simeq \downarrow \psi_1 \\
 Y_0 & \xleftarrow{d_\perp} & Y_1
 \end{array}$$

from the left face of the cube. Repeating the same argument for the diagram



provides the corresponding square $\psi_0 d_\perp \simeq d_\perp \psi_1$ and all remaining degeneracies.

Observe that due to the homotopical nature of the subject we could not recover a *strict* natural transformation $X \rightarrow Y$ (what we defined as a *simplicial map*), but a *pseudonatural one*. Essentially, the only difference is that naturality squares are commutative up to homotopy rather than strictly commutative.

This result is surprisingly strong, however. Recall that the particular case of Möbius categories was already general enough to be unable to provide an explicit isomorphism of categories. The difference in our case is that coefficients in linear combinations remain meaningful and that we preserve *much* more information by being able to still inspect fibres rather than simplifying them to coefficients c_{ab}^x .

5.5.2 Isomorphic numerical incidence algebras

In this section we finally work on the isomorphism problem for the numerical incidence algebra of a decomposition space. Most of the treatment will be possible thanks to the work from section 5.4, so we assume that X is a Möbius decomposition space throughout this section.

Since the treatment and the terminology of algebras is more common than coalgebras and we already worked with the incidence algebra in the section 3, we are going to focus on the incidence algebra rather than the coalgebra. This should not pose any additional difficulty since the passage to the convolution algebra is as simple as

$$\Delta(x) = a \otimes b \quad \rightsquigarrow \quad \chi_x = \chi_a * \chi_b$$

For simplicity we will often abuse notation by writing x instead of $[x]$ mostly when applying elements of the incidence algebra to them: $\phi([x]) = \phi(x)$. We will also consider that any sum in \mathbb{Q} indexed over a groupoid, like

$$\sum_{x \in X_1} c_{ab}^x \phi(a) \psi(b),$$

ranges over connected components rather than the entire collection of objects. In all cases the expression in the sum is going to be independent of the chosen representative of the isomorphism class. Using this notation, the formula we obtained for the comultiplication in Möbius decomposition spaces in 5.4 becomes

$$\Delta(x) = \delta_{d_{\top}x} \otimes \delta_x + \delta_x \otimes \delta_{d_{\perp}x} + \sum_{a,b \in X_1} c_{ab}^x \delta_a \otimes \delta_b$$

hence

$$(\phi * \psi)(x) = \phi(d_{\top}x) \psi(x) + \phi(x) \psi(d_{\perp}x) + \sum_{a,b \in X_1} c_{ab}^x \phi(a) \psi(b)$$

for ϕ, ψ in the incidence algebra.

We begin by noting a few more immediate consequences of section 5.4 by translating them from the coalgebra language.

Corollary 5.36. *For any $v \in X_0$ and $\phi, \psi \in I(X)$, $(\phi * \psi)(v) = \phi(v) \psi(v)$.*

Corollary 5.37. *For any $v \in X_0$ and $\phi \in I(X)$,*

$$(\phi * \chi_v)(x) = \begin{cases} \phi(x) & \text{if } d_{\perp}x \cong v \\ 0 & \text{otherwise.} \end{cases} \quad (\chi_v * \phi)(x) = \begin{cases} \phi(x) & \text{if } d_{\top}x \cong v \\ 0 & \text{otherwise.} \end{cases}$$

The proof of the following proposition is now a straightforward adaptation from the proof of Theorem 3.9 in [3, Theorem 1.1]. In particular this provides us with a simple formula for Möbius inversion.

Proposition 5.38. *An element $\phi \in I(X)$ is invertible if and only if $\phi(v) \neq 0$ for all $v \in X_0$.*

Proof. First, assume that ϕ is invertible. Then, there exists ϕ^{-1} such that

$$1 = \delta(v) = (\phi * \phi^{-1})(v) = \phi(v) \phi^{-1}(v),$$

for any $v \in X_0$, so $\phi(v) \neq 0$.

Conversely, define ϕ^{-1} recursively as

$$\phi^{-1}(x) = \begin{cases} \phi(x)^{-1} & \text{if } \ell(x) = 0, \\ -\phi(d_{\top x})^{-1} \left(\phi(x)\phi^{-1}(d_{\perp x}) + \sum_{a,b \in \vec{X}_1} c_{ab}^x \phi(a)\phi^{-1}(b) \right) & \text{if } \ell(x) > 0. \end{cases}$$

Note that it is well-defined because $c_{ab}^x \neq 0$ only for a and b with length $< \ell(x)$. It is then clearly the inverse of ϕ by definition. At vertices we have $(\phi * \phi^{-1})(v) = \phi(v)\phi^{-1}(v) = 1 = (\phi^{-1} * \phi)(v)$, while at nondegenerate edges we have

$$(\phi * \phi^{-1})(x) = \phi(x)\phi^{-1}(d_{\perp x}) + \phi(d_{\top x})\phi^{-1}(x) + \sum_{a,b \in \vec{X}_1} c_{ab}^x \phi(a)\phi^{-1}(b) = 0$$

for $x \in \vec{X}_1$. □

Corollary 5.39 (Möbius inversion). *The $|\zeta|$ function is invertible and its inverse is the Möbius function.*

The general scheme of this section is to imitate Leroux's proof for Möbius categories, but we are going to find some difficulties due to the lack of composition. This will force us to assume some additional hypotheses that a Möbius decomposition space may not satisfy. Since many proofs are extremely similar to their counterparts in section 3, we omit some of them or skip any arguments that remain valid in order to avoid too much redundancy.

First of all, as in the proof by Leroux, we relate the powers of the Jacobson radical with the ideals J_n , which classify simplices by length.

Definition 5.40. For X a Möbius decomposition space, define

$$J_n = \{\phi \in I(X) : \forall x \in X_1, \ell(x) < n \implies \phi(x) = 0\}$$

Most of the proof of Proposition 3.14 now holds verbatim thanks to Corollary 5.36 and Proposition 5.38.

Proposition 5.41. *In any Möbius decomposition space X , $J_0 = I(X)$ and $J_1 = J$, where J the Jacobson ideal of $I(X)$. Moreover,*

$$J_n / J_{n+1} \cong \mathbb{Q}^{\pi_0 X_{1,n}}$$

as vector spaces. In particular, $\mathbb{Q}^{\pi_0 X_0} \cong I(X) / J(X)$ as algebras with pointwise multiplication.

Similarly, lemmas 3.16 and 3.17 can be generalized simply by replacing identities and objects with vertices.

Lemma 5.42. $\chi_{\pi_0 X_0}$ is the unique maximal family of primitive orthogonal idempotents in $\mathbb{Q}^{\pi_0 X_0}$.

Lemma 5.43. If $\Psi : I(X) \rightarrow I(Y)$ is a morphism of algebras, then $\Psi(J) \subseteq J$.

Then, we obtain the same consequence as in section 3. Observe that this fact does not make use of any additional structure of the incidence algebra other than lemmas 5.42 and 5.43, so no changes are needed either.

Corollary 5.44. *Any morphism of algebras $\Psi : I(X) \rightarrow I(Y)$ induces an algebra morphism $\Psi_0 : \mathbb{Q}^{\pi_0 X_0} \rightarrow \mathbb{Q}^{\pi_0 Y_0}$ and a map of sets $\tau_\Psi : \pi_0 X_0 \rightarrow \pi_0 Y_0$.*

$$\begin{array}{ccccccc}
 I(X) & \longrightarrow & I(X)/J(X) & \xrightarrow{\cong} & \mathbb{Q}^{\pi_0 X_0} & \longleftarrow & \chi_{\pi_0 X_0} \longleftarrow \pi_0 X_0 \\
 \downarrow \Psi & & \downarrow & & \downarrow \Psi_0 & & \downarrow \tau_\Psi \\
 I(Y) & \longrightarrow & I(Y)/J(Y) & \xrightarrow{\cong} & \mathbb{Q}^{\pi_0 Y_0} & \longleftarrow & \chi_{\pi_0 Y_0} \longleftarrow \pi_0 Y_0
 \end{array}$$

Moreover, if Ψ is an isomorphism, so are Ψ_0 and τ_Ψ .

Using the same notation as before, let $e_X : \mathbb{Q}^{\pi_0 X_0} \rightarrow I(X)$ denote the extension of a map $\pi_0 X_0 \rightarrow \mathbb{Q}$ to a map $\pi_0 X_1 \rightarrow \mathbb{Q}$ by returning 0 outside of $\pi_0 X_0$. Clearly, it is still a section of the restriction map $r_X : I(X) \rightarrow \mathbb{Q}$, so Proposition 3.19 also generalizes correctly. We include the modified proof because some minor changes are required, but there are no major obstacles.

Proposition 5.45. *Given an isomorphism of algebras $\Psi : I(X) \rightarrow I(Y)$, there exists $\psi \in I(Y)$ such that $r_Y(\psi) = 1$ and $\gamma_\psi \Psi$ extends Ψ_0*

$$\begin{array}{ccccc}
 I(X) & \longleftarrow & \chi_{\pi_0 X_0} & \xleftarrow{e_X} & \chi_{\pi_0 X_0} \\
 \cong \downarrow \gamma_\psi \Psi & & \downarrow (\gamma_\psi \Psi_0)|_{\chi_{\pi_0 X_0}} & \cong & \downarrow \Psi_0|_{\chi_{\pi_0 X_0}} \\
 I(Y) & \longleftarrow & \chi_{\pi_0 Y_0} & \xleftarrow{e_Y} & \chi_{\pi_0 Y_0}
 \end{array}$$

Proof. Using the same notation as in the proof of 3.19, let $\psi_v = (\Psi e_X \Psi_0^{-1} r_Y)(\chi_v)$ for any $v \in \pi_0 Y_0$. Then recall that we have

$$\chi_v \xrightarrow{r_Y} \chi_v \xrightarrow{\Psi_0^{-1}} \chi_{\tau^{-1}v} \xrightarrow{e_X} \chi_{\tau^{-1}v} \xrightarrow{\Psi} \psi_v,$$

so $r_Y(\psi_v) = \chi_v$ remains true. Similarly, define $\psi \in I(Y)$ as $\psi(x) = \psi_{d_\perp x}(x)$ (note that d_\perp corresponds to the codomain for nerves) and $r_Y(\psi) = 1 \in \mathbb{Q}^{\pi_0 X_0}$ as well (so ψ is invertible). Now the main change is that factors have been reordered due to how composition is written for categories: we have

$$(\psi * \chi_v)(x) = \psi(x) \chi_v(d_\perp x) = \psi_{d_\perp x}(x) \chi_v(d_\perp x) = \begin{cases} \psi_v(x) & \text{if } d_\perp x \cong v \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned}
 (\psi_v * \psi)(x) &= \sum_{a,b \in X_1} c_{ab}^x \psi_v(a) \psi(b) = \sum_{a,b \in X_1} c_{ab}^x \psi_v(a) \psi_{d_\perp b}(b) = \sum_{a,b \in X_1} c_{ab}^x \psi_v(a) \psi_{d_\perp x}(b) \\
 &= (\psi_v * \psi_{d_\perp x})(x) = \Psi(\chi_{\tau^{-1}v} * \chi_{\tau^{-1}d_\perp x})(x) \\
 &= \begin{cases} \psi_v(x) & \text{if } d_\perp x \cong v \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Therefore $\psi * \chi_v = \psi_v * \psi$ and $\chi_v = \psi^{-1} * \psi_v * \psi$. This means that the inner automorphism is given by conjugation with ψ^{-1} , which still satisfies the required properties (this is easily verified by the inversion formula). \square

So far, everything related to vertices holds almost untouched. Difficulties start to appear once one studies the rest of X_1 , though. Firstly, there is a crucial fact that does not translate correctly and it is crucial in the proof for Möbius categories: the product formula of two characteristic functions.

Proposition 5.46. *Let X be a decomposition space, $a, b, c \in X_1$ and $\phi \in \mathbf{LIN}(\mathbf{Grpd}_{/X_1}, \mathbf{Grpd}_{/1})$. Then*

$$(\chi_a * \phi)(c) = \int^{b \in X_1} (X_2)_{c[a]b} \times \phi(b)$$

and, in particular,

$$(\chi_a * \chi_b)(c) = (X_2)_{c[a][b]}.$$

Proof. Observe that, in general, we have the cartesian square

$$\begin{array}{ccc} E_b & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ E_{[b]} & \longrightarrow & B_{[b]} \\ \downarrow & \lrcorner & \downarrow \\ E & \longrightarrow & B \end{array}$$

for any map $E \rightarrow B$ and $b \in B$. This implies that $E_{[b]} \simeq \int^{b' \in B_{[b]}} E_b$ by the fundamental equivalence and hence

$$\begin{aligned} (\chi_a * \phi)(c) &\simeq \int^{a', b \in X_1} (X_2)_{ca'b} \times \chi_a(a') \times \phi(b) \\ &\simeq \int^{a' \in (X_1)_{[a]}} \int^{b \in X_1} (X_2)_{ca'b} \times \phi(b) \\ &\simeq \int^{b \in X_1} \left(\int^{a' \in (X_1)_{[a]}} (X_2)_{ca'b} \right) \times \phi(b) \\ &\simeq \int^{b \in X_1} (X_2)_{c[a]b} \times \phi(b). \end{aligned}$$

Applying the same argument again yields result for the special case $\phi = \chi_b$. □

Corollary 5.47. *For X a Möbius decomposition space $a, b, c \in X_1$ and $\phi \in I(X)$ we have*

$$(\chi_a * \phi)(c) = \sum_{[b] \in \pi_0 X_1} c_{[a]b}^c \phi(b) \quad \text{and} \quad (\chi_a * \chi_b)(c) = c_{[a][b]}^c$$

where

$$c_{[a]b}^c = \frac{|(X_2)_{c[a]b}|}{|\Omega_b X_1|} \quad \text{and} \quad c_{[a][b]}^c = |(X_2)_{c[a][b]}|.$$

Observe that $(X_2)_{c[a][b]}$ is simply the fibre of the restriction $d_1 : (X_2)_{[a][b]} \rightarrow X_1$ at c , so it is zero if and only if there is no 2-simplex with principal edges isomorphic to a and b respectively and long arrow

isomorphic to c . There is nothing in the decomposition space axioms ensuring that $[a]$ and $[b]$ determine c even up to isomorphism, and this is indeed not the case for many cases.

Interestingly enough, the multiplication of characteristic functions defines a multivalued composition if we do not take coefficients into account. In order to be able to adapt the rest of the proof we need this operation to have a single output, i.e. $\chi_a * \chi_b = \lambda \chi_c$ or $\chi_a * \chi_b = 0$. One solution is to require X to be a Segal space, but we really only need uniqueness, not existence. Observe that this requirement is exactly the opposite of what happens in Example 5.11, where one could compose any two graphs in order to obtain a 2-simplex. The simplest case is $G_1 = G_2 = \bullet$. Then there are infinitely many (or exactly 2 if we do not allow multiple edges) graphs with two vertices whose partitions restrict to G_1 and G_2 .

Definition 5.48. We say that a decomposition space X has *unique compositions* if any two n -simplices with isomorphic principal edges have isomorphic long edges.

Firstly, this condition ensures that $(X_2)_{[a][b]}$ is connected, so indeed $\chi_a * \chi_b = c_{[a][b]}^c \chi_c$ for some $c \in X_1$ whenever the product is not zero. It also gives us that if c_{ab}^d is also nonzero for some other $d \in X_1$ then $c \cong d$, hence $(X_2)_{cab} \simeq (X_2)_{dab}$ and $(X_2)_{c[a][b]} \simeq (X_2)_{d[a][b]}$, so $c_{ab}^c = c_{ab}^d$ and $c_{[a][b]}^c = c_{[a][b]}^d$ as well.

There is either exactly one c (up to isomorphism) with $c_{ab}^c \neq 0$ or none, hence we omit the superscript altogether. As for Segal spaces, we write this operation as $c \cong a \cdot b$. This *partial composition* is associative due to the coassociativity of the comultiplication, so we are free to write $x \cong abc = (ab)c = a(bc)$ and $c_{[a][b][c]} = c_{[a][b]}c_{[ab][c]} = c_{a[bc]}c_{[b][c]}$. We also adopt the convention that $c_{[a]} = 1$ meaning that $\chi_a = c_{[a]}\chi_a$.

Many of these spaces can be obtained from existing Möbius Segal spaces by deleting simplices of length $\geq n$ for some fixed $n > 0$. In this case $X_2 \rightarrow X_1 \times_{X_0} X_1$ is no longer an equivalence as the composition of an edge of length $n - 1$ with an edge of length 1 would yield an edge of length $\geq n$ that does not exist. We now see a few examples of decomposition spaces which are not Segal but do have unique compositions. Both of the following examples are inspired by [2].

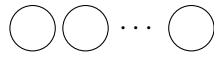
Example 5.49 (Partial monoids). A partial monoid is a set M together with a distinguished element $1 \in M$ and a partial operation $\cdot : M \times M \rightarrow M$ such that $1 \cdot m_1 = m_1 = m_1 \cdot 1$ and, whenever they are defined, $m_1 \cdot (m_2 \cdot m_3) = (m_1 \cdot m_2) \cdot m_3$, for any $m_1, m_2, m_3 \in M$. Then the *fat nerve* of a partial monoid has $X_0 = \{\bullet\}$ and invertible elements in M as isomorphisms. For $n > 0$, X_n is the groupoid with objects strings of n elements whose product exists in M and isomorphisms as in the fat nerve of a category. For instance, if $m_1 m_2 = m_2 m_1$ then

$$\begin{array}{ccc}
 \bullet & \xrightarrow{m_1} & \bullet \\
 \downarrow 1 & & \downarrow 1 \\
 \bullet & \xrightarrow{m_1} & \bullet
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \bullet & \xrightarrow{m_1} & \bullet \\
 \downarrow m_2 & & \downarrow m_2 \\
 \bullet & \xrightarrow{m_1} & \bullet
 \end{array}$$

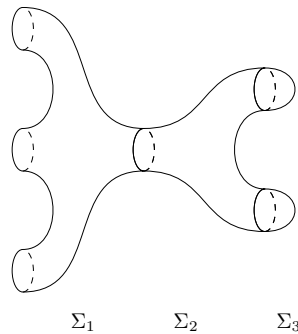
are two isomorphisms $(m_1, m_2) \rightarrow (m_2, m_1)$ in X_2 .

Face and degeneracy maps work exactly as in the fat nerve. The lack of some products shows that it can never be a Segal space unless the operation is *total*. To see that it is a decomposition space with unique compositions, we simply resort to Proposition 5.9. The first two diagrams follow from unitality ($1 \cdot m = m = m \cdot 1$), while the second pair is just stating that the operation is associative. Finally, it has unique compositions because the existence of a 2-simplex (m_1, m_2) with long edge m_3 means that $m_1 m_2$ exists and it is isomorphic to m_3 in X_1 .

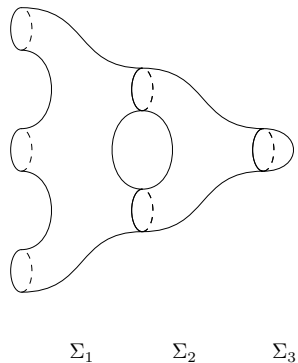
Example 5.50 (Restricted genus cobordisms). Given a fixed genus $g \geq 0$, consider all 2-dimensional cobordisms with genus no greater than g . Then we may consider the simplicial set with X_0 the set of 1-dimensional closed manifolds,



which are simply disjoint unions of circles. Then X_n for $n \geq 0$ is the set of chains $(\Sigma_1, \dots, \Sigma_n)$ of composable 2-dimensional cobordisms between elements of X_0 whose composition has genus no larger than g . The following is a 3-simplex of genus 0:



Again, face and degeneracy maps are similar to the ones in the nerve, where outer faces remove the first or the last cobordism and inner faces compose adjacent cobordisms. Degeneracies insert identity cobordisms at the appropriate index. As other examples, this is clearly a decomposition space (the fact that it is a simplicial set rather than a groupoid makes the verification even simpler) and has unique compositions (two composable cobordisms produce a unique cobordism). It is not a Segal space however, as two cobordisms of genus $\leq g$ may be composed as a cobordism of genus $> g$. For instance, one could compose two cobordisms of genus 0 to obtain a cobordism of genus 1.



Notice that this decomposition space corresponds to the idea that we expressed above: consider the category of all 2-dimensional cobordisms with objects disjoint unions of circumferences and morphisms cobordisms between them. Then the length of a simplex in the nerve of this category is described by its genus, so requiring genus $\leq g$ is tantamount to removing all simplices with genus $> g$.

Before continuing with the translation, we need to make a remark about the topology of $\mathbb{Q}^{\pi_0 X_1}$. As with the incidence algebra of Möbius categories, we consider the discrete topology on \mathbb{Q} and the product

topology on $\mathbb{Q}^{\pi_0 X_1}$. Then all the desirable properties of the topology in $\text{Hom}_{\text{Set}}(\text{Mor } \mathcal{C}, k)$ that we proved in section 3 are easily verified to be true for $\mathbb{Q}^{\pi_0 X_1}$ as well, including the formula

$$\phi = \sum_{x \in X_1} \phi(x) \chi_x.$$

We can now prove one of the key results in this section.

Proposition 5.51. *If X is a Möbius decomposition space with unique compositions, then $J_n = \overline{J^n}$.*

Proof. The proof about the closedness of J_n is identical to the one in Proposition 3.14. For the first inclusion we proceed similarly: let $\phi_1 * \cdots * \phi_n$ be a generator of J^n with $\phi_i \in J = J_1$ and $x \in X_1$ with $\ell(x) < n$. Then

$$\phi(x) = \sum_{a_1 \cdots a_n \cong x} c_{[a_1] \cdots [a_n]} \phi_1(a_1) \cdots \phi_n(a_n).$$

Since $n > \ell(x) \geq \ell(a_1) + \cdots + \ell(a_n)$, one of the a_i 's (for each decomposition) must be a vertex, hence $\phi(x) = 0$. This completes the proof that $J_n \supseteq \overline{J^n}$.

Conversely, let $\phi \in J_n$, $n > 0$ and observe that

$$\phi = \sum_{[x] \in U} \phi(x) \chi_x, \quad U = \{[x] \in \pi_0 X_1 : \ell(x) \geq n\}$$

because $\phi(x) = 0$ for any $x \in X_1$ with $\ell(x) < n$. Moreover, for any $x \in X_1$ of length at least n there exists a decomposition $x \cong a_1 \cdots a_n$ with nondegenerate a_i 's, so we have that $\chi_x = c_{[a_1] \cdots [a_n]} \chi_{a_1} * \cdots * \chi_{a_n}$ with $\chi_{a_i} \in J_1$. This shows that $\chi_x \in J^n$, so the previous sum belongs to $\overline{J^n}$. Finally, if $n = 0$ we simply have $J_0 = I(X) = \overline{I(X)} = \overline{J^0}$. \square

Observe that we are gradually shifting from fibres to connected components or full subgroupoids. This is mostly due to the nature of the problem, since the first step taken in the construction of the numerical incidence coalgebra is to simplify X_1 as $\pi_0 X_1$. The following definitions are just another step in that direction while mimicking more definitions from the context of Möbius categories.

Definition 5.52. For X a decomposition space and $u, v \in X_0$, define $[u, v]$ to be the full subgroupoid of X_0 of vertices w such that there exists some 2-simplex $\sigma \in X_2$ satisfying

$$d_{\top} d_{\top} \sigma \cong u \quad d_{\top} d_{\perp} \sigma \cong w \quad d_{\perp} d_{\perp} \sigma \cong v.$$

Definition 5.53. For X a complete decomposition space, $n, k \in \mathbb{N}$ and $u, v \in X_0$, we define

$$\begin{array}{ccc} X_{n,k} & \xrightarrow{d_1^{n-1}} & X_{1,k} \longrightarrow 1 \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ X_n & \xrightarrow{d_1^{n-1}} & X_1 \xrightarrow{\ell} \mathbb{N} \end{array} \quad \begin{array}{ccc} X_n(u, v) & \xrightarrow{d_1^{n-1}} & X_1(u, v) \longrightarrow 1 \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ X_n & \xrightarrow{d_1^{n-1}} & X_1 \xrightarrow{(d_{\top}, d_{\perp})} \pi_0 X_0 \times \pi_0 X_0 \end{array}$$

This is, $X_{n,k}$ is the full subgroupoid of X_n of simplices of length k and $X_n(u, v)$ is the one of simplices whose first and last vertex are u and v respectively. We also combine both to denote the subgroupoid of *length- k n -simplices from u to v* :

$$\begin{array}{ccc}
X_{n,k}(u, v) & \longrightarrow & 1 \\
\downarrow & \lrcorner & \downarrow \Gamma(k, u, v)^\top \\
X_n & \xrightarrow{(\ell, d_\top, d_\perp)} & \mathbb{N} \times X_0 \times X_0
\end{array}$$

As we did for categories, we can also consider length- n graphs in a decomposition space. The set of vertices is just $\pi_0 X_0$ and the set of edges between u and v is $\pi_0 X_{1,n}(u, v)$. Then we prove that any isomorphism of incidence algebras (as *topological* algebras) produces an isomorphism on length- n graphs. The proof remains a slight variation of Theorem 3.20.

Theorem 5.54. *Let X, Y be Möbius decomposition spaces with unique compositions and $\Psi : I(X) \rightarrow I(Y)$ an isomorphism of algebras which is a homeomorphism. If $X_{1,n}(u, v)$ has countably many connected components for all $u, v \in X_0$, then X and Y have isomorphic length- n graphs.*

Proof. We can assume that Ψ restricts to a bijection on vertices by Proposition 5.45, let $\tau : \pi_0 X_0 \rightarrow \pi_0 Y_0$ the bijection. Since Ψ is both an isomorphism of algebras and a homeomorphism, we also have

$$\Psi(J_n(X)) = \overline{\Psi(J^n(X))} = \overline{\Psi(J(X))^n} = \overline{J^n(Y)} = J_n(Y).$$

by Proposition 5.51. By the construction of τ in Corollary 5.44 we know that $\Psi(\chi_w) = \chi_{\tau w}$ for any $w \in X_0$, so Ψ restricts to an isomorphism of vector spaces

$$\chi_u * J_n(X) * \chi_v \rightarrow \chi_{\tau u} * J_n(Y) * \chi_{\tau v}$$

for each n . In particular, it induces an isomorphism of vector spaces

$$\frac{\chi_u * J_n(X) * \chi_v}{\chi_u * J_{n+1}(X) * \chi_v} \xrightarrow{\cong} \frac{\chi_{\tau u} * J_n(Y) * \chi_{\tau v}}{\chi_{\tau u} * J_{n+1}(Y) * \chi_{\tau v}}.$$

It is easy to verify that

$$\text{Hom}_{\text{Set}}(\pi_0 X_n(u, v), \mathbb{Q}) \cong \frac{\chi_u * J_n(X) * \chi_v}{\chi_u * J_{n+1}(X) * \chi_v}$$

as vector spaces by combining Corollary 5.37 and the argument of Proposition 5.41. Therefore we have an isomorphism

$$\text{Hom}_{\text{Set}}(\pi_0 X_n(u, v), \mathbb{Q}) \xrightarrow{\cong} \text{Hom}_{\text{Set}}(\pi_0 Y_n(\tau u, \tau v), \mathbb{Q}).$$

If both vector spaces have finite dimension, then the fact that their dimension is $|\pi_0 X_n(u, v)|$ provides a bijection $\pi_0 X_n(u, v) \leftrightarrow \pi_0 Y_n(\tau u, \tau v)$. Otherwise both have countably infinitely many elements and the same argument applies. Combining each of these bijections with τ on vertices provides the isomorphism of graphs. \square

This result clearly generalizes Theorem 3.20, so we must require the isomorphism to be continuous. Next we prove that similar finiteness conditions to the ones in section 3 to ensure that any isomorphism is continuous. The following are straightforward generalizations of Lemma 3.22 and Theorem 3.24.

Definition 5.55. A complete decomposition space X is said to be *finitely generated* if $\pi_0[u, v]$ and $\pi_0 X_{1,1}(u, v)$ are finite for all $u, v \in X_0$.

Lemma 5.56. *Let X be a finitely generated Möbius decomposition space with unique compositions. Then $\pi_0 X_{1,n}(u, v)$ is finite and $\chi_u * J_n(X) * \chi_v = \chi_u * J^n(X) * \chi_v$ for any $u, v \in X_0$ and $n \geq 1$.*

Proof. Fix $u, v \in X_0$ and $n \geq 1$. For each $[x] \in \pi_0 X_{1,n}(u, v)$ choose a decomposition $x \cong ab$ with $\ell(a) = 1$ and $\ell(b) = n - 1$. This choice defines a function

$$z : \pi_0 X_{1,n}(u, v) \rightarrow \coprod_{[w] \in \pi_0[u, v]} \pi_0 X_{1,1}(u, w) \times \pi_0 X_{1,n-1}(w, v)$$

which is injective because X has unique compositions. As for sum indices, we abuse notation by writing $z(x)$ as if z were a map between their respective sets of objects. Then the finiteness of $\pi_0 X_{1,n}(u, v)$ can be proved by induction. The base case is already given by hypothesis, and the inductive step is exactly the same as in Proposition 3.22

$$|\pi_0 X_{1,n}(u, v)| \leq \sum_{[w] \in \pi_0[u, v]} |\pi_0 X_{1,1}(u, w)| \cdot |\pi_0 X_{1,n-1}(w, v)| < \infty.$$

We prove the second part by induction as well. The base case is already given by $J_1 = J$ (Proposition 5.41), so assume that $\chi_u * J_{n-1} * \chi_v = \chi_u * J^{n-1} * \chi_v$ for any $u, v \in X_0$ for the inductive step. Fix $u, v \in X_0$ again and consider a map like z above for the new u, v and its components

$$\begin{aligned} z_1 : \pi_0 X_{1,n}(u, v) &\rightarrow \coprod_{[w] \in \pi_0[u, v]} \pi_0 X_{1,1}(u, w) \\ z_2 : \pi_0 X_{1,n}(u, v) &\rightarrow \coprod_{[w] \in \pi_0[u, v]} \pi_0 X_{1,n-1}(w, v). \end{aligned}$$

We also need to define

$$\begin{aligned} S_w &= \{[z_1(x)] : x \in X_{1,n}(u, v), d_{\perp} z_1(x) \cong w\}, & w \in [u, v] \\ V_a &= \{[z_2(x)] : x \in X_{1,n}(u, v), z_1(x) \cong a\}, & [a] \in S_w. \end{aligned}$$

Now $\phi \in \chi_u * J_n * \chi_v$, let

$$\psi_a(b) = \begin{cases} \frac{\phi(ab)}{c_{[a]b}} & \text{if } [b] \in V_a \\ 0 & \text{otherwise,} \end{cases} \quad [a] \in S_w, w \in [u, v]$$

which are well-defined because $[b] \in V_a$ ensures that $ab \in X_1$ exists (in fact, $z(ab) \cong (a, b)$). Then we have

$$\begin{aligned} \left(\sum_{w \in [u, v]} \sum_{[a] \in S_w} \chi_a * \psi_a \right)(x) &= \sum_{w \in [u, v]} \sum_{[a] \in S_w} (\chi_a * \psi_a)(x) \\ &= \sum_{w \in [u, v]} \sum_{[a] \in S_w} \sum_{y \cong ab} c_{[a]b} \psi_a(b) \\ &= \sum_{w \in [u, v]} \sum_{[a] \in S_w} \sum_{\substack{x \cong ab \\ z(y) \cong (a, b)}} \phi(ab) \\ &= \sum_{w \cong d_{\perp} z_1(x)} \sum_{a \cong z_1(x)} \phi(x) \\ &= \phi(x). \end{aligned}$$

Note that each $\psi_a = \chi_w * \psi_a * \chi_v$ belongs to $\chi_w * J_{n-1} * \chi_v$ (as it is either 0 or the evaluation of ϕ), so by the induction hypothesis there is $\psi'_a \in J^{n-1}$ such that $\chi_w * \psi_a * \chi_v = \chi_w * \psi'_a * \chi_v$. Using this fact we can rewrite ϕ as

$$\begin{aligned}
\phi &= \sum_{w \in [u, v]} \sum_{[a] \in V_a} \chi_a * \psi_a \\
&= \sum_{w \in [u, v]} \sum_{[a] \in V_a} \chi_a * \chi_w * \psi_a * \chi_v \\
&= \sum_{w \in [u, v]} \sum_{[a] \in V_a} \chi_a * \chi_w * \psi'_a * \chi_v \\
&= \chi_u * \left(\sum_{w \in [u, v]} \sum_{[a] \in V_a} \chi_a * \psi'_a \right) * \chi_v \in \chi_u * J^n * \chi_v,
\end{aligned}$$

thus completing the proof of $\chi_u * J_n * \chi_v \subseteq \chi_u * J^n * \chi_v$. The converse inclusion is true in general by Proposition 5.51. \square

Theorem 5.57. *Let X and Y be Möbius decomposition spaces with unique compositions. If X is finitely generated and $\Psi : I(X) \rightarrow I(Y)$ is an algebra morphism that restricts to a bijection $\tau : \chi_{X_0} \rightarrow \chi_{Y_0}$, then Ψ is continuous.*

Proof. As in the proof of Theorem 3.24, let $(\phi_\alpha)_{\alpha \in A}$ be a net in $I(X)$ with $\lim_\alpha \phi_\alpha = 0$. For each $x \in X_1$ we then have $\alpha_x \in A$ such that $\phi_\alpha(x) = 0$ for any $\alpha \geq \alpha_x$. Define the sets

$$U_n(u, v) = \prod_{k=1}^{n-1} \pi_0 X_{1,k}(u, v), \quad u, v \in X_0, \quad n \geq 1$$

which are finite by Lemma 5.56. Therefore by directedness there exists an upper bound $\alpha_n(u, v)$ for each $\{\alpha_x : [x] \in U_n(u, v)\}$. By definition of the α_x this means that any $\alpha \geq \alpha_n(u, v)$ satisfies $\phi_\alpha(x) = 0$ for any $[x] \in U_n(u, v)$. In other words, $\chi_u * \phi_\alpha * \chi_v \in \chi_u * J_n(X) * \chi_v$ and applying Ψ we obtain

$$\begin{aligned}
\chi_{\tau u} * \Psi(\phi_\alpha) * \chi_{\tau v} &\in \chi_{\tau u} * \Psi(J_n(X)) * \chi_{\tau v} \\
&= \chi_{\tau u} * \Psi(J_n(X)) * \chi_{\tau v} \\
&= \chi_{\tau u} * \Psi(J^n(X)) * \chi_{\tau v} \\
&\subseteq \chi_{\tau u} * J^n(Y) * \chi_{\tau v} \\
&\subseteq \chi_{\tau u} * J_n(Y) * \chi_{\tau v}.
\end{aligned}$$

Summing up, for any $n \geq 1$, $\tau u, \tau v \in Y_0$ and $y \in Y_1(\tau u, \tau v)$ there exists $\alpha_n(\tau u, \tau v)$ such that for any $\alpha \geq \alpha_n(\tau u, \tau v)$ one has $\Psi(\phi_\alpha)(y) = (\chi_{\tau u} * \Psi(\phi_\alpha) * \chi_{\tau v})(y) = 0$. This simply states that $\Psi(\phi_\alpha)(y)$ converges to 0 for any y , so $\lim_\alpha \Psi(\phi_\alpha) = 0$. \square

Then, we obtain the final result: given any isomorphism of (finitely generated) Möbius decomposition spaces with unique compositions, we have an isomorphism of length- n graphs.

Theorem 5.58. *Let X and Y be finitely generated Möbius decomposition spaces with unique compositions. If $I(X) \cong I(Y)$ as algebras, then X and Y have isomorphic length- n graphs for any $n \geq 0$.*

Proof. We may assume that the isomorphism $I(X) \rightarrow I(Y)$ is continuous by Proposition 5.45 and Theorem 5.57. The isomorphism of graphs is then given by Theorem 5.54. \square

This result is much stronger than simply a bijection $\pi_0 X_1 \rightarrow \pi_0 Y_1$, even one that preserves vertices. Recall that an isomorphism of graphs is simply a commutative diagram

$$\begin{array}{ccc} \pi_0 X_0 \times \pi_0 X_0 & \xleftarrow{(d_\top, d_\perp)} & \pi_0 X_{1,n} \\ \downarrow \cong & & \downarrow \cong \\ \pi_0 Y_0 \times \pi_0 Y_0 & \xleftarrow{(d_\top, d_\perp)} & \pi_0 Y_{1,n} \end{array}$$

Since $\pi_0 X_1 = \coprod_{n \geq 0} \pi_0 X_{1,n}$, we also have

$$\begin{array}{ccc} \pi_0 X_0 \times \pi_0 X_0 & \xleftarrow{(d_\top, d_\perp)} & \pi_0 X_1 \\ \downarrow \cong & & \downarrow \cong \\ \pi_0 Y_0 \times \pi_0 Y_0 & \xleftarrow{(d_\top, d_\perp)} & \pi_0 Y_1 \end{array}$$

Moreover, since vertices can be characterized as simplices with length 0, we have yet another commutative diagram

$$\begin{array}{ccccc} \pi_0 X_0 & \xrightarrow[\cong]{s_0} & \pi_0 X_{1,0} & \hookrightarrow & \pi_0 X_1 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \pi_0 Y_0 & \xrightarrow[\cong]{s_0} & \pi_0 Y_{1,0} & \hookrightarrow & \pi_0 Y_1 \end{array}$$

Together, these two commutative diagrams state that, for finitely generated Möbius decomposition spaces with unique compositions, an isomorphism of algebras induces an isomorphism between the simplicial sets $\{\pi_0 X_n\}_{n=0,1}$ and $\{\pi_0 Y_n\}_{n=0,1}$ having only degenerate simplices at $n \geq 2$. Compare this conclusion with Example 5.19: there is a clear length-preserving correspondence between the connected components of their vertices and edges.

In addition, note that it is hardly possible to recover any information about X_2 . Firstly because decomposition space axioms are not so strong as to reconstruct enough of X_2 from X_0 and $X_{1,n}$, and secondly (in the case in which X is Segal) because π_0 need not to preserve pullbacks and there is no way to escape $\pi_0 X$ once we restrict ourselves to the numerical incidence algebra.

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