VII International Conference on Textile Composites and Inflatable Structures
STRUCTURAL MEMBRANES 2015
E. Oñate, K.-U.Bletzinger and B. Kröplin (Eds)

ANALYSIS OF PNEUMATIC ENVELOPES ARRANGED ON RECTANGULAR PLAN

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Key words: Pneumatic, Envelope, Surface, ETFE, Least Squares

Summary. *Rectangular pneumatic envelopes, made of a polymer film, are considered. The envelopes are subjected to non-uniform external loads, which consist of symmetric and inverse-symmetric parts. The functional dependence describing the envelope's surface is given. The improved technique for obtaining geometrical parameters of envelopes is offered. It uses the differential equation of equilibrium, the method of least squares and the linearization procedure with Taylor series. The results of the work may be used for taking into account physical non-linear properties of polymer membranes, for estimation of changes of air-pressure in pneumatic cushions, for selection the most appropriate embodiment of the cushion from a set of possible variants.*

1 INTRODUCTION

Pneumatic envelopes, made of a polymer membrane, form air-inflated cushions. Together with their obvious advantages, there are several features, such as large deformations, severe dependence of their surface shape on internal stresses and external loadings, physical non-linear properties of the material and so on.

In spite of availability of computer programs and analytical approaches[1,2] for surface analysis, the purposes of optimization and the complex nature of pneumatic envelopes require elaboration of an updated technique for the analysis of such systems.

The technique is to take into account non-uniform external loads and relationship between stresses in longitudinal and transverse directions of the envelopes. It also should contribute to more precise calculations of volume of air, covered by the envelope, in order to obtain internal pressure more correctly.

The proposed technique is based on the differential equation of equilibrium[3], containing partial derivatives of the function of the envelope's surface. The technique uses the method of least squares and the linearization procedure with Taylor series.

2 EQUILIBRIUM OF PNEUMATIC ENVELOPES

The condition of equilibrium of a pneumatic envelope is described by the differential equation[3]:

$$Z_{xx} \cdot \sigma_x \cdot \frac{\sqrt{1+Z_y^2}}{\sqrt{1+Z_x^2}} + Z_{yy} \cdot \sigma_y \cdot \frac{\sqrt{1+Z_x^2}}{\sqrt{1+Z_y^2}} + 2 \cdot Z_{xy} \cdot \sigma_{xy} = -\left(p_z - p_x \cdot Z_x - p_y \cdot Z_y\right) \cdot \sqrt{1+Z_x^2+Z_y^2}, \qquad (1)$$

where Z_x, Z_y, Z_{xx}, Z_{yy}, Z_{xy} are partial derivatives of the function of surface $Z(x, y, \vec{Hs})$, defined in the Cartesian coordinate system (XYZ); $\vec{Hs}$ is a vector, containing parameters of the function; σ_x, σ_y, σ_{xy} are normal and tangential stresses at a point of the surface, kN/m; p_x, p_y, p_z are loads, acting on the surface, kN/m^2:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \frac{P_r}{\sqrt{1+Z_x^{\,2}+Z_y^{\,2}}} \cdot \begin{pmatrix} -Z_x \\ -Z_y \\ 1 \end{pmatrix} + \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}, \qquad (2)$$

where P_r is an air pressure; q_x, q_y, q_z are external loads, acting along the coordinate axis X, Y and Z.

Tangential stresses σ_{xy} are usually negligible, because of the shear stiffness of polymer membranes, of which pneumatic envelopes are made, is much less, than their tensile stiffness.

Normal stresses in the surface σ_x and σ_y are calculated from:

$$\sigma_{x,y} = \sigma \cdot \delta, \qquad (3)$$

where σ is the stress according to the relative deformation ε_x or ε_y of the material; δ is the thickness of the polymer membrane.

Pneumatic envelopes are usually made of ETFE films with non-linear stress-strain diagram. Consequently, for obtaining the stress σ, the secant modulus has to be used instead of a constant value of modulus of elasticity.

The relative deformations ε_x and ε_y are calculated from:

$$\varepsilon_\gamma = \frac{L_{c,\gamma} - L_{0,\gamma}}{L_{0,\gamma}} - \alpha \cdot \Delta t, \qquad (4)$$

where γ is a cross-section of the envelope by a plane, parallel to the plane XOZ or YOZ of the Cartesian coordinate system; $L_{0,\gamma}$ and $L_{c,\gamma}$ are initial and deformed lengths of the section; α is the coefficient of linear expansion of the material; Δt is a temperature variation.

Thus, for the solution of the equation (1) it is necessary to determine the surface shape of the envelope in deformed state under external loads.

3 THE FUNCTION OF THE SURFACE OF THE ENVELOPE

It is assumed that an external non-uniform load S, influencing the envelope, may be represented in the form of components $S_i, i = 1 \div 4$, each of which uniformly loads one of four quadrants, separated by the axes of symmetry (figure 1). Under this condition the components of the load may be decomposed as follows:

$$S_i = S_{sym} \pm S_x \pm S_y \pm S_{xy}, \quad (5)$$

where S_{sym} is a symmetric uniformly distributed part of the load; S_x, S_y, S_{xy} - are non-uniform inverse-symmetric parts of the load:

$$\begin{pmatrix} S_{sym} \\ S_x \\ S_y \\ S_{xy} \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} S_1 + S_2 + S_3 + S_4 \\ -S_1 + S_2 - S_3 + S_4 \\ S_1 + S_2 - S_3 - S_4 \\ -S_1 + S_2 + S_3 - S_4 \end{pmatrix}. \quad (6)$$

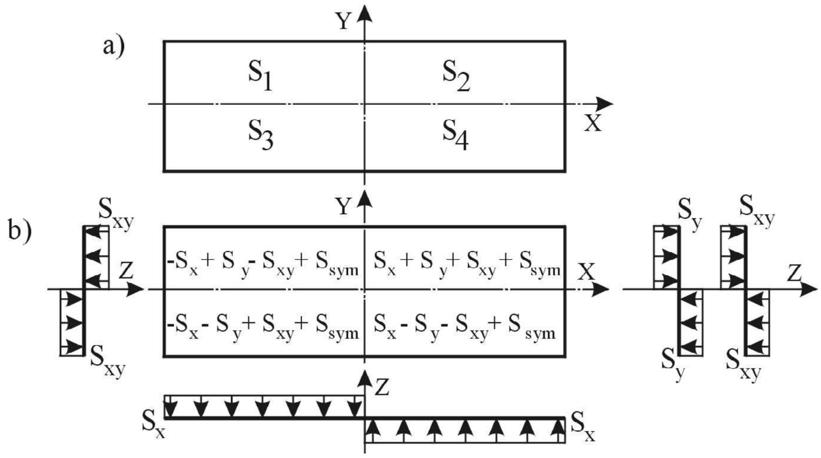

Figure 1. External loads, influencing the envelope:
a – the quadrants of the surface; b – decomposition of the load into symmetric and inverse-symmetric parts; positive value means that the load is acting in the direction of the vertical axis Z

If the symmetric part of the load influences the surface, an envelope's cross section (figure 2) parallel to the plane *XOZ* or *YOZ* may be represented with the following functions:

$$Z_{pol}(\gamma) = \sum_{i=0}^{4} U_i \cdot \gamma^i, \quad (7)$$

$$Z_{ch}(\gamma) = \frac{e^{\gamma \cdot U_3} + e^{-\gamma \cdot U_3}}{2} \cdot U_1 + U_2, \quad (8)$$

where γ is the coordinate along the axis *X* or *Y*; U_i are parameters of the functions, which have to pass through the points a – e (figure 2).

Functions (7) and (8) may be converted as follows:

$$Z_{pol}(\gamma, h_0, h_{05}, L) = \frac{64}{3} \cdot \frac{3 \cdot h_0 - 4 \cdot h_{05}}{L^4} \cdot \gamma^4 - \frac{4}{3} \cdot \frac{15 \cdot h_0 - 16 \cdot h_{05}}{L^2} \cdot \gamma^2 + h_0, \quad (9)$$

$$Z_{ch}(\gamma, h_0, h_{05}, L) = \frac{\tau^2 - \tau \cdot \left(e^{\gamma \cdot U} + e^{-\gamma \cdot U}\right) + 1}{(\tau - 1)^2} \cdot h_0, \tag{10}$$

where

$$\tau(h_0, h_{05}) = \left[\frac{2 \cdot h_{05} - h_0 + \sqrt{h_0 \cdot (4 \cdot h_{05} - 3 \cdot h_0)}}{2 \cdot (h_{05} - h_0)}\right]^2, \quad U(\tau, L) = 2 \cdot \frac{\ln(\tau)}{L}.$$

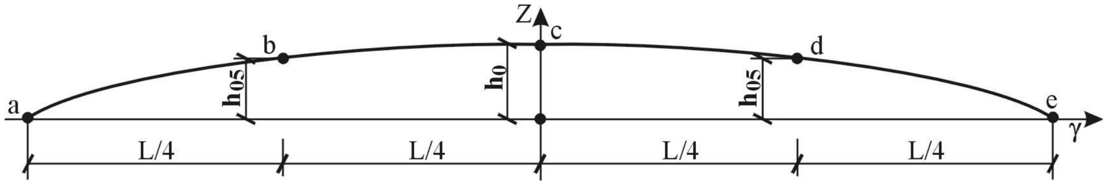

Figure 2. Cross section of the envelope influenced by the symmetric part of the load

The comparison of these two functions (Z_{pol} and Z_{ch}) is indicated in figure 3. They represent the envelope's cross sections quite well. The discrepancy between the function Z_{ch} and the reference curve is negligible. On the other hand, this function is only determined in the following range: $h_{05} \in (0.75 \cdot h_0, \quad h_0)$, while Z_{pol} is more universal.

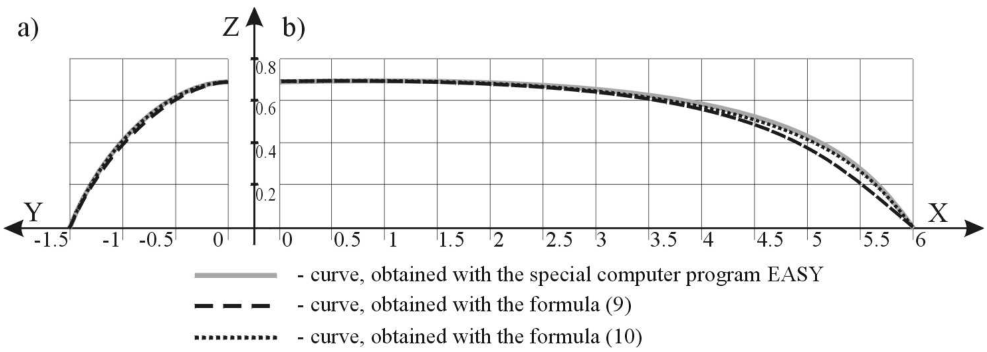

Figure 3. The comparison of cross sections of the envelope, which is influenced by internal air pressure and an external uniform load: a – section along Y-axis, b - section along X-axis

The whole surface of the uniformly loaded envelope may be represented as a set of generatrices, moving along two guides (figure 4). These two families of curves are determined by (9) or (10).

Under an inverse-symmetric part of the load, the central area of the envelope moves insignificantly, while the whole surface deforms according to the law of sine (figure 5):

$$f_{\sin}(\gamma) = V_\gamma \cdot \sin\left(\frac{2 \cdot \pi}{L} \cdot \gamma\right), \tag{11}$$

where V_γ is a parameter, describing the magnitude of the inverse-symmetric load.

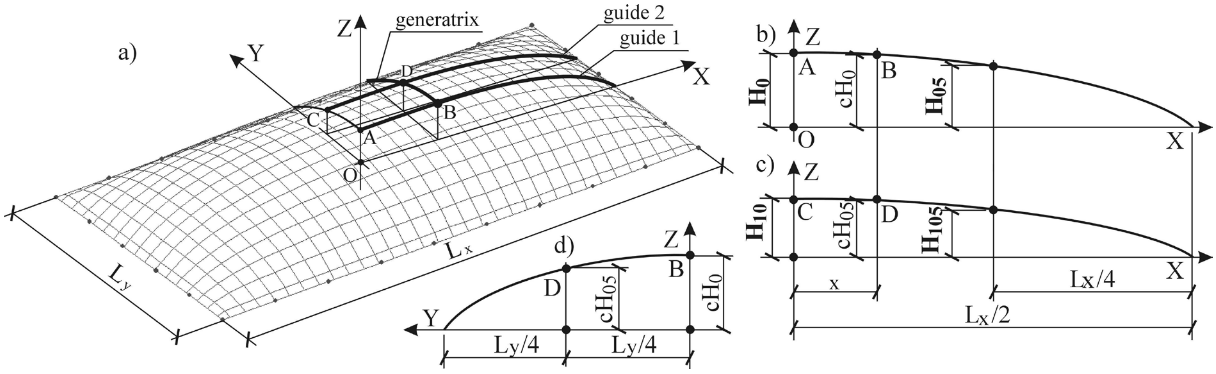

Figure 4. The surface of a uniformly loaded envelope:
a – axonometric scheme; b – the first guide; c – the second guide; d – the generatrix;
H_0, H_{05}, H_{10}, H_{105} are given heights at points $(0,\ \ 0)$, $(L_x/4,\ \ 0)$, $(0,\ \ L_y/4)$ and $(L_x/4,\ \ L_y/4)$

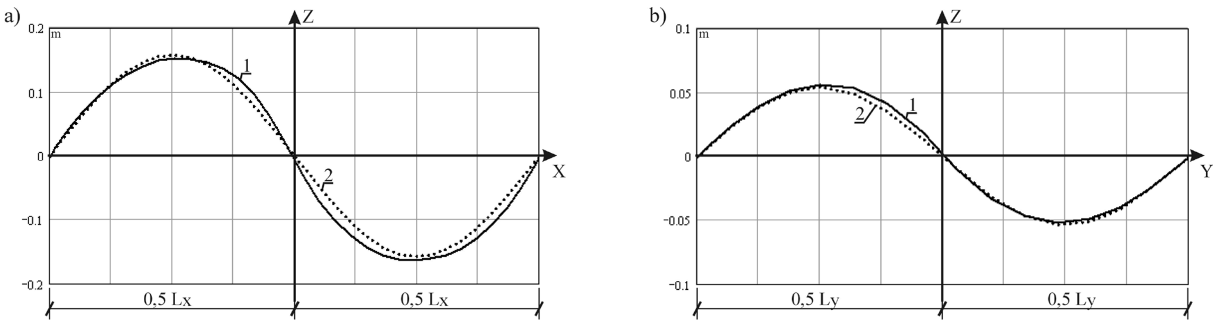

Figure 5. Displacements of the surface under an inverse-symmetric load: a – the section by the plane (XOZ), b – the section by the plane (YOZ); 1 – curves, obtained with the help of the special computer program EASY, 2 – curves, calculated with the formula (11)

The final formulation of the function of the envelope's surface may be written as follows:

$$Z(x,y,\vec{Hs}) = Q\left[y, cH_0(x), cH_{05}(x), L_y\right] + \left(V_y + V_{xy}\cdot sn(x)\right)\cdot cH_0(x)\cdot \sin\left(\frac{2\cdot\pi}{L_y}\cdot y\right), \tag{12}$$

where $\vec{Hs} = \left(H_0, H_{05}, H_{10}, H_{105}, V_x, V_y, V_{xy}\right)^T$ is a vector of parameters of the surface; $cH_0(x) = G\left(x, H_0, H_{05}\right)$, $cH_{05}(x) = G\left(x, H_{10}, H_{105}\right)$; $sn(x)$ is the function (13): $sn(x) \approx 1$ for $x > 0$ and $sn(x) \approx -1$ for $x < 0$; $G(x,h_0,h_{05})$ is the function (14); $Q\left(\gamma, h_0, h_{05}, L\right)$ is the function (15):

$$sn(x) = \frac{2}{1+e^{-x\cdot K}} - 1, \tag{13}$$

where K is a coefficient to be assumed equal to $K = 32/L_x$,

$$G(x,h_0,h_{05}) = Q\left(x, h_0, h_{05}, L_x\right) + V_x\cdot h_0\cdot \sin\left(\frac{2\cdot\pi}{L_x}\cdot x\right), \tag{14}$$

and

$$Q(\gamma, h_0, h_{05}, L) = \begin{Bmatrix} Z_{ch}, & if \quad 0.75 \cdot h_0 < h_{05} < h_0 \\ Z_{pol}, & otherwise \end{Bmatrix}, \tag{15}$$

where Z_{pol}, Z_{ch} are the functions (9) and (10), accordingly.

Parameters of the shape of the envelope's surface, contained in the vector $\vec{Hs}$, split into two groups. The first one consists of heights $(H_0, H_{05}, H_{10}, H_{105})$ of the envelope, influenced by uniformly distributed loads only, while the second group (V_x, V_y, V_{xy}) describes the magnitude of inverse-symmetric loads.

4 OBTAINING PARAMETERS OF THE ENVELOPE'S SURFACE

The equation[3] (1) may be represented as follows:

$$F = \frac{A}{B} = 1, \tag{16}$$

where F, A, B are the functions, depending on the parameters $\vec{Hs}$, calculated at any point of the envelope. The function A is the left side of (1), while the function B is its right side.

We propose to determine the parameters $\vec{Hs}$ by the method of least squares. Since the function of the envelope's surface is non-linear, the function (16) is linearized with the Taylor series:

$$F_t\left(x, y, \vec{Is}, \vec{Hs}\right) = \sum_{k=1}^{n_p} f_k\left(x, y, \vec{Is}\right) \cdot Hs_k + \left[F\left(x, y, \vec{Is}\right) - \sum_{k=1}^{n_p} f_k\left(x, y, \vec{Is}\right) \cdot Is_k\right], \tag{17}$$

where $\vec{Is}$ is an approximation to $\vec{Hs}$; $n_p = 7$ is the number of parameters to be determined; f_k is a partial derivative of the function F with respect to parameter *k* at point $\vec{Is}$:

$$f_k\left(x, y, \vec{Is}\right) = \frac{dF\left(x, y, \vec{Is}\right)}{dHs_k}. \tag{18}$$

All derivatives in (17) are calculated numerically. Required vector $\vec{Hs}$ is obtained as follows:

$$\vec{Hs} = M^{-1} \cdot \vec{W}, \tag{19}$$

where

$$M_{i,j} = \sum_{u=1}^{N} f_i\left(x_u, y_u, \vec{Is}\right) \cdot f_j\left(x_u, y_u, \vec{Is}\right), \tag{20}$$

$$W_i = \sum_{u=1}^{N} f_i\left(x_u, y_u, \vec{Is}\right) \cdot \left[Y_u - \left(F\left(x_u, y_u, \vec{Is}\right) - \sum_{k=1}^{n_p} f_k\left(x_u, y_u, \vec{Is}\right) \cdot Is_k\right)\right], \tag{21}$$

where N is the number of surface's points, for which the condition (16) is checked; $Y_u = 1$ is a constant, meaning the required value of the function F at every point of the surface; $i, j \in [1, n_p]$ are indexes, denoting the corresponding parameter of the vector $\overset{\rightarrow}{Hs}$.

The solution $\overset{\rightarrow}{Hs}$, found from (19), must be evaluated as follows:

$$\Omega = \sum_{u=1}^{N} \left| Y_u - F\left(x_u, y_u, \overset{\rightarrow}{Hs} \right) \right| \cdot \frac{1}{N} \cdot 100 \text{, \%.} \tag{22}$$

In case if the average error Ω exceeds a certain limiting value, the new initial approximation $\overset{\rightarrow}{Is} = \overset{\rightarrow}{Hs}$ has to be made and the calculations are to be done over again.

5 TAKING INTO ACCOUNT PHYSICAL NON-LINEAR PROPERTIES OF ENVELOPE'S MATERIAL

The secant modulus is to be used instead of a constant value of the modulus of elasticity, because the stress-strain diagram of the envelope's material is non-linear.

The values E of the secant modulus at any point of the surface of the envelope are obtained with the help of a given diagram[4] of dependence of stress vs. strain σ(ε). The relative strain ε is calculated from (4). The length of the corresponding cross section of the envelope is calculated by integration of the function of the surface (12).

The values E_g of the secant modulus, corresponding to the found parameters $\overset{\rightarrow}{Hs}$, may be used as initial data for any computer system of surface structure design, for example EASY. An average discrepancy between the given and obtained modulus has to be estimated as follows:

$$\Psi = \sum_{u=1}^{N} \left| \frac{E_{g,u} - E_{req,u}}{0.5 \cdot (E_{g,u} + E_{req,u})} \right| \cdot \frac{1}{N} \cdot 100 \text{, \%,} \tag{23}$$

where N is the number of points on the surface; E_{req} is the secant modulus, corresponding to the solution, obtained in the system of surface design; $E_{g,u}$, $E_{req,u}$ are values at the point u of the surface.

6 EXAMPLE

For an example, the pneumatic envelope, made of ETFE film with the thickness 0.25mm, is considered. The size of the envelope is 4 x 8 m. The film is initially flat, but when it is filled with air under pressure 2.0 kPa, it forms a curved surface. The external load is applied on the envelope: $\vec{S} = (S_1, S_2, S_3, S_4)^T = (-0.75, -1.25, +0.75, -0.75)^T$ kPa. Its positive direction coincides with the Z-axis. According to (6), the load may be represented as follows: $S_{sym} = -0.5$, $S_x = -0.5$, $S_y = -0.5$, $S_{xy} = 0.25$ kPa. It causes the pressure growth up to $P_g = 2.255$ kPa. The temperature is considered to be constant during the loading.

Parameters of the surface's function (12) are obtained with the help of the proposed technique: $\vec{Hs} = (0.654, \quad 0.580, \quad 0.500, \quad 0.453, \quad -0.179, \quad -0.066, \quad 0.016)^T$.

The final analysis of the envelope is performed in the computer system of surface structure design EASY 8.3. An average discrepancy calculated according to (23) is 3.7%.

The comparison of cross sections of the surface, obtained from (12) and with the help of the system EASY, is in figure 6.

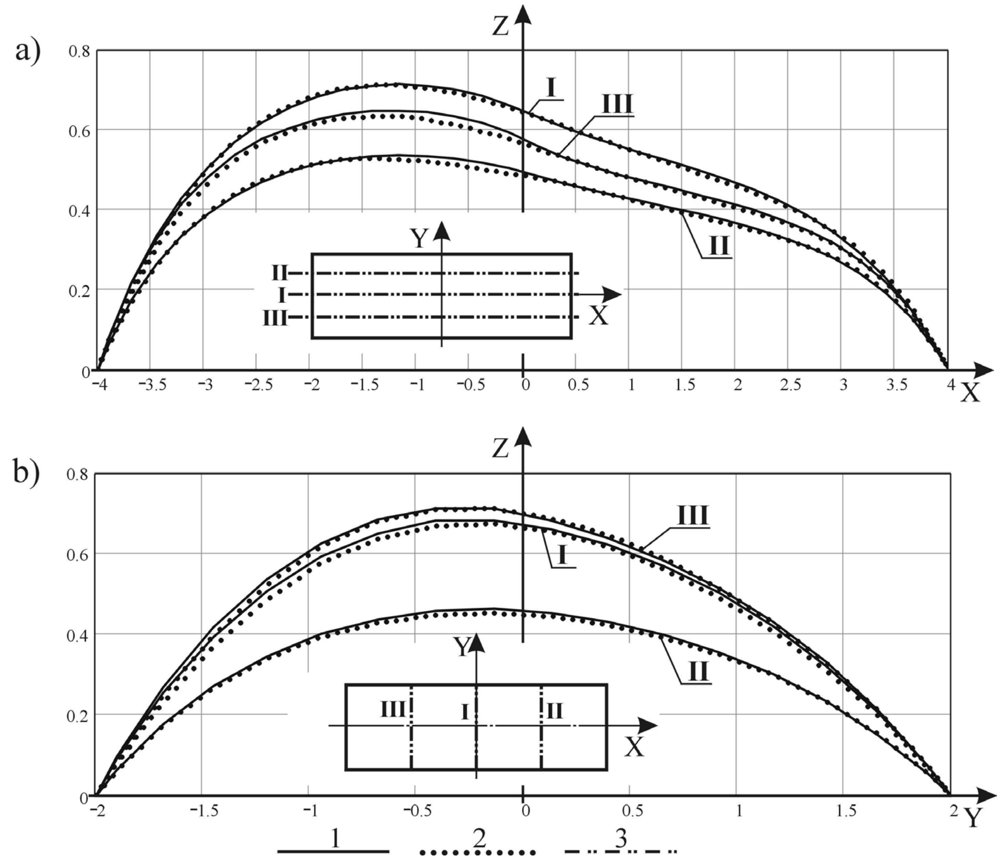

Figure 6. The surface of the envelope, influenced by internal pressure and external non-uniform loads: a – cross sections by planes, which are parallel to *XOZ*; b – cross sections by planes, which are parallel to *YOZ*; 1 – results of EASY 8.3; 2 – graphs of the function (12); 3 – disposition of cross sections

7 CONCLUSIONS

1. Rectangular pneumatic envelopes subjected to non-uniform external loads are considered. The function of envelopes' surface is proposed.
2. The technique for obtaining values of parameters of the function is offered. It uses the differential equation of equilibrium, the method of least squares and the linearization procedure with Taylor series.
3. The proposed technique is in a good agreement with data, obtained by the system EASY, intended for the surface structure design.

4. The results of the work may be used as follows:

- for taking into account physical non-linear properties of polymer membranes, of which pneumatic envelopes are made;
- for estimation of air-pressure changes in pneumatic cushions, which are influenced by non-uniform loads and temperature variations;
- for consideration of several embodiments of pneumatic cushions in order to select the best one.

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