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THE USE OF DYNAMIC RELAXATION TO SOLVE THE DIFFERENTIAL EQUATION DESCRIBING THE SHAPE OF THE TALLEST POSSIBLE BUILDING

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Abstract. The problem of finding the tallest possible column that can be constructed from a given volume of material without buckling under its own weight was finally solved by Keller and Niordson in 1966. The cross-sectional size of the column reduces with height so that there is less weight near the top and more bending stiffness near the base. Their theory can also be applied to tall buildings if the weight is adjusted to include floors, live load, cladding and finishes.

In this paper we simplify the Keller and Niordson derivation and extend the theory to materials with non-linear elasticity, effectively limiting the stress in the vertical structure of the building. The result is one highly non-linear ordinary differential equation which we solve using dynamic relaxation.

1 INTRODUCTION

The field of optimal structural design has been and still is a fertile area of research. One of the more interesting of such problems concerns the optimal design of the tallest possible elastic column.

Keller and Niordson [1] in 1966 solved the problem of finding the tallest column that can be constructed without buckling under its own weight, given a fixed volume of material and allowing the cross-sectional area to vary. Their work is based on the Euler-Bernoulli theory and involves maximising the lowest eigenvalue of a linear second order differential equation, or a Sturm-Liouville operator.

Further investigations of this problem were undertaken later by Cox and McCarthy [2] and again by McCarthy [3]. They formally prove the existence of the tallest column and solve the optimality conditions and both investigations used numerical iterative schemes to find the optimal design. This problem has also been approached by Atanackovic [4] and Egorov [5], confirming previous results.

However, there has not yet been any attempt to go outside the bounds of linearly elastic materials or to limit the compressive stress in the tallest column. This would become useful when considering optimal designs for super tall buildings.

Francis Reynolds Shanley [6, 7] showed that the buckling load of a column with non-linear material behaviour is determined by the tangent modulus, that is the slope of the tangent to the stress/strain curve during first loading. The increased stiffness during unloading of an elastic/plastic material does not influence the buckling load. This means that the buckling load of an elastic/plastic material can be obtained by treating the material as non-linear elastic, ignoring the different behaviour during unloading.

In this paper the Euler-Lagrange equations for a material described by its tangent modulus as a function of axial stress are derived using an energy approach which is equivalent to virtual work. The Euler-Lagrange equations are combined into a single 3rd order non-linear ordinary differential equation which is solved using dynamic relaxation.

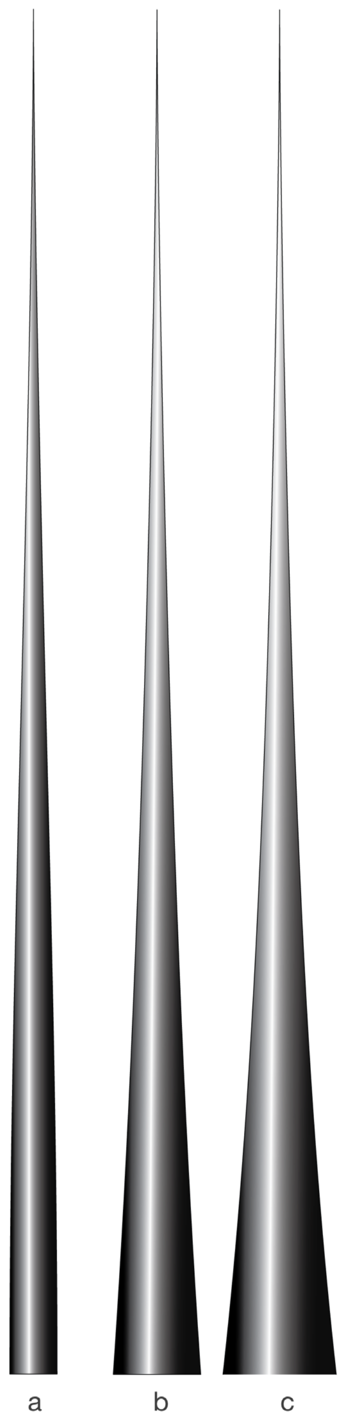

Figure 1: Tallest column profiles
(a) Linear elastic
(b) & (c) Non-linear elastic, stress limited

Figure 1 shows the result of this analysis. The 3 structures are made from a material with the same density and Young's modulus for low stress and hence they have the same profile near the top. However the material of structure (b) has a limited strength and that of structure (c) has an even lower strength. This causes the profile to get wider towards the base to limit the maximum stress. For a very weak material the profile away from the top increases exponentially with distance from the top, keeping the stress constant.

2 PROBLEM DEFINITION

Consider a vertical structure of height H that is clamped at the base and free at the top. Let s be the arc-length along the structure, measured from the top downwards such that $s = H$ at the base. The vertical coordinate, z, is also measured downwards from the top so that $z = H$ at the base in the undeformed configuration.

The cross-sectional area of the vertical structure is $A(s)$ and its second moment of area is assumed to be $I(A) = \alpha A^2$ in which α is a non-dimensional constant. For a solid circular section $\alpha = 1/4\pi$. The weight per unit height of the structure is assumed to be $\rho g A(s)$ in which ρg is a constant which is adjusted to include the weight of floors, cladding, live load etc. in the case of a building.

The volume of vertical structure above the level defined by s is

$$V(s) = \int_{s=0}^{s} A(s)\, ds \tag{1}$$

so that

$$A = \frac{dV}{ds}. \tag{2}$$

The axial load at that level is

$$P(s) = \rho g V(s) \tag{3}$$

and therefore the axial compressive stress is

$$\sigma(s) = P(s)/A(s). \tag{4}$$

The total volume of vertical structure is $V(H)$. At the top $V(0) = 0$ and the cross-sectional area is assumed to be zero, $A(0) = 0$.

It is assumed that the structure is sufficiently well braced for there to be no shear deformation and axial deformation is also ignored. Thus the only deformation is due to bending and the Euler-Bernoulli bending stiffness of the column is EI in which E is the tangent modulus. It is assumed that E is a known function of the axial stress, σ, and therefore EI is a known function of V and of A.

When the column loses stability and buckles sideways the lateral displacement is $u(s)$ and the rotation is

$$\varphi(s) = \frac{du}{ds}, \tag{5}$$

which is assumed small. As a result, bending stresses are also small, explaining why we assume that the tangent modulus is a function of the vertical stress only.

The drop in height due to buckling at level s is $w(s)$. The clamped condition at the base means that $\varphi(H) = 0$ and $w(H) = 0$.

Using the Maclaurin series expansion and the fact that φ is small,

$$\frac{dw}{ds} = -\left(1 - \cos\varphi\right) = -\left(1 - \left(1 - \frac{\varphi^2}{2} + ...\right)\right) = -\frac{\varphi^2}{2}. \tag{6}$$

3 ANALYSIS

3.1 Energy approach

The total change in gravitational potential energy due to sideways buckling is:

$$\begin{aligned} W &= -\rho g \int_{s=0}^{H} Aw\, ds = -\rho g \int_{s=0}^{H} \frac{dV}{ds} w\, ds \\ &= -\rho g \left[Vw\right]_{s=0}^{H} + \rho g \int_{s=0}^{H} V \frac{dw}{ds}\, ds \\ &= -\rho g \int_{s=0}^{H} V \frac{1}{2}\varphi^2\, ds \end{aligned} \tag{7}$$

in which we have used the boundary conditions $V(0) = 0$ and $w(H) = 0$.

The total strain energy due to bending is

$$U = \int_{s=0}^{H} \frac{1}{2} EI\kappa^2\, ds \tag{8}$$

in which the curvature

$$\kappa = \frac{d\varphi}{ds}. \tag{9}$$

Thus

$$U + W = \frac{1}{2} \int_{s=0}^{H} Q ds \tag{10}$$

in which

$$Q(A, V, \kappa, \varphi) = EI\kappa^2 - \rho g V \varphi^2. \tag{11}$$

The total energy $U + W$ can be minimised by varying φ so that $\varphi = \varphi(s,t)$. However, in order to also minimise the total volume, $V(H)$, the area and therefore the volume are also varied. Thus $V = V(s,t)$ and $\dfrac{\partial A}{\partial t} = \dfrac{\partial^2 V}{\partial s \partial t}$.

Thus, for $U + W$ to be a minimum

$$0 = \frac{d}{dt}(U + W) = \frac{1}{2} \int_{s=0}^{H} \left[\left(\frac{\partial (EI)}{\partial V} \frac{\partial V}{\partial t} + \frac{\partial (EI)}{\partial A} \frac{\partial^2 V}{\partial s \partial t} \right) \left(\frac{\partial \varphi}{\partial s} \right)^2 + 2EI \frac{\partial \varphi}{\partial s} \frac{\partial^2 \varphi}{\partial s \partial t} \right.$$
$$\left. - \rho g \frac{\partial V}{\partial t} \varphi^2 - 2\rho g V \varphi \frac{\partial \varphi}{\partial t} \right] ds$$

Integrating by parts,

$$0 = \frac{d}{dt}(U + W) = \frac{1}{2} \left[\frac{\partial (EI)}{\partial A} \left(\frac{\partial \varphi}{\partial s} \right)^2 \frac{\partial V}{\partial t} + 2EI \frac{\partial \varphi}{\partial s} \frac{\partial \varphi}{\partial t} \right]_{s=0}^{H}$$
$$- \frac{1}{2} \int_{s=0}^{H} \left[\left(\frac{\partial}{\partial s} \left[\frac{\partial (EI)}{\partial A} \left(\frac{\partial \varphi}{\partial s} \right)^2 \right] - \frac{\partial (EI)}{\partial V} \left(\frac{\partial \varphi}{\partial s} \right)^2 + \rho g \varphi^2 \right) \frac{\partial V}{\partial t} \right.$$
$$\left. + 2 \left(\frac{\partial}{\partial s} \left(EI \frac{\partial \varphi}{\partial s} \right) + \rho g V \varphi \right) \frac{\partial \varphi}{\partial t} \right] ds.$$

At the top where $s = 0$ the value of V remains constant at zero so that $\dfrac{\partial V}{\partial t} = 0$ and the bending moment $EI\kappa = EI\dfrac{\partial \varphi}{\partial s} = 0$. At the base where $s = H$ there is no rotation so that $\varphi = 0$ and minimisation of the total volume means that $\dfrac{\partial V}{\partial t} = 0$. Thus the term $\dfrac{1}{2}[...]_{s=0}^{H}$ is zero. The variations $\dfrac{\partial V}{\partial t}$ and $\dfrac{\partial \varphi}{\partial t}$ are arbitrary, subject to end constraints and hence we obtain the Euler-Lagrange equations

$$\frac{d}{ds} \left[\frac{\partial (EI)}{\partial A} \left(\frac{d\varphi}{ds} \right)^2 \right] - \frac{\partial (EI)}{\partial V} \left(\frac{d\varphi}{ds} \right)^2 + \rho g \varphi^2 = 0 \tag{12}$$

and

$$\frac{d}{ds}\left(EI\frac{d\varphi}{ds}\right)+\rho gV\varphi=0 \tag{13}$$

in which the partial derivatives of V and φ have been replaced by ordinary derivatives since the variation with the variable t is no longer needed. Note that, for example,

$$\frac{d}{ds}\left[\frac{\partial(EI)}{\partial A}\right]=\frac{\partial^2(EI)}{\partial A^2}\frac{dA}{ds}+\frac{\partial^2(EI)}{\partial A\partial V}\frac{dV}{ds}=\frac{\partial^2(EI)}{\partial A^2}\frac{d^2V}{ds^2}+\frac{\partial^2(EI)}{\partial A\partial V}\frac{dV}{ds} \tag{14}$$

in which $\dfrac{\partial^2(EI)}{\partial A^2}$ and $\dfrac{\partial^2(EI)}{\partial A\partial V}$ are known functions of V and A.

Alternatively, we could start from equation (11) and use a standard result from the calculus of variations [8] to minimise $U+W$ and $V(H)$ and produce

$$\frac{\partial Q}{\partial\varphi}-\frac{d}{ds}\left(\frac{\partial Q}{\partial\kappa}\right)=0 \qquad \text{and} \qquad \frac{\partial Q}{\partial V}-\frac{d}{ds}\left(\frac{\partial Q}{\partial A}\right)=0$$

which are identical to (12) and (13). If we set $E=$ constant in (12) and (13) for the linear elastic case we obtain the same equations as those derived by Keller and Niordson [1] using a somewhat more complicated argument.

We now have 2 equations in 2 unknowns, $V(s)$ and $\varphi(s)$, and to eliminate $\varphi(s)$ let us introduce a new variable $f(s)$ such that $\dfrac{d\varphi}{ds}=\dfrac{\varphi}{f}$. Equations (12) and (13) then become

$$\begin{aligned}0&=\frac{d}{ds}\left[\frac{\partial(EI)}{\partial A}\left(\frac{\varphi}{f}\right)^2\right]-\frac{\partial(EI)}{\partial V}\left(\frac{\varphi}{f}\right)^2+\rho g\varphi^2\\&=\frac{d}{ds}\left[\frac{1}{f^2}\frac{\partial(EI)}{\partial A}\right]\varphi^2+\frac{1}{f^2}\frac{\partial(EI)}{\partial A}2\varphi\frac{\varphi}{f}-\frac{\partial(EI)}{\partial V}\left(\frac{\varphi}{f}\right)^2+\rho g\varphi^2\end{aligned}$$

and

$$\begin{aligned}0&=\frac{d}{ds}\left(EI\frac{\varphi}{f}\right)+\rho gV\varphi\\&=\frac{d}{ds}\left(\frac{EI}{f}\right)\varphi+\frac{EI}{f}\frac{\varphi}{f}+\rho gV\varphi,\end{aligned}$$

which, after some manipulation produce

$$\frac{d}{ds}\left[\frac{\partial(EI)}{\partial A}\right]+\frac{2}{f}\left(1-\frac{df}{ds}\right)\frac{\partial(EI)}{\partial A}-\frac{\partial(EI)}{\partial V}+\rho gf^2=0 \tag{15}$$

and

$$\frac{d(EI)}{ds} + \frac{EI}{f}\left(1 - \frac{df}{ds}\right) + \rho g V f = 0. \tag{16}$$

Substituting $\frac{1}{f}\left(1 - \frac{df}{ds}\right)$ from (16) into (15) produces a quadratic in f:

$$f^2 - 2V\frac{\frac{\partial}{\partial A}\left(\frac{EI}{\rho g}\right)}{\left(\frac{EI}{\rho g}\right)} f + \frac{d}{ds}\left(\frac{\partial}{\partial A}\left(\frac{EI}{\rho g}\right)\right) - \frac{\partial}{\partial V}\left(\frac{EI}{\rho g}\right) - 2\frac{d}{ds}\left(\frac{EI}{\rho g}\right)\frac{\frac{\partial}{\partial A}\left(\frac{EI}{\rho g}\right)}{\left(\frac{EI}{\rho g}\right)} = 0 \tag{17}$$

which can be solved for f knowing that it must be negative. We can also differentiate equation (17) to find $\frac{df}{ds}$. Finally we can substitute f and $\frac{df}{ds}$ into either Equation (15) or (16) to produce one 3^{rd} order non-linear ordinary differential equation in V which can be solved numerically.

3.2 Material behaviour

The method and result obtained thus far are independent of the type of material used, provided that we have a unique relationship between the average axial stress and the tangent modulus. In the example which we will consider we assume a stress/strain relationship

$$\sigma = \sigma_{\max} \tanh\left(\frac{E_0}{\sigma_{\max}}\varepsilon\right)$$

where σ is the stress, ε is the strain and the material strength and stiffness are defined by the constants $\sigma_{\max}$ and E_0. $\sigma_{\max}$ is the maximum value of stress at large strain and E_0 is the linear elastic Young's modulus.

By definition the tangent modulus E is the slope of the stress/strain graph,

$$E = \frac{d\sigma}{d\varepsilon} = E_0\left(1 - \frac{\sigma^2}{\sigma_{\max}^2}\right)$$

and to obtain a linear elastic material we simply let $\sigma_{\max}$ tend to infinity.

3.3 Behaviour near the top

At the top $V(0) = 0$ and the moment is zero. If we also set $A(0) = 0$ then $\varphi \to \infty$ at the top because the bending stiffness tends to zero faster than the bending moment. If we write $V = as^\lambda$ and $\varphi = bs^\mu$, then upon substituting back into the Euler-Lagrange equations we obtain $\lambda = 4$ and $\mu = -2$ and therefore $a = 1/96$ and $A = s^3/24$. Even though $\varphi \to \infty$ at the top, the value of f is zero, $f(0) = 0$. Keller and Niordson [1] also arrive at a cubic taper for the area near the top.

In reality it is not possible to have a cubic taper since the tip would simply break off, and also we have made the assumption the φ is small. So a more realistic boundary condition would be $A(0) = A_{\text{tip}}$ in which A_{tip} is finite but whose value does not influence the overall shape of the column provided that A_{tip} is small.

3.4 Behaviour near the base

The column is clamped at the base, i.e. $\varphi(H) = 0$ and therefore $f(H) = 0$. From the quadratic solution for f, equation (17), it can be seen that at the base

$$\frac{d}{ds}\left(\frac{\partial}{\partial A}\left(\frac{EI}{\rho g}\right)\right) - \frac{\partial}{\partial V}\left(\frac{EI}{\rho g}\right) - 2\frac{d}{ds}\left(\frac{EI}{\rho g}\right)\frac{\frac{\partial}{\partial A}\left(\frac{EI}{\rho g}\right)}{\left(\frac{EI}{\rho g}\right)} = 0. \tag{18}$$

This gives an expression for $\dfrac{d^2V}{ds^2}$ and, in the linear elastic case this reduces to $\dfrac{d^2V}{ds^2} = 0$.

4 NUMERICAL SOLUTION

The above analysis eventually leads to one equation containing V, $A = \dfrac{dV}{ds}$, $\dfrac{dA}{ds}$ and $\dfrac{d^2A}{ds^2}$. We thus have a 3rd order differential equation in V or a 2nd order non-linear differential equation in A if we use numerical integration to find V.

The differentiations in equations such as (17) using rules of the form (14) mean that the differential equation is exceptionally long making it impossible to write it out explicitly in the constraints of a paper. However the differentiations follow the usual simple rules so that the derivation is tedious but not difficult.

Let the length of the column from $s = 0$ to $s = H$ be divided into n intervals of equal lengths δs with nodes numbered from $i = 0$ to $i = n$. An additional fictional node is required at the base ($i = n + 1$) in order to define the boundary behaviour. Using the boundary conditions and the trapezium rule

$$\begin{aligned} A_0 &= 0 \\ V_0 &= 0 \\ V_i &= V_{i-1} + \frac{\delta s}{2}\left(A_i + A_{i-1}\right). \end{aligned} \tag{19}$$

At every level from $i = 1$ to $i = n$

$$\left(\frac{dA}{ds}\right)_i = \frac{A_{i+1} - A_{i-1}}{2\delta s} \qquad \text{and} \qquad \left(\frac{d^2A}{ds^2}\right)_i = \frac{A_{i+1} - 2A_i + A_{i-1}}{\delta s^2}. \tag{20}$$

Substituting into the differential equation gives an expression for A_i in terms of A_{i-1}, A_{i+1} and V_i at all levels except when $i = 0$ at the top and when $i = n$ at the base where we use equation (18) to find A_{n+1}. The behaviour near the top is discussed in section 3.3 so that for small values of i we simply set $A_i = \dfrac{(i \times \delta s)^3}{24}$.

4.1 Use of dynamic relaxation

Dynamic relaxation is an explicit numerical method often used for finding the equilibrium shape of non-linear structures. This is done by moving the nodes of a structure under the influence of out-of-balance or residual forces, including some form of artificial damping, until the structure achieves equilibrium [9]. Dynamic relaxation is essentially the same as Vertlet or leapfrog integration used to integrate the equations of motion for dynamic problems

However, in our case the unknowns are not the displacements of the structure, but its shape as defined by A_i. We can still define an 'out of balance force', F_i, which is the error in the solution of the differential equation at node i. The dynamic relaxation algorithm is then

$$\begin{aligned} \left(\dot{A}_i\right)_{t+\frac{\delta t}{2}} &= (1-\eta)\left(\dot{A}_i\right)_{t-\frac{\delta t}{2}} + \frac{(F_i)_t}{m_i}\delta t \\ (A_i)_{t+\delta t} &= (A_i)_t + \left(\dot{A}_i\right)_{t+\frac{\delta t}{2}}\delta t \end{aligned} \tag{21}$$

in which δt is the time step, $\dot{A}_i$ is the rate of change of A_i, η is a small constant to produce damping and m_i is the 'mass' associated with the i^{th} nodal area. Because we are only interested in the final static solution we are free to choose m_i to get the best convergence. Therefore we choose m_i to be proportional to the coefficient of A_i in the finite difference version of the differential equation. For small values of δs this will be dominated by the $\dfrac{d^2A}{ds^2}$ term.

5 RESULTS

Our non-linear material is described by the 2 constants, E_0 and $\sigma_{\max}$. This leads to a 2 parameter family of solutions to the problem of the tallest possible column. The 2 non-dimensional parameters can be written

$$\beta = \frac{H}{L} \qquad \text{and} \qquad \gamma = \frac{\sigma_{\max}}{E_0} \tag{22}$$

where

$$L = \frac{\alpha E_0}{\rho g}. \tag{23}$$

The images in figure 1 were produced assuming material properties for steel $E_0 = 205$ GPa and $\rho g = 7850 \times 9.81$ N/m^3, and $\alpha = 1/4\pi$ for a solid circular section. The 3 images all share the same value of β, corresponding to a height, $H = 10,000$ m. Note however that the same value of β would correspond to a shorter structure if ρg were increased to account for floors etc. which do not contribute to the bending stiffness.

	(a) Linear	(b) Non-linear 90%	(c) Non-linear 95%
β	0.0472	0.0472	0.0472
γ	∞	0.866×10^{-3}	0.695×10^{-3}
$\sigma_{\max}$ (MPa)	∞	177.5	142.4
σ_H (MPa)	288.8	159.9	135.4
$\sigma_H/\sigma_{\max}$	0	0.90	0.95

Table 1: Non-dimensional parameters and stress values

The values of γ for the profiles shown in figure 1 are given in table 1. Setting $\sigma_{\max}$ to infinity corresponds to a linear elastic material for case (a). Cases (b) and (c) correspond to a non-linear material with different values of $\sigma_{\max}$. The area of the vertical structure automatically increases to limit the stress at the base σ_H to 90% and 95% of $\sigma_{\max}$.

Figure 2 shows a plot of the cross-sectional area against height on the vertical axis and figure 3 shows how the vertical stress varies with height.

6 CONCLUSIONS

This paper has addressed the classic problem of the tallest possible column. The analysis has been simplified and extended to the case of non-linear material behaviour in which the column automatically gets wider towards the base to limit the maximum stress.

Dynamic relaxation proved to be a powerful tool in solving the highly non-linear 2$^{\text{nd}}$ order ordinary differential equation that results from the analysis.

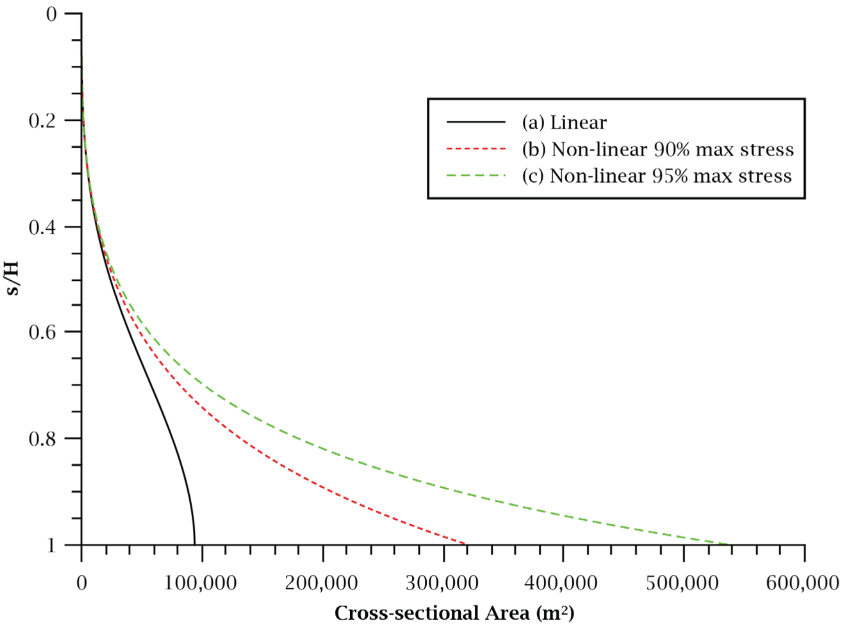

Figure 2: Cross-sectional Area plotted against the non-dimensional height, measured downwards

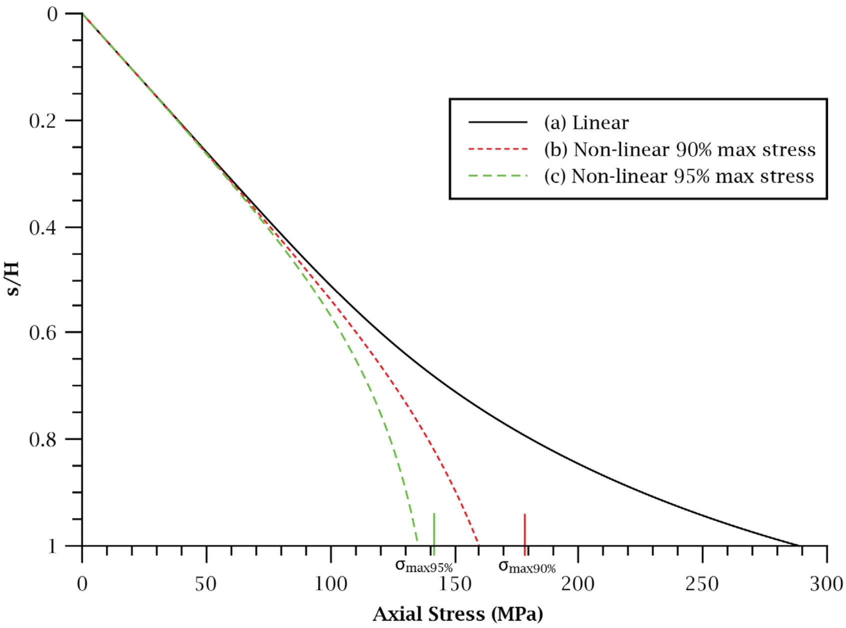

Figure 3: Axial stress plotted against the non-dimensional height, measured downwards; short vertical lines show values of $\sigma_{\max}$

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