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STRUCTURAL AND GEOMETRIC APPLICATIONS OF THE GEODESIC DYNAMIC RELAXATION METHOD

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Summary. The Geodesic Dynamic Relaxation method[1] is an extension of the existing Dynamic Relaxation method that allows the user to incorporate equality constraint conditions to minimization problems of strain energy functions. The existing Dynamic Relaxation method has been widely adopted as a form-finding method for mechanically and pneumatic pre-stressed tensile and bending active systems. While each structural component is usually modelled using an elastic material in the Dynamic Relaxation method, equality constraint conditions are introduced in the Geodesic Dynamic Relaxation Method as an alternative way to model some of the structural components in form-finding problems.

While the Geodesic Dynamic Relaxation method directly relates to the structural behavior of systems, the algorithm can also be used in a purely geometric context. More specifically, it allows the user to construct a geodesic line on an implicit surface.

This paper explains the Geodesic Dynamic Relaxation method briefly, and demonstrates both its structural and geometric applications. The structural applications relate to pre-stressed tensile structures, whereas the geometric application demonstrates the generation of fractal trees with geodesic branches on given implicit surfaces. The paper concludes and makes suggestions for future works. This paper will be of interest to structural and architectural engineers with an interest in computational design as well as computer scientists.

1 INTRODUCTION AND BACKGROUND

The Dynamic Relaxation (DR) method was first introduced by A. Day[2] to the engineering community in 1965. It has been recognized as a powerful computational method particularly applicable to form-finding problems of prestressed tensile structures[5,9]. However, the method itself can be classified as a gradient-based direct minimization approach[7], similar to the steepest descent or conjugate gradient methods. Hence, it can be applied to various nonlinear problems such as geometrically nonlinear problems and tracking large deformations of structures[3,6,8]. In the recent trend involving the combination of Grasshopper® and Rhinoceros®, it is noteworthy that the DR method has been employed as a key technique in Kangaroo, a plug-in developed by D. Piker[10]. The Kangaroo plug-in is known as the component that opened the door for architectural designers to perform basic structural or form-finding analyses.

As a gradient-based direct minimization approach, the hysteresis of minimization does not represent anything; however, it is possible to consider the DR method as a simple time integrator of the equation of motion. It has been observed that the total energy remains conserved in the DR process. This is a very important characteristic that every time integrator should provide.

An extension of the existing DR method that can incorporate equality constraint conditions to minimization problems was presented by the authors[1]. This extension was named the Geodesic Dynamic Relaxation Method. As a form-finding tool, this extension allows us to model some of the structural components of a structure as constraint conditions instead of modeling them as elastic materials. On the other hand, as a time integrator, this extension allows us to track the dynamic motion of a particle constrained on an implicit surface or a higher dimensional implicit surface. One important feature of the Geodesic Dynamic Relaxation Method is that it allows the user to draw geodesics on implicit surfaces. Although the geodesics are usually defined in a purely geometric context, this makes sense because the geodesics are also understood as the dynamic motion of a particle constrained to a surface to which no force is applied.

For these reasons, the Geodesic Dynamic Relaxation Method is considered to be a useful technique in both structural and geometric contexts. In this paper, both structural (i.e., form-finding) and geometric (i.e., geodesics generation) applications of the Geodesic Dynamic Relaxation Method are presented.

2 GEODESIC DYNAMIC RELAXATION METHOD

This section summarizes the Geodesic Dynamic Relaxation Method.

In the existing DR process, a structural system is discretized in such a way that it is represented by a finite number of independent variables. Those variables are typically x, y, and z coordinates of the nodes of linear members or triangular elements that approximate a membrane surface. If we denote the total number of such independent variables as n, we can encapsulate all independent variables into $\boldsymbol{x} \in \boldsymbol{R}^n$ and consider this vector as representing the position of a particle moving in $\boldsymbol{R}^n$. Internal (i.e., stress) and external forces that act on the structures are redistributed to the unknown variables (i.e., as a lumped nodal force) and represented by a vector with the same dimension as $\boldsymbol{x}$. We denote this force as $\boldsymbol{\omega} \in \boldsymbol{R}^n$. As a result, we have an n-dimensional vector force acting on a particle moving within an n-dimensional Euclidean space. In many cases, this force might be represented as $\boldsymbol{\omega} = \nabla f$, where f is a function of $\boldsymbol{x}$ and usually represents the total energy of the system and ∇ is a gradient operator with respect to $\boldsymbol{x}$.

In DR, we consider not only force and position, but also the velocity of the particle. Let us denote this velocity as $\boldsymbol{q} \in \boldsymbol{R}^n$. The DR method iteratively updates position and velocity based on Newton's second law of motion such that the system oscillates stably. We admit that a rigorous mathematical proof is difficult to submit; however, the law of conservation of energy seems to be established via the DR process because this oscillation neither diverges nor converges. When a damping effect is applied to the system, the system quickly stops oscillating and converges. The resulting configuration then represents a structural system that is in a state of equilibrium. This is the essential idea upon which DR is based.

When a certain number of equality constraint conditions are added to the problem, the feasible space (i.e., the points on which all conditions are satisfied) becomes a subspace of $\boldsymbol{R}^n$. We denote this subspace as $\bar{S}$. Although $\boldsymbol{R}^n$ is usually considered a Euclidean space, $\bar{S}$ becomes, in general, a curved subspace. Ideally, particle $\boldsymbol{x}$ should be constrained on $\bar{S}$ and both velocity $\boldsymbol{q}$ and force $\boldsymbol{\omega}$ should be tangent vectors of $\bar{S}$.

Because it is rather difficult to strictly satisfy the above hard constraints, we decided to relax the conditions as follows. The constraint conditions are usually given in the form

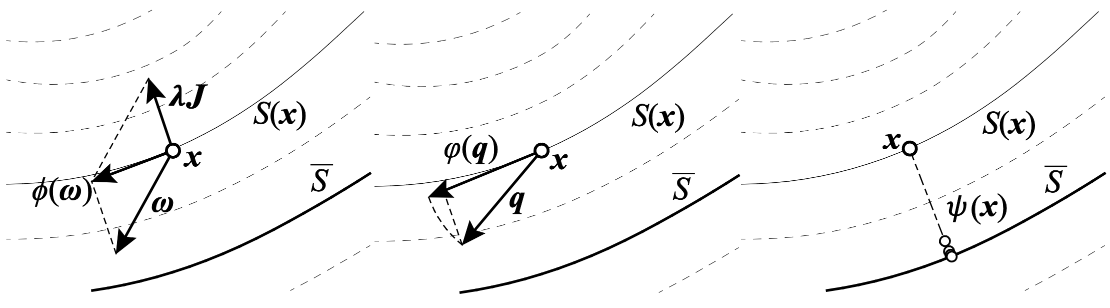
$$\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} g_1(\boldsymbol{x}) \\ \vdots \\ g_r(\boldsymbol{x}) \end{bmatrix} = \boldsymbol{0}. \tag{1}$$

Thus, $\bar{S}$ can be defined as the complete set of points that satisfies $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}$. For an $\boldsymbol{x}$ that is not supposed to satisfy $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}$, it is still possible to define a complete set of points that yield the same value of $\boldsymbol{g}(\boldsymbol{x})$ as $\boldsymbol{x}$. We denote this isomanifold (or isosurface in a special case) as $S(\boldsymbol{x})$. When $S(\boldsymbol{x})$ overlays $\bar{S}$, the constraint conditions are satisfied.

In the Geodesic Dynamic Relaxation Method, we proposed to project force $\boldsymbol{\omega}$ and velocity $\boldsymbol{q}$ on the tangent space (or the tangent plane in a special case) of $S(\boldsymbol{x})$ at each step. In our original publication (and as shown in Figure 1), these projection operators are called the projected gradient and the discrete parallel transportation operators, respectively. Note that different projection operators are chosen for each entity. Furthermore, it is not strictly assumed that $S(\boldsymbol{x}) = \bar{S}$ at this moment (i.e., the problem is relaxed).

Next, an attempt to project $\boldsymbol{x}$ onto $\bar{S}$ is made at the end of each step of the DR process. This trial is an iterative procedure that is terminated after few steps before $\boldsymbol{x}$ rigorously reaches $\bar{S}$. Even if we do so, $\boldsymbol{x}$ gradually approaches $\bar{S}$ through the DR process. As shown in Figure 1 (a), this iterative projection operator is called pull-back in our original publication[1].

When the system converges, $S(\boldsymbol{x})$ is overlaid with $\bar{S}$ and $\boldsymbol{\omega}$ has only a component orthogonal to the tangent space of $\bar{S}$. The former indicates that all constraint conditions are satisfied, while the latter means that the residual force can be canceled out by reaction forces produced by the constraint conditions. Hence, when the Geodesic Dynamic Relaxation Method converges, we obtain a structural system in equilibrium.



(a) the projected gradient operator constrained to prescribed values. (b) the discrete parallel transportation operator (c) the pull-back operator

Figure 1: projection operators, which first appeared in our previous publication[1].

An interesting byproduct, i.e., geodesics generation, can be obtained as follows. When the number of independent variable is three and the number of equality constraint conditions is one, the feasible space becomes a two-dimensional subspace of $\boldsymbol{R}^3$, i.e., a surface. When both damping effect and forces are ignored, but a non-zero initial velocity is given at the initial step of the DR process, the Geodesic Dynamic Relaxation Method generates a point series in a straightforward manner such that it approximates a geodesic line on the surface.

3 STRUCTURAL APPLICATIONS

As our first example of applications of the Geodesic Dynamic Relaxation Method, we present a form-finding problem of tension structures that comprises tensile cables and compression bars. In particular, the system we consider here is a structural system that comprises a series of five equispaced copper compression bars (Length = 30 [cm], Diameter = 6 [mm], Thickness = 0.5 [mm], EA = 1172 [N]), at one end pinned to the foundation and at the other end attached to a nylon continuous cable. As shown in Figure 2, the continuous cable (Diameter = 0.66 [mm], EA = 1368 [N]) runs from one pinned connection at the foundation over the ends of the five compression bars back to a pinned connection at the foundation. The continuous cable is prestressed to a value of 20 [N].

For form-finding, we chose the summation of squared lengths of tension cables as strain energy and regard the compression bars as constraint conditions of lengths between points. This problem can therefore be expressed as follows:

$$\sum_j L_j^2(\boldsymbol{x}) \rightarrow \min, \quad (j \in \text{tension cables}) \tag{2}$$

$$\text{s.t.}\, L_k(\boldsymbol{x}) - \bar{L}_k = 0, (\forall k \in \text{compression bars}). \tag{3}$$

The sum of squared lengths of tension cables typically becomes quadratic in the x, y, and z coordinates of both ends of the cables; thus, we can solve this by a single computation using an inverse matrix. Hence, rather than using DR, some might prefer this inverse matrix approach. This technique has been called the Force Density Method[4]. Although the original publication of the Force Density Method describes a well-considered procedure to consider some typical constraint conditions, such as those represented by Eq. (3), it requires us to perform computationally complex calculations, including differentiation of constraint conditions by force densities. Hence, we think that the Geodesic Dynamic Relaxation Method is a computationally less expensive alternative when one performs a form-finding analysis that considers equality constraint conditions.

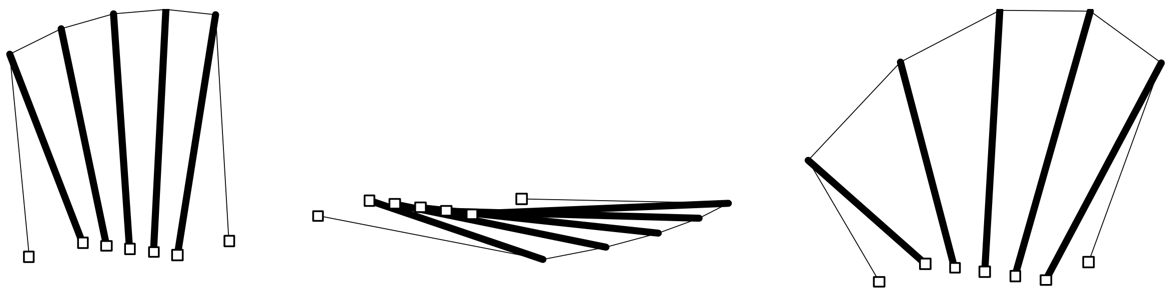
Another advantage of the Geodesic Dynamic Relaxation Method, as compared with the Force Density Method, is that one can replace the exponent that appears in Eq. (2) with numbers greater than two. In practice, the above minimization problem results in an undesired solution in most cases. For example, Figure 2 (a) seems to represent a desired prestressed structure; however, it quickly starts to lay down during the DR process and form the configuration shown in Figure 2 (b). In contrast, when we replace the exponent in Eq. (2) with four, the DR process converges to the shape shown in Figure 2 (c) in only a few steps. This solution is stable and does not fall into another global minimum of the total energy.

When four is used as an exponent, the proportions of prestress of the cables can be calculated as follows:

$$n_j = 4L_j^3, \quad (j \in \text{tension cables}). \tag{4}$$

On the other hand, the proportions of the compression forces in the bars are stored as Lagrange multipliers (please refer to the original publication[1] for further details regarding Lagrange multipliers).

Figure 3 shows a physical model that was created on the basis of the computational results discussed in this section. The model shown is made of nylon strings and copper bars. It was possible to stabilize the structure by prestressing it using turnbuckles.



(a) a result of minimization of the summation of the squared lengths of cables under which lengths of the compression bars are constrained to prescribed values.

(b) the figure on the left is not a rigorous solution and quickly moves toward the global minimum of the problem shown here.

(c) when four is used as the exponent of the energy function, a stable and desired solution is successfully obtained.

Figure 2: Form-finding results using our Geodesic Dynamic Relaxation Method.

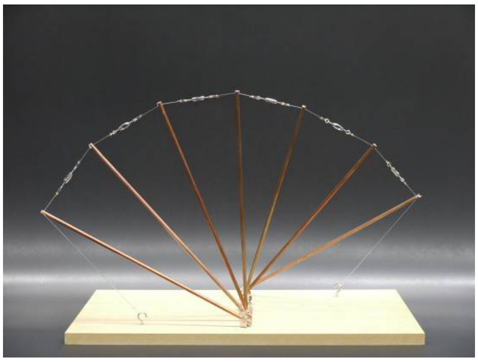


Figure 3: A photograph of the prestressed tension structure.

4 PURELY GEOMETRIC APPLICATIONS

4.1 Geodesics

In this section, we present two simple implicit surfaces. The first is a torus; an implicit representation of a torus is given as follows:

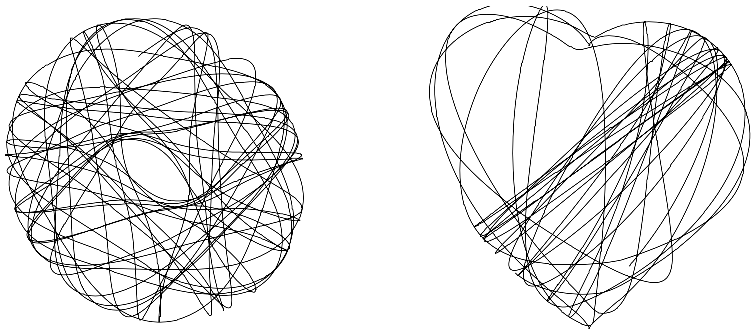
$$g(x,y,z) = \left(\boldsymbol{x} - \boldsymbol{R}\frac{x}{\sqrt{x^2+z^2}}\right)^{\boldsymbol{2}} - \left(\boldsymbol{z} - \boldsymbol{R}\frac{z}{\sqrt{x^2+z^2}}\right)^{\boldsymbol{2}} + \boldsymbol{y}^{\boldsymbol{2}} - \boldsymbol{r}^{\boldsymbol{2}} = 0. \qquad (5)$$

Here, R and r represent the major and minor radii of the torus, respectively.

Second, a heart-shaped surface is defined as follows:

$$g(x, y, z) = \left(x^2 + \frac{9}{4}y^2 + z^2 - 1\right)^3 - x^2z^3 - \frac{9}{80}y^2z^3 = 0. \quad (6)$$

Geodesics of these surfaces can be generated using the Geodesic Dynamic Relaxation Method, with results shown in Figure 4.



(a) torus (b) heart-shaped function

Figure 4: Geodesics of implicit surfaces.

4.2 Fractal tree

Figure 5 shows a typical fractal tree. This geometry would typically be constructed by recursive calls of a subroutine that takes two endpoints as input and draws a line between them; however, we implemented the program in a different way such that it generates a point series that represents a straight line based on the following iterative formula:

$$\boldsymbol{v}_t = \boldsymbol{v}_{t-1}, \quad (7)$$

$$\boldsymbol{x}_t = \boldsymbol{x}_{t-1} + \alpha\boldsymbol{v}_t. \quad (8)$$

Note that $\boldsymbol{v}_t$ remains constant such that the generated point series becomes a straight line. When $\boldsymbol{x}_t$ reaches one third of the specified total length, two branches are created by the program. For those two branches, new velocities and a new total length are assigned. The new velocities are rotated 30° to the left and 30° to the right; and the new total length is two thirds of the previous total length.

Although complex coding is needed and the geometry shown in Figure 5 can be constructed using a way simpler technique, this treatment enables us to draw a fractal tree on an implicit surface, of which all the branches are geodesics.

Figure 5: A fractal tree generated using our technique.

4.3 Geodesic fractal

Because the basic structure of Equations (7) and (8) is very similar to the DR process, it is possible to replace straight line generations in the fractal tree generation implementation with geodesics generation. Figure 6 shows such geodesic fractals drawn on the same implicit surfaces discussed in Section 4.1 above.

Figure 7 shows a three-dimensional (3-D) printed model of the computational results shown in Figure 6 (b), processed by Shapeways®. The thinnest curves in the 3-D printed model have 1mm diameters. Although 3-D printing is not necessarily in the scope of this work, we note that it is quite remarkable that such thin curves can keep their shapes as designed due to their elastic strength.

(a) torus (b) heart-shaped function

Figure 6: Geodesic fractals drawn using our approach.

Figure 7: A photograph of a 3-D printed geodesic fractal.

6 CONCLUSIONS

In this paper, we first presented a short summary of the Geodesic Dynamic Relaxation Method. The method allows us to incorporate equality constraint conditions to the existing DR method. Simultaneously, the method itself can be used to generate geodesics on implicit surfaces. The extension preserves the basic structure of the DR and comprises projection operators that project vectors in DR to appropriate subspaces.

We also presented a form-finding problem of a tensile structure as an application of our method. The structure comprised tensile cables and compression bars. Tensile cables were modeled using a form-finding specific elastic material, while compression bars were treated as constraint conditions of lengths between points.

Next, we provided an example of structural design using geodesics of implicit surfaces by employing a fractal tree as a global design in which each branch of the tree was composed of geodesics drawn using our Geodesic Dynamic Relaxation Method.

For our future work, apart from form-finding analyses, the Geodesic Dynamic Relaxation Method might be useful for analyzing large deformations of a structure that contains stretchy materials as opposed to stiff materials. In such cases, we would aim to model such stiff materials using constraint conditions to achieve numerical stability.

Furthermore, geodesics of surfaces are natural extension of straight lines in Euclidean spaces to surfaces; therefore, such "straight" curves can be excellent candidates as the most basic curves when one designs a structure of which global shape is determined by a surface. The major disadvantage of our proposed geodesics generation algorithm is that the surface must be represented using an implicit representation. The use of implicit surfaces in architectural design has rarely been considered. This might not be because of the limited design freedom, but rather the lack of a design interface. Hence, we expect the development of an architectural design-specific user-interface to be beneficial.

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