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# PHRAGMÉN-LINDELÖF ALTERNATIVE FOR THE LAPLACE EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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**Abstract** This paper investigates the spatial behavior of the solutions of the Laplace equation on a semi-infinite cylinder when dynamical nonlinear boundary conditions are imposed on the lateral side of the cylinder. We prove a Phragmén-Lindelöf alternative for the solutions. To be precise, we see that the solutions increase in an exponential way or they decay as a polynomial. To give a complete description of the decay in this last case we also obtain an upper bound for the amplitude term by means of the boundary conditions. In the last section we sketch how to generalize the results for the case of a system of two elliptic equations related with the heat conduction in mixtures.

**Keywords** Phragmén-Lindelöf alternative, dynamic boundary conditions, Laplace equation

**MSC(2010)** 35B30, 35B40, 35B53, 35J65.

## 1. Introduction

Spatial stability of solutions for the equilibrium equations in elasticity is related with Saint-Venant's principle. It is a relevant aspect when one studies the deformations of a cylinder. For this reason, the study of the spatial stability is a topic which has deserved a big interest in the recent years and many scientists have direct their point of view to this topic. With the time, several people draw their attention to dynamical elastic problems or dynamical thermal problems and they try to extend the spatial decay results to the new situations.

In fact, these studies have exceed the interest from the thermomechanical point of view and spatial behavior of solutions for different types of partial differential equations and/or systems are currently studying. It is worth noting that the mathematical framework where such results are considered is the Phragmén-Lindelöf principle which propose an increase/decay alternative for the solutions.

The spatial behavior of elliptic [3], parabolic [6, 8], hyperbolic [1, 4, 9] equations and/or combinations of them [12] have been obtained in the last years. However, there are many aspects yet which need to be studied and clarified. In this note we want to pay attention to the case of the Laplace equation with nonlinear dynamical boundary conditions. That is, when we assume that a certain nonlinear ordinary dynamical differential equation is satisfied at the lateral boundary of the cylinder where the Laplace equation is satisfied. It is worth noting that, as far as the authors know, there are no results in the literature on spatial behavior of solutions when such

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kind of boundary conditions are assumed. We will obtain a Phragmén-Lindelöf type alternative for the solutions of the problem. In fact, we will see that the solutions either blow-up in an exponential type when the large variable becomes unbounded, or they decay as a polynomial when the large variable is increasing. It is appropriate to recall several results when nonlinear boundary conditions are imposed (see the papers by Horgan etc [7] and Leseduarte etc [11]). In our approach we try to follow a similar way in order to obtain our results. However, our results only apply when the nonlinear term is super-linear, but not in the sub-linear case.

The plan of the paper is the following. In the next section we propose the problem we will work. A Phragmén-Lindelöf alternative is obtained in Section 3. When the solutions decay, our estimate is impractical if we do not have some information on the amplitude term. In Section 4 we obtain an upper bound for the amplitude term for the case when the solutions decay. In the last section we propose an extension of the results for the case when we consider a system of two linear elliptic equations that are related with the heat conduction in mixtures.

## 2. Preliminaries

This paper is concerned with investigating the spatial asymptotic behavior of the solutions of the Laplace equation with nonlinear dynamic boundary conditions. Therefore, we consider a semi-infinite cylinder $R = [0, \infty) \times D$, where $D$ is a two-dimensional bounded domain smooth enough to apply the divergence theorem.

We consider a problem related with the Laplace equation

$$\Delta u = 0 \text{ on } R \times (0, t). \tag{2.1}$$

To define the boundary conditions, we suppose that $\partial D = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2 = \emptyset$ and such that the measure of $\Omega_2$ is positive. On $\Omega_1$, we impose that

$$\frac{\partial u}{\partial n} + f_1(u) = 0 \text{ on } [0, \infty) \times \Omega_1 \times (0, t); \tag{2.2}$$

on $\Omega_2$ we suppose that

$$\frac{\partial u}{\partial n} + s(u)u_t + f_2(u) = 0 \text{ on } [0, \infty) \times \Omega_2 \times (0, t); \tag{2.3}$$

and on the finite end of the cylinder we impose that

$$u(0, x_2, x_3, \tau) = g(x_2, x_3, \tau) \text{ on } \{0\} \times D \times (0, t). \tag{2.4}$$

From now on, we assume that

$$f_1(u)u \geq 0, \tag{2.5}$$

for every $u$ and that there exists a positive constant $C$ such that

$$f_2(u)u + \omega S_1(u) \geq C|u|^{2p}, \tag{2.6}$$

where $p \geq 1$, $\omega$ is a strictly positive constant large enough and

$$S_1(u) = \int_0^u \eta s(\eta)\, d\eta \geq 0. \tag{2.7}$$

As we impose dynamic boundary conditions on $\Omega_2$, we need to assume initial conditions on $\Omega_2$. We suppose that

$$u = 0 \quad \text{on} \quad [0,\infty) \times \Omega_2 \times \{0\}. \tag{2.8}$$

It is worth giving examples where conditions (2.5)–(2.7) hold. We can take, for instance, $f_1(u) = a_1(1-\cos u)u^{-1}$, where $a_1 \geq 0$; $f_2(u) = a_2(1-\cos u)u^{-1} - b_2|u|^k u$, where $a_2 \geq 0$, $b_2 \geq 0$ and $k \geq 0$ and $s(u) = s_1|u|^k$, with $s_1 > 0$. In this case we have that $f_1(u)u = a_1(1-\cos u) \geq 0$ and (2.5) holds. Moreover,

$$S_1(u) = \frac{s_1}{k+2}|u|^{k+2}$$

and condition (2.7) holds. Regard to the condition (2.6), if we take $\bar{C} > 0$ and

$$\omega = \frac{(k+2)b_2}{s_1} + \bar{C},$$

we have

$$\begin{aligned} f_2(u)u + \omega S_1(u) &= a_2(1-\cos u) - b_2|u|^{k+2} + \left[\frac{(k+2)b_2}{s_1} + \bar{C}\right]\frac{s_1}{k+2}|u|^{k+2} \\ &= a_2(1-\cos u) + \frac{s_1\bar{C}}{k+2}|u|^{k+2} \geq C|u|^{k+2} \geq C|u|^{2p}, \end{aligned}$$

where $C = \dfrac{s_1\bar{C}}{k+2}$ and $p = \dfrac{k+2}{2}$. So, (2.6) holds.

In this paper we will use the following notation:

$$D(z) = \{z\} \times D; \quad \Omega_1(z) = \{z\} \times \Omega_1; \quad \Omega_2(z) = \{z\} \times \Omega_2;$$

$$R(z) = \{\boldsymbol{x} \in R, x_1 \geq z\}; \quad \Omega_i^*(z) = \{\boldsymbol{x} \in [0,\infty) \times \Omega_i, x_1 \geq z\}.$$

In the analysis we will need to use the generalized Poincaré inequality. We recall that there exists a positive constant $C_1$ such that (see [2], p. 281)

$$\int_D |u|^2\, da \leq C_1 \left[\int_D |\nabla u|^2\, da + \left|\int_{\Omega_2} u\, dl\right|^2\right], \tag{2.9}$$

for every smooth function $u$ and for a two dimensional domain $D$. It is worth noting that the precise value of the constant $C_1$ depends on the domain $D$ and the subset of the boundary $\Omega_2$.

## 3. Spatial Estimates

In this section we obtain an alternative of the Phragmén-Lindelöf type for the solutions of the problem determined by (2.1)–(2.4) and (2.8). From now on, we consider $\omega$ a positive constant such that condition (2.6) is satisfied. We define the function

$$\Phi(z,t) = -\int_0^t \int_{D(z)} \exp(-\omega\tau) u u_{,1}\, da\, d\tau. \tag{3.1}$$

We note that for $z \geq z_0$, $\Phi(z,t)$ may be expressed as

$$\begin{aligned}\Phi(z,t) - \Phi(z_0,t) = &- \int_0^t \int_{z_0}^z \int_D \exp(-\omega\tau)|\nabla u|^2 \, dx \, d\tau \\ &- \int_0^t \int_{z_0}^z \int_{\Omega_1} \exp(-\omega\tau) f_1(u) u \, da \, d\tau \\ &- \int_0^t \int_{z_0}^z \int_{\Omega_2} \exp(-\omega\tau) \left[f_2(u)u + \omega S_1(u)\right] da \, d\tau \\ &- \int_{z_0}^z \int_{\Omega_2} \exp(-\omega t) S_1(u) \, da.\end{aligned} \tag{3.2}$$

We note that in case that

$$\lim_{z\to\infty} \Phi(z,t) = 0, \tag{3.3}$$

then the relation (3.2) implies that

$$\begin{aligned}\Phi(z,t) = &\int_0^t \int_{R(z)} \exp(-\omega\tau)|\nabla u|^2 \, dx \, d\tau + \int_0^t \int_{\Omega_1^*(z)} \exp(-\omega\tau) f_1(u) u \, da \, d\tau \\ &+ \int_0^t \int_{\Omega_2^*(z)} \exp(-\omega\tau) \left[f_2(u)u + \omega S_1(u)\right] da \, d\tau \\ &+ \int_{\Omega_2^*(z)} \exp(-\omega t) S_1(u) \, da.\end{aligned} \tag{3.4}$$

From (3.2) we also see that

$$\begin{aligned}\frac{\partial \Phi}{\partial z} = &- \int_0^t \int_{D(z)} \exp(-\omega\tau)|\nabla u|^2 \, da \, d\tau - \int_0^t \int_{\Omega_1(z)} \exp(-\omega\tau) f_1(u) u \, dl \, d\tau \\ &- \int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau) \left[f_2(u)u + \omega S_1(u)\right] dl \, d\tau \\ &- \exp(-\omega t) \int_{\Omega_2(z)} S_1(u) \, dl.\end{aligned} \tag{3.5}$$

In view of the Schwarz inequality, from (3.1) we find

$$\begin{aligned}|\Phi(z,t)| \leq &\left( \int_0^t \int_{D(z)} \exp(-\omega\tau) u^2 \, da \, d\tau \right)^{1/2} \\ &\times \left( \int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,1}^2 \, da \, d\tau \right)^{1/2}.\end{aligned} \tag{3.6}$$

We have that

$$\begin{aligned}\int_0^t \int_{D(z)} \exp(-\omega\tau) u^2 \, da \, d\tau \leq &\int_0^t \int_{D(z)} u^2 \, da \, d\tau \\ \leq &C_1 \int_0^t \left[ \int_{D(z)} u_{,\alpha} u_{,\alpha} \, da + \left| \int_{\Omega_2(z)} u \, dl \right|^2 \right] d\tau \\ \leq &C_1 \int_0^t \left[ \int_{D(z)} u_{,\alpha} u_{,\alpha} \, da + M_1 \int_{\Omega_2(z)} u^2 \, dl \right] d\tau,\end{aligned} \tag{3.7}$$

where

$$M_1 = \big[\text{measure } (\Omega_2)\big]^{1/2},$$

and Greek sub-indices are restricted to two and three.

From (3.6) and (3.7), it follows that

$$\begin{aligned} |\Phi(z,t)| \leq & C_1^{1/2}\left[\int_0^t \int_{D(z)} u_{,\alpha}u_{,\alpha}\, da\, d\tau + M_1 \int_0^t \int_{\Omega_2(z)} u^2\, dl\, d\tau\right]^{1/2} \\ & \times \left[\int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,1}^2\, da\, d\tau\right]^{1/2}. \end{aligned} \tag{3.8}$$

But

$$\int_0^t \int_{\Omega_2(z)} |u|^2\, dl\, d\tau \leq M_2 \left(\int_0^t \int_{\Omega_2(z)} |u|^{2p}\, dl\, d\tau\right)^{1/p}, \tag{3.9}$$

where $p \geq 1$ and

$$M_2 = \big[t \text{ measure } (\Omega_2)\big]^{p/(p-1)}.$$

We obtain that

$$\begin{aligned} & \left(\int_0^t \int_{\Omega_2(z)} |u|^2\, dl\, d\tau\right)^{1/2} \\ & \leq \exp(\omega t)^{1/(2p)} M_2^{1/2} \left(\int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau)|u|^{2p}\, dl\, d\tau\right)^{1/(2p)} \\ & \leq M_3 \left(\int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau)\left[f_2(u)u + \omega S_1(u)\right] dl\, d\tau\right)^{1/(2p)}, \end{aligned} \tag{3.10}$$

where

$$M_3 = C^{-1/(2p)} \exp(\omega t)^{1/(2p)} M_2^{1/2}.$$

It then follows

$$\begin{aligned} \Phi(z,t) \leq & \Bigg[2M_4 \left(\int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,\alpha}u_{,\alpha}\, da\, d\tau\right)^{1/2} \\ & + M_5 \left(\int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau)\left[f_2(u)u + \omega S_1(u)\right]\, dl\, d\tau\right)^{1/(2p)}\Bigg] \\ & \times \left[\int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,1}^2\, da\, d\tau\right]^{1/2}, \end{aligned} \tag{3.11}$$

where

$$M_4 = \frac{1}{2} C_1^{1/2} \exp(\omega t), \quad M_5 = C_1^{1/2} M_1 M_3.$$

After some standard manipulations we arrive at (see [7], p. 128)

$$|\Phi(z,t)| \leq M_4 \left[-\frac{\partial\Phi}{\partial z}\right] + M_6 \left[-\frac{\partial\Phi}{\partial z}\right]^{(p+1)/(2p)}, \tag{3.12}$$

where

$$M_6 = p^{1/2}(p+1)^{-(p+1)/(2p)} M_5.$$

Consequences of the estimate (3.12) have been studied by Horgan etc (see [7, p134]). It can be proved that either there exists a positive constant $Q_1$ (see [7, p135]) such that

$$-\Phi(z,t) \geq \hat{C}_1 Q_1 \exp\left(\frac{z-z_0}{\hat{C}_1}\right), \quad z \geq z_0, \tag{3.13}$$

where

$$\hat{C}_1 = M_4 + M_6(2-\beta)\beta^{-1}\hat{\sigma}_2, \quad \beta = \frac{2p}{p+1}$$

and $\hat{\sigma}_2$ is an arbitrary positive constant, or the decay estimate (see [7, p136])

$$\begin{aligned} \Phi(z,t) \leq & \hat{C}_2 \left\{ \left[2\hat{C}_3(p+1)\right]^{-1}(p-1)\left[z+\hat{Q}(0)\right]\right\}^{-(p+1)/(p-1)} \\ & + \hat{C}_3 \left\{ \left[2\hat{C}_3(p+1)\right]^{-1}(p-1)\left[z+\hat{Q}(0)\right]\right\}^{-2(p+1)/(p-1)} \end{aligned} \tag{3.14}$$

holds, where

$$\hat{C}_2 = M_4(2-\beta)\hat{\sigma}_1^{-(\beta-1)/(2-\beta)} + M_6, \quad \hat{C}_3 = M_4(\beta-1)\hat{\sigma}_1$$

and $\hat{\sigma}_1$ is an arbitrary positive constant and

$$\begin{aligned} \hat{Q}(0) = & 2\hat{C}_3(p+1)\left\{\left[\Phi(0,t)\hat{C}_3^{-1} + \frac{\hat{C}_2^2}{4\hat{C}_3^2}\right]^{1/2} - \frac{\hat{C}_2}{2\hat{C}_3}\right\}^{-(p-1)/(p+1)} \\ & - \hat{C}_3(p+1)\left\{\left[\Phi(0,t)\hat{C}_3^{-1} + \frac{\hat{C}_2^2}{4\hat{C}_3^2}\right]^{1/2} - \frac{\hat{C}_2}{2\hat{C}_3}\right\}^{2/(p+1)}. \end{aligned}$$

We note that estimate (3.14) implies that (3.3) holds and then the function $\Phi(z,t)$ is determined by (3.4).

Our results can be summarize by means of the following theorem.

**Theorem 3.1.** *Let $u(\boldsymbol{x},t)$ be a solution of the problem determined by (2.1)–( 2.4) and (2.8). Then either the function*

$$\begin{aligned} & \int_0^t \int_0^z \int_D |\nabla u|^2 dx\, d\tau + \int_0^t \int_0^z \int_{\Omega_1} f_1(u)u\, da\, d\tau \\ & + \int_0^t \int_0^z \int_{\Omega_2} [f_2(u)u + \omega S_1(u)]\, da\, d\tau + \int_0^z \int_{\Omega_2} S_1(u)\, da \end{aligned}$$

*becomes unbounded in an exponential way when $z$ tends to infinite, or the function*

$$\begin{aligned} & \int_0^t \int_{R(z)} |\nabla u|^2 dx\, d\tau + \int_0^t \int_{\Omega_1^*(z)} f_1(u)u\, da\, d\tau \\ & + \int_0^t \int_{\Omega_2^*(z)} [f_2(u)u + \omega S_1(u)]\, da\, d\tau + \int_{\Omega_2^*(z)} S_1(u)\, da \end{aligned}$$

*decays at least as fast as $z^{-(p+1)/(p-1)}$ when $z$ tends to infinite.*

In the particular case that $p = 1$, we can improve the estimates. From the estimate (3.12), we see that

$$|\Phi(z,t)| \leq (M_4 + M_6)\left(-\frac{\partial \Phi}{\partial z}\right). \tag{3.15}$$

It is well known that this inequality implies an alternative of the type (see [3]):

The function $\Phi(z,t)$ satisfies the estimate

$$-\Phi(z,t) \geq Q_1^* \exp\left(\frac{z - z_0}{M_4 + M_6}\right),\ z \geq z_0, \tag{3.16}$$

where $Q_1^*$ is a positive constant, or the decay estimate

$$\Phi(z,t) \leq \Phi(0,t) \exp\left(-\frac{z}{M_4 + M_6}\right),\ z \geq 0 \tag{3.17}$$

is satisfied. We note that estimates (3.16) and (3.17) give an alternative of exponential type.

# 4. The amplitude term

To make clear the estimates obtained in the previous section, we require a bound for $\Phi(0,t)$ in terms of the boundary conditions at the end $x_3 = 0$. Otherwise, the decay estimate obtained at (3.14) would be impractical because the dependence of the amplitude on the data would not be explicit. To make calculations easier, we assume in this section that $\Omega_1 = \emptyset$ and that $f_2(u) = 0$. Furthermore, we impose that

$$mS_1(u) \geq |S_2(u)|^{p_1},\ m > 0,\ p_1 > 1, \tag{4.1}$$

where

$$S_2(u) = \int_0^u s(\eta)\, d\eta. \tag{4.2}$$

It is worth giving an example where condition (4.1) holds. The case when $s(u) = s_1|u|^k$, with $s_1 > 0$ and $k \geq 0$ always satisfies the required condition. In fact, in this case we have that

$$|S_2(u)| = \frac{s_1}{k+1}|u|^{k+1}.$$

If we take

$$p_1 = \frac{k+2}{k+1} > 1 \quad \text{and} \quad m = \frac{s_1^{p_1-1}(k+2)}{(k+1)^{p_1}} > 0,$$

we get

$$m\,\frac{s_1}{k+2}|u|^{k+2} \geq \left(\frac{s_1}{k+1}\right)^{p_1} |u|^{(k+1)p_1}$$

and condition (4.1) holds.

We note that

$$\begin{aligned}\Phi(0,t) = &\int_0^t \int_R \exp(-\omega\tau)|\nabla u|^2\, dx\, d\tau + \omega \int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau) S_1(u)\, da\, d\tau \\ &+ \exp(-\omega t) \int_{\Omega_2^*(0)} S_1(u)\, da = -\int_0^t \int_{D(0)} \exp(-\omega\tau) g u_{,1}\, da\, d\tau,\end{aligned} \tag{4.3}$$

where $g(x_2,x_3,t)$ was considered at (2.4).

We now define

$$h(x_1,x_2,x_3,s)=g(x_2,x_3,s)\exp(-bx_1), \tag{4.4}$$

where $b$ is an arbitrary positive constant. We have that

$$\begin{aligned}\Phi(0,t)&=\int_0^t\int_R \exp(-\omega\tau)h_{,i}u_{,i}\,dx\,d\tau+\int_0^t\int_{\Omega_2^*(0)}\exp(-\omega\tau)hS(u)u_\tau\,da\,d\tau\\&=\int_0^t\int_R \exp(-\omega\tau)h_{,i}u_{,i}\,dx\,d\tau+\omega\int_0^t\int_{\Omega_2^*(0)}\exp(-\omega\tau)hS_2(u)\,da\,d\tau\\&\quad-\int_0^t\int_{\Omega_2^*(0)}\exp(-\omega\tau)h_sS_2(u)\,da\,d\tau+\exp(-\omega t)\int_{\Omega_2^*(0)}hS_2(u)\,da.\end{aligned} \tag{4.5}$$

We see that

$$\begin{aligned}&\Phi(0,t)\\&\le\left[\int_0^t\int_R \exp(-\omega\tau)h_{,i}h_{,i}dx\,d\tau\right]^{1/2}\left[\int_0^t\int_R \exp(-\omega\tau)u_{,i}u_{,i}dx\,d\tau\right]^{1/2}\\&\quad+\omega\left[\int_0^t\int_{\Omega_2^*(0)}\exp(-\omega\tau)h^{q_1}\,da\,d\tau\right]^{1/q_1}\left[\int_0^t\int_{\Omega_2^*(0)}\exp(-\omega\tau)|S_2(u)|^{p_1}\,da\,d\tau\right]^{1/p_1}\\&\quad+\left[\int_0^t\int_{\Omega_2^*(0)}\exp(-\omega\tau)h_\tau^{q_1}\,da\,d\tau\right]^{1/q_1}\left[\int_0^t\int_{\Omega_2^*(0)}\exp(-\omega\tau)|S_2(u)|^{p_1}\,da\,d\tau\right]^{1/p_1}\\&\quad+\exp(-\omega t)\left[\int_{\Omega_2^*(0)}h^{q_1}da\right]^{1/q_1}\left[\int_{\Omega_2^*(0)}|S_2(u)|^{p_1}da\right]^{1/p_1},\end{aligned} \tag{4.6}$$

where $q_1^{-1}+p_1^{-1}=1$.

Using the arithmetic-geometric mean inequality and Young's inequality, we find that

$$\begin{aligned}\Phi(0,t)\le&\frac12\int_0^t\int_R \exp(-\omega\tau)u_{,i}u_{,i}\,dx\,d\tau+\frac12\int_0^t\int_R \exp(-\omega\tau)h_{,i}h_{,i}\,dx\,d\tau\\&+\frac12\int_0^t\int_{\Omega_1^*(0)}\exp(-\omega\tau)S_1(u)\,da\,d\tau\\&+\left(\frac{4m}{p_1}\right)^{q_1/p_1}\frac{\omega^{q_1}}{p_1}\int_0^t\int_{\Omega_1^*(0)}\exp(-\omega\tau)h^{q_1}\,da\,d\tau\\&+\left(\frac{4m}{p_1}\right)^{q_1/p_1}\frac{1}{p_1}\int_0^t\int_{\Omega_1^*(0)}\exp(-\omega\tau)h_\tau^{q_1}\,da\,d\tau\\&+\frac12\exp(-\omega_1 t)\int_{\Omega_1^*(0)}S_1(u)\,da\\&+\left(\frac{2m}{p_1}\right)^{q_1/p_1}\frac{1}{p_1}\int_{\Omega_1^*(0)}\exp(-\omega t)h^{q_1}\,da.\end{aligned} \tag{4.7}$$

We obtain that

$$
\begin{aligned}
\Phi(0,t) \leq & \int_0^t \int_R \exp(-\omega\tau) h_{,i} h_{,i} dx\, d\tau \\
& + 2\left(\frac{4m}{p_1}\right)^{q_1/p_1} \frac{\omega^{q_1}}{p_1} \int_0^t \int_{\Omega_1^*(0)} \exp(-\omega\tau) h^{q_1}\, da\, d\tau \\
& + 2\left(\frac{4m}{p_1}\right)^{q_1/p_1} \frac{1}{p_1} \int_0^t \int_{\Omega_1^*(0)} \exp(-\omega\tau) h_\tau^{q_1}\, da\, d\tau \\
& + 2\left(\frac{2m}{p_1}\right)^{q_1/p_1} \frac{1}{p_1} \exp(-\omega_1 t) \int_{\Omega_1^*(0)} h^{q_1}\, da.
\end{aligned} \tag{4.8}
$$

We have

$$
h_{,1}(x_1,x_2,x_3,\tau) = -bg(x_2,x_3,\tau)\exp(-bx_1), \tag{4.9}
$$

$$
h_{,\alpha}(x_1,x_2,x_3,\tau) = g_{,\alpha}(x_2,x_3,\tau)\exp(-bx_1), \tag{4.10}
$$

$$
h_\tau(x_1,x_2,x_3,\tau) = g_\tau(x_2,x_3,\tau)\exp(-bx_1). \tag{4.11}
$$

Therefore,

$$
\int_0^t \int_R \exp(-\omega\tau) h_{,i} h_{,i}\, dx\, d\tau = \int_0^t \int_{D(0)} \exp(-\omega\tau) \left(\frac{g_{,\alpha} g_{,\alpha}}{2b} + \frac{b}{2} g^2\right) da\, d\tau, \tag{4.12}
$$

$$
\int_0^t \int_{\Omega_1^*(0)} \exp(-\omega\tau) h^{q_1}\, da\, d\tau = \int_0^t \int_{\partial D(0)} \exp(-\omega\tau) \frac{g^{q_1}}{q_1 b}\, dl\, d\tau, \tag{4.13}
$$

$$
\int_0^t \int_{\Omega_1^*(0)} \exp(-\omega\tau) h_\tau^{q_1}\, da\, d\tau = \int_0^t \int_{\partial D(0)} \exp(-\omega\tau) \frac{g_\tau^{q_1}}{q_1 b}\, dl\, d\tau \tag{4.14}
$$

and

$$
\int_{\Omega_1^*(0)} h^{q_1}\, da = \int_{\partial D(0)} \frac{g^{q_1}}{q_1 b}\, dl. \tag{4.15}
$$

We then obtain

$$
\begin{aligned}
\Phi(0,t) \leq & \int_0^t \int_{D(0)} \exp(-\omega\tau) \left(\frac{g_{,\alpha} g_{,\alpha}}{2b} + \frac{b}{2} g^2\right) da\, d\tau \\
& + 2\left(\frac{4m}{p_1}\right)^{q_1/p_1} \frac{\omega^{q_1}}{p_1 q_1 b} \int_0^t \int_{\partial D(0)} \exp(-\omega\tau) |g|^{q_1}\, dl\, d\tau \\
& + 2\left(\frac{4m}{p_1}\right)^{q_1/p_1} \frac{1}{p_1 q_1 b} \int_0^t \int_{\partial D(0)} \exp(-\omega\tau) |g_{,\tau}|^{q_1}\, dl\, d\tau \\
& + 2\left(\frac{2m}{p_1}\right)^{q_1/p_1} \frac{1}{p_1 q_1 b} \exp(-\omega t) \int_{\partial D(0)} |g|^{q_1}\, dl.
\end{aligned} \tag{4.16}
$$

We can optimize the right hand side of (4.16) with respect to $b$, but it does seem an easy task.

# 5. Extension to the system

In this section we sketch how to extend the Phragmén-Lindelöf alternative to the case of a system of equations. In place of the equation (2.1) we will consider the system

$$\begin{aligned} k_{11}\Delta u_1 + k_{12}\Delta u_2 - \alpha(u_1 - u_2) = 0 \\ k_{21}\Delta u_1 + k_{22}\Delta u_2 - \alpha(u_1 - u_2) = 0 \end{aligned} \tag{5.1}$$

where we asume that the matrix

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \tag{5.2}$$

is positive definite and that $\alpha$ is a positive constant.

It is worth recalling that this system determines the temperatures in a mixture of isotropic and homogeneous heat conducting materials (see [5,10,13]). To define the boundary conditions, we assume that

$$u_1 - u_2 = 0 \ \text{ on } \ (0,\infty) \times \partial D \times (0,t) \tag{5.3}$$

together with

$$q_i n_i + f_1^*(u_1,u_2) = 0 \ \text{ on } \ [0,\infty) \times \Omega_1 \times (0,t) \tag{5.4}$$

and on $\Omega_2$ we assume that

$$q_i n_i + m_1(u_1,u_2)u_{1,t} + m_2(u_1,u_2)u_{2,t} + f_2^*(u_1,u_2) = 0 \ \text{ on } \ [0,\infty)\times\Omega_2\times(0,t), \tag{5.5}$$

where

$$q_i = q_i^{(1)} + q_i^{(2)}, \quad q_i^{(1)} = k_{11}u_{1,i} + k_{12}u_{2,i}, \quad q_i^{(2)} = k_{21}u_{1,i} + k_{22}u_{2,i} \tag{5.6}$$

and

$$f_1^*(u,u)u \geq 0, \quad f_2^*(u,u)u + \omega M_1(u) \geq C|u|^{2p}, \tag{5.7}$$

with

$$M_1 = \int_0^u \eta\big[m_1(\eta,\eta) + m_2(\eta,\eta)\big]\,d\eta \geq 0$$

and $C$ a positive constant. In this situation, the analysis starts by considering the function

$$\begin{aligned} \Phi(z,t) = -\int_0^t \int_{D(z)} & \exp(-\omega\tau)\big[\,(k_{11}u_{1,1} + k_{12}u_{2,1})\,u_1 \\ & + (k_{21}u_{1,2} + k_{22}u_{2,2})\,u_2\big]\,da\,d\tau. \end{aligned} \tag{5.8}$$

We note that, for $z \geq z_0$,

$$\begin{aligned} \Phi(z,t) - \Phi(z_0,t) = & -\int_0^t\int_{z_0}^z\int_D \exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2 + (k_{12}+k_{21})\nabla u_1\nabla u_2 \\ & + k_{22}|\nabla u_2|^2 + \alpha(u_1-u_2)^2\big]\,dx\,d\tau \\ & - \int_0^t\int_{z_0}^z\int_{\Omega_1} \exp(-\omega\tau)f_1^*(u,u)u\,da\,d\tau \\ & - \int_0^t\int_{z_0}^z\int_{\Omega_2} \exp(-\omega\tau)\,[f_2^*(u,u)u + \omega M_1(u)]\,da\,d\tau \\ & - \int_{z_0}^z\int_{\Omega_2} \exp(-\omega t)M_1(u)\,da. \end{aligned} \tag{5.9}$$

It is worth noting that

$$\begin{aligned}\frac{\partial\Phi}{\partial z} = &-\int_0^t\int_{D(z)}\exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2+(k_{12}+k_{21})\nabla u_1\nabla u_2\\ &+k_{22}|\nabla u_2|^2+\alpha(u_1-u_2)^2\big]\,da\,d\tau\\ &-\int_0^t\int_{\Omega_1(z)}\exp(-\omega\tau)f_1^*(u,u)u\,dl\,d\tau\\ &-\int_0^t\int_{\Omega_2(z)}\exp(-\omega\tau)\big[f_2^*(u,u)u+\omega M_1(u)\big]\,dl\,d\tau\\ &-\int_{\Omega_2(z)}\exp(-\omega t)M_1(u)\,dl.\end{aligned} \tag{5.10}$$

An analysis similar to the one proposed at Section 3 allows us to obtain the inequality

$$|\Phi(z,t)|\leq M_4^*\left[-\frac{\partial\Phi}{\partial z}\right]+M_6^*\left[-\frac{\partial\Phi}{\partial z}\right]^{(p+1)/(2p)}, \tag{5.11}$$

where $M_4^*$ and $M_6^*$ are calculable positive constants depending on the parameters of the problem and the time. This estimate allows us to get an alternative of the type (3.13) and (3.14). Therefore, we can obtain a similar result to the Theorem 3.1. To be precise, we can prove that either the function

$$\begin{aligned}&\int_0^t\int_{R(z)}\exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2+(k_{12}+k_{21})\nabla u_1\nabla u_2\\ &+k_{22}|\nabla u_2|^2+\alpha(u_1-u_2)^2\big]\,dx\,d\tau+\int_0^t\int_{\Omega_1^*(z)}\exp(-\omega\tau)f_1^*(u,u)u\,da\,d\tau\\ &+\int_0^t\int_{\Omega_2^*(z)}\exp(-\omega\tau)\,[f_2^*(u,u)u+\omega M_1(u)]\,da\,d\tau+\int_{\Omega_2^*(z)}\exp(-\omega t)M_1(u)\,da\end{aligned}$$

decays as a polynomial, or the function

$$\begin{aligned}&\int_0^t\int_{z_0}^z\int_D\exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2+(k_{12}+k_{21})\nabla u_1\nabla u_2\\ &+k_{22}|\nabla u_2|^2+\alpha(u_1-u_2)^2\big]\,dx\,d\tau+\int_0^t\int_{z_0}^z\int_{\Omega_1}\exp(-\omega\tau)f_1^*(u,u)u\,da\,d\tau\\ &+\int_0^t\int_{z_0}^z\int_{\Omega_2}\exp(-\omega\tau)\,[f_2^*(u,u)u+\omega M_1(u)]\,da\,d\tau+\int_{z_0}^z\int_{\Omega_2}\exp(-\omega t)M_1(u)\,da\end{aligned}$$

increases in an exponential way.

Estimates for the amplitude term can be obtained in a similar way as in Section 4.

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