Wildness of the problems of classifying two-dimensional spaces of commuting linear operators and certain Lie algebras

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Abstract

For each two-dimensional vector space V of commuting $n\times n$ matrices over a field $\mathbb{F}$ with at least 3 elements, we denote by $\widetilde{V}$ the vector space of all $(n+1)\times(n+1)$ matrices of the form $\left[\begin{smallmatrix}A&*\\0&0\end{smallmatrix}\right]$ with $A\in V$. We prove the wildness of the problem of classifying Lie algebras $\widetilde{V}$ with the bracket operation $[u,v]:=uv-vu$. We also prove the wildness of the problem of classifying two-dimensional vector spaces consisting of commuting linear operators on a vector space over a field.

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1. Introduction

Let $\mathbb{F}$ be a field that is not the field with 2 elements. We prove the wildness of the problems of classifying

- two-dimensional vector spaces consisting of commuting linear operators on a vector space over $\mathbb{F}$ (see Section 2), and

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- Lie algebras $\mathrm{L}(V)$ with bracket $[u,v] := uv - vu$ of matrices of the form

$$\begin{bmatrix} & & & \alpha_1 \\ & A & & \vdots \\ & & & \alpha_n \\ 0 & \dots & 0 & 0 \end{bmatrix}, \qquad \text{in which } A \in V, \quad \alpha_1, \dots, \alpha_n \in \mathbb{F}, \tag{1}$$

  in which V is any two-dimensional vector space of $n \times n$ commuting matrices over $\mathbb{F}$ (see Section 3).

A classification problem is called *wild* if it contains the problem of classifying pairs of $n \times n$ matrices up to similarity transformations

$$(M, N) \mapsto S^{-1}(M, N)S := (S^{-1}MS, S^{-1}NS)$$

with nonsingular S. This notion was introduced by Donovan and Freislich [8, 9]. Each wild problem is considered as hopeless since it contains the problem of classifying an arbitrary system of linear mappings, that is, representations of an arbitrary quiver (see [13, 5]).

Let $\mathcal{U}$ be an n-dimensional vector space over $\mathbb{F}$. The problem of classifying linear operators $\mathcal{A} : \mathcal{U} \to \mathcal{U}$ is the problem of classifying matrices $A \in \mathbb{F}^{n\times n}$ up to similarity transformations $A \mapsto S^{-1}AS$ with nonsingular $S \in \mathbb{F}^{n\times n}$. In the same way, the problem of classifying vector spaces $\mathcal{V}$ of linear operators on $\mathcal{U}$ is the problem of classifying matrix vector spaces $V \subset \mathbb{F}^{n\times n}$ up to similarity transformations

$$V \mapsto S^{-1}VS := \{S^{-1}AS \mid A \in V\} \tag{2}$$

with nonsingular $S \in \mathbb{F}^{n\times n}$ (the spaces V and $S^{-1}VS$ are *matrix isomorphic*; see [14]). In Theorem 1(a), we prove the wildness of the problem of classifying two-dimensional vector spaces $V \subset \mathbb{F}^{n\times n}$ of commuting matrices up to transformations (2).

Each two-dimensional vector space $V \subset \mathbb{F}^{n\times n}$ is given by its basis $A, B \in V$ that is determined up to transformations $(A, B) \mapsto (\alpha A + \beta B, \gamma A + \delta B)$, in which $\left[\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right] \in \mathbb{F}^{2\times 2}$ is a change-of-basis matrix. Thus, the problem of classifying two-dimensional vector spaces $V \subset \mathbb{F}^{n\times n}$ up to transformations (2) is the problem of classifying pairs of linear independent matrices $A, B \in \mathbb{F}^{n\times n}$ up to transformations

$$(A, B) \mapsto (A', B') := S^{-1}(\alpha A + \beta B, \gamma A + \delta B)S, \tag{3}$$

in which both $S \in \mathbb{F}^{n\times n}$ and $\left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right] \in \mathbb{F}^{2\times 2}$ are nonsingular matrices. We say that the matrix pairs (A,B) and (A',B') from (3) are *weakly similar*.

In Theorem 1(b), we prove that the problem of classifying pairs of commuting matrices up to weak similarity is wild, which ensures Theorem 1(a).

The analogous problem of classifying matrix pairs (A,B) up to weak congruence $S^T(\alpha A+\beta B, \gamma A+\delta B)S$ appears in the problem of classifying finite p-groups of nilpotency class 2 with commutator subgroup of type (p,p), in the problem of classifying commutative associative algebras with zero cube radical, and in the problem of classifying Lie algebras with central commutator subalgebra; see [3, 4, 6, 18]. The problem of classifying matrix pairs up to weak equivalence $R(\alpha A+\beta B, \gamma A+\delta B)S$ appears in the theory of tensors [2].

Note that the group of $(n+1)\times(n+1)$ matrices

$$\begin{bmatrix} A & v\\ 0 & 1\end{bmatrix}, \qquad \text{in which } A \in \mathbb{F}^{n\times n} \text{ is nonsingular and } v \in \mathbb{F}^n$$

is called the *general affine group*; it is the group of all invertible affine transformations of an affine space; see [15]. If $\mathbb{F} = \mathbb{R}$, then this group is a Lie group, its Lie algebra consists of all $(n+1)\times(n+1)$ matrices

$$\begin{bmatrix} A & v\\ 0 & 0\end{bmatrix}, \qquad \text{in which } A \in \mathbb{R}^{n\times n} \text{ is nonsingular and } v \in \mathbb{R}^n,$$

and each Lie algebra $\mathrm{L}(V)$ of matrices of the form (1) with $\mathbb{F} = \mathbb{R}$ is its subalgebra.

The abstract version of the construction of Lie algebras $\mathrm{L}(V)$ of matrices of the form (1) is the following. Let $\mathbb{F}[x,y]$ be the polynomial ring, and let $\mathcal{W}_{\mathbb{F}[x,y]}$ be a left $\mathbb{F}[x,y]$-module given by a finite dimensional vector space $\mathcal{W}_{\mathbb{F}}$ and two commuting linear operators $P : w \mapsto xw$ and $Q : w \mapsto yw$ on $\mathcal{W}_{\mathbb{F}}$ that are linearly independent. The $(2+\dim_{\mathbb{F}}\mathcal{W})$-dimensional vector space $L_{\mathcal{W}} := \mathbb{F}x \oplus_{\mathbb{F}} \mathbb{F}y \oplus_{\mathbb{F}} \mathcal{W}$ is the metabelian Lie algebra with the bracket operation defined by $[x,v] := Pv$, $[y,v] := Qv$, and $[x,y] = [v,w] := 0$ for all $v,w \in \mathcal{W}$. If $\mathcal{W} = \mathbb{F}^n$ and V is the two-dimensional vector space generated by P and Q, then the Lie algebra $L_{\mathcal{W}}$ coincides with the Lie algebra $\mathrm{L}(V)$ of all matrices (1). By [16, Corollary 1] and Theorem 1, *the problem of classifying metabelian Lie algebras $L_{\mathcal{W}}$ is wild.*

We use the following definition of wild problems (see more formal definitions in [1, 10, 11]). Every matrix problem $\mathcal{M}$ is given by a set $\mathcal{M}_1$ of tuples of matrices over a field $\mathbb{F}$ and a set $\mathcal{M}_2$ of admissible transformations with them. A matrix problem $\mathcal{M}$ is *wild* if there exists a t-tuple

$$M(x, y) = (M_1(x, y), \dots, M_t(x, y)) \tag{4}$$

of matrices, whose entries are noncommutative polynomials in x and y over $\mathbb{F}$, such that

(i) $M(A, B) \in \mathcal{M}_1$ for all $A, B \in \mathbb{F}^{n\times n}$ and $n = 1, 2, \dots$ (in particular, each scalar entry α of $M_i(x, y)$ is replaced by αI_n),

(ii) $M(A, B)$ is reduced to $M(A', B')$ by transformations $\mathcal{M}_2$ if and only if (A, B) is similar to (A', B').

2. Spaces of linear operators

Theorem 1. (a) *The problem of classifying up to similarity* (2) *of two-dimensional vector spaces of commuting matrices over a field* $\mathbb{F}$ *is wild. If* $\mathbb{F}$ *is not the field of two elements, then the problem of classifying up to similarity of two-dimensional vector spaces of commuting matrices over* $\mathbb{F}$ *that contain nonsingular matrices is wild.*

(b) *The problem of classifying up to weak similarity* (3) *of pairs of commuting matrices over a field* $\mathbb{F}$ *is wild. If* $\mathbb{F}$ *is not the field of two elements, then the problem of classifying up to weak similarity of pairs* (A, B) *of commuting matrices over* $\mathbb{F}$ *such that* $\alpha A+\beta B$ *is nonsingular for some* $\alpha, \beta \in \mathbb{F}$ *is wild.*

Proof. (a) This statement follows from statement (b) since each two-dimensional vector space $V \subset \mathbb{F}^{n\times n}$ determined up to similarity is given by its basis $A, B \in V$ that is determined up to transformations (3).

(b) *Step 1: We prove that the problem of classifying pairs of commuting and nilpotent matrices up to similarity is wild.* This statement was proved by Gelfand and Ponomarev [13]; it was extended in [7] to matrix pairs under consimilarity. By analogy with [7, Section 3], we consider two commuting and nilpotent $5n \times 5n$ matrices

$$J := \left[\begin{array}{cccc|c} 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad K_{XY} := \left[\begin{array}{cccc|c} 0 & 0 & X & 0 & Y \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_n & 0 \end{array}\right] \tag{5}$$

that are partitioned into $n \times n$ blocks, in which $X, Y \in \mathbb{F}^{n\times n}$ are arbitrary.

Let us prove that

$$\begin{array}{l}\text{two pairs } (X,Y) \text{ and } (X',Y') \text{ of } n\times n \text{ matrices are similar} \\ \Longleftrightarrow \text{ two pairs of commuting and nilpotent matrices } (J, K_{XY}) \\ \text{and } (J, K_{X'Y'}) \text{ are similar.}\end{array} \tag{6}$$

$\Longrightarrow$. If $(X,Y)S = S(X',Y')$, then $(J,K_{XY})R = R(J,K_{X'Y'})$ with $R := S \oplus S \oplus S \oplus S \oplus S$.

$\Longleftarrow$. Let $(J,K_{XY})R = R(J,K_{X'Y'})$ with nonsingular R. All matrices commuting with a given Jordan matrix are described in [12, Section VIII, § 2]. Since R commutes with J, we analogously find that

$$R = \left[\begin{array}{cccc|c} C & C_1 & C_2 & C_3 & D \\ 0 & C & C_1 & C_2 & 0 \\ 0 & 0 & C & C_1 & 0 \\ 0 & 0 & 0 & C & 0 \\ \hline 0 & 0 & 0 & E & F \end{array}\right].$$

The equality $K_{XY}R = RK_{X'Y'}$ implies that

$$\left[\begin{array}{cccc|c} 0 & 0 & XC & XC_1 + YE & YF \\ 0 & 0 & 0 & XC & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & CX' & C_1X' + D & CY' \\ 0 & 0 & 0 & CX' & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & F & 0 \end{array}\right],$$

and so $(X,Y)C = C(X',Y')$.

Step 2: We prove that the problem of classifying matrix pairs up to weak similarity is wild. If the field $\mathbb{F}$ has at least 3 elements, we fix any $\lambda \in \mathbb{F}$ such that $\lambda \neq 0$ and $\lambda \neq -1$. If $\mathbb{F}$ consists of two elements, we take $\lambda = 1$.

For each pair (A,B) of $m \times m$ matrices with $m \geqslant 1$ over $\mathbb{F}$, define the matrix pair $(M_1(A), M_2(B))$ as follows:

$$\begin{aligned} M_1(A) &:= I_{2m+2} \oplus 0_{3m+3} \oplus I_{m+1} \oplus A, \\ M_2(B) &:= 0_{2m+2} \oplus I_{3m+3} \oplus \lambda I_{m+1} \oplus B. \end{aligned}$$

(Analogous constructions are used in [3, 4].)

Let us prove that $(M_1(A), M_2(B))$ can be used in (4) in order to prove the wildness of the problem of classifying matrix pairs up to weak similarity. We should prove that

> arbitrary pairs (A, B) and (A', B') of $m\times m$ matrices are similar $\iff$ $(M_1(A), M_2(B))$ and $(M_1(A'), M_2(B'))$ are weakly similar. (7)

$\Longrightarrow$. If $S^{-1}(A, B)S = (A', B')$, then

$$(I_{6m+6} \oplus S)^{-1}(M_1(A), M_2(B))(I_{6m+6} \oplus S) = (M_1(A'), M_2(B')).$$

$\Longleftarrow$. Let

$$S^{-1}(\alpha M_1(A) + \beta M_2(B), \gamma M_1(A) + \delta M_2(B))S = (M_1(A'), M_2(B'))$$

with a nonsingular $\left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right]$. Then

$$\begin{aligned}\operatorname{rank}(\alpha M_1(A) + \beta M_2(B)) &= \operatorname{rank} M_1(A'),\\ \operatorname{rank}(\gamma M_1(A) + \delta M_2(B)) &= \operatorname{rank} M_2(B').\end{aligned}$$

If $\beta \neq 0$, then

$$\operatorname{rank}(\alpha M_1(A) + \beta M_2(B)) > 4m + 3 \geqslant \operatorname{rank} M_1(A').$$

Hence $\beta = 0$. Since $\left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right]$ is nonsingular, $\delta \neq 0$. If $\gamma \neq 0$, then

$$\operatorname{rank}(\gamma M_1(A) + \delta M_2(B)) > 5m + 4 \geqslant \operatorname{rank} M_2(B').$$

Hence $\gamma = 0$.

Thus

$$S^{-1}(\alpha M_1(A), \delta M_2(B))S = (M_1(A'), M_2(B')),$$

and so the pairs

$$\begin{aligned}(\alpha M_1(A), \delta M_2(B)) &= (\alpha I_{2m+2}, 0_{2m+2}) \oplus (0_{3m+3}, \delta I_{3m+3})\\ &\qquad \oplus (\alpha I_{m+1}, \delta\lambda I_{m+1}) \oplus (\alpha A, \delta B),\\ (M_1(A'), M_2(B')) &= (I_{2m+2}, 0_{2m+2}) \oplus (0_{3m+3}, I_{3m+3})\\ &\qquad \oplus (I_{m+1}, \lambda I_{m+1}) \oplus (A', B')\end{aligned} \tag{8}$$

give isomorphic representations of the quiver $\circlearrowleft\!\bullet\!\circlearrowright$. By the Krull–Schmidt theorem for quiver representations (see [17, Theorem 1.2]), every representation of a quiver is isomorphic to a direct sum of indecomposable representations, and this sum is uniquely determined up to replacements of direct summands by isomorphic representations and permutations of direct summands.

If we delete in (8) the summands $(\alpha I_{2m+2}, 0_{2m+2})$ and $(0_{3m+3}, \delta I_{3m+3})$ of $(\alpha M_1(A), \delta M_2(B))$ and the corresponding isomorphic summands $(I_{2m+2}, 0_{2m+2})$ and $(0_{3m+3}, I_{3m+3})$ of $(M_1(A'), M_2(B'))$, we find that the remaining pairs

$$(\alpha I_{m+1}, \delta\lambda I_{m+1}) \oplus (\alpha A, \delta B), \qquad (I_{m+1}, \lambda I_{m+1}) \oplus (A', B')$$

give isomorphic representations of the quiver $\circlearrowleft\!\bullet\!\circlearrowright$. The first pair has $m+1$ direct summands $(\alpha, \delta\lambda)$ and the second pair has $m+1$ direct summands $(1, \lambda)$. By the Krull–Schmidt theorem, these summands give isomorphic representations, hence $\alpha = \delta = 1$, and so the pairs (A, B) and (A', B') give isomorphic representations too. Therefore, the pairs (A, B) and (A', B') are similar.

Step 3. By Steps 1 and 2, the following equivalences hold for arbitrary pairs (X, Y) and (X', Y') of $n \times n$ matrices over $\mathbb{F}$:

(X, Y) and (X', Y') are similar

$\Longleftrightarrow$ (J, K_{XY}) and $(J, K_{X'Y'})$ are similar

$\Longleftrightarrow$ $(\lambda I + J, K_{XY})$ and $(\lambda I + J, K_{X'Y'})$ are similar

$\Longleftrightarrow$ $(M_1(\lambda I + J), M_2(K_{XY}))$ and $(M_1(\lambda I + J), M_2(K_{X'Y'}))$ are weakly similar.

Note that $(M_1(\lambda I + J), M_2(K_{XY}))$ and $(M_1(\lambda I + J), M_2(K_{X'Y'}))$ are pairs of commuting matrices. If $\mathbb{F}$ has at least 3 elements, then the matrix $M_1(\lambda I + J) + M_2(K_{XY})$ is nonsingular. $\square$

3. Lie algebras

For each vector space $V \subset \mathbb{F}^{n\times n}$ of commuting matrices over a field $\mathbb{F}$, we denote by $\widetilde{V}$ the vector space of all $(n+1)\times(n+1)$ matrices of the form

$$(A|a) := \begin{bmatrix} & & & \alpha_1 \\ & A & & \vdots \\ & & & \alpha_n \\ 0 & \dots & 0 & 0 \end{bmatrix}, \qquad \text{in which } A \in V \text{ and } a := \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{F}^n.$$

We consider the space $\widetilde{V}$ as the Lie algebra $\mathrm{L}(V)$ with the Lie bracket operation

$$[(A|a), (B|b)] := (A|a)(B|b) - (B|b)(A|a) = (0|Ab - Ba). \tag{9}$$

Theorem 2. *Let a field $\mathbb{F}$ be not the field with 2 elements.*

(a) *Let $V \subset \mathbb{F}^{n\times n}$ and $V' \subset \mathbb{F}^{n'\times n'}$ be two vector spaces of commuting matrices that contain nonsingular matrices. Then the following statements are equivalent:*

   (i) *The Lie algebras $\mathrm{L}(V)$ and $\mathrm{L}(V')$ are isomorphic.*

   (ii) *$n = n'$ and V is similar to V' (i.e., $SVS^{-1} = V'$ for some nonsingular $S \in \mathbb{F}^{n\times n}$),*

   (iii) *$n = n'$ and $\widetilde{V}$ is similar to $\widetilde{V}'$.*

(b) *The problem of classifying Lie algebras $\mathrm{L}(V)$ with $\dim_{\mathbb{F}} V = 2$ up to isomorphism is wild.*

Proof. (a) Let us prove the equivalence of (i)–(iii).

(i)⇒(ii) Let $\varphi : \mathrm{L}(V) \xrightarrow{\sim} \mathrm{L}(V')$ be an isomorphism of Lie algebras. Then $\varphi[\widetilde{V}, \widetilde{V}] = [\widetilde{V}', \widetilde{V}']$. By (9), $[\widetilde{V}, \widetilde{V}] \subset (0|\mathbb{F}^n)$. Since V contains a nonsingular matrix A, $[(A|0), (0|\mathbb{F}^n)] = (0|\mathbb{F}^n)$, and so $[\widetilde{V}, \widetilde{V}] = (0|\mathbb{F}^n)$. Hence $\varphi(0|\mathbb{F}^n) = (0|\mathbb{F}^{n'})$ and $n = n'$.

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{F}^n$, and let $(0|f_i) := \varphi(0|e_i)$. Since $\varphi(0|\mathbb{F}^n) = (0|\mathbb{F}^n)$, $f_1, \dots, f_n$ is also a basis of $\mathbb{F}^n$. Denote by S the nonsingular matrix whose columns are $f_1, \dots, f_n$. Then

$$f_i = Se_i. \tag{10}$$

Let $A \in V$ and write $(B|b) := \varphi(A|0)$. Let $A = [\alpha_{ij}]_{i,j=1}^{n}$, i.e., $Ae_j = \sum_i \alpha_{ij} e_i$. Then

$$\begin{aligned}(0|Bf_j) &= [(B|b),(0|f_j)] = [\varphi(A|0),\varphi(0|e_j)] = \varphi[(A|0),(0|e_j)]\\ &= \varphi(0|Ae_j) = \varphi(0,\textstyle\sum_i \alpha_{ij} e_i) = \varphi\textstyle\sum_i \alpha_{ij}(0|e_i)\\ &= \textstyle\sum_i \alpha_{ij}\varphi(0|e_i) = \textstyle\sum_i \alpha_{ij}(0|f_i) = (0|\textstyle\sum_i \alpha_{ij} f_i)\end{aligned}$$

and so $Bf_j = \sum_i \alpha_{ij} f_i$. By (10),

$$BSe_j = \textstyle\sum_i \alpha_{ij} Se_i = S\textstyle\sum_i \alpha_{ij} e_i = SAe_j.$$

Therefore, $BS = SA$ and so $V'S = SV$.

(ii)⇒(iii) If V and V' are similar via S, then $\widetilde{V}$ and $\widetilde{V}'$ are similar via $S \oplus I_1$.

(iii)⇒(i) If $R\widetilde{V}R^{-1} = \widetilde{V}'$ for some nonsingular $R \in \mathbb{F}^{(n+1)\times(n+1)}$, then $X \mapsto RXR^{-1}$ is an isomorphism $\mathrm{L}(V)\tilde{\to}\mathrm{L}(V')$.

(b) This statement follows from the equivalence (i)⇔(ii) and Theorem 1(a). □

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