Prospective teachers' interpretative knowledge: giving sense to subtraction algorithms

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The process of interpretation and assessment of students' mathematical productions represents a crucial aspect of teachers' practices. In such processes, teachers rely on the so-called interpretative knowledge, which includes particular aspects of their mathematical and pedagogical knowledge, their view of mathematics, and their values. In this paper, we analyze and discuss prospective primary teachers' interpretative knowledge gained through their assessment of different subtraction algorithms.

Keywords: prospective teachers' interpretative knowledge; prospective teachers' beliefs; subtraction algorithms.

INTRODUCTION

In practice, teachers are required to continuously interpret students' mathematical behaviors, speech, and productions. In this process of interpretation, teachers introduce some *affective* aspects, including their beliefs, their values (for example, in the evaluation of the seriousness of an error), and their expectations (Liljedahl and Oesterle, 2014). Closely linked with the affective aspects, teachers' knowledge is also exhibited through interpretation and evaluation processes. In previous studies, this type of knowledge was referred to as *interpretative knowledge* (e.g., Ribeiro, Mellone, & Jakobsen, 2013; Ribeiro, Mellone & Jakobsen, to appear) and has emerged as a potentially significant construct both for researchers in mathematics education and for teacher educators.

Indeed, through the observation and discussion of teachers' interpretations of students' mathematical productions (and comments), researchers can gain insight into teachers' mathematical and pedagogical knowledge, beliefs, values, and expectations. On other hand, it allows teacher educators to develop significant discussions and mathematical knowledge with prospective teachers by highlighting the potential for mathematical exploration through students' productions, especially those containing errors or proposing non-standard solutions (Borasi, 1996).

Within this framework, we have developed a wider project aiming to access mathematical teachers' knowledge, beliefs, values, and expectations implicit in these processes of interpretation—during the initial as well as continuous education. Imbedded in such project, a particular kind of tasks has been conceptualized and implemented. One of the core aspects of the nature of such a task is rooted in asking (prospective) teachers to give sense to pupils' productions (some of which can be considered incomplete, containing errors, or simply based on non-standard reasoning) in response to a posed problem, as well as provide them with constructive feedback (e.g., Ribeiro et al., 2013). The work we have conducted to date has mainly focused on mathematical teachers' knowledge. Here, on the other hand, also teachers' beliefs are explored, in order to broaden our understanding of the nature and factors that influence prospective teachers' reasoning and argumentation when giving meaning to students' productions. In particular our analysis shows how some prospective teachers' beliefs about mathematics, together with their lack of knowledge about the mathematical proprieties at the roots of algorithm, prevent them to appreciate the correctness of an algorithm different from the "traditional one".

The study of the arithmetical operations and the relative algorithms is one core aspect of most primary school curricula around the world (e.g., NCTM, 2000). Nevertheless, the approach, the focus, and the algorithms related to the whole number arithmetic, in some cases, differ from one country to another. Such diversity of algorithms and of the mathematical rationality sustaining them can be perceived as a source for deepening teachers' beliefs and understanding of not only the algorithms, but also the whole number arithmetic in general. Indeed, if from one side we agree with Bass (2015) when he mention that "*A numerical computation, of a say a sum of two numbers, is not about understanding what the sum means. Instead, give two numbers A and B in notation system S, a calculation is a construction of a representation of A+B in same notation system S.*" (p. 11). On other side, we consider that the navigation among (between) different algorithms of one same operation can enhance the opportunity to unpack both the different meanings of the operation as well as the features of the notation system of representation. This was one of the reasons that motivated us to conduct inquiry into the subtraction algorithm(s). Indeed, the knowledge and awareness of the mathematical aspects (such as the properties of the arithmetical operations or the decimal positional representation of numbers) involved in arithmetic operations, as well as the relative algorithms, are perceived as a crucial aspect of (primary) mathematic teachers' knowledge.

THEORETICAL FRAMEWORK

In the last decades, the research in mathematics education has emphasized the need to consider affect in the interpretation of the teaching/learning process of the mathematics. In particular, Thompson (1992) underlines the role of teachers' beliefs in classroom practices: beliefs that Philipp (2007, p. 258) defines as "the lenses through which one looks when interpreting the world". In this context, Grootenboer (2008, p. 479) refers explicitly to "the pervasive influence of beliefs on teaching practice" and scholars debate about how to recognize their central role also in programs devoted to mathematics teachers' development.

In their overview of the literature, Liljedahl and Oesterle (2014) underline as on the one hand beliefs are organized in systems, on the other hand the different types of beliefs systems that may affect teaching: beliefs about mathematics, beliefs about the teaching of mathematics, beliefs about the learning of mathematics, beliefs about students, beliefs about teachers' own ability to do mathematics, to teach mathematics, etc.

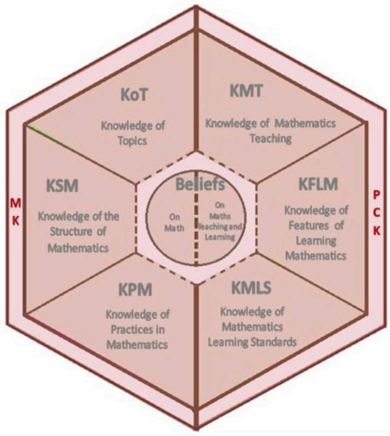

Figure 1. MTSK conceptualization.

Complementary to the role of beliefs and values in teachers' practices and mathematical understanding (as well as knowledge development), interpretative knowledge is perceived as one core element of the content of teachers' knowledge. Such interpretative knowledge is deemed to support teachers in giving sense to students' productions, always perceiving such productions as learning opportunities, even when they are non-standard or contain errors (e.g., Ribeiro et al., 2013). Such knowledge would allow teachers to develop and implement ways to lead students in building knowledge, starting from their own reasoning, even when it differs from that expected by the teacher.

The development of pupils' mathematical knowledge starting from their own reasoning, in our view, is possible only if the teacher activates a real process of interpretation, shifting from an *evaluative listening* and to a more *flexible hermeneutic* listening activity (Davis, 1997). In particular, in our framework the teacher's evaluative listening is conceived as process trough which the teacher sees if there is a fitting between pupils' productions and the mathematical scheme of correct answers he/she has. While a real interpretation process, linked also with Davis's notion of *hermeneutic listening* (1997), is linked to teacher's flexible attempt of redrawing a mathematical learning path that embodies pupils' productions. This vision makes our notion of interpretative knowledge different from other mathematical teachers' knowledge conceptualization, in the sense that errors/non standard reasoning are not conceived as something to avoid. Rather this framework puts errors/non standard reasoning at the core of mathematical teachers' knowledge as source to capitalize that really shapes the dynamics in mathematics educational process (Borasi, 1996).

Aimed at framing the relationships between interpretative knowledge and beliefs, we ground our work in the Mathematics Teachers Specialized Knowledge—MTSK—conceptualization (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013; see Figure 1). Indeed, in accordance with such approach, all of the teachers' knowledge is specialized, and teachers' beliefs are considered a core aspect influencing, and being influenced by, teachers' knowledge. Such beliefs are rooted in all their previous experiences, both as students and as teachers undergoing their initial (and continuous) teachers' education. Moreover, these beliefs not only affect teachers' attitudes and actions, but also have a direct and crucial link with their mathematical knowledge (MK, considered in the left part in Figure 1), thus shaping their perception of the mathematics education process(es) (pedagogical dimension, depicted in the right part of Figure 1).

Although the MTSK considers six sub-domains of teachers' knowledge, in the scope of this work, we only address two of the MK sub-domains. In particular, concerning the context of subtraction in the set of natural numbers N, we focus on the Knowledge of Mathematical Topics (KoT) and the Knowledge of the Structure of Mathematical (KSM).

KoT includes teachers' knowledge pertaining to the definition and justification of the mathematical content (e.g., the difference between a and b, when $a > b$, corresponds to the search of the unique c that satisfies the equation $c + b = a$); properties, issues, and associated procedures such as algorithms; different forms of representation (e.g., decimal positional representation of numbers, columns or linear arrangement of algorithms); phenomenology (e.g., comparing, taking away, and compensation, see, for example, Fuson et al., 1997).

The KSM refers to teachers' knowledge of an integrated system of connections. Such system allows teachers to understand and develop advanced concepts from an elemental standpoint, as well as elemental concepts from approaches considering an advanced mathematical standpoint. Concerning subtraction in N, it is related to, for example, the same operation in other number sets; subtraction involving other mathematical entities (e.g., algebraic variables, vectors, matrices, functions); the potential transition from the elemental aspects of subtraction in N to other advanced aspects such as, for example, the use of finite-difference methods in finding the solution of differential equations.

When considering KoT and KSM pertaining to interpretative knowledge, the content of such sub-domains should allow teachers to look for the potentialities embedded in students' productions and comments (even if students are unaware of them). For example, when giving meaning to different subtraction algorithms, such knowledge should allow teachers to perceive, understand, and appreciate each of the different mathematical aspects required to explain the different steps followed by the student to find the solution.

Obviously, when teachers' beliefs about mathematics (Liljedahl & Oesterle, 2014) are exclusively linked with a procedural, instrumental (Skemp, 1971) approach to/view of mathematics (also due to the set of experiences they have been immersed in), such beliefs implicitly shape the ways they perceive the content of their own KoT and KSM and what they deem necessary to be included in these sub-domains. The aim of this study is to explore the relationship between teachers' beliefs and their revealed KoT and KSM in the light of the interpretative knowledge. We hypothesize that teachers' beliefs related to an instrumental vision of mathematics (Skemp, 1971) can be an obstacle to their interpretation of students' productions if these differ from that anticipated by the teacher.

METHOD

In this study, we explore the nature of beliefs, KoT, and KSM revealed by a group of Italian prospective primary teachers when solving a particular interpretation task in the scope of a Mathematics Education course in which the second and the third author were the educators.

In particular, our sample included 40 prospective primary teachers in the third year of the five-year professional primary teacher training program provided in Italy. The task was administered during one of the first sessions of the course. It commenced by instructing the prospective teachers to find a solution to a given subtraction and afterwards to pose problems involving such operation (Figure 2).

> Consider the following subtraction: 51−17.
>
> a) Find the result and explain verbally how you obtained it
>
> b) Pose two problems that involve this operation

Figure 2. First part of the task.

After completing this first part of the task, prospective teachers were given another sheet containing seven pupils' productions to the same problem. The prospective teachers were asked to reflect and comment on the mathematical correctness (and adequacy) of these productions, and to propose possible feedback that could be given to each of the seven pupils in order to support their mathematical learning. For brevity, we focus our attention on three of the pupils' algorithms only (Figure 3), along with the corresponding prospective teachers' comments and reactions.

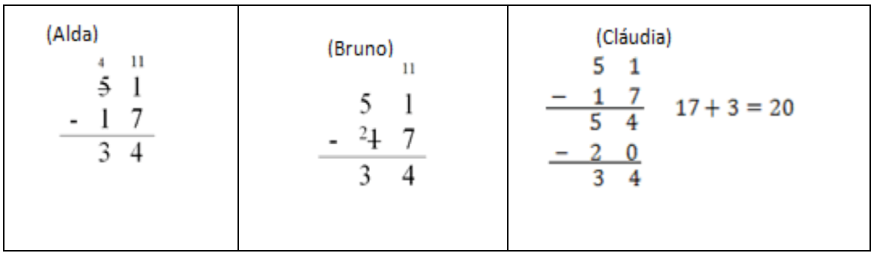

Figure 3. Three subtraction algorithms/representations.

Each of the pupils' algorithms included in the task have been selected with a particular rationale. In particular, concerning the three discussed in this work, Alda's algorithm (the one traditionally used in Italian schools) was included in order to access prospective teachers' beliefs and aspects included in their KoT and KSM when discussing and giving meaning to such "traditional" algorithms. Bruno's and Claudia's algorithms (the first is essentially rooted in the decimal representation of numbers and the properties of subtraction, whereas the second is grounded in the handiness of working with tens) were included to discuss prospective teachers' knowledge and ability to interpret and grasp the correctness of algorithms that differ from their preferred solution (Alda's algorithm) and the emerging beliefs in this process of interpretation and sense given.

We commence the analysis with a qualitative discussion on teachers' beliefs that emerged in the evaluation of the pupils' algorithms shown in Figure 2. Next, we intertwine this discussion with the contents of their KoT and KSM, whereby our analysis is grounded in the argumentation they present when giving meaning to the algorithms provided (e.g., reference to subtraction properties, definitions, representation issues or advanced mathematical aspects). Finally, we present a more quantitative analyses of the links between teachers' KoT and KSM that sustain their ability to interpret students' solutions.

DISCUSSION

All the 40 prospective teachers' answers converge on considering Alda's solution as "mathematically adequate." In ten several cases, the judgment of *adequateness* is related only with the consideration that Alda has solved the subtraction in the same way the prospective teachers would, as noted in the following comment:

> *"Alda's solution is based on correct mathematical reasoning, and is the same as the one provided by me."*

> *"For me, the adequate solution is Alda's solution because it is also how I perform the subtraction"*

Moreover, in ten cases, the adequateness is attributed to the fact that Alda's algorithm is the "traditional" one, i.e., the one "learned at school," as evident in the following answer:

> *"Alda solved the subtraction in an adequate way. She firstly subtracted the 11 from the 7 (by borrowing a ten) and then she subtracted 1 from the 4 (the 5 became 4 because it loaned a ten to the units) I think that the reasoning is 'adequate' because the procedure followed to solve the problem is the traditional one /the one taught in the school."*

This last prospective teacher's answer is based on considering Alda's algorithm adequate. It is rooted in recognizing it as the "traditional" approach, expressing it using the same wording used when learning it at primary school. These two facts provide this prospective teacher the guaranty of correctness– no references to subtraction properties or number representation issues are considered important.

None of prospective teachers' interpretations of Alda's algorithm provides a reference to the subtraction definition, properties, or potential different meanings. In this sense, the provided interpretations of Alda's algorithm allow us to recognize a very basic prospective teachers' KoT, as they do not seem to know the actual rationale underpinning the algorithm (as all are using the mnemonic they have learned while primary students). Alternatively, it is possible that they find referring to such rationales unimportant or irrelevant (as seen, in some cases, they just mention that, as Alda's algorithm is the same as their solution, it must be an adequate one).

On the other hand, it is important to highlight that none of the other algorithms is assessed as *adequate*, even if some are deemed *correct.* This discrepancy suggests that prospective teachers do not consider *correct* and *adequate* as synonymous. In particular, 15 teachers considered Bruno's and Claudia's solutions *inadequate* or even *wrong*, for various reasons, mostly because they differ from the approach they know (they named traditional), as noted below:

> *"The answers given by the other children are inadequate because they don't reflect the traditional solving method for the subtraction."*
>
> *"I think [Bruno] doesn't understand the action of taking away."*

In this last comment, the prospective teacher refers only to one of the subtraction meanings (taking away), thus revealing the need for more extensive work on developing prospective teachers' KoT, revealing also her beliefs about mathematics concerning the uniqueness of a process to find the solution. In this sense, the link between beliefs about mathematics stemming from their previous experiences ("there is only one correct answer—the traditional one") and their revealed KoT is evident.

In some other cases, alternative solutions are considered *confusing*:

> *"The calculation is very personal. The result is correct, but the solving method is not very clear [Claudia]."*
>
> *"Alda performed the calculation correctly. Claudia and Bruno confused me, I don't know . . ."*

These comments point to the low interpretative ability rooted in the belief that only "the traditional algorithm is correct" (probably related to a more general belief concerning the uniqueness of a correct mathematical answer). Moreover, in all 15 answers that consider Claudia's and Bruno's solutions inadequate or wrong, no references to issues that could be include into the content of KoT are made. Finally, in these answers, very few attempts were made to recognize the student's purpose or strategy.

In fact, only 17 of the 40 prospective teachers provided an answer accepting, without negative comments, Bruno's or Claudia's productions. Nine of these answers mainly focus on the validity of the algorithms based on the use of the so-called "invariantive"[1] property of subtraction in order to justify the proposed procedure:

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[1] The invariantive property refers to the fact that the difference between two numbers does not change if the same number is added or subtracted to both the original numbers.

"Claudia added 8 units to 51 and to 17; she therefore used the invariantive property."

"I think that Claudia's reasoning is correct because it is based on the invariantive property in order to make the calculation easier, owing to the use of 'rounded' numbers."

Another critical point of our inquiry concerns prospective teachers' proposals of possible feedback to these students. It is interesting to note that very few teachers suggested any feedback and those that did (feedback perceived as explaining the "correct" process) tended to provide a set of actions aimed at explaining the traditional algorithm, or using the meaning of subtraction as "taking away," as noted below:

"I could help the children with stimulating questions as: What is the subtraction? What does taking away mean?"

Moreover, regarding the KSM, which we consider an important knowledge to refer in mathematics teacher education, it is important to underline that, in prospective teachers' answers, we recognized none references to KSM. Finally, our analyses showed that in solving this task in the complex system of beliefs (Liljedahl & Oesterle, 2014) the beliefs about mathematics seemed crucial, in particular the pne according with only "the traditional algorithm is correct". As we observed probably this is related to a more general belief concerning the uniqueness of a correct mathematical answer and in our analysis it appeared always intertwined with a presence of a poor KoT.

CONCLUSION

Aimed at deepening our understanding of the type and nature of potential links between prospective teachers' interpretative knowledge and beliefs, we designed a particular mathematical task. This task required the prospective teachers to interpret different students' subtraction algorithms and provide feedback to those they consider incorrect. The choice of the task follows the MTSK theoretical framework, which underscores the importance of ensuring that tasks utilized in teacher education are directly connected to the work of teaching. Moreover, we recognized the potential of arithmetic operations algorithms to bring out insights about the prospective teachers' views and understanding of mathematics and its teaching.

One of the findings stemming from our work pertains to the fact that majority of the prospective teachers' answers are rooted in a very firm belief about mathematics that only the "traditional" algorithm—the one they are familiar with (Alda's one)—should be considered correct. Moreover, even in this case, the revealed KoT is at the level of description of the different steps of the procedure, thus revealing prospective teachers' difficulties in arguing about the mathematical reasons of the correctness of the traditional algorithm.

Almost half of the prospective teachers deemed the traditional algorithm as the only correct one, as we could in statements like "*Alda's algorithm is THE right one*." The use of the definite article "the" highlights the prospective teachers' believes on the uniqueness way for finding the answer to the given subtraction: it should be interesting to investigate if perspective teachers believe that – in general – a mathematical task has an unique answer and an unique way to be solved. Such belief about mathematics is present in prospective teachers who also show a poor KoT, which determines a very narrow space of solutions to the posed problem (Ribeiro *et al.*, 2016). This example evidences the role of mathematical knowledge and the view/belief of what means mathematically epistemologically incorrect answers, are a relevant obstacles in the development of a strong interpretative knowledge.

On the other hand, in many of the prospective teachers answers which consider Bruno's and Claudia's algorithms inadequate, we found that subtraction is described assuming only one of its three possible meanings, like in the statements "*I think he [Bruno] doesn't understand the action of taking away.*". This finding supports, once again, the idea that the failure of the interpretation process is intertwined with the content of teachers' KoT. In that sense, it will also be important (and interesting) to deepen our knowledge of the meanings of subtraction presented in the problems teachers posed in the first part of the task. It was also interesting to observe the differences in use (and the corresponding beliefs) of the terms "correct" and "adequate" by some prospective teachers. Indeed, according to some study participants, Bruno's and Claudia's algorithms are correct but not adequate. This finding clearly requires further investigation.

Moreover, it is important to underline that almost half of the prospective teachers who grasped the correctness of Bruno's and Claudia's algorithms used the elements of KoT (in particular the invariantive property) to justify their correctness. This finding, yet again, reveals the essential role of KoT in the development of the interpretative knowledge.

We showed that the absence of key elements of KoT together with particular beliefs about mathematics prevent (prospective) teachers to trigger these processes. Our proposal to place the idea of interpretative knowledge as the core of the Mathematical Knowledge for Teaching, underlines a need of a different mathematics education culture that induces teachers to activate real interpretation processes and to use/capitalize pupils' answers as sources. A possible further step is to clarify the essence of the interpretative knowledge and to identify possible key experiences that trigger in teachers' practices attitudes oriented to a real listening and interpreting of pupils' answers. In this direction, it is important to highlight that our proposal is also conceived as a potentially effective approach to working on teachers' beliefs and knowledge. Indeed in our classes analyzing and discussing with prospective teachers the kind of task we presented here, recognizing with them the correctness of pupils' answers previously labeled as incorrect, reflecting with them upon their own different reactions and evaluations, we have often been observing interesting (prospective) teachers' changes in beliefs and knowledge developments on mathematical critical issues. But further research is needed for analyze and document this other part of our work as teacher educators.

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