Solution of Three-Constraint Entropy-Based Velocity Distribution

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Abstract: A two-dimensional velocity profile based upon the principle of maximum entropy (POME) for wide open channel flows is presented. The derivation is based on the conservation of mass and momentum. The resulting profile involves three parameters that are determined from observations of mean velocity and the velocity at the water surface. The velocity profile is verified using field data in a river with a live bed. A comparison with three existing methods shows that the profile presented is the most accurate of the three, especially near the bed.

Introduction

There are presently two basic methods and a recently proposed method to obtain a time-averaged horizontal velocity profile: the logarithmic distribution law, the power law, and the two constraint entropy methods by Chiu (1987).

The entropy method uses the principle of maximum entropy (POME) to maximize the information content of the data. The entropy method produces four integral equations in four unknowns. These equations are derived from the physical constraints on the system and are not solvable by exact analytical means except for the two constraint cases (Chiu 1987).

Here, the entropy method with three constraints is used, and an approximate solution to the resulting integral equations is determined. The resulting equation is then compared to the other methods stated previously using actual field data.

Derivation of Velocity Distribution Using POME

The concept of entropy can be applied in modeling the vertical profile of the horizontal velocity in open channel flow. Four constraints can be developed for use in the entropy method, namely, constraints for probability, continuity, momentum, and energy. Chiu (1987, 1989) used this method to derive a velocity distribution for the horizontal velocity in a wide, open channel with uniform flow. Chiu (1987) obtained an exact solution for the entropy method when only two of the constraints on the system were used. The constraints he used were the probability and the continuity constraints. Here the constraint obtained from momentum consideration proposed by Chiu (1989) is added and the approximate solution to the resulting integral equations is obtained.

From boundary shear considerations, the classical method of describing

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the two-dimensional velocity profile in wide channels (von Karman 1935) is by relating it to the depth. In wide, open channel flow with depth D, the velocity monotonously increases from zero at the bed to a maximum value at the surface when the water-air interface shear is neglected. Let u be the velocity at a distance y above the channel bed. Then, the probability of the velocity being less than or equal to u is y/D and the cumulative distribution function is

$$F(u) = \frac{y}{D} \qquad (1)$$

and the probability density function is

$$f(\mathrm{u}) = \left(\frac{1}{D}\right)\left(\frac{dy}{du}\right) \qquad (2)$$

Chiu (1987, 1989) used the POME and the constraints on the system based on probability and the three conservation principles, namely conservation of mass, momentum, and energy, to obtain the probability density function of u as

$$f(\mathrm{u}) = \exp(A + L_2u + L_3u^2 + L_4u^3) \qquad (3)$$

and the velocity profile as

$$\int \exp(A + L_2u + L_3u^2 + L_4u^3)\, du = \frac{y}{D} + C \qquad (4)$$

where $A = L_1 - 1$; L_1, L_2, L_3, and L_4 = Lagrange multipliers; and C = the constant of integration to be evaluated by the boundary condition, u = 0 at $y = 0$.

Chiu (1987) then let $L_3 = L_4 = 0$ and solved this equation using only the first two constraints to obtain

$$\mathrm{u} = \frac{1}{L_2} \ln\left\{1 + [\exp(L_2\, u_D) - 1]\frac{y}{D}\right\} \qquad (5)$$

where u_D and L_2 = the parameters. u_D and L_2 are related to u_m by

$$\mathrm{u}_m = \mathrm{u}_D \exp(L_2\, u_D)\,[\exp(L_2\, u_D) - 1]^{-1} - \frac{1}{L_2} \qquad (6)$$

Eq. (5) is the entropy-based, two-constraint, velocity profile equation, for flow in a wide channel, developed by Chiu (1987).

Approximation of Entropy Distribution

Chiu (1989) showed that the value for L_4 was small for the data that he used in his analysis. If in general $L_4 = 0$, the entropy-based, three-constraint, velocity distribution based on momentum is obtained [(3) with $L_4 = 0$]. This is evaluated using Lagrange multipliers to maximize the entropy subject to the three constraints representing probability, continuity, and momentum. These equations are not solvable by exact analytic means.

An exponential function can be approximated by a Maclaurin series. An approximate solution for the three-constraint entropy method is obtained by expansion of the term involving the third parameter, L_3. Using the first

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two terms of the Maclaurin series expansion, and solving the constraints by integration by parts gives three equations in the three unknown Lagrange multipliers, A, L_2, and L_3. Solving these equations simultaneously leads to the velocity profile equation for the two-term Maclaurin series expansion of the three-constraint entropy method based on momentum as

$$\exp(A)\left\{\exp(L_2u) + L_3\left[\exp(L_2u)\left(u^2 - \frac{2u}{L_2} + \frac{2}{L_2^2}\right)\right]\right\} = \left[\frac{y}{D} + \exp(A)\left(\frac{1}{L_2} + \frac{2L_3}{L_2^3}\right)\right]L_2 \quad \text{(7)}$$

where L_2 is obtained from the following equation

$$\left[\exp(L_2u_D)\left(u_D - \frac{1}{L_2} - u_m\right) + \frac{1}{L_2} + u_m\right] \times \left[\exp(L_2u_D)\left(K_1u_D^2 - \frac{2u_DK_1}{L_2} + \frac{2K_1}{L_2^2} - u_D^4 + \frac{4u_D^3}{L_2} - \frac{12u_D^2}{L_2^2} + \frac{24u_D}{L_2^3} - \frac{24}{L_2^4}\right) - \frac{2K_1}{L_2^2} + \frac{24}{L_2^4}\right]$$

$$= \left[\exp(L_2u_D)\left(u_D^2 - \frac{2u_D}{L_2} + \frac{2}{L_2^2} - K_1\right) - \frac{2}{L_2^2} + K_1\right] \times \left[\exp(L_2u_D)\left(u_mu_D^2 - \frac{2u_mu_D}{L_2} + \frac{2u_m}{L_2^2} - u_D^3 + \frac{3u_D^2}{L_2} - \frac{6u_D}{L_2^2} + \frac{6}{L_2^3}\right) - \frac{2u_m}{L_2^2} - \frac{6}{L_2^3}\right] \quad \text{(8)}$$

and $K_1 = M/(\rho D) = \beta u_m^2$, ($\beta$ = the momentum coefficient).

The value of L_3 is obtained by substitution of L_2 into the equation

$$L_3 = \frac{\exp(L_2u_D)\left(u_D - \frac{1}{L_2} - u_m\right) + \frac{1}{L_2} + u_m}{\exp(L_2u_D)\left(u_mu_D^2 - \frac{2u_mu_D}{L_2} + \frac{2u_m}{L_2^2} - u_D^3 + \frac{3u_D^2}{L_2} - \frac{6u_D}{L_2^2} + \frac{6}{L_2^3}\right) - \frac{2u_m}{L_2^2} - \frac{6}{L_2^3}} \quad \text{(9)}$$

The coefficient A is then obtained by substitution of L_2 and L_3 into the equation from the solution of the probability constraint:

$$L_2\exp(-A) = [\exp(L_2u_D) - 1] + L_3\left[\exp(L_2u_D)\left(u_D^2 - \frac{2u_D}{L_2} + \frac{2}{L_2^2}\right) - \frac{2}{L_2^2}\right] \quad \text{(10)}$$

Comparison of Velocity Profile Methods

The basic methods that are compared are: (1) Logarithmic distribution law of Prandtl–von Karman; (2) power law; (3) two-constraint entropy method [(5)]; and (4) three-constraint entropy method based on momentum [(7)]. In each comparison, it will be assumed that only the average and mean velocities are known.

All the profiles are computed for the variables and compared to the data presented in the work of Davoren (1985). This is actual field data for a river with a live bed. Davoren measured velocities of the flow downstream from a hydropower plant. This provided steady uniform flows over several hours for his measurements. A comparison to Davoren's run 1 will be shown for all four methods. Also, the two-constraint entropy method [(5)] and the three-constraint entropy method based on momentum [(7)] are compared to Davoren's run 6 and run 10.

The Prandtl–von Karman universal logarithmic velocity distribution can be stated as follows (Daugherty and Franzini 1977):

$$u = u_D + \frac{(gDS)^{1/2}}{K} \ln\left(\frac{y}{D}\right) \quad \text{(11)}$$

where S = the slope of the energy grade line; and K = the von Karman constant, having a value of about 0.40 for clear water and a value as low as 0.2 for sediment laden water (Daugherty and Franzini 1977). This is a distribution with one parameter that is determined by the maximum or the mean velocity only. In practice, the value of K and S are not known. Therefore to compare the different methods, the bed slope is used as an estimate of S for uniform flow and the range of K from Daugherty and Franzini (1977) is shown in Fig. 1.

The power-law velocity distribution for flow in an open channel can be stated as follows (Sarma et al. 1983):

$$\frac{u}{u_D} = \left(\frac{y}{D}\right)^{(1/n)} \quad \text{(12)}$$

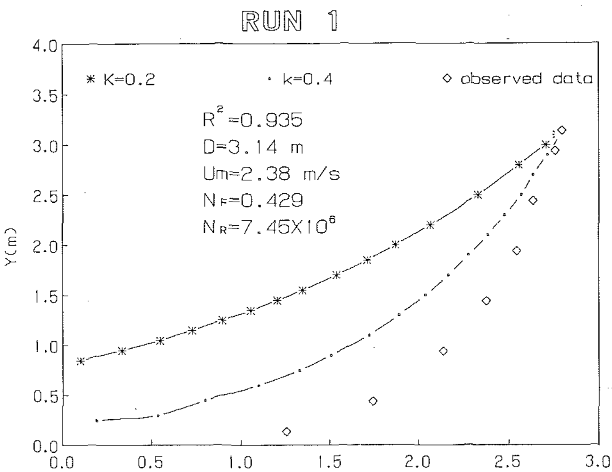

FIG. 1. Prandtl–von Karman Velocity Distribution Plotted Against Observed Profile (Run 1)

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where n = a parameter determined by the frictional resistance at the bed that usually is in the range of 6–7 (Karim and Kennedy 1987). In practice, n is not known and its value is often estimated from sources as stated previously. This is also a distribution with one parameter that is determined by the maximum velocity only. The velocity distribution obtained from this equation is plotted against the observed profile in Fig. 2.

The fit for the log and power distributions could be improved if the values of K and n are known. In cases where observed profiles are available, optimum values of K and n could be obtained by least-squares methods. However, in actual practice these values are not known, therefore generally accepted estimates of their value are used for comparison.

To use the three-constraint entropy method based on momentum a value of $\beta = 1$ is used for a first estimate of K_1 in (8). The first iteration of the velocity profile is then obtained using (7), (8), (9), and (10). The velocity profile is then used to obtained a second estimate of β and therefore K_1. An iterative process yields the velocity profile of the three-constraint entropy method based on momentum. The velocity profile obtained from the three-constraint entropy method [(7)] and the two-constraint entropy method [(5)] are plotted against the observed profile in Figs. 3, 4, and 5.

Remarks and Conclusions

The fit of the Prandtl–von Karman logarithmic velocity profile was only good close to the surface of the flow. As the value of y is decreased, the departure from the observed data of the equation becomes apparent (R^2 = 0.935). The logarithmic velocity profile was totally unacceptable near the bed of the channel. Therefore it seems reasonable to conclude that this velocity profile would not be appropriate for any evaluation of near-bed processes such as scour. However, a modified version such as by Christensen (1972) does not have this shortcoming. Still, a value for K is needed for its

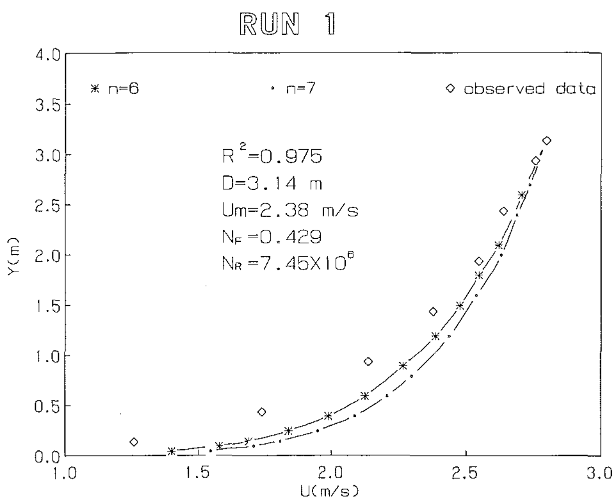

FIG. 2. Power Law Velocity Distribution Plotted Against Observed Profile (Run 1)

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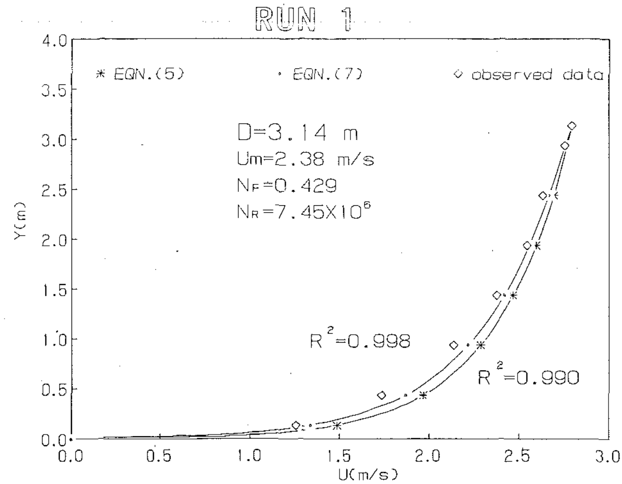

FIG. 3. Entropy Velocity Distributions Plotted Against Observed Profile (Run 1)

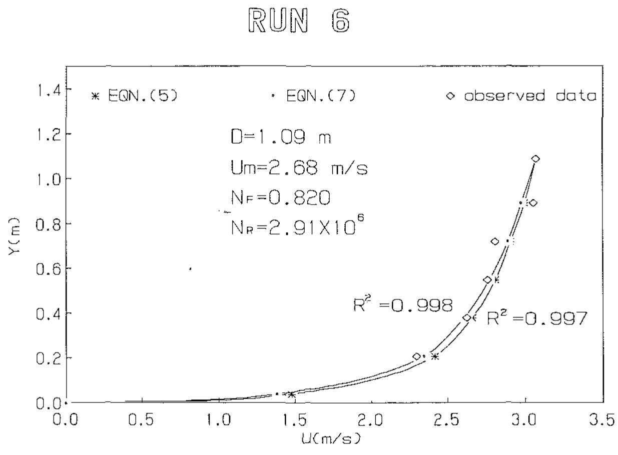

FIG. 4. Entropy Velocity Distributions Plotted Against Observed Profile (Run 6)

use. Of course, the fit could have been improved by obtaining K by least squares.

The velocity profile for the power law was good for a value of $n = 6$ ($R^2 = 0.975$). Again, this profile fit the observed data best near the channel surface and departed from this fit as the value of y is decreased. Even though the fit of the power-law velocity distribution was better than that of the logarithmic velocity distribution near the channel bed, more accuracy for use in the determination of near-bed processes is still desirable.

The two-parameter entropy velocity profile [(5] had a superior fit to the observed data than the two previously mentioned velocity profiles ($R^2 =$

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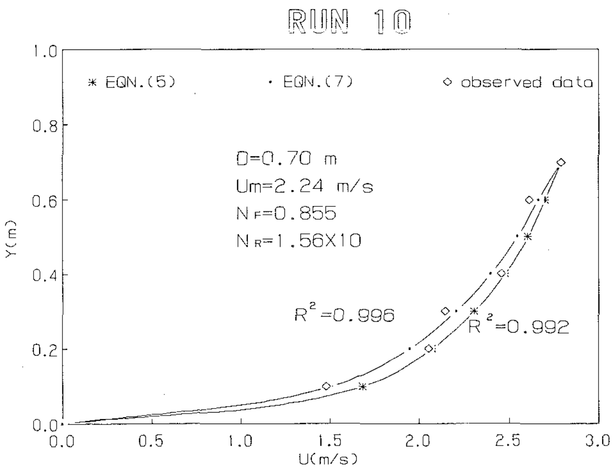

FIG. 5. Entropy Velocity Distributions Plotted Against Observed Profile (Run 10)

0.990 for run 1). The fit of this velocity profile was not only good at the surface of the channel but also near the channel bed.

The velocity profile obtained by the three-constraint entropy method based on momentum [(7)] had the best fit to the observed data of any of the methods compared ($R^2 = 0.998$ for run 1). The fit of this profile was particularly good near the bed of the channel. It is of interest to compare the two- and three-constraint entry methods in order to determine the relative value of the third constraint. In the two-constraint method, only the probability and mass-conservation constraints are observed. In the three-constraint method, the hydrodynamics of the flow are introduced in the form of the momentum conservation principle. Then, the question arises as the relative benefit of this improvement in the case of uniform flow.

Of course, this question cannot be conclusively answered based upon the limited data samples analyzed in this study. However, taken as a whole, it appears that the three-constraint method does not offer a significant improvement to the fit of the overall profile in any of the cases analyzed. As expected, the maximum improvement is greatest near the bottom of the profile, or near the channel bed. This may prove to be significant if near-bed processes, such as scour or sediment transport in the form of bed load, are to be analyzed.

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Appendix II. Notation

The following symbols are used in this paper:

A, L_1, L_2, L_3, and L_4 = Lagrange multipliers;
C = constant of integration;
D = depth of flow in the channel;
g = acceleration due to gravity;
K = von Karman universal constant, which has value of 0.40 for clear water and value as low as 0.2 in flows with heavy sediment loads;
K_1 = $M/(\rho D) = \beta u_m^2$;
M = the momentum flux per unit width;
n = parameter determined by frictional resistance at bed (n is usually in range of 6–7);
S = slope of energy grade line;
u = horizontal velocity at distance y from channel bed;
u_D = maximum velocity of flow;
u_m = mean velocity (depth averaged);
β = momentum coefficient; and
ρ = mass density of water.

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