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Published online: 21 Nov 2011.

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Research paper

Entropy approach for 2D velocity distribution in open-channel flow

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ABSTRACT
In this paper, considering time-averaged velocity as a random variable, two-dimensional (2D) velocity distributions in open-channel flow have been derived based on the Shannon entropy concept and the principle of maximum entropy. The velocity distributions so derived have limited practical use, since they contain too many parameters that need to be experimentally calibrated and hence are not convenient to apply. This work develops a new entropy-based approach for deriving a 2D velocity distribution in open-channel flow, thereby investigating a rectangular geometric domain. The derived distribution is parsimonious, and the values determined using the proposed distribution are found to be in good agreement with the experimentally-measured velocity values.

Keywords: Entropy, flow measurement, open-channel flow, Shannon entropy, streamflow, velocity distribution

1 Introduction

The velocity distribution is fundamental in the hydraulic modelling of natural rivers, including modelling of sediment and contaminant transport, the design of channels, river training works, and hydraulic structures, or the development of rating curves. For an accurate velocity distribution over the full range of flow stages, velocity measurements must be carried out during flood events, which result in considerable difficulties and danger. Therefore, it is desirable to define in a simple and quick way the velocity distribution based only a few sampling points.

Considering that the velocity is subjected to uncertainties arising from natural and/or man-made causes, Chiu (1987) considered time-averaged velocity as a probabilistic variable, unlike traditional approaches which assume it to be a deterministic variable (Tang and Knight 2009). Since then, one-dimensional (1D) velocity distributions have been derived and employed in hydraulics (Chiu 1987, 1989, Barbé *et al.* 1991, Chiu and Said 1995, Kirkgoz *et al.* 2009), whereas two-dimensional (2D) velocity distribution has resulted from the extension of the entropy-based method (Chiu 1988, Chiu and Hsu 2006). Although the latter distribution has

been used in theoretical investigations, its practical use is limited.

Therefore, the objective of this study is to derive a 2D velocity distribution using the Shannon entropy, so that the coordinate system is mathematically sound and the distribution is parsimonious, and to interpret the distribution parameters in terms of hydraulic characteristics.

2   Derivation of 2D velocity distribution using entropy

2.1   *The principle of maximum entropy*

To obtain the least biased probability density function (PDF) of velocity, the entropy of the velocity distribution must be maximized subject to the specified constraints. The reason for entropy maximization stems from the principle of maximum entropy (POME) postulated by Jaynes (1957), stating that any system in the equilibrium state under steady constraints tends to maximize its entropy. As a river reaches a dynamic (or quasi-dynamic) equilibrium, the entropy must attain its maximum value (Singh *et al.* 2003). The POME states that the velocity distribution obtained by entropy maximization will be least biased towards what is not known about the velocity and most biased towards what is known via constraints. This principle is also consistent with the theory of minimum energy dissipation proposed by Yang (1972, 1976). The constraints have physical meaning as discussed below. Therefore, the derivation of the 2D velocity distribution involves the following steps: (1) definition of the continuous form of the Shannon entropy, (2) specification of velocity information in terms of constraints, (3) maximization of entropy and derivation of the probability distribution of velocity, (4) derivation of the 2D velocity distribution, (5) expression of the 2D probability distribution, and (6) estimation of parameters.

2.2   *The Shannon entropy*

Let temporally-averaged velocity u be treated as a random variable with $f(u)$ as its PDF. Then, the Shannon (1948) entropy of u or $f(u)$ is defined as

$$H(u) = -\int f(u) \ln[f(u)] \mathrm{d}u \qquad (1)$$

in which $f(u)\mathrm{d}u$ represents the probability of u between u and $u + \mathrm{d}u$. Entropy $H(u)$ quantifies the uncertainty associated with u or its PDF. Here, the objective is to derive $f(u)$ by applying the POME (Jaynes 1957), subject to specified constraints on velocity.

2.3   *Specification of constraints*

Information on velocity can be derived from the laws of conservation of mass, momentum, and energy. Chiu (1987) and later Barbé *et al.* (1991) observed that for purposes of deriving time-averaged velocity distributions, it is sufficient to use only mass

conservation. Therefore, constraints for the PDF of velocity $f(u)$ are expressed with $u_{\max}$ as the maximum value of u at or below the water surface and $\bar{u}$ as the mean velocity as

$$\int_0^{u_{\max}} f(u)\,\mathrm{d}u = 1 \qquad (2)$$

and

$$\int_0^{u_{\max}} u f(u)\,\mathrm{d}u = \bar{u} \qquad (3)$$

The average velocity constraint in Eq. (3) is derived from, and hence satisfies, mass conservation (Barbé *et al.* 1991). This formulation does not include energy or momentum conservation.

2.4   *Maximization of entropy and probability distribution*

In accordance with the POME, $H(u)$ can be maximized subject to Eqs. (2) and (3) using the method of Lagrange multipliers and recalling the calculus of variation, resulting in

$$f(u) = \exp(\lambda_1 + \lambda_2 u - 1) \qquad (4)$$

where λ_1 and λ_2 are Lagrange multipliers.

2.5   *The 2D velocity distribution*

Consider a 2D domain (x, y), with x as the transverse direction and y as the vertical direction measured from the bed upward positive. Thus, $u = u(x, y)$ and its PDF is $f[u(x, y)]$ and the cumulative probability distribution function (CDF) is $F[u(x, y)]$. Taking the partial derivatives of $F(u)$ with respect to x and y,

$$\frac{\partial F(u)}{\partial x} = \frac{\mathrm{d}F(u)}{\mathrm{d}u}\frac{\partial u}{\partial x} = f(u)\frac{\partial u}{\partial x}$$
$$\frac{\partial F(u)}{\partial y} = \frac{\mathrm{d}F(u)}{\mathrm{d}u}\frac{\partial u}{\partial y} = f(u)\frac{\partial u}{\partial y} \qquad (5)$$

From Eq. (4), Eqs. (5) are rewritten as

$$\exp(\lambda_2 u)\frac{\partial u}{\partial x} = \exp(1 - \lambda_1)\frac{\partial F(u)}{\partial x}$$
$$\exp(\lambda_2 u)\frac{\partial u}{\partial y} = \exp(1 - \lambda_1)\frac{\partial F(u)}{\partial y} \qquad (6)$$

Let quantity $\exp(\lambda_2 u)$ be w so that the partial derivatives of w with respect to x and y are

$$\frac{\partial w}{\partial x} = \frac{\partial \exp(\lambda_2 u)}{\partial x} = \lambda_2 \exp(\lambda_2 u)\frac{\partial u}{\partial x}$$
$$\frac{\partial w}{\partial y} = \frac{\partial \exp(\lambda_2 u)}{\partial y} = \lambda_2 \exp(\lambda_2 u)\frac{\partial u}{\partial y} \qquad (7)$$

Substitution of Eqs. (7) into Eqs. (6) gives the following system of equations:

$$\frac{\partial w}{\partial x} = \lambda_2 \exp(1 - \lambda_1)\frac{\partial F(u)}{\partial x}$$
$$\frac{\partial w}{\partial y} = \lambda_2 \exp(1 - \lambda_1)\frac{\partial F(u)}{\partial y} \tag{8}$$

Equations (8) can be integrated using the Leibniz rule that states

$$\int_{0,0}^{(x,y)} \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy = w(x,y) - w(0,0) \tag{9}$$

Because the point (0,0) lies on solution domain's contour, u at this point is equal to 0, and the right-hand side of Eq. (9) becomes

$$w(x,y) - w(0,0) = w(x,y) - \exp(\lambda_2 u)$$
$$= w(x,y) - \exp(0) = w(x,y) - 1 \tag{10}$$

The definite integral of the first part of Eq. (9) is calculated at a generic point $(\bar{x}, \bar{y})$ identified by means of a polygonal curve that starts from (0,0), passes across $(\bar{x}, 0)$, and ends at $(\bar{x}, \bar{y})$ so that

$$\int_{0,0}^{(\bar{x},\bar{y})} \frac{\partial F(u)}{\partial y}\lambda_2 \exp(1 - \lambda_1)dy + \frac{\partial F(u)}{\partial x}\lambda_2 \exp(1 - \lambda_1)dx$$
$$= \int_{0}^{\bar{y}} \frac{\partial F(u)}{\partial y}\lambda_2 \exp(1 - \lambda_1)dy = \lambda_2 \exp(1 - \lambda_1)F(u) \tag{11}$$

in which $(\bar{x}, \bar{y})$ represents a point of the domain.

The right-hand side of Eq. (11) can be equated to the right-hand side of Eq. (10) to obtain

$$w(x,y) = 1 + \lambda_2 \exp(1 - \lambda_1)F(u) \tag{12}$$

Because $w(x,y) = \exp(\lambda_2 u)$, Eq. (12) is rewritten as

$$\exp[\lambda_2 u(x,y)] = 1 + \lambda_2 \exp(1 - \lambda_1)F(u(x,y)) \tag{13}$$

resulting for $u(x,y)$ in

$$u(x,y) = \frac{1}{\lambda_2} \ln[1 + \lambda_2 \exp(1 - \lambda_1)F(u(x,y))] \tag{14}$$

Equation (14) contains the two Lagrange multipliers λ_1 and λ_2, which are determined using Eqs. (2) and (3). Integration of Eq. (2) yields

$$\int_{0}^{u_{max}} \exp(\lambda_1 - 1 + \lambda_2 u)\,du = 1 \Rightarrow \lambda_2 \exp(\lambda_1 - 1)$$
$$= \exp(\lambda_2 u_{max}) - 1 \tag{15}$$

Considering Eq. (15) and with $\lambda_2 u_{max} = G$ as entropic parameter (Chiu 1988), Eq. (14) becomes

$$u(x,y) = \frac{u_{max}}{G} \ln[1 + (\exp(G) - 1) \cdot F(u(x,y))] \tag{16}$$

in which parameter G is determined using the constraint expressed by Eq. (3). Equation (16) is the 2D velocity distribution in terms of u_{max}, G, and 2D CDF. Substitution of Eq. (16) in Eq. (4) yields the PDF of velocity $f(u)$, which results in the uncertainty associated with the 2D velocity distribution or the PDF of u, when inserted into Eq. (1).

Using the PDF $f(u)$ defined by Eq. (4), the ratio between the mean and the maximum velocities can be derived as an exponential function of G only. A constraint equation used for deriving velocity distributions, for example Eq. (3), is often referred to as the average value of velocity, for example, the average velocity in the geometric space u_{av}. This average value is, however, different from $\bar{u}$ in Eq. (3), a point often overlooked in the literature. Therefore, for the cross-sectional average velocity, one must use with A as the total area of the 2D domain

$$u_{av} = \frac{1}{A} \int_A \frac{u_{max}}{G} \ln[1 + (\exp(G) - 1) \cdot F(u)]\,dA \tag{17}$$

Equation (17) can be solved to obtain G. Referring to a geometric domain in which u_{av} and u_{max} are known, one can determine first G using Eq. (17) and then calculate the velocity distribution using Eq. (16), once the CDF in 2D is defined.

In practice, the method requires u_{av} and u_{max} to be measured. In particular, u_{av} is calculated from the discharge and the flow area or inferred from the ratio u_{av}/u_{max}, which was proved to be constant over time by Chiu and Said (1995), Xia (1997), and Moramarco *et al*. (2004). Consequently, if the stage-discharge curve is available at a certain station, only u_{max}, generally simple to measure, is required to apply the 2D velocity model. This model invokes that the cross-section be practically invariant with time which is not always true because of water hyacinth growth, accumulation of debris, or erosion and deposition affecting the average velocity.

2.6   *The CDF in the 2D case*

The CDF depends on the geometry of the domain, which must have the following properties: (1) defined between 0 and 1, (2) continuous and differentiable, and (3) its value on the borders must be 0 and have just one point in which it reaches 1. Consider a rectangular channel in which the distribution of velocity is symmetrical with respect to the vertical axis. One can distinguish the position of coordinates, the location of u_{max} occurring on or below the water surface depending on distance y_0 of the point at the maximum velocity from the channel bed, and the size of domain H (height) and $B/2$ (half width).

It is convenient to convert this domain into a dimensionless form using the normalizing quantities B and H as $\psi = y/H$,
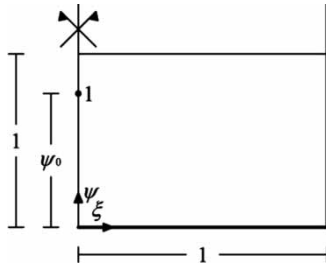
Figure 1   Symmetrical rectangular dimensionless domain

$\xi = 2x/B$, and $\psi_0 = y_0/H$, in which the ratio u/u_{max} instead of u is considered (Fig. 1). Using these variables and geometrical considerations (Appendix), $F(u)$ is obtained as

$$F(u) = (1 - \xi^2)^{H/B} \cdot 4$$
$$\cdot \left[\left(\frac{\psi}{2} \right)^{\ln 2/(\ln 2 - \ln(\psi_0))} - \left(\frac{\psi}{2} \right)^{2 \ln 2/(\ln 2 - \ln(\psi_0))} \right] \quad (18)$$

Equation (18) satisfies all the aforementioned properties: it is continuous and differentiable, varies between 0 and 1, and reaches the value 1 as $\xi = 0$ and $\psi = \psi_0$.

The CDF $F(u)$ given by Eq. (18) has two parts: $(1 - \xi^2)^{H/B}$ expresses the dependence on ξ and the second part expresses the dependence on ψ. In the first part, the ratio H/B appears as an exponent, meaning that as the domain is very wide and, consequently, as $H/B \rightarrow 0$, the first part of $F(u)$ tends to 1 and so $F(u)$ depends only on ψ. Therefore, as the domain becomes very wide, $F(u)$ must depend just on ψ, following also a physical intuition. Then, Eq. (18) becomes

$$F(u) = 4 \cdot \left[\left(\frac{\psi}{2} \right)^{\ln 2/(\ln 2 - \ln(\psi_0))} - \left(\frac{\psi}{2} \right)^{2 \ln 2/(\ln 2 - \ln(\psi_0))} \right]$$
$$(19)$$

Equation (19) gives the 1D velocity distribution if the maximum velocity is on or below the water surface. This case was discussed by Chiu (1988), and the equations are comparable, suggesting that the 2D theory proposed here, when applied to the 1D case, represents nearly his formulation.

2.7   *Parameter G*

To apply the 2D velocity distribution equation given by Eq. (16), parameter G must be evaluated by means of Eq. (17). Using Eq. (18) for $F(u)$, one can specify Eq. (17) as

$$\frac{u_{av}}{u_{max}} = \int_0^1 d\psi \int_0^1 \frac{1}{G} \ln \left\{ 1 + (\exp(G) - 1) \cdot (1 - \xi^2)^{H/B} \cdot 4 \right.$$
$$\left. \cdot \left[\left(\frac{\psi}{2} \right)^{\ln 2/(\ln 2 - \ln(\psi_0))} - \left(\frac{\psi}{2} \right)^{2 \ln 2/(\ln 2 - \ln(\psi_0))} \right] \right\} d\xi$$
$$(20)$$

To obtain G, Eq. (20) needs to be solved numerically because of its implicit form, depending on H/B, ψ_0, and u_{av}/u_{max}. The entropic parameter tends to be invariant at a channel section, whether the flow is steady or unsteady (Chiu *et al.* 2005). Moramarco and Singh (2010) noted that the entropic parameter is also independent of the energy or the water surface slope.

3   Comparison with Chiu's 2D distribution

3.1   *Chiu's distribution*

Chiu (1987, 1988, 1989) derived a general expression for CDF of velocity by introducing equations for isovels where each curve in the physical space has a constant value of velocity. For estimating discharge during floods, Moramarco *et al.* (2004) developed a practical method by assuming that the 2D Chiu equation, written for the vertical where the maximum velocity occurs, also applies to other verticals as (Chiu 1988, 1989, Chiu and Said 1995, Greco 1999)

$$\frac{u}{u_{max,i}} = \frac{1}{M_i} \ln \left[1 + (\exp(M_i) - 1) \frac{\psi}{\psi_{0,i}} \left(1 - \frac{\psi}{\psi_{0,i}} \right) \right],$$
$$i = 1, 2, \ldots, N_v, \quad (21)$$

where i is the ith vertical; ψ, the vertical distance measured from the channel bed divided by the water depth; $\psi_{0,i}$, the value of ψ with the maximum velocity; and N_v the number of verticals sampled. Weaknesses of the above approach include requirement of too many parameters and the physical basis for using isovels. The modification of Moramarco *et al.* (2004) is simpler, but it still requires information on average velocity, position, and magnitude of maximum velocity for each vertical.

3.2   *Experimental measurements*

To evaluate the proposed 2D velocity distribution, experimental velocity data were collected. Table 1 presents the following characteristics: reference, number of data, channel width, water depth, maximum velocity u_{max}, average velocity u_{av}, relative vertical position of maximum velocity $\psi_0 = y_0/H$, and value of entropy parameter G from Eq. (20). Each series contains velocity observations distributed along various verticals, so that for each a 2D distribution of experimental velocity is obtained. Only a preliminary validation is discussed herein for the lack of more data, which is a limitation on the validation of the proposed model.

3.3   *Validation*

For each data series, velocity profiles were estimated using the proposed 2D distribution and by means of the work of Chiu (1988) and as revised by Chiu and Hsu (2006). For the data of Chiu and Hsu (2006), the observed velocity distribution, the proposed 2D distribution, and Chiu's 2D velocity distribution are shown in Fig. 2. The profiles based on the proposed velocity

Table 1   Characteristics of velocity data

Reference	Number of data	Channel width [cm]	Water depth [cm]	u_{max} [cm/s]	u_{av} [cm/s]	ψ_0 [−]	G [−]
Bortz (1989)	29	121.9	18.3	78.3	71.6	0.95	7.032
Chiu and Hsu (2006)	29	61	18.3	126.3	90.0	0.62	0.759
Guo (1990)	13	10	2.3	37.9	29.0	0.72	0.469
Run14 Guy *et al.* (1966)	24	243.8	27.4	125.3	112.7	0.90	4.978
Run28 Guy *et al.* (1966)	25	243.8	18.3	131.7	117.7	0.84	3.716
Run1 Steffler *et al.* (1983)	194	114.3	14.6	84.4	75.5	1.00	7.364
Run2 Steffler *et al.* (1983)	307	114.3	22.5	54.3	49.0	1.00	6.353
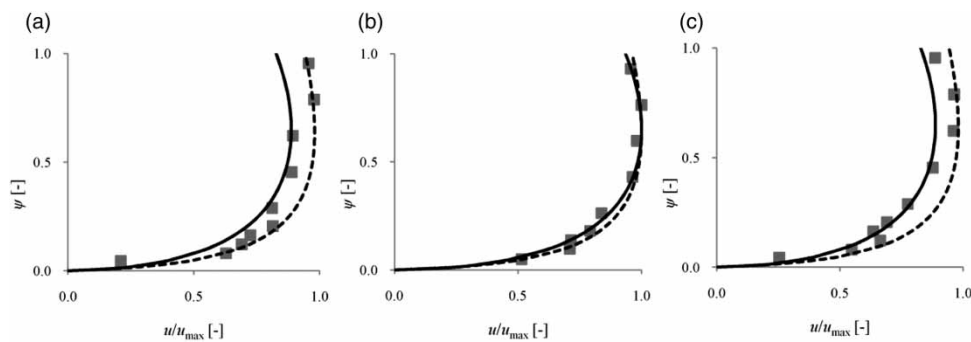
Figure 2   Comparison between experimental data and theoretical velocity profiles at section $\xi = $ (a) −25%, (b) 0%, and (c) +25%: (■) experiment, (—) theory, and (- -) Chiu and Hsu (2006)

distribution compare better with experimental data than do the profiles of Chiu. In addition, the average d_{av} and maximum d_{max} values of these differences and three validation indices were calculated. The difference is defined as $|u - u^*|/u_{max}$, in which u^* is calculated by means of the proposed or Chiu's distribution. The validation indices of Krause *et al.* (2005) used were the coefficient of determination r^2, the Nash–Sutcliffe efficiency E, and the index of agreement ia.

The values of differences and indices are given in Table 2 for each series. Again, the proposed 2D distribution yields smaller differences and validation indices better than Chiu's distribution.

As stated above, Chiu's distribution needs many parameters, whereas the proposed distribution needs just the average and maximum velocities and the position of u_{max}.

For the sake of brevity, the velocity profiles for other series are not presented; Fig. 3 shows the differences between the experimental and theoretical profiles inferred from the proposed 2D distribution. A similar figure for Chiu's distribution is omitted. The results given in Table 2 confirm the largest scatter of the points related to Chiu's approach with respect to Fig. 3.

For a few points, there are large differences between experimental and theoretical values. These are related to lower velocity

Table 2   Deviations and validation indices for all series

| Reference	Approach	d_{av} [−]	d_{max} [−]	r^2 [−]	E [−]	ia [−]
Bortz (1989)	Proposed	0.0493	0.1727	0.8192	0.6609	0.9181
	Chiu (1988)	0.0528	0.1925	0.7668	0.5688	0.9131
Chiu and Hsu (2006)	Proposed	0.0542	0.1930	0.8936	0.8728	0.9633
	Chiu (1988)	0.0757	0.2801	0.8908	0.7368	0.9220
Guo (1990)	Proposed	0.0864	0.2146	0.8741	0.8399	0.9550
	Chiu (1988)	0.0556	0.3875	0.9594	0.8193	0.9339
Run14 Guy *et al.* (1966)	Proposed	0.0783	0.3853	0.8709	0.6519	0.8892
	Chiu (1988)	0.0828	0.3726	0.8719	0.6396	0.8894
Run28 Guy *et al.* (1966)	Proposed	0.0203	0.0553	0.9768	0.9469	0.9883
	Chiu (1988)	0.0315	0.0723	0.9711	0.8882	0.9740
Run1 Steffler *et al.* (1983)	Proposed	0.0555	0.2016	0.9363	0.8379	0.9683
	Chiu (1988)	0.0651	0.2341	0.9153	0.7780	0.9578
Run2 Steffler *et al.* (1983)	Proposed	0.0544	0.3691	0.9647	0.9647	0.9909
	Chiu (1988)	0.1063	0.4553	0.9484	0.5988	0.9318
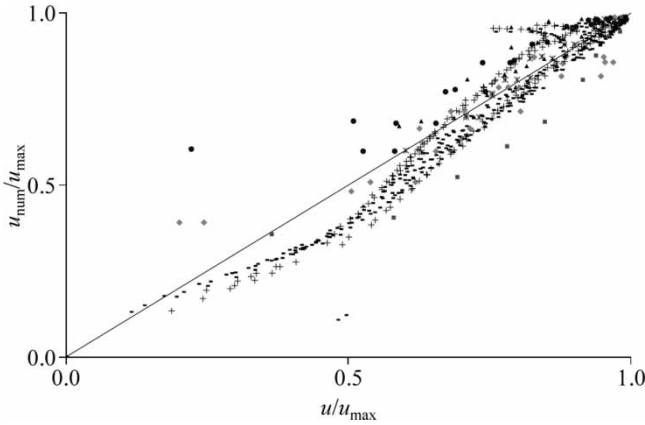
Figure 3　Comparison between experimental and theoretical data (2D proposal) with (▲) Bortz (1989), (◆) Chiu and Hsu (2006), (■) Guo (1990), (●) Run14 Guy *et al.* (1966), (∗) Run28 Guy *et al.* (1966), (+) Run1 Steffler *et al.* (1983), and (-) Run2 Steffler *et al.* (1983)

areas or to boundary areas, near either the bottom or the banks. These differences are large for both Chiu's and the proposed distributions. These differences can be explained as follows: (1) Experimental data near the boundary suffer from greater uncertainty than those obtained from other areas. Thus, it is likely that these data are not as precise as expected. (2) Banks and the channel bottom significantly affect the velocity and both the proposed and Chiu's models are based on only one constraint which is based on mass conservation and do not explicitly account for boundary effects. Scatter related to points not near the boundary limits the method. Table 2 shows that the proposed 2D velocity distribution is remarkably accurate and superior to Chiu's distribution.

4　Conclusions

This research has developed a new 2D velocity distribution using entropy. The derivation uses a probability function defined in a 2D symmetric rectangular domain. The agreement between experimental velocity measurements and the velocity computed from the 2D distribution data is quite close. Neither parameter calibration nor isovel equation is required; only the average velocity and the maximum velocity and its position have to be known: while the first can be inferred from a stage-discharge curve or from the maximum velocity, the second has to be measured. The proposed plane velocity distribution is parsimonious and superior to existing velocity distributions.

Appendix

Consider a rectangular channel with the CDF $F[u(x, y)]$ to be searched of properties defined in Section 2.5. A CDF should be based on available experimental data and satisfy similarity in the velocity distribution, so that suitable parameters appear in the final equation depending on both ξ and ψ. Assume that the first part depends on ξ and the second part on ψ only; each

part must vary from 0 to 1 so that the product yields 0 on the border and 1 at $(0, \psi_0)$. The first part depending only on ξ has to be symmetrical on the ψ axis and its derivative has to be nil. The simplest equation satisfying these properties is the parabolic equation

$$1 - \xi^2 \tag{A1}$$

If the rectangular domain is very wide ($H \ll B$), F should depend not on ξ, but only on ψ, and the first part should be equal to 1. This is satisfied by the exponent H/B to obtain

$$(1 - \xi^2)^{H/B} \tag{A2}$$

The second part of $F(u)$ must depend only on ψ, and it must be $\psi(0) = 0$, and $\psi(\psi_0) = 1$. Starting from the simple case $\psi_0 = 1$ and using the parabolic equation, one obtains

$$4 \left[\frac{\psi}{2} - \left(\frac{\psi}{2} \right)^2 \right] \tag{A3}$$

Since ψ_0 can be <1, Eq. (A3) exponent a for each term is introduced, with a as a function of ψ_0

$$4 \left[\frac{\psi}{2}^a - \left(\frac{\psi}{2} \right)^{2a} \right] \tag{A4}$$

Because $a(\psi_0)$ must have the properties <1 for ψ_0 between 0 and 1, $=1$ as $\psi_0 = 1$, and as $\psi = \psi_0 \psi(1) = \psi_0$, $a(\psi_0)$ is determined from

$$\left[\frac{\psi_0}{2}^a - \left(\frac{\psi_0}{2} \right)^{2a} \right] = \frac{1}{4} \tag{A5}$$

Insertion of the expression for a into Eq. (A4) gives

$$4 \left[\frac{\psi}{2}^{\ln(2)/(\ln(2) - \ln(\psi_0))} - \left(\frac{\psi}{2} \right)^{2 \ln(2)/(\ln(2) - \ln(\psi_0))} \right] \tag{A6}$$

The product of Eqs. (A2) and (A6) finally gives Eq. (18).

Notation

a	=	function of ψ_0 [−]
A	=	total area of cross-section [L²]
B	=	width of cross-section [L]
d_{av}	=	average difference between experimental and theoretical data [−]
d_{max}	=	maximum difference between experimental and theoretical data [−]
E	=	Nash–Sutcliffe efficiency validation index [−]
$f(u)$	=	PDF of u [−]
$F(u)$	=	CDF of u [−]
G	=	entropic parameter [−]
H	=	height of cross-section or water depth [L]

$H(u)$ = Shannon entropy of u [−]
ia = index of agreement [−]
M = entropic parameter in Chiu's 2D approach [−]
M_i = entropic parameter for ith vertical of Moramarco *et al.* (2004) [−]
N_v = number of verticals sampled by Moramarco *et al.* (2004) [−]
r^2 = coefficient of determination [−]
u = temporally-averaged velocity [LT^{-1}]
u^* = value of velocity calculated [LT^{-1}]
u_{av} = cross-sectional average velocity [LT^{-1}]
u_{max} = maximum velocity [LT^{-1}]
$u_{max,i}$ = value of u_{max} for ith vertical given by Moramarco *et al.* (2004) [LT^{-1}]
u_{num} = velocity from a numerical model [LT^{-1}]
$\bar{u}$ = mean velocity [LT^{-1}]
w = $\exp(\lambda_2 u_{max})$ [−]
x, y = coordinates in a 2D domain [L]
$\bar{x}, \bar{y}$ = coordinates of a point defined in a 2D domain [L]
y_0 = vertical coordinate for $u = u_{max}$ in a 2D domain [L]
λ_1, λ_2 = Lagrange multipliers [−]
ξ, ψ = coordinates in a 2D dimensionless domain [−]
ψ_0 = vertical coordinate for $u = u_{max}$ in a 2D dimensionless domain [−]
$\psi_{0,i}$ = value of ψ_0 for ith vertical given by Moramarco *et al.* (2004) [−]

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