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Citation

Cellular Bucket Brigades on U-Lines with Discrete Work Stations

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Abstract

It is challenging to maximize and maintain productivity of a U-line with discrete stations under the impact of variability. This is because maximizing productivity requires assigning workers to suitable tasks and maintaining productivity requires sufficient flexibility in task assignment to absorb the impact of variability. To achieve this goal, we propose an operating protocol to coordinate workers on the U-line. Under the protocol the system can be configured such that its productivity is maximized. Workers are allowed to dynamically share work so that the system can effectively absorb the impact of variability. Analysis based on a deterministic model shows that the system always converges to a fixed point or a period-2 orbit. We identify a sufficient condition for the system to converge to the fixed point. Increasing the number of stations improves productivity only under certain circumstances. The improvement is most significant when the number of stations in each stage increases from one to two, but further dividing the U-line into more stations has diminishing return. Simulations based on random work velocities suggest that our approach significantly outperforms an optimized, static work-allocation policy if variability in velocity is large.

1   Introduction

Consider a U-shaped assembly line with M stations shown in Figure 1. There are three stages in the U-line. Stages 1 and 3 are separated by an aisle and stage 2 spans across the aisle. Stage 1 consists of m_1 stations $S_1(1), \ldots, S_1(m_1)$ located on one side of the aisle. Stage 2 has m_2 stations $S_2(1), \ldots, S_2(m_2)$ located across the aisle. Stage 3 consists of m_3 stations $S_3(1), \ldots, S_3(m_3)$ located on the other side of the aisle. We assume $m_1 + m_2 + m_3 = M$. Each item (an instance of the product) is initiated at the start of $S_1(1)$ and is progressively assembled in the same sequence of stations until it is completed at the end of $S_3(m_3)$. We assume the work content on each station is deterministic and is uniformly distributed on the station. Thus, each worker continuously moves along a station as he progressively works on the station.

We consider a team of two workers W_1 and W_2. Each worker is cross-trained to work on any station of the U-line, and he assembles only a single item at a time. We say the stations are *discrete* because at most one worker is allowed to work on a station at any time (for example, to avoid interference due to limited equipment or space in the station). As a result, a worker might be idle while his colleague is working on a station. We first assume W_i works with a constant, deterministic velocity $v_{ij}(k)$ on $S_j(k)$. We will investigate the impact of random velocities in Section 6. Since the travel time between stations is short compared to the time to assemble an item, we neglect the time to walk from one station to another.

We have seen an example of such a system in porcelain painting. Each item (for example, vase) is painted with a specific theme of colors on each station, which is a rectangular table equipped with painting tools. Each item is progressively painted with different colors at different locations on a station when the item moves along a track on the station's longitudinal axis. After the item is done with a theme of colors, it is removed from the track and is carried to the next station for another theme of colors. Since porcelain painting is a complicated process, at most one worker is allowed to work on a station at any time to avoid interference. To facilitate coaching and learning, an experienced worker is often paired with a relatively new worker to form a team in such a system. The entire flow line is commonly configured in U-shape to provide
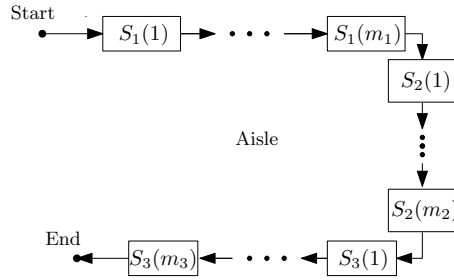
Figure 1: **A U-shaped line.** Each item is initiated at the start of $S_1(1)$ and is progressively assembled in the same sequence of stations until it is completed at the end of $S_3(m_3)$.

better communication and to reduce travel.

U-lines are common in manufacturing because they possess several advantages over straight lines. These include providing better visibility and communication that leads to better quality control. The travel of workers is reduced as they can execute nonconsecutive tasks that are physically close to each other, especially if the aisle is narrow. Many firms adopt a U-shaped layout also because of space constraints. For a discussion on the advantages of U-lines, see Miltenburg and Wijingaard (1994).

Our objective for the above system is to maximize and maintain its productivity under the impact of variability (for example, in work velocity). This is challenging because maximizing productivity requires assigning workers to the stations where they work fast; and maintaining productivity requires sufficient flexibility in task assignment to absorb the impact of variability. To achieve this objective, we propose an operating protocol to coordinate the workers. Under the protocol the system can be configured such that its productivity is maximized. We also allow the workers to dynamically share work so that they are not assigned and restricted to a fixed set of stations. This flexibility can effectively absorb the impact of variability.

The protocol also reduces travel of workers by allowing each of them to work on nonconsecutive stations that are physically close to each other. Although we neglect the travel time between stations in our model, we believe allowing each worker to work on stations close to each other may boost productivity in the actual implementation. In addition, the protocol is straightforward to implement in practice because workers just follow simple rules.

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Specifically, we adapt the basic ideas of *cellular bucket brigades* introduced by Lim (2011) to coordinate workers on the U-line. Under the design of a cellular bucket brigade, the work content of an assembly line is distributed on both sides of an aisle. Each worker works on one side when he proceeds in one direction and works on the other side when he proceeds in the reverse direction. Two workers on different sides of the aisle may approach each other as they work in opposite directions. The workers will exchange their work when they meet. By applying similar coordination rules, which will be discussed in detail later, each item in the U-line is initiated at the start of $S_1(1)$ typically by W_1. The item is passed to W_2 at some point in stage 1. W_2 then finishes the remaining work of stage 1 and continues to assemble the item in stage 2, before he passes it back to W_1 in stage 3. W_1 then completes the item at the end of $S_3(m_3)$. We analyze the dynamics and determine the throughput (number of items produced per unit time) of the U-line under the coordination rules proposed.

The paper by Lim (2011) analyzes a system where work content is continuously distributed along an assembly line and workers can work arbitrarily close to each other. In contrast, the analysis of the above U-line is much more challenging due to the discrete stations. The system dynamics are complex because a worker cannot enter a station while his colleague is working on the station. The problem is further complicated by the facts that each stage may contain any number of stations, different stations may have different amounts of work, and workers may have different velocities on different stations.

We first study U-lines with three stations and two workers in Section 4. We define simple rules for the workers to share work. Under these rules, we provide a complete analysis of the dynamics and determine the throughput of the system. We study U-lines with M stations and two workers in Section 5. We then compare our approach with two other work-allocation policies for random velocities in Section 6 before we make concluding remarks.

2 Related literature

Most papers in the literature study *serial bucket brigades* where each item is handed off from one worker to another in a straight-line layout (see Bartholdi et al. (2010) for a review). Each worker processes an item at most once in such a setting. In contrast, a worker may process an item twice in different stages of assembly in a cellular bucket brigade. We first discuss the literature on serial bucket brigades.

Bartholdi and Eisenstein (1996a) introduce (serial) bucket brigades as a way to coordinate workers along an assembly line with more stations than workers. When workers form a bucket brigade on an assembly line, each worker assembles his item until it is taken over by a downstream colleague or he completes his item if he is the last worker of the line. After that the worker walks back to take over an item from an upstream colleague or to initiate a new item at the start of the line if he is the first worker. The authors consider a model with deterministic work content. Each worker has a deterministic, finite work velocity and an infinite walk-back velocity. They show that if workers are sequenced from slowest to fastest according to their work velocities in the direction of production flow, then the system will *self-balance*: The hand-off locations between any two neighboring workers will converge to a *fixed point* and every worker repeatedly works on a fixed portion of the line. Furthermore, the long-run average throughput will achieve the maximum possible for the system if the work content is continuously and uniformly distributed on the assembly line.

The most widely known application of bucket brigades is order-picking in distribution centers (Bartholdi et al. 1996b, 2001). Bucket brigades are also used in the production of garments, packaging of cellular phones, and assembly of tractors, large-screen televisions, and automotive electrical harnesses (see Bartholdi and Eisenstein (1996a,b, 2005) and Villalobos et al. (1999a,b)). Bucket brigades are effective for the following reasons: (1) The rule is simple for workers to learn and follow. (2) Due to their self-balancing property, we need neither a work-content model nor computation for work balance, which are required by any static work-allocation policy. (3) Since workers dynamically and constantly balance their work, the system

can restore balance from temporary disruptions and is adaptive to changes in work content.

Based on the same model, Bartholdi et al. (1999) study the dynamics of two- and three-worker bucket brigades with workers not necessarily sequenced from slowest to fastest. Bartholdi et al. (2001) consider stochastic work content on work stations. They find that the dynamics and throughput of the stochastic system will be similar to that of the deterministic system when there is sufficient work distributed among sufficiently many stations.

Bartholdi and Eisenstein (2005) extend the basic model of bucket brigades to capture walk-back time and hand-off time. Bartholdi et al. (2009) consider the case where workers are allowed to overtake or pass each other and they walk back with finite velocities. The authors show that the system may exhibit chaotic behavior that causes the inter-completion times of items to be effectively random, even though the model is purely deterministic. The system can avoid such pathologies if workers are indexed from most impeded by work to least impeded by work. Bartholdi et al. (2006) extend the ideas of bucket brigades to a network of subassembly lines so that all subassembly lines are synchronized to produce at the same rate and items are completed at regular, predictable intervals.

Armbruster and Gel (2006) assume workers' work velocities do not dominate each other along the entire line. They study the dynamics and throughput of a two-worker system. Armbruster et al. (2007) consider a model where workers improve their work velocities as they learn. Webster et al. (2012) examine the performance of a bucket brigade order-picking system by changing the distribution of products along an aisle. They identify conditions where product distribution has large impact on throughput.

Lim and Yang (2009) analyze the dynamics of bucket brigades on discrete work stations and identify the best policies that maximize the system's throughput. They show that the policy that fully cross-trains the workers and sequences them from slowest to fastest is not always the best for the system, even though it outperforms other policies for most work-content distributions. Gurumoorthy et al. (2009) study an M-station, two-worker bucket brigade. They determine the asymptotic dynamic behavior and the throughput of the system using an algorithmic approach.

Kirkizlar et al. (2012) study tandem lines with more stations than workers. They consider buffers between stations. The authors find flexibility structures and worker assignment policies that maximize the system's throughput. For an excellent review of workforce cross-training and coordination, see Hopp and Van Oyen (2004).

Bischak (1996) considers a U-shaped manufacturing module with fewer workers than stations. She proposes rules that are suitable for a straight-line layout for workers to move in the module. The throughput and flow time of this moving-worker module are compared with a system with one dedicated worker per station through simulation studies. Chand and Zeng (2001) consider static work allocation and compare U-lines with straight-line layouts under the impact of stochastic task times. Geismar et al. (2008) study a U-shaped manufacturing cell with multiple stages. Items are moved from one stage to another by dual gripper robots. Each robot visits a cyclic sequence of stages in a manner similar to that considered in our paper. However, they assume the assignment of stages to each robot is predetermined and fixed.

The ideas of cellular bucket brigades are first introduced by Lim (2011), who presents an alternative design that may provide significant improvement in throughput. Under the new design, each worker works on one side of an aisle when he proceeds in one direction and works on the other side of the aisle when he proceeds in the reverse direction. The author proposes the cellular bucket brigade rules to coordinate workers under the new design. He also finds a sufficient condition for the system to self-balance. Numerical examples suggest that the system under the new design can be 30% more productive than a traditional, serial bucket brigade. Lim (2012) provides a case study of order-picking by cellular bucket brigades using data from a distribution center in North America.

In this paper, we adapt the basic ideas of Lim (2011) and propose rules to coordinate workers on the U-line with discrete stations. Under our assumption on discrete stations, a worker cannot enter a station if his colleague is working on the station. This constraint makes the analysis significantly more challenging than that of Lim (2011), where work content is continuously distributed along the line and workers can work arbitrarily close to each other. We believe our
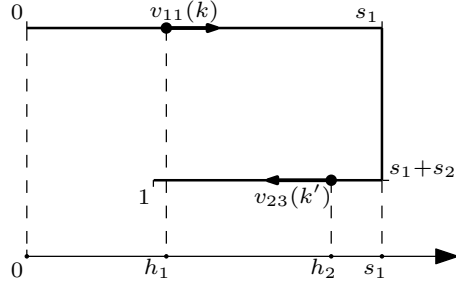
Figure 2: **A path.** The U-line is conceptualized as a path with length 1. In this graph, W_1 and W_2 work on $S_1(k)$ and $S_3(k')$ respectively. The horizontal position h_i is determined by projecting the point on the path where worker i is located to the horizontal axis.

work is the first to analytically address dynamic work-sharing on U-lines with discrete stations.

3   A path

Let s_j^k denote the work content of $S_j(k)$ and define $s_j = \sum_{k=1}^{m_j} s_j^k$, for $j = 1, 2, 3$. We normalize the total work content of the line such that $\sum_{j=1}^{3} s_j = 1$. The assembly line can be conceptualized as a path with length 1. Figure 2 shows such a path, which is represented by a bold solid line. The start and the end of the path are represented by points 0 and 1 respectively. The intervals $[0, s_1]$, $(s_1, s_1 + s_2]$, and $(s_1 + s_2, 1]$ correspond to the work content of stages 1, 2, and 3 respectively. The horizontal line segments $[0, s_1]$ and $(s_1 + s_2, 1]$ are parallel to each other, and the line segment $(s_1, s_1 + s_2]$ is perpendicular to them.

Define h_i as the *horizontal position* of W_i. This horizontal position is determined by projecting the point on the path where the worker is located to the horizontal axis. Figure 2 shows the relationship between the point on the path where each worker is located and his horizontal position. To distinguish these two coordinate systems, we call any location on the path a *point* and any location on the horizontal axis a *position*.

We set the origin of the horizontal axis to be the projection of point 0 (the start of stage 1) to the axis. Note that a horizontal position can be negative if $s_1 < s_3$. Since stage 2 runs vertically across the aisle, we have $h_i \leq s_1$, for $i = 1, 2$. We require the workers to remain in a fixed sequence along the horizontal axis such that $h_1 \leq h_2$ at any point in time.

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4 The three-station, two-worker U-lines

In this section, we discuss a special case where $m_1 = m_2 = m_3 = 1$. Thus, the notation $v_{ij}(k)$ and $S_j(k)$ can be simplified as v_{ij} and S_j, respectively, for $i = 1, 2$ and $j = 1, 2, 3$. We fully analyze the dynamics of this special case and determine the asymptotic behavior and throughput of the system in closed-form expressions. Understanding the behavior of the three-station system will help us in the analysis of the M-station case.

4.1 Definitions and rules

We say W_1, who is working on S_1, *meets* W_2, who is working on S_3, when their horizontal positions coincide (that is, $h_1 = h_2$). When W_1 meets W_2, a *hand-off* between the two workers occurs: Each worker relinquishes his item, walks across the aisle, and takes over each other's item. After the hand-off, W_1 works on S_3, while W_2 proceeds on S_1.

Figure 3 shows how the two workers move on the U-line. Let x_n denote the n-th hand-off position. At the n-th hand-off, the two workers first relinquish their work and then walk across the aisle. After they exchange their work, W_1 works on S_3 with velocity v_{13}. When he finishes his work on S_3, he walks instantaneously to the start of S_1, initiates a new item, and works on S_1 as soon as the station is free. Meanwhile, W_2 works on S_1 with velocity v_{21}. After he reaches the end of S_1, he continues to work on S_2. W_2 then works on S_3, as soon as the station is free, until he meets W_1 again at position x_{n+1}.

Specifically, the workers follow the simple rules below:

Rule for W_1:

- If you are on S_1, assemble your item until you meet W_2. Then exchange work with W_2 and work on S_3.
- If you are on S_3, assemble your item until you complete it. Then initiate a new item and work on S_1.

Rule for W_2:

- Assemble your item along the assembly line until you meet W_1. Then exchange work with W_1 and work on S_1.
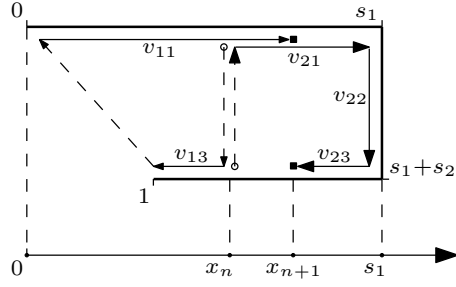
Figure 3: **Movements of workers on three stations.** This figure shows the movements of the two workers between the n-th and $(n+1)$-st hand-offs on a three-station U-line. The solid arrows correspond to working, while the dashed arrows correspond to instantaneous walk. The start and the end of each worker's movement are represented by a circle and a square respectively.

We call the above the cellular bucket brigade rules.

W_1 is *blocked* at point 0 if he reaches the start of S_1 while his colleague is still working on S_1. Similarly, W_2 is blocked at point $s_1 + s_2$ if he reaches the start of S_3 while his colleague is still working on S_3.

If W_1 reaches the end of S_1 before he meets his colleague, then W_1 is *halted* at point s_1. If W_1 is halted, he remains idle until a hand-off occurs immediately after W_2 finishes his work on S_2. After the hand-off, W_1 works on S_3 while W_2 reenters S_2. On the other hand, if W_2 reaches the end of S_3 before he meets his colleague, then W_2 is halted at point 1. Note that W_2 can be halted only if $s_1 > s_3$. If W_2 is halted, he remains idle until a hand-off occurs when the horizontal positions of the two workers coincide.

4.2   Dynamics and throughput

Given the stations' work content and the workers' work velocities, we determine the asymptotic dynamic behavior and the throughput of the system for any initial state. According to the cellular bucket brigade rules, if $s_1 > s_3$ then $h_1 \in [0, s_1]$ and $h_2 \in [s_1 - s_3, s_1]$. Otherwise, $h_1 \in [s_1 - s_3, s_1]$ and $h_2 \in [0, s_1]$. Thus, any hand-off position falls in the interval $I = [\max\{s_1 - s_3, 0\}, s_1]$ on the horizontal axis. Let $f : I \mapsto I$ be a function such that $x_{n+1} = f(x_n)$. The sequence of iterates $x_1, x_2, x_3, \ldots$ is called the *orbit* of an initial iterate x_0 under f (Alligood et

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al. 1996). We say x^* is a fixed point if $x^* = f(x^*)$. A *period-2 orbit* is an orbit that alternates between p and q, where $p = f(q)$ and $q = f(p)$. Note that $f(f(p)) = p$ and $f(f(q)) = q$.

We first construct the function f and then determine the asymptotic behavior of the cellular bucket brigade by analyzing f (see Appendix A). We show that the system either converges to a fixed point or a period-2 orbit. We find closed-form expressions of the fixed point, the period-2 orbit, and the corresponding throughput. Some of these expressions are too complex and thus we only summarize the main results in the following paragraphs. The details can be found in Appendix A. The dynamics of the system can be characterized by the following parameter:

$$\varphi = \frac{1/v_{21} - 1/v_{23}}{1/v_{11} - 1/v_{13}}.$$

There are two cases: (1) $\varphi \leq 1$ and (2) $\varphi > 1$, which give rise to distinct dynamics and are discussed as follows.

4.2.1 Case 1: $\varphi \leq 1$

Figure 4(a) summarizes the asymptotic behavior of the three-station, two-worker system for $\varphi \leq 1$. Each point (s_1, s_3) in Figure 4(a) represents a work-content distribution on the stations. Figure 4(a) shows that, for any given work velocities, the work-content distributions can be grouped into five regions. Each region corresponds to a distinct asymptotic behavior. For convenience, let $\mu_{ij} = v_{22}/v_{ij}$ denote the unit work time of worker i on stage j normalized by the unit work time of worker 2 on stage 2, for $i = 1, 2$ and $j = 1, 3$. For brevity, define

$$\eta_1 = \frac{1 + (\mu_{23} - 1)s_1 - s_3}{\mu_{11} + \mu_{23}};$$

$$\eta_2 = \frac{\mu_{23}s_1}{\mu_{11} + \mu_{23}};$$

$$\eta_3 = \frac{1 + (\mu_{13} + \mu_{21} + \mu_{23} - 1)s_1 - (\mu_{13} + 1)s_3}{\mu_{11} + \mu_{13} + \mu_{21} + \mu_{23}};$$

$$\gamma(x) = \frac{1 + (\mu_{13} + \mu_{21} + \mu_{23} - 1)s_1 - (\mu_{13} + 1)s_3 - (\mu_{13} + \mu_{21})x}{\mu_{11} + \mu_{23}}.$$

For any initial workers' locations on the U-line, the asymptotic behavior in each region is summarized as follows.
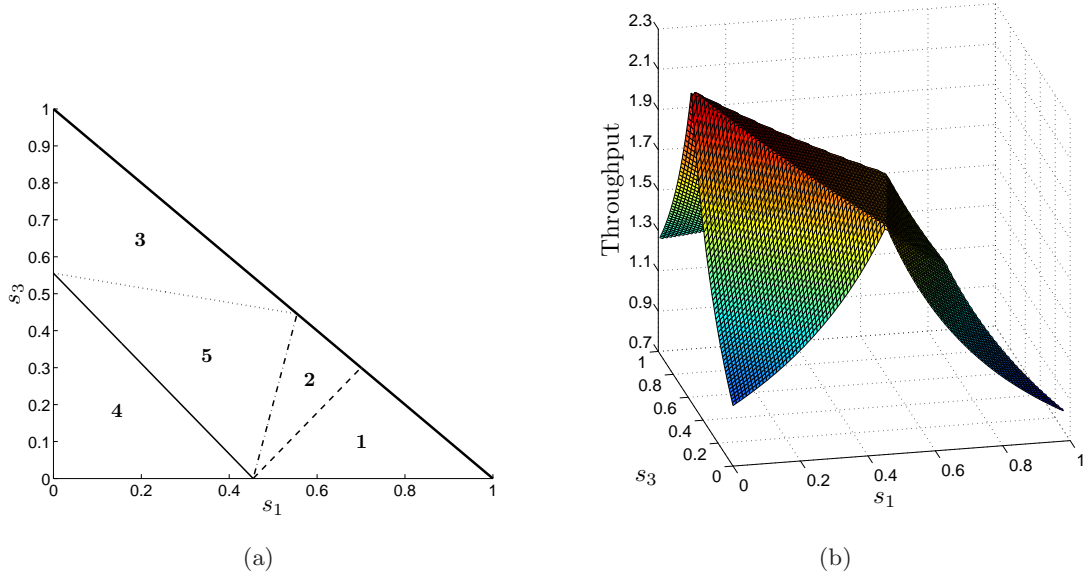
Figure 4: **Asymptotic behaviors and throughput ($\varphi \leq 1$).** (a) The cellular bucket brigade has different asymptotic behaviors in different regions. (b) The throughput in each region has a different expression. For both graphs, we set $v_{11} = 5/6$, $v_{13} = 5/4$, $v_{21} = 10/7$, $v_{22} = 1$, and $v_{23} = 5/8$.

Region 1 (Blocking and halting): This region corresponds to systems with "long" S_1 and "short" S_3. The system converges to a fixed point $x^* = s_1 - s_3$. At the fixed point, W_1 is blocked at point 0 and W_2 is halted at point 1 in each iteration.

Region 2 (Blocking): The system converges to a fixed point $x^* = \eta_1$. At the fixed point, W_1 is blocked at point 0 in each iteration.

Region 3 (Blocking): This region corresponds to systems with "short" S_1 and "long" S_3. The system converges to a fixed point $x^* = \eta_2$. At the fixed point, W_2 is blocked at point $s_1 + s_2$ in each iteration.

Region 4 (Halting): Both S_1 and S_3 are "short" in this region. The system converges to a fixed point $x^* = s_1$. At the fixed point, W_1 is halted at point s_1 in each iteration.

Region 5 (Nonidling): If $\varphi < 1$, the system converges to a fixed point $x^* = \eta_3$. If $\varphi = 1$, the system converges to a period-2 orbit: x and $\gamma(x)$, where x depends on the initial workers'

locations on the line. Neither blocking nor halting occurs in this region.

It is noteworthy that if $\varphi < 1$, the system always converges to a unique fixed point in each region. Thus, given a work-content distribution on the stations, the system always converges to a unique fixed point if the following condition is satisfied.

Convergence Condition (3-Station U-Lines):

$$1/v_{11} - 1/v_{13} > 1/v_{21} - 1/v_{23}. \tag{1}$$

The condition above can be interpreted as follows: The term $1/v_{i1} - 1/v_{i3}$ represents the extra work time worker i needs on stage 1 compared to stage 3 to complete a unit of work. According to the condition, a worker with larger extra work time on stage 1 should be assigned a lower index. In other words, *a worker who is slower in stage 1 but faster in stage 3 than his colleague should be assigned a lower index.*

When the system operates on a fixed point, W_1 repeatedly works in a loop on the left of Figure 3, while W_2 covers a loop on the right that includes stage 2. Convergence to a fixed point could be desirable because each worker repeats the same portion of work on each item produced. This allows workers to learn more efficiently as each of them concentrates on a set of possibly nonconsecutive tasks. Furthermore, each worker covers a set of tasks that are physically close to each other especially if the aisle is narrow. This reduces travel of workers and thus may substantially boost productivity in practice. All other attractive characteristics of traditional bucket brigades on a straight-line layout are preserved under the U-line layout. For example, the system constantly seeks balance and the output is regular.

Figure 4(b) shows the long-run average throughput in each region. The throughput in each region has a different expression, which can be found in Appendix A. Even though there is neither blocking nor halting in Region 5, the throughput in this region may be lower than that of other regions. This is because each worker has different work velocities on different stations. Although there is no idleness in Region 5, a worker may repeatedly work on a station where he is slow. On the other hand, although a worker may be blocked or halted in other regions, he

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may repeatedly work on stations where he is fast. This phenomenon does not exist in a system where each worker has a constant velocity over all stations. In that case, it is guaranteed that a region with no idleness will have the highest throughput.

4.2.2 Case 2: $\varphi > 1$

If $\varphi > 1$, the asymptotic behaviors and the expressions of throughput remain the same in all regions except for Region 5, which is now partitioned into seven subregions as shown in Figure 5(a). We summarize the system's asymptotic behavior in each subregion as follows.

Region 5a: The system converges to a period-2 orbit: η_1 and $\gamma(\eta_1)$. On the period-2 orbit, W_1 is blocked at point 0 in every other iteration.

Region 5b: The system converges to a period-2 orbit: η_2 and $\gamma(\eta_2)$. On the period-2 orbit, W_2 is blocked at point $s_1 + s_2$ in every other iteration.

Region 5c: The system converges to a period-2 orbit: s_1 and $\gamma(s_1)$. On the period-2 orbit, W_1 is halted at point s_1 in every other iteration.

Region 5d: The system converges to a period-2 orbit: η_1 and $s_1 - s_3$. On the period-2 orbit, W_1 and W_2 are repeatedly idle in alternative iterations: W_1 is blocked at point 0 in one iteration, and W_2 is halted at point 1 in the next iteration.

Region 5e: The system converges to a period-2 orbit: η_1 and η_2. On the period-2 orbit, W_1 and W_2 are repeatedly idle in alternative iterations: W_1 is blocked at point 0 in one iteration, and W_2 is blocked at point $s_1 + s_2$ in the next iteration.

Region 5f: The system converges to a period-2 orbit: s_1 and η_2. On the period-2 orbit, W_1 and W_2 are repeatedly idle in alternative iterations: W_1 is halted at point s_1 in one iteration, and W_2 is blocked at point $s_1 + s_2$ in the next iteration.

Region 5g: The system converges to a period-2 orbit: s_1 and $s_1 - s_3$. On the period-2 orbit, W_1 and W_2 are repeatedly idle in alternative iterations: W_1 is first blocked at point 0 and
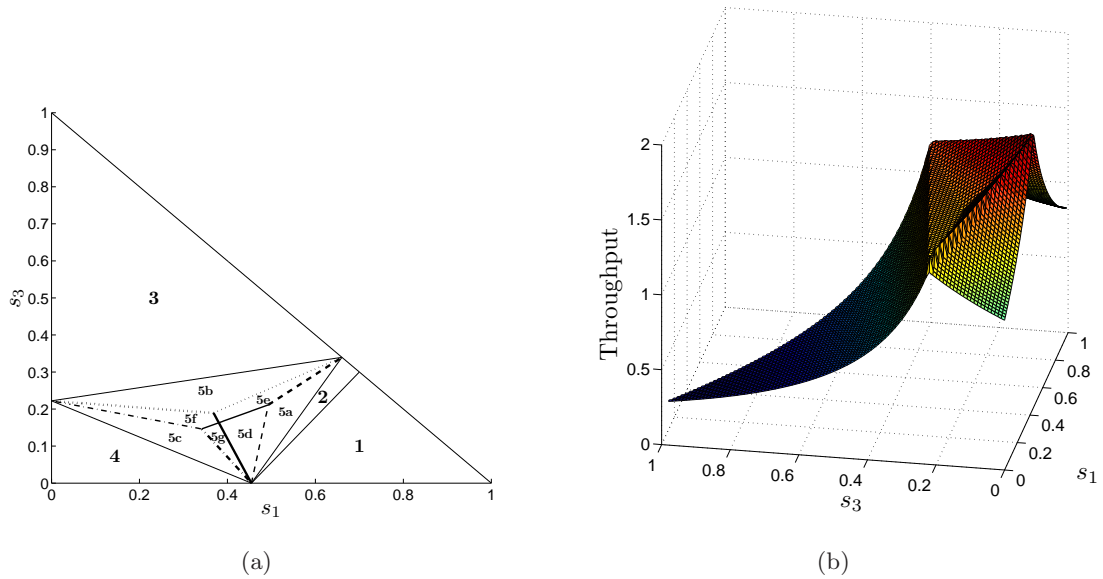
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Figure 5: **Asymptotic behaviors and throughput ($\varphi > 1$).** (a) Region 5 is partitioned into seven subregions. (b) The throughput in each subregion of Region 5 has a different expression. For both graphs, $v_{11} = 5/6$, $v_{13} = 2/7$, $v_{21} = 10/7$, $v_{22} = 1$, and $v_{23} = 5/8$.

then halted at point s_1 in one iteration, and W_2 is halted at point 1 in the next iteration.

Figure 5(b) shows that each subregion of Region 5 may have different throughput, which may be lower than that of other regions.

4.2.3 Summary

There is blocking and halting in Regions 1 to 4 independent of the value of φ. This is because the three stages of the U-line are less balanced in these regions: Stages 1, 2, and 3 have relatively more work content than other stages in Regions 1–2, 4, and 3 respectively. In contrast, the work content of the three stages is more balanced in Region 5, which allows the system to avoid blocking and halting if the workers are sequenced properly. Specifically, if the workers are sequenced such that $\varphi < 1$, then the system always converges to a fixed point with no blocking or halting in Region 5. If $\varphi = 1$, the system converges to a period-2 orbit, which depends on the initial workers' locations, with no blocking or halting. If $\varphi > 1$, the system converges to different period-2 orbits with blocking and/or halting in different parts of Region 5.

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4.3 Choosing the best worker sequence

Suppose the velocities of workers and the work content of stations are given, what is the best sequence of workers to maximize throughput? Since there are only two possible sequences for the two-worker system, we can calculate their respective values of φ. For each sequence, the throughput under the given work-content distribution can be obtained from the results of Section 4.2.1 (if $\varphi \leq 1$) or Section 4.2.2 (if $\varphi > 1$). The sequence that gives a higher throughput should be chosen. For example, there are two possible sequences given workers A and B: W_1 is worker A and W_2 is worker B (denoted by AB), or W_1 is worker B and W_2 is worker A (denoted by BA). Figure 6(a) shows the best sequence of workers for each work-content distribution with $v_{A1} < v_{B1}$, $v_{A2} = v_{B2}$, and $v_{A3} > v_{B3}$.

The sequences AB and BA perform equally well near the bottom-left corner of Figure 6(a). This is because the bottom-left corner of the figure corresponds to Region 4 in Section 4.2 (see Figures 4(a) and 5(a)). In Region 4, stage 2 is a bottleneck because it has relatively more work content than other stages. Thus, the throughput of a worker sequence is determined by the worker that works on stage 2. Since both workers A and B have the same velocity on stage 2, both sequences AB and BA have the same throughput in Region 4.

In other areas of Figure 6(a), the preferred worker sequence depends on the workers' velocities and the relative amounts of work in the three stages. For example, stage 1 has relatively more work content than other stages near the bottom-right corner of Figure 6(a). We should assign a worker who is fast on stage 1 to the left. The sequence BA is preferred because $v_{A1} < v_{B1}$. Similarly, stage 3 has relatively more work content than other stages near the top-left corner of Figure 6(a). The sequence AB is preferred because $v_{A3} > v_{B3}$.

Figure 6(b) represents a case where worker B is faster than worker A in both stages 1 and 3. Apparently, the sequence BA dominates as it allows the faster worker to do more work. We summarize the above findings as follows.

1. If stage 2 has relatively more work content than other stages, then assign a worker who is faster than his colleague in stage 2 to the right of the U-line.
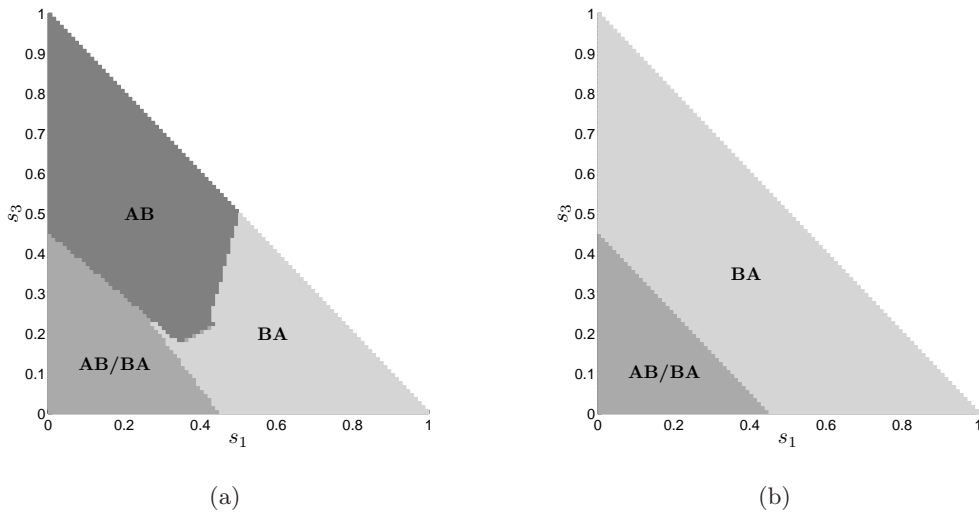
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Figure 6: **Best sequences.** The sequence AB dominates in the dark-shaded area, whereas the sequence BA dominates in the light-shaded area. Both sequences perform equally well in the remaining area. (a) $v_{A1} = 4/5$, $v_{A2} = 1$, $v_{A3} = 5/4$, $v_{B1} = 5/4$, $v_{B2} = 1$, and $v_{B3} = 4/5$. (b) $v_{A1} = 4/5$, $v_{A2} = 1$, $v_{A3} = 4/5$, $v_{B1} = 5/4$, $v_{B2} = 1$, and $v_{B3} = 5/4$.

2. If stage 1 (stage 3) has relatively more work content than other stages, then assign a worker who is faster than his colleague in stage 1 (stage 3) to the left of the U-line.

5 The M-station, two-worker U-lines

In this section we analyze the dynamics and determine the throughput of U-lines with M stations and two workers. Unlike in the three-station case, we cannot find a closed-form expression of the dynamic function f. However, we show that the system converges to either a fixed point or a period-2 orbit. We determine the fixed point and the corresponding throughput using an algorithmic approach.

5.1 Definitions and rules

Recall that a worker is in stage $j \in \{1, 2, 3\}$ if he is on station $S_j(k)$ for any $k \in \{1, \ldots, m_j\}$. We say W_1, who is working on stage 1, meets W_2, who is working on stage 3, when their horizontal positions coincide (that is, $h_1 = h_2$). There are three different types of hand-offs in the M-

station U-line. A hand-off of type I occurs when W_1 meets W_2. Each worker first relinquishes his item, walks across the aisle, and then takes over each other's item. After the hand-off, W_1 works on stage 3 while W_2 proceeds on stage 1. The movements of the two workers beginning with a type I hand-off are similar to that shown in Figure 3.

Since there are multiple stations in stage 3, W_2 can enter stage 3 as long as he is not blocked by W_1. The two workers can work on different stations in stage 3 simultaneously. A type II hand-off occurs when W_1 completes an item at the end of stage 3 and W_2 has passed the origin of the horizontal axis ($h_2 < 0$). In this case W_1 walks back, takes over work from W_2 at the horizontal position $h_2 < 0$, and continues the work on stage 3. Meanwhile, W_2 walks across the aisle and initiates a new item in stage 1. Note that type II hand-offs are possible only if $s_1 < s_3$. Figure 7(a) shows the movements of the two workers beginning with a type II hand-off.

Similarly, W_1 can enter stage 2 as long as he is not blocked by W_2. The two workers can work on different stations in stage 2 simultaneously. A type III hand-off occurs when W_2 reaches the end of stage 2 while W_1 is working on stage 2. The two workers exchange work so that W_1 works on stage 3 and W_2 reenters stage 2 after the hand-off. Figure 7(b) shows the movements of the two workers beginning with a type III hand-off. To map a hand-off point in stage 2 to a unique position on the horizontal axis, we rotate counterclockwise the path segment in stage 2 by 90° (see the dotted arc in Figure 7(b)). The rightmost dashed vertical line in Figure 7(b) shows how the horizontal position x_n is determined after a type III hand-off point in stage 2 is projected to the horizontal axis. Note that type II and type III hand-offs do not occur in the three-station system.

The cellular bucket brigade rules for the M-station U-line are given as follows.

Rule for W_1:

- If you are in stage 1 or 2, assemble your item until you exchange work with W_2. Then work in stage 3.
- If you are in stage 3, assemble your item until you complete it. Upon completion,
  1. if the horizontal position of W_2 is nonnegative then initiate a new item and work in stage 1;
  2. otherwise, take over work from W_2 and continue the work in stage 3.
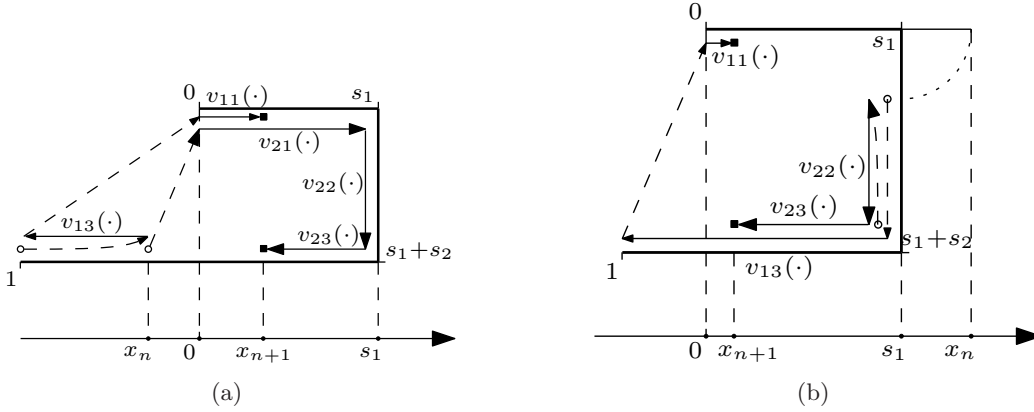
Figure 7: **Movements of workers on M stations.** Both graphs show the movements of the two workers between the n-th and $(n+1)$-st hand-offs on an M-station U-line. The arrows, the circles, and the squares are interpreted in the same way as in Figure 3. (a) Type II hand-off: The n-th hand-off occurs when W_1 completes an item at point 1. He then takes over work from W_2, whose horizontal position is negative. (b) Type III hand-off: The n-th hand-off occurs when W_2 reaches the end of stage 2. He then exchanges work with W_1, who is also in stage 2.

Rule for W_2: Assemble your item along the assembly line until

- you exchange work with W_1, who is in stage 1 (2), then work in stage 1 (2); or
- your item is taken over by W_1, then initiate a new item and work in stage 1.

In the M-station U-line, W_1 can be blocked at the start of any station in stages 1 and 2, whereas W_2 can be blocked at the start of any station in stage 3. If W_2 reaches the end of stage 3 before he meets his colleague, then W_2 is halted at point 1. Note that W_2 can be halted only if $s_1 > s_3$. If W_2 is halted, he remains idle until a hand-off occurs when the horizontal positions of the two workers coincide.

5.2 Dynamics

According to the cellular bucket brigade rules for the M-station U-line, any hand-off position falls in the interval $I = [s_1 - s_3, s_1 + s_2 - s_2^{m_2}]$ on the horizontal axis. Note that the position of a type II hand-off is negative (see Figure 7(a)), and the position of a type III hand-off falls in $(s_1, s_1 + s_2 - s_2^{m_2}]$ (see Figure 7(b)). Let $f : I \mapsto I$ be a function such that $x_{n+1} = f(x_n)$.

Due to numerous combinations of numbers of stations in the three stages of the U-line, we cannot enumerate each possible case and determine the dynamic function f in closed form (such

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as the one for the three-station case in Appendix A.1). However, we prove that f is continuous, non-increasing, and piecewise linear (see Appendix B.1). These properties of f enable us to determine the asymptotic behavior of the M-station system.

Specifically, we show that the system has a unique fixed point x^* and has no periodic orbits of period greater than 2 (see Lemma 9 in Appendix B.2). We say a hand-off position x is *interior* if the points corresponding to x on the U-line fall in the interior of some stations. We find a sufficient condition for the system to converge to the fixed point x^*, independent of the initial workers' locations on the U-line:

Convergence Condition (M-Station U-Lines): For any pair of interior hand-off positions x and $f(x)$, one of the following cases should hold:

1. $0 < x < s_1$, $0 < f(x) < s_1$, where x falls in $S_1(k_1)$ and $S_3(k_2)$ while $f(x)$ falls in $S_1(k_3)$ and $S_3(k_4)$, and

$$1/v_{11}(k_3) - 1/v_{13}(k_2) > 1/v_{21}(k_1) - 1/v_{23}(k_4). \tag{2}$$

2. $0 < x < s_1$, $f(x) < 0$, where x falls in $S_1(k_1)$ and $S_3(k_2)$ while $f(x)$ falls in $S_3(k_4)$, and

$$-1/v_{13}(k_2) > 1/v_{21}(k_1) - 1/v_{23}(k_4). \tag{3}$$

3. $x < 0$, $0 < f(x) < s_1$, where x falls in $S_3(k_2)$ while $f(x)$ falls in $S_1(k_3)$ and $S_3(k_4)$, and

$$1/v_{11}(k_3) - 1/v_{13}(k_2) > -1/v_{23}(k_4). \tag{4}$$

4. $x < 0$, $f(x) < 0$, where x falls in $S_3(k_2)$ while $f(x)$ falls in $S_3(k_4)$, and

$$-1/v_{13}(k_2) > -1/v_{23}(k_4). \tag{5}$$

5. $x > s_1$, $0 < f(x) < s_1$, where x falls in $S_2(k_1)$ while $f(x)$ falls in $S_1(k_3)$ and $S_3(k_4)$, and

$$1/v_{11}(k_3) > 1/v_{22}(k_1) - 1/v_{23}(k_4). \tag{6}$$

6. $x > s_1$, $f(x) < 0$, where x falls in $S_2(k_1)$ while $f(x)$ falls in $S_3(k_4)$, and

$$0 > 1/v_{22}(k_1) - 1/v_{23}(k_4). \tag{7}$$

7. $0 < x < s_1$, $f(x) > s_1$, where x falls in $S_1(k_1)$ and $S_3(k_2)$ while $f(x)$ falls in $S_2(k_3)$, and

$$1/v_{12}(k_3) - 1/v_{13}(k_2) > 1/v_{21}(k_1). \tag{8}$$

8. $x < 0$, $f(x) > s_1$, where x falls in $S_3(k_2)$ while $f(x)$ falls in $S_2(k_3)$, and

$$1/v_{12}(k_3) - 1/v_{13}(k_2) > 0. \tag{9}$$

9. $x > s_1$, $f(x) > s_1$, where x falls in $S_2(k_1)$ while $f(x)$ falls in $S_2(k_3)$, and

$$1/v_{12}(k_3) > 1/v_{22}(k_1). \tag{10}$$

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We show that the system always converges to the fixed point x^* if the above Convergence Condition holds (see Lemma 10 in Appendix B.2 for details). Note that if $s_1 \geq s_3$ then both x and $f(x)$ are non-negative, and so only the inequalities (2), (6), (8), and (10) are needed to ensure convergence.

It is noteworthy that inequality (2) is a generalized version of inequality (1). Thus, the Convergence Condition for M-station U-lines is a generalization of that for three-station U-lines. However, in an M-station system we need to consider workers' velocities on specific stations in a stage because each stage may contain multiple stations. If at least one of the hand-off positions x and $f(x)$ is negative, some terms are missing from inequality (2) because the stations corresponding to those terms do not exist. This leads to inequalities (3)–(5). If x falls in stage 2 (due to a type III hand-off), then $v_{21}(k_1)$ is replaced by $v_{22}(k_1)$. Similarly, if $f(x)$ falls in stage 2, then $v_{11}(k_3)$ is replaced by $v_{12}(k_3)$. These lead to inequalities (6)–(10).

For a three-station U-line, only type I hand-offs are possible and their horizontal positions are always non-negative. As a result, the Convergence Condition for M-station U-lines reduces to condition (1) for the three-station case. The Convergence Condition for M-station U-lines can be interpreted in a similar manner using the intuition derived from condition (1).

Checking the Convergence Condition for the M-station system only requires enumeration of all possible values of $k_i, i = 1, \ldots, 4$. It is straightforward to check this condition as there are at most $(m_1 + m_2 + m_3)^2$ combinations of $k_i, i = 1, \ldots, 4$. Furthermore, the condition can be checked easily if $v_{ij}(k) = v_{ij}$, where v_{ij} is a constant, for $k = 1, \ldots, m_j$.

We develop algorithms to calculate the fixed point x^* and its throughput for the M-station U-line. Based on these algorithms, we investigate the impact of the number of stations in each stage on throughput.
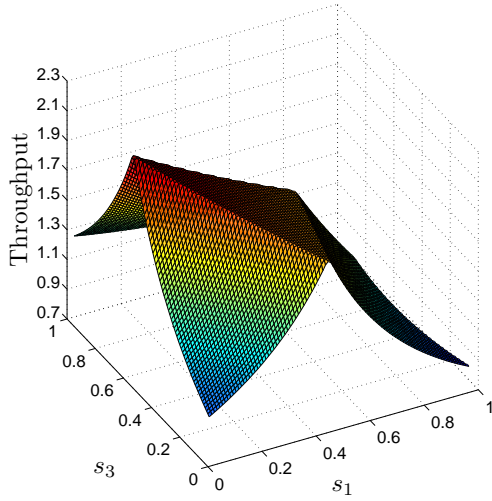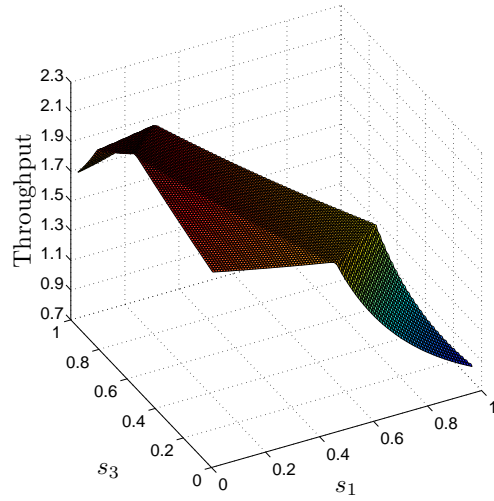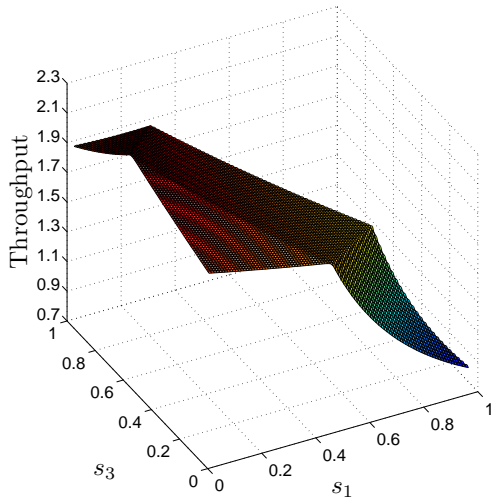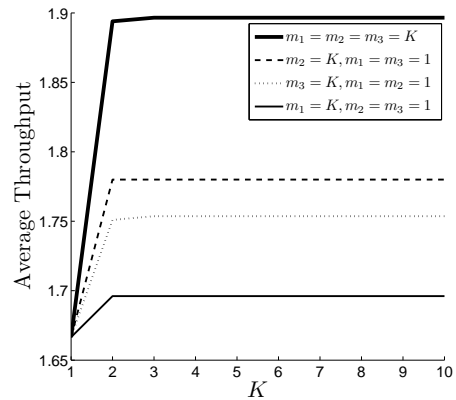
5.3 Impact of number of stations on throughput

Increasing the number of stations in each stage makes the system more flexible because a worker can enter a station in a stage as long as he is not blocked by his colleague. This reduces the

idling time of workers and may potentially result in higher throughput than the three-station U-line. We examine the throughput by increasing the number of stations in each stage.

Figure 8(a) shows the system's throughput with $m_1 = m_2 = m_3 = 1$ for different work-content distributions. Figure 8(b) shows the throughput when the number of stations in each stage increases to 2. The throughput increases in Regions 1 and 2 (see Figure 4(a)). This is because in the three-station system W_1 is repeatedly blocked at the start of stage 1 when the system operates on the corresponding fixed point. Dividing stage 1 into more stations reduces the time of W_1 being blocked and so increases the throughput. Similarly, dividing stage 3 into more stations reduces the time of W_2 being blocked at the start of stage 3. This increases the throughput in Region 3. Likewise, dividing stage 2 into more stations reduces the time of W_1 being halted at the start of stage 2. This increases the throughput in Region 4. Finally, the throughput remains unchanged in Region 5. This is because in the three-station system there is neither blocking nor halting in this region. Thus, increasing the number of stations in each stage will not improve the throughput in this region.

Does productivity in Regions 1 to 4 continue to increase as the number of stations in each stage increases? Figure 8(c) shows the throughput when $m_1 = m_2 = m_3 = 10$. In comparison with Figure 8(b), the throughput is significantly improved only in Region 3. Given the number of stations m_j in each stage j, Figure 8(d) shows the average throughput of all work-content distributions (s_1, s_2, s_3) over the three stages. We assume $s_j^k = s_j/m_j$, for $j = 1, 2, 3$, $k = 1, \ldots, m_j$. The average throughput significantly increases if the number of stations in each stage increases from 1 to 2. However, the average throughput is only marginally improved and soon becomes constant if each stage is further divided into more stations.

(a) K = 1

(b) K = 2

(c) K = 10

(d)

Figure 8: **Impact of number of stations.** (a–c) We set $m_1 = m_2 = m_3 = K$. The throughput remains unchanged in Region 5, but may increase in other regions as K increases. (d) The average throughput increases significantly when the number of stations in any stage increases from 1 to 2, but soon becomes constant as the number of stations further increases. For all graphs, we set $v_{11}(k) = 5/6$, $v_{21}(k) = 10/7$, $v_{12}(k) = v_{22}(k) = 1$, $v_{13}(k) = 5/4$, $v_{23}(k) = 5/8$, and $s_j^k = s_j/m_j$, for $k = 1, \ldots, m_j$, $j = 1, 2, 3$.

Sometimes it could be expensive to divide a stage into more stations. In this situation knowing where flexibility can add the most value becomes important. Our analysis allows us to identify an attractive candidate to be divided into more stations. For example, Figure 8(d) suggests that, for this particular velocity setting, increasing m_2 from 1 to 2 gives the largest improvement in average throughput compared to increasing m_1 or m_3. Figure 8(d) also suggests that the throughput is significantly improved if we increase the number of stations in *every* stage from 1 to 2. Note that we still have blocking and halting for the systems with $m_1 = m_2 = m_3 = K \geq 2$ in Figure 8(d). However, the idle time of workers in these systems is reduced due to the systems' extra flexibility (more stations in each stage). This results in a significant gap between the top curve (with $m_1 = m_2 = m_3 = K$) and the lower curves in Figure 8(d).

6   Performance under random work velocities

Sections 4 and 5 assume no variability in each $v_{ij}(k)$: Each worker has a constant and deterministic work velocity on each station. In practice, the work velocity of each worker on each station is usually neither constant nor deterministic. To demonstrate that a cellular bucket brigade can absorb the impact of variability in work velocity, we assume the workers have random work velocities in this section. We compare numerically the throughput of the cellular bucket brigade with that of a team based on static allocation of work (called the *static team*).

For the static team, we assume a worker can only exchange work with his colleague after he finishes his work on a station. Thus, the work of a worker cannot be preempted within a station. Each worker in the static team repeats a fixed loop in the U-line. Figure 9 shows an example: W_1 works on stations $S_1(1), \ldots, S_1(\alpha_1)$ in stage 1 and stations $S_3(\alpha_3 + 1), \ldots, S_3(m_3)$ in stage 3; whereas W_2 works on stations $S_1(\alpha_1 + 1), \ldots, S_1(m_1)$ in stage 1, all the stations in stage 2, and stations $S_3(1), \ldots, S_3(\alpha_3)$ in stage 3. See Geismar et al. (2008) for a similar allocation.

Workers in a static team such as the one shown in Figure 9 follow the rules below.
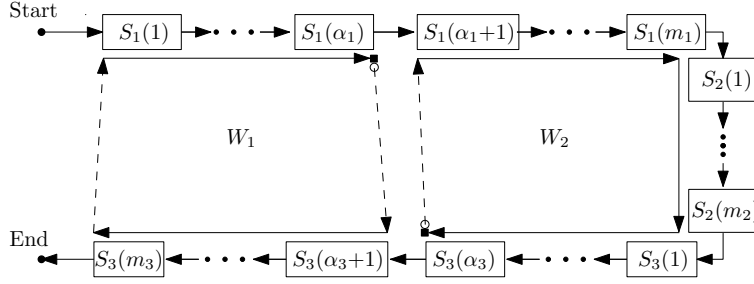
Rule for W_1:

Figure 9: **Static allocation of work.** Each worker repeats a fixed loop under static allocation of work. The solid arrows correspond to working, while the dashed arrows correspond to instantaneous walk.

- If you are in stage 1, assemble your item until you reach the end of $S_1(\alpha_1)$. Then exchange work with W_2 once he finishes the work on $S_3(\alpha_3)$, and work in stage 3.
- If you are in stage 3, assemble your item until you complete it. Then initiate a new item and work in stage 1.

Rule for W_2:

- Assemble your item along the assembly line until you reach the end of $S_3(\alpha_3)$. Then exchange work with W_1 once he finishes the work on $S_1(\alpha_1)$, and work in stage 1.

In Figure 9, W_1 covers the first segment of stage 1 and the second segment of stage 3, while W_2 works on the rest of the stations. We consider $\alpha_1 \in [0, m_1 + m_2]$ and $\alpha_3 \in [-m_2, m_3]$ such that other kinds of work allocation are possible. If $\alpha_1 = 0$, then W_1 will not work on stage 1. If $m_1 \leq \alpha_1 \leq m_1 + m_2$, then W_2 will not work on stage 1 and W_1 relinquishes his work for W_2 in stage 2. For example, $\alpha_1 = m_1$ and $\alpha_1 = m_1 + m_2$ correspond to cases where W_1 relinquishes his work for W_2 at the start and the end of stage 2 respectively. If $-m_2 \leq \alpha_3 \leq 0$, then W_2 will not work on stage 3 and he relinquishes his work for W_1 in stage 2. For example, $\alpha_3 = -m_2$ and $\alpha_3 = 0$ correspond to cases where W_2 relinquishes his work for W_1 at the start and the end of stage 2 respectively. If $\alpha_3 = m_3$, then W_1 will not work on stage 3.

For benchmarking, we also consider an *alternative dynamic team* in which each worker works individually over all stations. At the start of each station, if a faster worker (with a larger mean velocity on the station) is blocked by a slower worker, they will exchange work and so the slower worker will be blocked at the station. Note that this policy does not facilitate learning.

For all the three policies considered, we assume that when W_i works on $S_j(k)$ for the t-

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th time his velocity is $\tilde{v}_{ij}^t(k)$. We define $\tilde{v}_{ij}^t(k) = v_{ij}(k)/(1 + \varepsilon_{ij}^t(k))$, where $v_{ij}(k)$ (velocities in the deterministic model) serve as parameters and $\varepsilon_{ij}^t(k)$ are independent and identically distributed random variables, for $i = 1, 2$, $j = 1, 2, 3$, $k = 1, \ldots, m_j$, and $t = 1, 2, 3, \ldots$. Under this definition, a worker generally has a different velocity every time he revisits a station. We report the results of the case where each $\varepsilon_{ij}^t(k)$ follows a normal distribution $\mathcal{N}(0, \sigma^2)$, where the standard deviation σ is a parameter. Similar results are observed if each $\varepsilon_{ij}^t(k)$ follows a uniform distribution with mean 0.

We compare the average throughput of the three policies by simulations. Given any σ, an *experiment* corresponds to a specific set of values for the parameters s_j^k and $v_{ij}(k)$, $i = 1, 2$, $j = 1, 2, 3$, $k = 1, \ldots, m_j$. Each experiment consists of 10 simulation runs. In each simulation run, 500 items are produced after the system is stabilized. In each experiment, the indices α_1 and α_3 for the static team are determined such that its average throughput with deterministic work velocities $v_{ij}(k)$ is maximized. This can be done by enumerating all possible locations in stages 1, 2, and 3 for exchanging work.

We consider each stage has two stations with evenly distributed work content, and the parameter $v_{ij}(k)$ equals a constant v_{ij} for all $k = 1, \ldots, m_j$. Similar results are observed when we consider more complex situations such as more stations per stage, uneven work content on the stations, or general $v_{ij}(k)$. For each σ, we set $v_{12} = v_{22} = 1.0$ and consider $v_{ij} \in \{2/3, 1, 2\}$, for $i = 1, 2$, $j = 1, 3$, and (s_1, s_2, s_3) are chosen from the combinations in Table 1.

Table 1: **Different work-content distributions on the three stages of the U-line.**

Combinations	1	2	3	4	5	6	7
s_1	0.6	0.2	0.2	0.4	0.4	0.2	1/3
s_2	0.2	0.6	0.2	0.4	0.2	0.4	1/3
s_3	0.2	0.2	0.6	0.2	0.4	0.4	1/3

This results in 567 (that is, $3^4 \times 7$) experiments for each σ.

Define the *percentage efficiency* of a policy as

$$\frac{\text{Average throughput of the policy over all experiments}}{\text{Average throughput of the static team over all experiments}} \times 100\%.$$

A policy, on average, outperforms the static team if the former's percentage efficiency is larger than 100%. Figure 10(a) shows the cellular bucket brigade's percentage efficiency for various values of σ, incremented by 0.025, in $[0, 0.5]$. This efficiency is always larger than 100% and it increases with σ. The cellular bucket brigade can better absorb the impact of variability in work velocity and its performance relative to the static team improves as variability increases. The performance of the static team is close to that of the cellular bucket brigade when variability is low. This is because we optimize the work allocation for workers in the static team: For each experiment, we optimize α_1 and α_3 such that the average throughput of the static team with deterministic work velocities $v_{ij}(k)$ is maximized. In contrast, we do not optimize the worker sequence of the cellular bucket brigade in each experiment.

The alternative dynamic team is the least productive among the three policies when variability is low. However, its performance improves and becomes similar to that of the cellular bucket brigade as variability increases. The problem of the alternative dynamic team is that even if a faster worker exchanges work with a slower worker at a station, the slower worker will still need to work on the station (where he is slow) after the faster worker leaves the station.

Figure 10(b) shows the percentage of experiments where the average throughput of the cellular bucket brigade is greater than or equal to that of the static team for various σ. This percentage generally increases with σ. The cellular bucket brigade dominates in at least 53% of the 567 experiments for each σ. The curve given by the alternative dynamic team overlaps with the curve of the cellular bucket brigade. Although the percentage of experiments that the alternative dynamic team outperforms the static team is the same as that of the cellular bucket brigade, the average throughput of the alternative dynamic team is significantly lower than that of the cellular bucket brigade if σ is small (see Figure 10(a)).

Figure 10 suggests that the cellular bucket brigade is promising because it does not require optimization on work allocation, unlike the static team where we need to find the best loops for the workers in advance. By following the simple coordination rules, the cellular bucket brigade can better accommodate the impact of random work velocities and maintain higher efficiency.
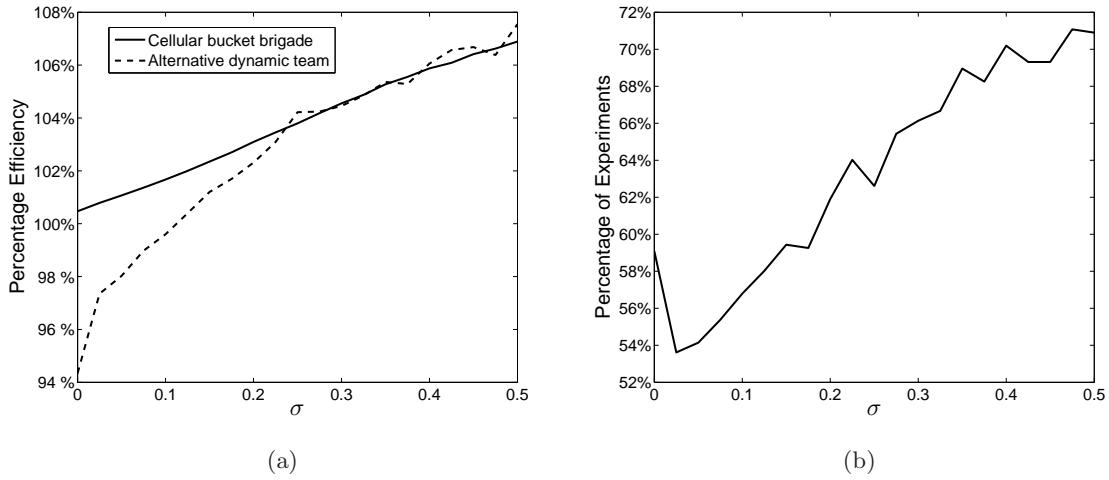
Figure 10: **Average throughput comparison.** (a) The percentage efficiency of both the cellular bucket brigade and the alternative dynamic team increases with variability in work velocity. (b) The percentage of experiments in which the throughput of the cellular bucket brigade is greater than or equal to that of the static team generally increases with variability.

7   Conclusions

Maximizing and maintaining productivity of a U-line with discrete stations under the impact of variability can be challenging. This is because maximizing productivity requires assigning workers to suitable tasks and maintaining productivity requires sufficient flexibility in task assignment to absorb the impact of variability.

To achieve this goal, we propose the cellular bucket brigade rules to coordinate workers on the U-line with discrete stations. Under these rules the system's productivity can be maximized by properly choosing the worker sequence. Workers are allowed to dynamically share work (they are not assigned and restricted to a fixed set of stations) such that the system can effectively absorb the impact of variability. Our approach also reduces travel by allowing each worker to work on nonconsecutive stations that are physically close to each other. We believe this may boost productivity in the actual implementation even though we neglect the travel time between stations in our model. In addition, the rules are easy to follow and implement in practice.

We analyze a deterministic model with two workers and multiple stations. Each worker handles a single item at a time, and at most one worker is allowed to work on a station at

any time. We assume worker- and station-dependent work velocities. The analysis of the system dynamics is nontrivial due to possible blocking and halting of workers and numerous combinations of numbers of stations, work-content distributions, and work velocities.

The three-station system always converges to a fixed point or a period-2 orbit for a given work-content distribution. For the system to converge to the fixed point, a worker who is slower in stage 1 but faster in stage 3 than his colleague should be assigned a lower index. Based on closed-form expressions of the fixed point, the period-2 orbit, and the corresponding throughput, we identify the best worker sequence that maximizes the system's throughput.

Convergence to a fixed point could be desirable because each worker repeatedly works in the same loop on the U-line, which facilitates learning. The travel of workers is also reduced as each worker executes tasks that are physically close to each other especially if the aisle is narrow. All other attractive characteristics of traditional bucket brigades on a straight-line layout are preserved under the U-line layout. For example, the system has regular output on the fixed point and it can restore balance after disruptions.

Similarly, the M-station system always converges to a fixed point or a period-2 orbit for a given number of stations in each stage and a given work-content distribution on the stations. We identify a sufficient condition for the system to converge to the fixed point. This condition is a generalization of that of the three-station case. It can be interpreted in a similar manner using the intuition derived from the three-station system and can be tested efficiently. We find that dividing each stage into more stations will improve the throughput if the system falls in Regions 1 to 4, but will not boost productivity if the system falls in Region 5. The throughput is significantly improved as the number of stations in each stage increases from 1 to 2, but there is diminishing return if we further divide each stage into more stations.

To evaluate the performance of the cellular bucket brigade under random work velocities, we compare it with a team based on optimized, static allocation of work. Our simulation results suggest that the cellular bucket brigade is more productive and its performance relative to the static team improves as variability in work velocity increases. The cellular bucket brigade is

promising because, by following the simple coordination rules, it can better absorb the impact of random work velocities and maintain higher efficiency.

We should emphasize that it is easier to identify the hand-off position under the cellular bucket brigade rules if each worker continuously moves along a station when he progressively works on the station. It is straightforward to implement the rules in such a setting, which can be found, for example, in porcelain painting.

The cellular bucket brigade rules proposed in this paper are based on the assumption that the work on each station can be preempted. For nonpreemptive work content, we will need to redefine hand-offs. It will be interesting to investigate the performance of cellular bucket brigades in such an environment. It is also interesting to study the impact of travel time for long, U-shaped assembly lines.

Acknowledgments

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A Technical details for the three-station system

A.1 Constructing the function f
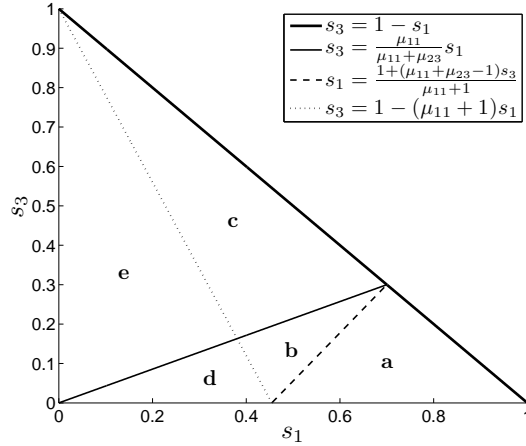
Figure 11: **Five distinct forms of the function f.** Each region corresponds to a distinct form of the function f. We set $\mu_{11} = 1.2$, $\mu_{13} = 0.8$, $\mu_{21} = 0.7$, and $\mu_{23} = 1.6$ in this example.

To study the dynamics of the three-station, two-worker system, we first construct the function f. Figure 11 shows five work-content regions. Each region corresponds to a distinct form of the function f, which is determined in Lemma 1. Let

$$
\begin{aligned}
\theta_1 &= s_1 - \frac{\mu_{13}}{\mu_{13} + \mu_{21}} s_3; \\
\theta_2 &= \frac{1 + (\mu_{13} + \mu_{21} - \mu_{11} - 1)s_1 - (\mu_{13} + 1)s_3}{\mu_{13} + \mu_{21}}; \\
\theta_3 &= \frac{1 + (\mu_{13} + \mu_{21} - \mu_{11} - 1)s_1 + (\mu_{11} + \mu_{23} - \mu_{13} - 1)s_3}{\mu_{13} + \mu_{21}}; \\
\theta_4 &= \frac{1 + (\mu_{13} + \mu_{21} - 1)s_1 - (\mu_{13} + 1)s_3}{\mu_{13} + \mu_{21}}.
\end{aligned}
$$

Lemma 1. *The function f is given as follows.*

Region a $\left(s_1 > \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1} \right)$:

$$
f(x_n) \;=\; s_1 - s_3.
$$

Region b $\left(s_3 \geq 1 - (\mu_{11} + 1)s_1, \ s_1 \leq \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1}, \ and \ s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1\right)$:

$$f(x_n) \ = \ \begin{cases} \eta_1, & \text{if } x_n \in [s_1 - s_3, \ \theta_1); \\ \gamma(x_n), & \text{if } x_n \in [\theta_1, \ \theta_3]; \\ s_1 - s_3, & \text{otherwise.} \end{cases}$$

Region c $\left(s_3 \geq 1 - (\mu_{11} + 1)s_1 \ and \ s_3 \geq \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1\right)$:

$$f(x_n) \ = \ \begin{cases} \eta_1, & \text{if } x_n \in [\max\{0, s_1 - s_3\}, \ \theta_1); \\ \gamma(x_n), & \text{if } x_n \in [\theta_1, \ \theta_4]; \\ \eta_2, & \text{otherwise.} \end{cases}$$

Region d $\left(s_3 < 1 - (\mu_{11} + 1)s_1 \ and \ s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1\right)$:

$$f(x_n) \ = \ \begin{cases} s_1, & \text{if } x_n \in [s_1 - s_3, \ \theta_2); \\ \gamma(x_n), & \text{if } x_n \in [\theta_2, \ \theta_3]; \\ s_1 - s_3, & \text{otherwise.} \end{cases}$$

Region e $\left(s_3 < 1 - (\mu_{11} + 1)s_1 \ and \ s_3 \geq \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1\right)$:

$$f(x_n) \ = \ \begin{cases} s_1, & \text{if } x_n \in [\max\{0, s_1 - s_3\}, \ \theta_2); \\ \gamma(x_n), & \text{if } x_n \in [\theta_2, \ \theta_4]; \\ \eta_2, & \text{otherwise.} \end{cases}$$

Proof. We construct the function f for the following two cases separately: (I) $s_1 > s_3$ and (II) $s_1 \leq s_3$.

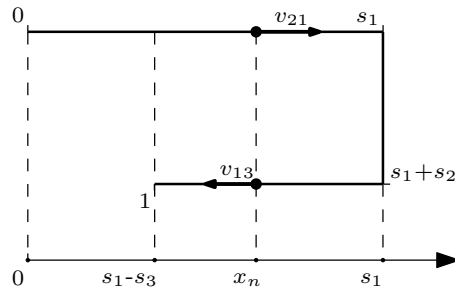Figure 12: **Case (I)** $(s_1 > s_3)$**.** The hand-off position x_n falls in the interval $[s_1 - s_3, s_1]$. The points on the path where workers are located immediately after the n-th hand-off are shown.

For Case (I), the hand-off position x_n falls in the interval $[s_1 - s_3, s_1]$ on the horizontal axis. Figure 12 shows the path for Case (I). Note that the points on the path where workers are

located immediately *after* the n-th hand-off are shown in the figure. In this case, W_1 may be blocked at point 0 or halted at point s_1 on the line, and W_2 may be blocked at point $s_1 + s_2$ or halted at point 1. We determine the next hand-off position x_{n+1} by considering all possible combinations of blocking and halting events.

(I) $s_1 > s_3$ $(x_n \in [s_1 - s_3, s_1])$:

(A) W_1 is not blocked at point 0 if $\frac{s_1 - x_n}{v_{21}} \leq \frac{x_n - s_1 + s_3}{v_{13}} \Leftrightarrow x_n \geq \theta_1$.

(1) W_1 is not halted at point s_1 if $\frac{s_1 - x_n}{v_{21}} + \frac{1 - s_1 - s_3}{v_{22}} \leq \frac{x_n - s_1 + s_3}{v_{13}} + \frac{s_1}{v_{11}} \Leftrightarrow x_n \geq \theta_2$.

(a) W_2 is not blocked at point $s_1 + s_2$ if $\frac{s_1 - x_n}{v_{21}} + \frac{1 - s_1 - s_3}{v_{22}} \geq \frac{x_n - s_1 + s_3}{v_{13}} \Leftrightarrow x_n \leq \theta_4$.

(i) W_2 is not halted at point 1 if

$$\frac{s_1 - x_n}{v_{21}} + \frac{1 - s_1 - s_3}{v_{22}} + \frac{s_3}{v_{23}} \geq \frac{x_n - s_1 + s_3}{v_{13}} + \frac{s_1 - s_3}{v_{11}} \Leftrightarrow x_n \leq \theta_3.$$

— In this case, $x_{n+1} = \gamma(x_n)$.

(ii) W_2 is halted at point 1 if $x_n > \theta_3$.

— In this case, $x_{n+1} = s_1 - s_3$.

(b) W_2 is blocked at point $s_1 + s_2$ if $x_n > \theta_4$.

(i) W_2 is not halted at point 1 if $\frac{s_3}{v_{23}} \geq \frac{s_1 - s_3}{v_{11}} \Leftrightarrow s_3 \geq \frac{\mu_{11}}{\mu_{11} + \mu_{23}} s_1$.

— In this case, $x_{n+1} = \eta_2$.

(ii) W_2 is halted at point 1 if $s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}} s_1$.

— In this case, $x_{n+1} = s_1 - s_3$.

(2) W_1 is halted at point s_1 if $x_n < \theta_2$.

— In this case, $x_{n+1} = s_1$.

(B) W_1 is blocked at point 0 if $x_n < \theta_1$.

(1) W_1 is not halted at point s_1 if $\frac{1 - s_1 - s_3}{v_{22}} \leq \frac{s_1}{v_{11}} \Leftrightarrow s_3 \geq 1 - (\mu_{11} + 1)s_1$.

(a) W_2 is not halted at point 1 if $\frac{1 - s_1 - s_3}{v_{22}} + \frac{s_3}{v_{23}} \geq \frac{s_1 - s_3}{v_{11}} \Leftrightarrow s_1 \leq \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1}$.

— In this case, $x_{n+1} = \eta_1$.

(b) W_2 is halted at point 1 if $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$.

— In this case, $x_{n+1} = s_1 - s_3$.

(2) W_1 is halted at point s_1 if $s_3 < 1 - (\mu_{11}+1)s_1$.

— In this case, $x_{n+1} = s_1$.
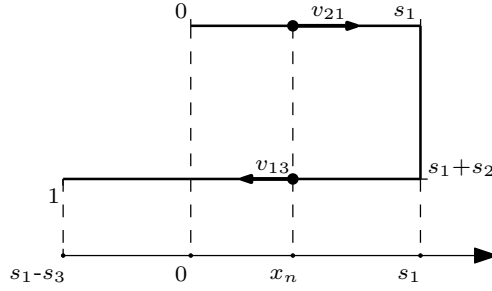
Figure 13: **Case (II)** $(s_1 \leq s_3)$. The hand-off position x_n falls in the interval $[0, s_1]$ on the horizontal axis. The actual locations of workers on the line immediately after the n-th hand-off are shown.

For Case (II), the hand-off position x_n falls in the interval $[0, s_1]$ on the horizontal axis.

Figure 13 shows the path for Case (II). The actual locations of workers on the line immediately

after the n-th hand-off are shown in the figure. In this case, W_1 may be blocked at point 0 or

halted at point s_1 on the line, and W_2 may be blocked at point $s_1 + s_2$. We determine the next

hand-off position x_{n+1} by considering all possible combinations of blocking and halting events.

(II) $s_1 \leq s_3$ $(x_n \in [0, s_1])$:

(A) W_1 is not blocked at point 0 if $x_n \geq \theta_1$.

(1) W_1 is not halted at point s_1 if $x_n \geq \theta_2$.

(a) W_2 is not blocked at point $s_1 + s_2$ if $x_n \leq \theta_4$.

— In this case, $x_{n+1} = \gamma(x_n)$.

(b) W_2 is blocked at point $s_1 + s_2$ if $x_n > \theta_4$.

— In this case, $x_{n+1} = \eta_2$.

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(2) W_1 is halted at point s_1 if $x_n < \theta_2$.

     — In this case, $x_{n+1} = s_1$.

(B) W_1 is blocked at point 0 if $x_n < \theta_1$.

     (1) W_1 is not halted at point s_1 if $s_3 \geq 1 - (\mu_{11} + 1)s_1$.

         — In this case, $x_{n+1} = \eta_1$.

     (2) W_1 is halted at point s_1 if $s_3 < 1 - (\mu_{11} + 1)s_1$.

         — In this case, $x_{n+1} = s_1$.

Now, we check the function f in each region of Figure 11 using the above results. Note that $\theta_3 > \theta_2$, $\theta_4 > \theta_1$, and $\theta_4 > \theta_2$.

Region a: In this region, we have $s_1 > \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1}$, which implies $\theta_1 > \theta_3 > \theta_2$, $s_3 > 1 - (\mu_{11} + 1)s_1$, and $s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1$. (The last inequality is implied by $s_1 + s_3 < 1$: The lines $s_1 = \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1}$ and $s_3 = \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1$ always intersect at point $(\frac{\mu_{11} + \mu_{23}}{2\mu_{11} + \mu_{23}}, \frac{\mu_{11}}{2\mu_{11} + \mu_{23}})$ on the line $s_1 + s_3 = 1$. See Figure 11.)

Since $s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1$, we have $s_1 > s_3$. Thus, this region corresponds to Case (I). If $x_n < \theta_1$ then, because of inequalities $s_3 > 1 - (\mu_{11} + 1)s_1$ and $s_1 > \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1}$, we have Case (I)(B)(1)(b): $x_{n+1} = s_1 - s_3$. Otherwise, we have $x_n \geq \theta_1 > \theta_3 > \theta_2$ and so the region corresponds to Case (I)(A)(1). In addition, we have $s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1 \Leftrightarrow \theta_4 > \theta_3$. Thus, we have either Case (I)(A)(1)(a)(ii) due to the inequality $x_n > \theta_3$ or Case (I)(A)(1)(b)(ii) due to the inequality $s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1$. Both cases imply $x_{n+1} = s_1 - s_3$. Therefore, for any x_n, we have $x_{n+1} = s_1 - s_3$ in this region.

Region b: In this region, we have $s_3 \geq 1 - (\mu_{11} + 1)s_1$, $s_1 \leq \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1}$, and $s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1$. The last inequality implies $s_1 > s_3$, and thus this region corresponds to Case (I). If $x_n < \theta_1$ then, because of the inequalities $s_3 \geq 1 - (\mu_{11} + 1)s_1$ and $s_1 \leq \frac{1 + (\mu_{11} + \mu_{23} - 1)s_3}{\mu_{11} + 1}$, we have Case (I)(B)(1)(a): $x_{n+1} = \eta_1$. Otherwise, we have $x_n \geq \theta_1$. Since $s_3 \geq 1 - (\mu_{11} + 1)s_1 \Leftrightarrow \theta_1 \geq \theta_2$, this region corresponds to Case (I)(A)(1). In addition, $s_3 < \frac{\mu_{11}}{\mu_{11} + \mu_{23}}s_1 \Leftrightarrow \theta_4 > \theta_3$. Thus, if $x_n \leq \theta_3 < \theta_4$, we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_3$,

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and thus we have either Case (I)(A)(1)(a)(ii) or Case (I)(A)(1)(b)(ii) due to the inequality $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$. Both cases imply $x_{n+1} = s_1 - s_3$.

Region c: In this region, we have $s_3 \geq 1 - (\mu_{11}+1)s_1$ and $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, which imply $\theta_1 \geq \theta_2$ and $\theta_3 \geq \theta_4$ respectively. In addition, as shown in Region a, if $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, then $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$. Thus, in this region we have $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1 \Rightarrow s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$. Both Cases (I) and (II) are possible in this region.

For Case (I), if $x_n < \theta_1$ then, because of the inequalities $s_3 \geq 1 - (\mu_{11}+1)s_1$ and $s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, we have Case (I)(B)(1)(a): $x_{n+1} = \eta_1$. Otherwise, we have $x_n \geq \theta_1 \geq \theta_2$, and thus this region corresponds to Case (I)(A)(1). If $x_n \leq \theta_4$ then, because of the inequality $\theta_3 \geq \theta_4$, we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_4$. Since $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, we have Case (I)(A)(1)(b)(i): $x_{n+1} = \eta_2$.

For Case (II), if $x_n < \theta_1$ then, because of the inequality $s_3 \geq 1 - (\mu_{11}+1)s_1$, we have Case (II)(B)(1): $x_{n+1} = \eta_1$. Otherwise, we have $x_n \geq \theta_1 \geq \theta_2$, and thus this region corresponds to Case (II)(A)(1). If $x_n \leq \theta_4$, then we have Case (II)(A)(1)(a): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_4$, and thus we have Case (II)(A)(1)(b): $x_{n+1} = \eta_2$.

Region d: In this region, we have $s_3 < 1 - (\mu_{11}+1)s_1$ and $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$. The first inequality implies $\theta_2 > \theta_1$, and the second inequality implies $s_1 > s_3$ and $\theta_4 > \theta_3$. Thus, this region corresponds to Case (I). If $x_n < \theta_2$, then we have either Case (I)(B)(2) due to the inequality $s_3 < 1 - (\mu_{11}+1)s_1$ or Case (I)(A)(2). Both cases imply $x_{n+1} = s_1$. If $\theta_2 \leq x_n \leq \theta_3$, then we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_3$, and thus we have either Case (I)(A)(1)(a)(ii) or Case (I)(A)(1)(b)(ii) due to the inequality $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$. Both cases imply $x_{n+1} = s_1 - s_3$.

Region e: In this region, we have $s_3 < 1 - (\mu_{11}+1)s_1$ and $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, which imply $\theta_2 > \theta_1$ and $\theta_3 \geq \theta_4$ respectively. Both Cases (I) and (II) are possible in this region.

For Case (I), if $x_n < \theta_2$, then we have either Case (I)(B)(2) due to the inequality $s_3 < 1 - (\mu_{11}+1)s_1$ or Case (I)(A)(2). Both cases imply $x_{n+1} = s_1$. If $\theta_2 \leq x_n \leq \theta_4$ then, because of the inequality $\theta_3 \geq \theta_4$, we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have

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$x_n > \theta_4$. Since $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, we have Case (I)(A)(1)(b)(i): $x_{n+1} = \eta_2$.

For Case (II), if $x_n < \theta_2$, then we have either Case (II)(B)(2) due to inequality $s_3 < 1 - (\mu_{11} + 1)s_1$ or Case (II)(A)(2). Both cases imply $x_{n+1} = s_1$. If $\theta_2 \leq x_n \leq \theta_4$, then we have Case (II)(A)(1)(a): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_4$, and thus we have Case (II)(A)(1)(b): $x_{n+1} = \eta_2$. $\qquad\square$

A.2 Dynamics of a piecewise-linear function

We need the following lemma to determine the asymptotic behaviors of the three-station U-line.

Lemma 2. *For any $\rho > 0$, suppose $x_{n+1} = g(x_n)$ and $g : [A, B] \mapsto [A, B]$ $(0 \leq A < B)$ has the following form*

$$g(x) \;=\; \begin{cases} Y, & \text{if } x \in [A, X_1); \\[2mm] Y + \rho X_1 - \rho x, & \text{if } x \in [X_1, X_2]; \\[2mm] Y + \rho X_1 - \rho X_2, & \text{otherwise}; \end{cases}$$

where Y, X_1, and X_2 are constants. The asymptotic behaviors of the system can be summarized as follows.

(I) $Y \leq X_1$**:** *The system converges to a fixed point Y.*

(II) $X_1 < Y < (1 + \rho)X_2 - \rho X_1$**:** *There are three cases:*

    (1) $\rho < 1$**:** *The system converges to a fixed point $\frac{Y+\rho X_1}{1+\rho}$;*

    (2) $\rho = 1$**:** *The system converges to a period-2 orbit: x and $Y + \rho X_1 - \rho x$, where x depends on the initial point of the orbit;*

    (3) $\rho > 1$**:** *The system converges to a period-2 orbit:*

        a. $X_1 < Y \leq X_2$**:** *Period-2 orbit: Y and $(1 - \rho)Y + \rho X_1$;*

        b. $X_2 < Y < X_2 + (\rho - 1)(X_2 - X_1)$**:** *Period-2 orbit: Y and $Y + \rho X_1 - \rho X_2$;*

        c. $X_2 + (\rho - 1)(X_2 - X_1) \leq Y < (1 + \rho)X_2 - \rho X_1$**:** *Period-2 orbit: $Y + \rho X_1 - \rho X_2$ and $\rho^2 X_2 - (\rho - 1)Y - \rho(\rho - 1)X_1$.*

(III) $Y \geq (1 + \rho)X_2 - \rho X_1$**:** *The system converges to a fixed point* $Y + \rho X_1 - \rho X_2$.

Proof. We first prove case (I). If $Y \leq X_1$, then for any initial point $x \in [A, B]$ we have $g(x) \leq Y \leq X_1$. Thus, $g(g(x)) = Y$ and the system stays at the fixed point Y. Similarly, we can prove case (III) as follows. If $Y \geq (1 + \rho)X_2 - \rho X_1$, then for any initial point $x \in [A, B]$ we have $g(x) \geq Y + \rho X_1 - \rho X_2 \geq X_2$. Thus, $g(g(x)) = Y + \rho X_1 - \rho X_2$ and the system stays at the fixed point $Y + \rho X_1 - \rho X_2$.

Now, we prove case (II). If $X_1 < Y < (\rho + 1)X_2 - \rho X_1$, there are three possible cases: (1) $\rho < 1$, (2) $\rho = 1$, and (3) $\rho > 1$. We analyze each case as follows.

For case (1), we have $\rho < 1$. For any initial point $x_0 \in [A, B]$, we have $|f(x_n) - \eta_0| \leq \rho^n |x_0 - \eta_0|$. Since $\rho < 1$, the system converges to the fixed point η_0.

For case (2), we have $\rho = 1$. For any initial point $x_0 \in [A, B]$, there are three possible cases:

a. If $x_0 < X_1$, then $g(x_0) = Y$.

- If $X_1 < Y \leq X_2$, then $g(Y) = Y + X_1 - Y = X_1$. Thus, $g(g(Y)) = Y$, which means the system converges to a period-2 orbit: Y and X_1.

- If $X_2 < Y < 2X_2 - X_1$, then $g(Y) = Y + X_1 - X_2$. We have $g(Y) \in [X_1, X_2]$. Thus, $g(g(Y)) = Y + X_1 - g(Y) = X_2$, which implies $g(g(g(Y))) = Y + X_1 - X_2 = g(Y)$. The system converges to a period-2 orbit: $Y + X_1 - X_2$ and X_2.

b. If $x_0 > X_2$, then $g(x_0) = Y + X_1 - X_2$. Let $Y' = Y + X_1 - X_2$. We have $2X_1 - X_2 < Y' < X_2$.

- If $2X_1 - X_2 < Y' < X_1$, then $g(Y') = Y$. We have $g(Y') \in [X_1, X_2]$. Thus, $g(g(Y')) = Y + X_1 - g(Y') = X_1$, which implies $g(g(g(Y'))) = Y + X_1 - X_1 = Y = g(Y')$. The system converges to a period-2 orbit: Y and X_1.

- If $X_1 \leq Y' < X_2$, then $g(Y') = Y + X_1 - Y' = X_2$. Thus, $g(g(Y')) = Y + X_1 - X_2 = Y'$. The system converges to a period-2 orbit: Y' and X_2.

c. If $X_1 \leq x_0 \leq X_2$, then $g(x_0) = Y + X_1 - x_0$.

- If $X_1 \leq g(x_0) \leq X_2$, then $g(g(x_0)) = x_0$. The system converges to a period-2 orbit: x_0 and $Y + X_1 - x_0$.

- If $g(x_0) < X_1$, then $g(g(x_0)) = Y$ and this reduces to case a.

- If $g(x_0) > X_2$, then $g(g(x_0)) = Y + X_1 - X_2$ and this reduces to case b.

For case (3), we have $\rho > 1$. We first prove by contradiction that for any initial point $x_0 \in [A, B]$, such that $x_0 \neq \eta_0$, the orbit under g contains at least one endpoint Y or $Y + \rho X_1 - \rho X_2$. If not, then $x_n \in (X_1, X_2)$ for all $n = 0, 1, 2, \ldots$. However, since $|x_n - \eta_0| = \rho^n |x_0 - \eta_0|$, there exists a n' such that $x_{n'} \notin (X_1, X_2)$. This contradicts our assumption. Thus, any orbit under g contains at least one endpoint Y or $Y + \rho X_1 - \rho X_2$. As a result, we can focus our analysis on orbits starting from Y or $Y + \rho X_1 - \rho X_2$. There are three cases:

a. If $X_1 < Y \leq X_2$, then $Y < X_2 + (\rho - 1)(X_2 - X_1) \Leftrightarrow Y + \rho X_1 - \rho X_2 < X_1$, which implies $g(Y + \rho X_1 - \rho X_2) = Y$. Thus, we only need to analyze orbits starting from Y. $g(Y) = (1-\rho)Y + \rho X_1 < X_1$, which implies $g(g(Y)) = Y$. Therefore, the system converges to a period-2 orbit: Y and $(1 - \rho)Y + \rho X_1$.

b. If $X_2 < Y < X_2 + (\rho-1)(X_2 - X_1)$, then $g(Y) = Y + \rho X_1 - \rho X_2$ and $g(Y + \rho X_1 - \rho X_2) = Y$. Thus, the system converges to a period-2 orbit: Y and $Y + \rho X_1 - \rho X_2$.

c. If $X_2 + (\rho - 1)(X_2 - X_1) \leq Y < (\rho + 1)X_2 - \rho X_1$, then $Y > X_2$, which implies $g(Y) = Y + \rho X_1 - \rho X_2$. Thus, we only need to analyze orbits starting from $Y + \rho X_1 - \rho X_2$. The inequalities $X_2 + (\rho-1)(X_2 - X_1) \leq Y < (\rho+1)X_2 - \rho X_1$ imply $X_1 \leq Y + \rho X_1 - \rho X_2 < X_2$, and so $g(Y + \rho X_1 - \rho X_2) = Y + \rho X_1 - \rho(Y + \rho X_1 - \rho X_2) = \rho^2 X_2 - (\rho-1)Y - \rho(\rho-1)X_1 > X_2$. The last inequality is implied by $Y < (\rho + 1)X_2 - \rho X_1$. Thus, $g(g(Y + \rho X_1 - \rho X_2)) = Y + \rho X_1 - \rho X_2$. Therefore, the system converges to a period-2 orbit: $Y + \rho X_1 - \rho X_2$ and $\rho^2 X_2 - (\rho - 1)Y - \rho(\rho - 1)X_1$.

$\square$

Note that the function g in Lemma 2 represents a general form of the function f in Lemma 1.

We will use the properties of the function g described in Lemma 2 to determine the asymptotic behaviors of the three-station, two-worker system.

A.3 Asymptotic behaviors and throughput

Lemma 3. *If $\varphi \leq 1$, the two-worker cellular bucket brigade on a three-station u-line has a distinct asymptotic behavior in each of the following five regions.*

Region 1: *This region is defined by $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$. The system converges to a fixed point $x^* = s_1 - s_3$. At the fixed point, W_1 is constantly blocked at point 0 and W_2 is constantly halted at point 1. The average throughput is $\mathcal{T} = \left(\frac{s_1-s_3}{v_{11}} + \frac{s_3}{v_{21}} \right)^{-1}$.*

Region 2: *This region is defined by $s_1 < \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$ and $s_1 > \frac{1}{\mu_{11}+1} + \frac{\mu_{11}\mu_{13}+\mu_{13}\mu_{23}-\mu_{13}-\mu_{21}}{(\mu_{11}+1)(\mu_{13}+\mu_{21})} \cdot s_3$. The system converges to a fixed point $x^* = \eta_1$. At the fixed point, W_1 is constantly blocked at point 0. The average throughput is $\mathcal{T} = \left(\frac{\eta_1}{v_{11}} + \frac{s_1-\eta_1}{v_{21}} \right)^{-1}$.*

Region 3: *This region is defined by $s_3 > \frac{1}{\mu_{13}+1} + \frac{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}-\mu_{11}-\mu_{23}}{(\mu_{13}+1)(\mu_{11}+\mu_{23})} \cdot s_1$. The system converges to a fixed point $x^* = \eta_2$. At the fixed point, W_2 is constantly blocked at point s_1+s_2. The average throughput is $\mathcal{T} = \left(\frac{\eta_2}{v_{11}} + \frac{s_3-s_1+\eta_2}{v_{13}} \right)^{-1}$.*

Region 4: *This region is defined by $s_3 < \frac{1-(\mu_{11}+1)s_1}{\mu_{13}+1}$. The system converges to a fixed point $x^* = s_1$. At the fixed point, W_1 is constantly halted at point s_1. The average throughput is $\mathcal{T} = \left(\frac{1-s_1-s_3}{v_{22}} \right)^{-1}$.*

Region 5: *This region is defined by $s_1 < \frac{1}{\mu_{11}+1} + \frac{\mu_{11}\mu_{13}+\mu_{13}\mu_{23}-\mu_{13}-\mu_{21}}{(\mu_{11}+1)(\mu_{13}+\mu_{21})} \cdot s_3$, $s_3 < \frac{1}{\mu_{13}+1} + \frac{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}-\mu_{11}-\mu_{23}}{(\mu_{13}+1)(\mu_{11}+\mu_{23})} \cdot s_1$, and $s_3 > \frac{1-(\mu_{11}+1)s_1}{\mu_{13}+1}$. If $\varphi < 1$, the system converges to a fixed point η_3. If $\varphi = 1$, the system converges to a period-2 orbit: x and $\gamma(x)$, where x depends on the initial locations of the workers on the line. Neither blocking nor halting occurs in this region. The average throughput is $\mathcal{T} = \left(\frac{\eta_3}{v_{11}} + \frac{s_3-s_1+\eta_3}{v_{13}} \right)^{-1}$.*

Proof. We partition the entire feasible work-content area into five regions shown in Figure 4(a). We determine the asymptotic behavior and throughput of the system in each region separately.

Region 1: This region is identical to Region a in Figure 11. Lemma 1 shows that the system always converges to the fixed point $x^* = s_1 - s_3$. When the system operates on the fixed point, according to the proof of Lemma 1 (see Region a), W_1 is constantly blocked at point 0 and W_2 is constantly halted at point 1. The average throughput is $\mathcal{T} = \left(\frac{s_3}{v_{21}} + \frac{s_1-s_3}{v_{11}} \right)^{-1}$.

Region 2: This region falls in Regions b and c in Figure 11. Since $s_1 > \frac{1}{\mu_{11}+1} + \frac{\mu_{11}\mu_{13}+\mu_{13}\mu_{23}-\mu_{13}-\mu_{21}}{(\mu_{11}+1)(\mu_{13}+\mu_{21})} \cdot s_3 \Leftrightarrow \eta_1 < \theta_1$, according to the function f in Regions b and c as well as Lemma 2, the system converges to the fixed point $x^* = \eta_1$. When the system operates on the fixed point, according to the proof of Lemma 1 (see Regions b and c), W_1 is constantly blocked at point 0. The average throughput is $\mathcal{T} = \left(\frac{s_1-\eta_1}{v_{21}} + \frac{\eta_1}{v_{11}} \right)^{-1}$.

Region 3: This region falls in Regions c and e in Figure 11. Since $s_3 > \frac{1}{\mu_{13}+1} + \frac{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}-\mu_{11}-\mu_{23}}{(\mu_{13}+1)(\mu_{11}+\mu_{23})} \cdot s_1 \Leftrightarrow \eta_2 > \theta_4$, according to the function f in Regions c and e as well as Lemma 2, the system converges to the fixed point $x^* = \eta_2$. When the system operates on the fixed point, according to the proof of Lemma 1 (see Regions c and e), W_2 is constantly blocked at point $s_1 + s_2$. The average throughput is $\mathcal{T} = \left(\frac{s_3-s_1+\eta_2}{v_{13}} + \frac{\eta_2}{v_{11}} \right)^{-1}$.

Region 4: Since $s_3 < \frac{1-(\mu_{11}+1)s_1}{\mu_{13}+1} \Leftrightarrow \frac{s_3}{v_{13}} + \frac{s_1}{v_{11}} < \frac{1-s_1-s_3}{v_{22}}$, W_1 is constantly halted at point s_1. The system converges to the fixed point $x^* = s_1$. The average throughput is $\mathcal{T} = \left(\frac{1-s_1-s_3}{v_{22}} \right)^{-1}$.

Region 5: This region falls in Regions b, c, d, and e in Figure 11. According to the function f in Regions b, c, d, and e as well as Lemma 2, we have (1) if $\varphi < 1$, the system converges to a fixed point η_3, and (2) if $\varphi = 1$, the system converges to a period-2 orbit: x and $\gamma(x)$, where x depends on the initial locations of the workers on the line. Neither blocking nor halting occurs in this region. The average throughput is $\mathcal{T} = \left(\frac{\eta_3}{v_{11}} + \frac{s_3-s_1+\eta_3}{v_{13}} \right)^{-1}$. $\qquad\square$

According to the proof of Lemma 3, the asymptotic behaviors in Regions 1 to 4 are independent of φ. Thus, if $\varphi > 1$, the asymptotic behaviors and the expressions of the throughput remain the same in all regions except for Region 5. Due to page limitation, the detailed analysis of the asymptotic behaviors and the throughput for the case with $\varphi > 1$ is only available upon request.

B Technical details for the M-station system

In the M-station U-line, a hand-off position falls in the range $[\underline{x}, \bar{x}]$, where $\underline{x} = s_1 - s_3$ and $\bar{x} = s_1 + s_2 - s_2^{m_2}$. Recall that after a hand-off W_1 first works on stage 3 before he works on stages 1 and 2, and W_2 first works on stage 1 and then works on stages 2 and 3 (see Figure 3). Note that W_1 can only be blocked at the start of a station in stages 1 and 2. On the other hand, W_2 can only be blocked at the start of a station in stage 3, and can only be halted at point 1. Let $L_j(k)$ denote the point at the start of $S_j(k)$ on the path, for $k = 1, \dots, m_j$ and $j = 1, 2, 3$.

Consider W_i starts from a hand-off position x. For convenience, we say W_i is *blocked at* $L_j(k)$ *from* x if he is blocked at $L_j(k)$ before the next hand-off. Similarly, we say W_i is *halted at point L from* x if he is halted at point L before the next hand-off.

We have the following properties:

Property 1. *For any x and x' in $[\underline{x}, \bar{x}]$, if W_i is blocked at $L_j(k)$ from both x and x', then $f(x) = f(x')$.*

Property 2. *For any x and x' in $[\underline{x}, \bar{x}]$, if W_i is halted at point L from both x and x', then $f(x) = f(x')$.*

Property 3. *For any x and x' in $[\underline{x}, \bar{x}]$ such that $x > x'$, if W_1 is blocked at $L_1(k)$ from x then W_1 is blocked at $L_1(k)$ from x', and if W_1 is halted at point s_1 from x then W_1 is halted at point s_1 from x'.*

Property 4. *For any x and x' in $[\underline{x}, \bar{x}]$ such that $x < x'$, if W_2 is blocked at $L_3(k)$ from x then W_2 is blocked at $L_3(k)$ from x', and if W_2 is halted at point 1 from x then W_2 is halted at point 1 from x'.*

B.1 Characterizing the function f

From the above properties, we have the following results.

Lemma 4. *There exists a constant c_1 such that W_1 is blocked or halted from any $x \in [\underline{x}, c_1)$, but he is neither blocked nor halted from any $x \in [c_1, \bar{x})$.*

Proof. The lemma claims that W_1 is blocked or halted from any $x < c_1$. We can find c_1 in each of the following three cases: (1) If W_1 is neither blocked nor halted from $\underline{x}$, then according to Property 3, W_1 is neither blocked nor halted from any $x \in [\underline{x}, \bar{x}]$. Thus, we have $c_1 = \underline{x}$. (2) If W_1 is blocked or halted from $\bar{x}$, then according to Property 3, W_1 is blocked or halted from any $x \in [\underline{x}, \bar{x}]$. Thus, we have $c_1 = \bar{x}$. (3) Otherwise, according to Property 3, there exists a hand-off position c_1 such that W_1 is blocked or halted from any $x \in [\underline{x}, c_1)$, but he is neither blocked nor halted from any $x \in [c_1, \bar{x}]$. The three cases above imply that W_1 is blocked or halted from any $x \in [\underline{x}, c_1)$, but he is neither blocked nor halted from any $x \in [c_1, \bar{x})$. $\square$

Using Property 4, the proof of the following lemma is similar and is omitted.

Lemma 5. *There exists a constant c_2 such that W_2 is blocked or halted from any $x \in (c_2, \bar{x}]$, but he is neither blocked nor halted from any $x \in (\underline{x}, c_2]$.*

Together with Properties 1 and 2, Lemmas 4 and 5 imply the following result.

Corollary 1. *There exist constants Y_1 and Y_2 such that for any $x \in [\underline{x}, c_1)$, $f(x) = Y_1$, and for any $x \in (c_2, \bar{x}]$, $f(x) = Y_2$.*

Lemma 6. *If $c_1 < c_2$ then f is strictly decreasing in $[c_1, c_2]$.*

Proof. According to Lemmas 4 and 5, both workers are neither blocked nor halted from any $x \in [c_1, c_2]$. For any hand-off positions χ_1 and χ_2 such that $c_1 \leq \chi_1 < \chi_2 \leq c_2$, we will show that $f(\chi_1) > f(\chi_2)$. There are three cases: (1) $0 \leq f(\chi_2) \leq s_1$; (2) $f(\chi_2) < 0$; and (3) $f(\chi_2) > s_1$.

For case (1), it is sufficient to prove that after a hand-off at χ_1, when W_1 works in stage 1 and arrives at position $f(\chi_2)$ he has not met W_2. For any hand-off position $x \in [\underline{x}, \bar{x}]$, let $t_1(x)$ denote the total time for W_1 to start from point $\min\{s_1, x\}$, finish his item at the end of stage 3, work on a new item in stage 1 (and possibly stage 2), and reach position $f(\chi_2)$. Let $t_2(x)$ denote the total time for W_2 to start from $\max\{0, x\}$, work on his item in stages 1, 2, and 3, and reach position $f(\chi_2)$. Since $\chi_1 < \chi_2$, if $\chi_1 < s_1$ we have $t_1(\chi_1) < t_1(\chi_2)$ and $t_2(\chi_1) \geq t_2(\chi_2)$; otherwise, we have $t_1(\chi_1) \leq t_1(\chi_2)$ and $t_2(\chi_1) > t_2(\chi_2)$. In addition, we know that $t_1(\chi_2) = t_2(\chi_2)$. Thus,

we have $t_1(\chi_1) < t_1(\chi_2) = t_2(\chi_2) \le t_2(\chi_1)$ or $t_1(\chi_1) \le t_1(\chi_2) = t_2(\chi_2) < t_2(\chi_1)$, which imply $f(\chi_1) > f(\chi_2)$.

For case (2), it is sufficient to prove that after a hand-off at position χ_1, when W_1 arrives at point 1, W_2 has not reached position $f(\chi_2)$. For any hand-off position $x \in [\underline{x}, \bar{x}]$, let $t_1(x)$ be the total time for W_1 to start from point $\min\{s_1, x\}$ and finish his item at the end of stage 3 (reach point 1). Let $t_2(x)$ denote the total time for W_2 to start from $\max\{0, x\}$, work on his item in stages 1, 2, and 3, and reach position $f(\chi_2)$. Since $\chi_1 < \chi_2$, if $\chi_1 < s_1$ we have $t_1(\chi_1) < t_1(\chi_2)$ and $t_2(\chi_1) \ge t_2(\chi_2)$; otherwise, we have $t_1(\chi_1) \le t_1(\chi_2)$ and $t_2(\chi_1) > t_2(\chi_2)$. In addition, we know that $t_1(\chi_2) = t_2(\chi_2)$. Thus, we have $t_1(\chi_1) < t_1(\chi_2) = t_2(\chi_2) \le t_2(\chi_1)$ or $t_1(\chi_1) \le t_1(\chi_2) = t_2(\chi_2) < t_2(\chi_1)$, which imply $f(\chi_1) > f(\chi_2)$.

For case (3), it is sufficient to prove that after a hand-off at position χ_1, when W_2 arrives at point $s_1 + s_2$, W_1 has passed position $f(\chi_2)$. For any hand-off position $x \in [\underline{x}, \bar{x}]$, let $t_2(x)$ denote the total time for W_2 to start from point $\max\{0, x\}$ and reach position $s_1 + s_2$. Let $t_1(x)$ be the total time for W_1 to start from point $\min\{s_1, x\}$, and finish his item at the end of stage 3, work on a new item in stages 1 and 2, and reach position $f(\chi_2)$. Since $\chi_1 < \chi_2$, if $\chi_1 < s_1$ we have $t_1(\chi_1) < t_1(\chi_2)$ and $t_2(\chi_1) \ge t_2(\chi_2)$; otherwise, we have $t_1(\chi_1) \le t_1(\chi_2)$ and $t_2(\chi_1) > t_2(\chi_2)$. In addition, we know that $t_1(\chi_2) = t_2(\chi_2)$. Thus, we have $t_1(\chi_1) < t_1(\chi_2) = t_2(\chi_2) \le t_2(\chi_1)$ or $t_1(\chi_1) \le t_1(\chi_2) = t_2(\chi_2) < t_2(\chi_1)$, which imply $f(\chi_1) > f(\chi_2)$.

$\square$

Lemma 7. *f is continuous.*

Proof. According to the proofs of Lemmas 4 and 5, W_1 is almost blocked or halted from c_1 and W_2 is almost blocked or halted from c_2. Together with Corollary 1, we have $f(c_1) = Y_1$ and $f(c_2) = Y_2$. Thus, it is sufficient to prove that f is continuous in $[c_1, c_2]$.

For convenience, define $v_{i,\max} = \max_{j,k} v_{ij}(k)$, $v_{i,\min} = \min_{j,k} v_{ij}(k)$ for $i = 1, 2$, and $v_{\max} = \max\{v_{1,\max}, v_{2,\max}\}$. Consider any hand-off positions χ_1 and χ_2, where $c_1 \le \chi_1 < \chi_2 \le c_2$ such that $\chi_2 - \chi_1 < \delta$ for a small δ. There are three cases: (1) $0 \le f(\chi_2) \le s_1$; (2) $f(\chi_2) < 0$; (3) $f(\chi_2) > s_1$. For each case, we adopt the same definitions of $t_1(x)$ and $t_2(x)$ as those in the proof

of Lemma 6.

For case (1), the proof of Lemma 6 shows that after a hand-off at position χ_1, when W_1 works in stage 1 and arrives at position $f(\chi_2)$, he has not met W_2. Meanwhile, W_2 reaches the position $h_2 \le f(\chi_2) + v_{2,\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$. Thus, we have $f(\chi_1) < h_2 \le f(\chi_2) + v_{2,\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$.

For case (2), the proof of Lemma 6 shows that after a hand-off at position χ_1, when W_1 arrives at point 1, W_2 has not reached position $f(\chi_2)$. Instead, W_2 reaches the position $f(\chi_1) \le f(\chi_2) + v_{2,\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$.

For case (3), the proof of Lemma 6 shows that after a hand-off at position χ_1, when W_2 arrives at point $s_1 + s_2$, W_1 has passed position $f(\chi_2)$, and reaches the position $f(\chi_1) \le f(\chi_2) + v_{1,\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$.

Combining cases (1), (2), and (3), we have $f(\chi_1) \le f(\chi_2) + v_{\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$, which implies $f(\chi_1) - f(\chi_2) \le v_{\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$. Since $t_2(\chi_1) - t_1(\chi_1) = (t_2(\chi_1) - t_2(\chi_2)) + (t_1(\chi_2) - t_1(\chi_1)) \le \frac{\chi_2 - \chi_1}{v_{2,\min}} + \frac{\chi_2 - \chi_1}{v_{1,\min}} = (\chi_2 - \chi_1) \cdot \left(\frac{1}{v_{1,\min}} + \frac{1}{v_{2,\min}} \right) < \left(\frac{1}{v_{1,\min}} + \frac{1}{v_{2,\min}} \right) \cdot \delta$, we have $f(\chi_1) - f(\chi_2) < v_{\max} \cdot \left(\frac{1}{v_{1,\min}} + \frac{1}{v_{2,\min}} \right) \cdot \delta$.

Thus, for any $\varepsilon > 0$, there exists $\delta = \varepsilon \cdot \left(v_{\max} \cdot \left(\frac{1}{v_{1,\min}} + \frac{1}{v_{2,\min}} \right) \right)^{-1} > 0$ such that for any hand-off positions χ_1 and χ_2, if $\chi_2 - \chi_1 < \delta$ then $f(\chi_1) - f(\chi_2) < \varepsilon$. Therefore, $f(x)$ is continuous in $[c_1, c_2]$. $\qquad\square$

Recall that a hand-off position x is an interior hand-off position if the locations corresponding to x on the U-line fall in the interior of some stations.

Lemma 8. *f is piecewise linear.*

Proof. According to Corollary 1, it is sufficient to prove that f is piecewise linear in $[c_1, c_2]$. Consider any hand-off position $\chi \in (c_1, c_2)$, such that both χ and $f(\chi)$ are interior hand-off positions. We will show that f is linear in the neighborhood of such χ.

For convenience, define u_1 as the velocity of W_1 at position χ when he works in stage 3. If $f(\chi) \ge 0$, then define v_1 as the velocity of W_1 at position $f(\chi)$ when he works in stage 1 or 2.

If $\chi \geq 0$, then define v_2 as the velocity of W_2 at position χ when he works in stage 1 or 2. We also define u_2 as the velocity of W_2 at position $f(\chi)$ when he works in stage 3.

For any $x = \chi \pm \Delta x$, where Δx is a small positive number, we have nine cases: (1) $0 < \chi < s_1$ and $0 < f(\chi) < s_1$; (2) $0 < \chi < s_1$ and $f(\chi) < 0$; (3) $\chi < 0$ and $0 < f(\chi) < s_1$; (4) $\chi < 0$ and $f(\chi) < 0$; (5) $\chi > s_1$ and $0 < f(\chi) < s_1$; (6) $\chi > s_1$ and $f(\chi) < 0$; (7) $0 < \chi < s_1$ and $f(\chi) > s_1$; (8) $\chi < 0$ and $f(\chi) > s_1$; (9) $\chi > s_1$ and $f(\chi) > s_1$.

For case (1), we have $f(x) = f(\chi) \mp \frac{v_1 u_2}{v_1 + u_2} \cdot \left(\frac{1}{u_1} + \frac{1}{v_2} \right) \cdot \Delta x$. For case (2), we have $f(x) = f(\chi) \mp u_2 \cdot \left(\frac{1}{u_1} + \frac{1}{v_2} \right) \cdot \Delta x$. For case (3), we have $f(x) = f(\chi) \mp \frac{v_1 u_2}{v_1 + u_2} \cdot \frac{1}{u_1} \cdot \Delta x$. For case (4), we have $f(x) = f(\chi) \mp \frac{u_2}{u_1} \cdot \Delta x$. For case (5), we have $f(x) = f(\chi) \mp \frac{v_1 u_2}{v_1 + u_2} \cdot \frac{1}{v_2} \cdot \Delta x$. For case (6), we have $f(x) = f(\chi) \mp \frac{u_2}{v_2} \cdot \Delta x$. For case (7), we have $f(x) = f(\chi) \mp v_1 \cdot \left(\frac{1}{u_1} + \frac{1}{v_2} \right) \cdot \Delta x$. For case (8), we have $f(x) = f(\chi) \mp \frac{v_1}{u_1} \cdot \Delta x$. For case (9), we have $f(x) = f(\chi) \mp \frac{v_1}{v_2} \cdot \Delta x$. Thus, f is linear in the neighborhood of χ, and so f is piecewise linear in $[c_1, c_2]$. □

The following corollary summarizes the above results.

Corollary 2. *f is continuous, non-increasing, and has the following form*

$$
f(x) = \begin{cases} Y_1, & \text{if } x \in [\underline{x}, c_1); \\ F(x), & \text{if } x \in [c_1, c_2]; \\ Y_2, & \text{otherwise}; \end{cases}
$$

where F is strictly decreasing and piecewise linear.

B.2 Asymptotic behaviors

Corollary 2 implies the following lemma.

Lemma 9. *There exists a unique fixed point and there are no periodic orbits of period greater than 2 in the system.*

Proof. According to Brouwer's fixed point theorem, there exists a fixed point because f is continuous. Since f is also non-increasing, the fixed point is unique.

We then prove by contradiction that there are no periodic orbits of period greater than 2. Suppose there exists a periodic orbit of period $\pi > 2$: $x_1, x_2, \ldots, x_\pi$. For convenience, define $X = \{x_1, x_2, \ldots, x_\pi\}$. First note that for any $x_i \in X$, we have $x_i \neq x^*$. Without loss of generality, assume that $x_1 < x^*$. Since f is non-increasing, for any $x_i \in X$, if $x_i < x^*$, then $f(x_i) > x^*$, and if $x_i > x^*$, then $f(x_i) < x^*$. As a result, we have $f^{2n-1}(x_1) > x^*$ and $f^{2n}(x_1) < x^*$, for $n = 1, 2, \ldots$. Thus, π is even because $f^\pi(x_1) = x_1 < x^*$.

Since $f^2(\cdot)$ is non-decreasing, if $x_1 < x_3$ then we have $x_1 < x_3 = f^2(x_1) < x_5 = f^2(x^3) < \cdots < x_1 = f^2(x_{\pi-1})$, which is a contradiction; otherwise, we have $x_1 > x_3 = f^2(x_1) > x_5 = f^2(x^3) > \cdots > x_1 = f^2(x_{\pi-1})$, which is also a contradiction. Therefore, there does not exist a periodic orbit of period $\pi > 2$. $\square$

The following lemma provides a sufficient condition for the system to converge to the fixed point x^*, independent of the initial workers' locations on the U-line. This condition can be tested easily. Let $P_j(k)$ denote the horizontal position of $L_j(k)$, for $k = 1, \ldots, m_j$ and $j = 1, 2, 3$. Note that we project points in stage 2 except the last station $S_2(m_2)$ onto $[s_1, s_1 + s_2 - s_2^{m_2}]$, that is $P_2(k) = s_1 + \sum_{l=1}^{k-1} s_2^l$. For convenience, define $P_1(m_1 + 1) = s_1$ and $P_3(m_3 + 1) = \underline{x}$.

Lemma 10. *The system converges to a fixed point x^* if for any pair of interior hand-off positions x and $f(x)$, one of the following conditions is satisfied:*

1. *$0 < x < s_1$, $0 < f(x) < s_1$, where $x \in (P_1(k_1), P_1(k_1 + 1)) \bigcap (P_3(k_2 + 1), P_3(k_2))$ and $f(x) \in (P_1(k_3), P_1(k_3 + 1)) \bigcap (P_3(k_4 + 1), P_3(k_4))$, and*

$$\frac{1}{v_{11}(k_3)} - \frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)} - \frac{1}{v_{23}(k_4)}.$$

2. *$0 < x < s_1$, $f(x) < 0$, where $x \in (P_1(k_1), P_1(k_1 + 1)) \bigcap (P_3(k_2 + 1), P_3(k_2))$ and $f(x) \in (P_3(k_4 + 1), P_3(k_4))$, and*

$$-\frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)} - \frac{1}{v_{23}(k_4)}.$$

3. $x < 0$, $0 < f(x) < s_1$, where $x \in (P_3(k_2 + 1), P_3(k_2))$ and $f(x) \in (P_1(k_3), P_1(k_3 + 1)) \bigcap (P_3(k_4 + 1), P_3(k_4))$, and

$$\frac{1}{v_{11}(k_3)} - \frac{1}{v_{13}(k_2)} > -\frac{1}{v_{23}(k_4)}.$$

4. $x < 0$, $f(x) < 0$, where $x \in (P_3(k_2 + 1), P_3(k_2))$ and $f(x) \in (P_3(k_4 + 1), P_3(k_4))$, and

$$-\frac{1}{v_{13}(k_2)} > -\frac{1}{v_{23}(k_4)}.$$

5. $x > s_1$, $0 < f(x) < s_1$, where $x \in (P_2(k_1), P_2(k_1 + 1))$ and $f(x) \in (P_1(k_3), P_1(k_3 + 1)) \bigcap (P_3(k_4 + 1), P_3(k_4))$, and

$$\frac{1}{v_{11}(k_3)} > \frac{1}{v_{22}(k_1)} - \frac{1}{v_{23}(k_4)}.$$

6. $x > s_1$, $f(x) < 0$, where $x \in (P_2(k_1), P_2(k_1 + 1))$ and $f(x) \in (P_3(k_4 + 1), P_3(k_4))$, and

$$0 > \frac{1}{v_{22}(k_1)} - \frac{1}{v_{23}(k_4)}.$$

7. $0 < x < s_1$, $f(x) > s_1$, where $x \in (P_1(k_1), P_1(k_1 + 1)) \bigcap (P_3(k_2 + 1), P_3(k_2))$ and $f(x) \in (P_2(k_3), P_2(k_3 + 1))$, and

$$\frac{1}{v_{12}(k_3)} - \frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)}.$$

8. $x < 0$, $f(x) > s_1$, where $x \in (P_3(k_2 + 1), P_3(k_2))$ and $f(x) \in (P_2(k_3), P_2(k_3 + 1))$, and

$$\frac{1}{v_{12}(k_3)} - \frac{1}{v_{13}(k_2)} > 0.$$

9. $x > s_1$, $f(x) > s_1$, where $x \in (P_2(k_1), P_2(k_1 + 1))$ and $f(x) \in (P_2(k_3), P_2(k_3 + 1))$, and

$$\frac{1}{v_{12}(k_3)} > \frac{1}{v_{22}(k_1)}.$$

Proof. According to the proof of Lemma 8, the four conditions in Lemma 10 ensure that the absolute value of the derivative of f, where f is differentiable, is smaller than 1. Let $\rho \in [0, 1)$ denote the largest absolute value of the slope of f. For any $x \in [\underline{x}, \bar{x}]$, we have $|f(x) - f(x^*)| \leq \rho|x - x^*|$. Since $f(x^*) = x^*$, we have $|f^n(x) - x^*| \leq \rho^n|x - x^*|$, and thus $\lim_{n \to \infty} f^n(x) = x^*$. Therefore, the system converges to the fixed point. $\qquad \square$