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A Reliable Algorithm for a Local Fractional Tricomi Equation Arising in Fractal Transonic Flow

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Abstract: The pivotal proposal of this work is to present a reliable algorithm based on the local fractional homotopy perturbation Sumudu transform technique for solving a local fractional Tricomi equation occurring in fractal transonic flow. The proposed technique provides the results without any transformation of the equation into discrete counterparts or imposing restrictive assumptions and is completely free of round-off errors. The results of the scheme show that the approach is straightforward to apply and computationally very user-friendly and accurate.

Keywords: Tricomi equation; fractal transonic flow; local fractional derivative; homotopy perturbation method; local fractional Sumudu transform method

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1. Introduction

Partial differential equations of mixed type with boundary conditions have played an important role in describing real world problems such as in elementary research conducted by Tricomi [1]. Mixed type partial differential equations [2,3] are used to investigate transonic flow and they produce special boundary value problems, known as Tricomi and Frankl problems [1,4]. Transonic flows include a change from the subsonic to the hypersonic region [4,5] via the sonic curve. Consequently, transonic flows are very attractive phenomena occurring in aeronautics and hydraulics. The familiar mixed type partial differential equation is known as a Tricomi equation, $yu_{xx} + u_{yy} = 0$, because of Tricomi, who found this mathematical model, for the function $u = u(x, y)$ of two variables x and y. It acts as a basis for the mathematical modelling of the transonic flows, since it is of elliptic and hyperbolic type, where the coefficient y of the second order partial differential coefficient of the required function $u = u(x, y)$ with respect to x changes sign. This mathematical equation is also parabolic at the points where y vanishes.

The Tricomi equation [1] is a mixed type of linear partial differential equation of the second order, which has been used to narrate the theory of plane transonic flow [6–9]. The Tricomi equation was used to recount differentiable problems for the theory of plane transonic flow. However, for the fractal theory of plane transonic flow with non-differentiable expressions, the Tricomi equation is not registered to report them. Recently, local fractional calculus [10] was tried for non-differentiable problems, for instance fractal heat conduction [10,11], damped and dissipative wave equations in fractal strings [12], local fractional Laplace equations [13], the Helmholtz equation associated with local fractional derivatives [14], the wave equation on Cantor sets [15], Navier–Stokes equations on Cantor sets [16], local fractional Schrödinger equations [17], Korteweg–de Vries equations involving local fractional derivative [18], *etc.* In recent times, the local fractional model of the Tricomi equation in fractal transonic flows was recommended in the form [19,20]

$$\frac{y^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{2\beta}u(x,y)}{\partial x^{2\beta}}+\frac{\partial^{2\beta}u(x,y)}{\partial y^{2\beta}}=0, \tag{1}$$

where the size $u(x,y)$ is the non-differentiable function, and the local fractional operator indicates [10]

$$\frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}}=\frac{\Delta^{\beta}\left(u(x,t)-u(x,t_0)\right)}{(t-t_0)}, \tag{2}$$

where

$$\Delta^{\beta}\left(u(x,t)-u(x,t_0)\right)\cong\Gamma(1+\beta)\left[u(x,t)-u(x,t_0)\right] \tag{3}$$

The Tricomi equation finds its application in modeling transonic flow [21–23]. Inspired and motivated by the ongoing research in this area and wide applications of local fractional differential equations, we propose the local fractional homotopy perturbation Sumudu transform method (LFHPSTM) to solve the local fractional model of the Tricomi equation appearing in fractal transonic flow pertaining to the local fractional derivative boundary value conditions. The LFHPSTM is a conjunction of the classical homotopy perturbation method (HPM) [24–26] and the local fractional Sumudu transform technique. The formation of this article is as developed. In Section 2, the local fractional integrals and derivatives are initiated. In Section 3, the local fractional homotopy perturbation Sumudu transform method is proposed. In Section 4, the non-differentiable numerical solutions for local fractional Tricomi equation along the local fractional derivative boundary value conditions are specified. Finally, in Section 5, the conclusions are discussed.

2. Local Fractional Integrals and Derivatives

In this section, we review the basic theory of local fractional calculus, which is applied in this research article.

Definition 1 ([10–20])**.** For the relation $|x-x_0|<\delta$, when $\varepsilon,\delta>0$ and $\varepsilon>R$, we permit the function $f(x)\in C_{\beta}(a,b)$, while

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\beta}, 0<\beta\leqslant 1, \tag{4}$$

exists.

Definition 2 ([10–20])**.** Consider the interval [a, b] and (t_j,t_{j+1}), $j=0,\ldots,N-1$, $t_0=a$, and $t_N=b$ with $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_0,\Delta t_1,\Delta t_2,\ldots\}$ a partition of this interval. Then, the local fractional integral of $f(x)$ is explained as

$$ {}_aI_b^{(\beta)}{}_{f(x)}=\frac{1}{\Gamma(1+\beta)}\int_a^b f(t)(dt)^{\beta}=\frac{1}{\Gamma(1+\beta)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f(t_j)(\Delta t_j)^{\beta}, \tag{5}$$

Definition 3 ([10–20])**.** Suppose the function $f(x)$ fulfill state in Equation (4), then the inverse formula of Equation (5) is defined as follows:

$$\frac{d^{\beta}f(x_0)}{dx^{\beta}}=D_x^{(\beta)}f(x_0)=\frac{\Delta^{\beta}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\beta}}, \tag{6}$$

where

$$\Delta^{\beta}\left(f(x)-f(x_0)\right)\cong\Gamma(1+\beta)[f(x)-f(x_0)]. \tag{7}$$

The formula of the local fractional derivative, employed in this paper, is given as follows [10]:

$$\frac{d^{\beta}}{dx^{\beta}}\frac{x^{n\beta}}{\Gamma(1+n\beta)}=\frac{x^{(n-1)\beta}}{\Gamma(1+(n-1)\beta)},n\in N. \tag{8}$$

3. Local Fractional Sumudu Transform

The Sumudu transform was initially proposed and developed by Watugala [27] and some of its important properties were discovered and investigated by Belgacem *et al.* [28] and Belgacem and Karaballi [29]. Katatbeh and Belgacem [30] employed the Sumudu transform to solve fractional differential equations. Gupta *et al.* [31] used the Sumudu transform to solve generalized fractional kinetic equations. Belgacem [32] investigated the applications of the Sumudu transform to Bessel functions and equations. Belgacem [33] introduced and analyzed deeper Sumudu properties. Bulut *et al.* [34] obtained the analytical solutions of some fractional ordinary differential equations by using the Sumudu transform technique. The Sumudu transform method is also coupled with HPM to investigate the fractional biological population model [35]. The local fractional Sumudu transform of a function $f(x)$ is first introduced and defined by Srivastava *et al.* [36] in the following manner:

$$\begin{aligned} LFS_{\beta}\{f(x)\} &= F_{\beta}(z) \\ &= \frac{1}{\Gamma(1+\beta)}\int_{0}^{\infty}E_{\beta}(-z^{-\beta}x^{\beta})\frac{f(x)}{z^{\beta}}(dx)^{\beta},0<\beta\leqslant 1 \end{aligned} \tag{9}$$

and the inverse formula is expressed as follows

$$LFS_{\beta}^{-1}\{F_{\beta}(z)\}=f(x),0<\beta\leqslant 1. \tag{10}$$

4. Local Fractional Homotopy Perturbation Sumudu Transform Method

In order to establish the basic idea of the LFHPSTM, we assume the following linear differential equation with a local fractional derivative

$$L_{\beta}u(x,t)+R_{\beta}u(x,t)=h(x,t), \tag{11}$$

where L_{β} denotes the linear local fractional differential operator, R_{β} is the remaining linear operator and $h(x,t)$ is a source term.

Using the local Sumudu transform on Equation (11) yields

$$\begin{aligned} U_{\beta}(x,z)= \;& u(x,0)+z^{\beta}u^{\beta}(x,0)+z^{2\beta}u^{2\beta}(x,0)+\ldots \\ & +z^{(k-1)\beta}u^{(k-1)\beta}(x,0)-z^{k\beta}LFS_{\beta}\left[R_{\beta}u(x,t)\right]+z^{k\beta}LFS_{\beta}\left[h(x,t)\right]. \end{aligned} \tag{12}$$

Applying the inverse of the local fractional Sumudu transform on Equation (12), we have the following result

$$\begin{aligned} u(x,t)= \;& u(x,0)+\frac{t^{\beta}}{\Gamma(1+\beta)}u^{\beta}(x,0)+\frac{t^{2\beta}}{\Gamma(1+2\beta)}u^{2\beta}(x,0)+\ldots \\ & +\frac{t^{(k-1)\beta}}{\Gamma(1+(k-1)\beta)}u^{(k-1)\beta}(x,0)-LFS_{\beta}^{-1}\left[z^{k\beta}LFS_{\beta}\left[R_{\beta}u(x,t)\right]\right] \\ & +LFS_{\beta}^{-1}\left[z^{k\beta}LFS_{\beta}\left[h(x,t)\right]\right]. \end{aligned} \tag{13}$$

Now we use the HPM [24–26]

$$u(x,t)=\sum_{n=0}^{\infty}p^{n}u_{n}(x,t). \tag{14}$$

Substituting Equation (14) in Equation (13), we get the following result:

$$\begin{aligned}\sum_{n=0}^{\infty} p^n u_n(x,t) = \quad & u(x,0) + \tfrac{t^{\beta}}{\Gamma(1+\beta)} u^{\beta}(x,0) + \tfrac{t^{2\beta}}{\Gamma(1+2\beta)} u^{2\beta}(x,0) + \dots \\ & + \tfrac{t^{(k-1)\beta}}{\Gamma(1+(k-1)\beta)} u^{(k-1)\beta}(x,0) - pLFS_{\beta}^{-1}\left[z^{k\beta} LFS_{\beta}\left[R_{\beta} \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right] \\ & + LFS_{\beta}^{-1}\left[z^{k\beta} LFS_{\beta}\left[h(x,t)\right]\right].\end{aligned} \tag{15}$$

which is a mixture of the local fractional Sumudu transform technique and HPM. Comparing the coefficients of like powers of p, we get

$$\begin{aligned} p^0 : u_0(x,t) &= u(x,0) + \tfrac{t^{\beta}}{\Gamma(1+\beta)} u^{\beta}(x,0) + \tfrac{t^{2\beta}}{\Gamma(1+2\beta)} u^{2\beta}(x,0) + \dots \\ & \quad + \tfrac{t^{(k-1)\beta}}{\Gamma(1+(k-1)\beta)} u^{(k-1)\beta} u(x,0) + LFS_{\beta}^{-1}\left[z^{k\beta} LFS_{\beta}\left[h(x,t)\right]\right], \\ p^1 : u_1(x,t) &= -LFS_{\alpha}^{-1}\left[z^{k\beta} LFS_{\beta}\left[R_{\beta} u_0(x,t)\right]\right], \\ p^2 : u_2(x,t) &= -LFS_{\beta}^{-1}\left[z^{k\beta} LFS_{\beta}\left[R_{\beta} u_1(x,t)\right]\right], \\ &\vdots \end{aligned} \tag{16}$$

Therefore, the solution of Equation (11) is given by

$$u(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,t) \tag{17}$$

5. Nondifferential Solutions for the Local Fractional Tricomi Equation

In this section, we present the nondifferential solutions for the Tricomi equation pertaining to the local fractional derivative occurring in fractal transonic flow with local fractional derivative boundary value conditions.

Example 1. Firstly, we investigate the following local fractional Tricomi equation

$$\frac{y^{\beta}}{\Gamma(1+\beta)} \frac{\partial^{2\beta} u(x,y)}{\partial x^{2\beta}} + \frac{\partial^{2\beta} u(x,y)}{\partial y^{2\beta}} = 0 \tag{18}$$

subject to the initial-boundary value conditions

$$\begin{aligned} u(x,0) &= 0, \\ u(h,l) &= 0, \\ u(x,0) &= \tfrac{x^{2\beta}}{\Gamma(1+2\beta)}, \\ \tfrac{\partial^{\beta} u(x,0)}{\partial y^{\beta}} &= 0. \end{aligned} \tag{19}$$

Applying the local fractional Sumudu transform on Equation (18), we get

$$U_{\beta}(x,z) = u(x,0) + z^{\beta} u^{\beta}(x,0) + z^{2\beta} LFS_{\beta}\left[-\frac{y^{\beta}}{\Gamma(1+\beta)} \frac{\partial^{2\beta} u(x,y)}{\partial x^{2\beta}}\right] \tag{20}$$

which implies

$$U_{\beta}(x,z) = \frac{x^{2\beta}}{\Gamma(1+2\beta)} + z^{2\beta} LFS_{\beta}\left[-\frac{y^{\beta}}{\Gamma(1+\beta)} \frac{\partial^{2\beta} u(x,y)}{\partial x^{2\beta}}\right] \tag{21}$$

Applying the inverse local fractional Sumudu transform to Equation (21) gives

$$u(x,y)=\frac{x^{2\beta}}{\Gamma(1+2\beta)}+LFS_{\beta}^{-1}\left[z^{2\beta}LFS_{\beta}\left[-\frac{y^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{2\beta}u(x,y)}{\partial x^{2\beta}}\right]\right] \tag{22}$$

Now using HPM [24–26] we get

$$\sum_{n=0}^{\infty}p^{n}u_{n}(x,y)=\frac{x^{2\beta}}{\Gamma(1+2\beta)}+pLFS_{\beta}^{-1}\left[z^{2\beta}LFS_{\beta}\left[-\frac{y^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{2\beta}(\sum\limits_{n=0}^{\infty}p^{n}u_{n}(x,y))}{\partial x^{2\beta}}\right]\right] \tag{23}$$

Comparing the like powers of p, we get the following components of the series solution

$$\begin{aligned} p^{0}:u_{0}(x,y) &= \tfrac{x^{2\beta}}{\Gamma(1+2\beta)}, \\ p^{1}:u_{1}(x,y) &= -\tfrac{y^{3\beta}}{\Gamma(1+3\beta)}, \\ p^{2}:u_{2}(x,y) &= 0, \\ &\vdots \end{aligned} \tag{24}$$

Finally, we get the exact solution of Equation (18) with the local fractional derivative boundary value conditions Equation (19), namely

$$\begin{aligned} u(x,y) &= \lim_{N\to\infty}\sum_{n=0}^{N}u_{n}(x,y) \\ &= \tfrac{x^{2\beta}}{\Gamma(1+2\beta)}-\tfrac{y^{3\beta}}{\Gamma(1+3\beta)} \end{aligned} \tag{25}$$

Example 2. Next, we study the local fractional Tricomi equation of the form

$$\frac{y^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{2\beta}u(x,y)}{\partial x^{2\beta}}+\frac{\partial^{2\beta}u(x,y)}{\partial y^{2\beta}}=0 \tag{26}$$

and the initial conditions are presented as

$$\begin{aligned} u(x,0) &= \tfrac{x^{\beta}}{\Gamma(1+\beta)}, \\ \tfrac{\partial^{\beta}u(x,0)}{\partial y^{\beta}} &= \tfrac{x^{\beta}}{\Gamma(1+\beta)}, \end{aligned} \tag{27}$$

Applying the local fractional Sumudu transform to Equation (26), we get

$$U_{\beta}(x,z)=u(x,0)+z^{\beta}u^{\beta}(x,0)+z^{2\beta}LFS_{\beta}\left[-\frac{y^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{2\beta}u(x,y)}{\partial x^{2\beta}}\right] \tag{28}$$

which gives

$$U_{\beta}(x,z)=\frac{x^{\beta}}{\Gamma(1+\beta)}+z^{\beta}\frac{x^{\beta}}{\Gamma(1+\beta)}+z^{2\beta}LFS_{\beta}\left[-\frac{y^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{2\beta}u(x,y)}{\partial x^{2\beta}}\right] \tag{29}$$

Applying the inverse local fractional Sumudu transform to Equation (29), we have

$$u(x,y)=\frac{x^{\beta}}{\Gamma(1+\beta)}\left[1+\frac{y^{\beta}}{\Gamma(1+\beta)}\right]+LFS_{\beta}^{-1}\left[z^{2\beta}LFS_{\beta}\left[-\frac{y^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{2\beta}u(x,y)}{\partial x^{2\beta}}\right]\right] \tag{30}$$

Now using HPM [24–26], we get

$$\sum_{n=0}^{\infty} p^n u_n(x,y) = \frac{x^\beta}{\Gamma(1+\beta)}\left[1+\frac{y^\beta}{\Gamma(1+\beta)}\right] + pLFS_\beta^{-1}\left[z^{2\beta} LFS_\beta\left[-\frac{y^\beta}{\Gamma(1+\beta)}\frac{\partial^{2\beta}\left(\sum_{n=0}^{\infty} p^n u_n(x,y)\right)}{\partial x^{2\beta}}\right]\right] \tag{31}$$

Comparing the coefficients of like powers of p provides

$$\begin{aligned} p^0 : u_0(x,y) &= \frac{x^\beta}{\Gamma(1+\beta)}\left[1+\frac{y^\beta}{\Gamma(1+\beta)}\right], \\ p^1 : u_1(x,y) &= 0, \\ p^2 : u_2(x,y) &= 0, \\ &\vdots \end{aligned} \tag{32}$$

Hence, we get the exact solution with non-differential term, as follows:

$$u(x,y) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,y) = \frac{x^\beta}{\Gamma(1+\beta)}\left[1+\frac{y^\beta}{\Gamma(1+\beta)}\right] \tag{33}$$

6. Conclusions

In the present paper, the local fractional Tricomi equation with its applications in fractal transonic flow is discussed by using the local fractional homotopy perturbation Sumudu transform technique. We obtain the solution with non-differential terms by applying this approach. The results show that the proposed technique is very efficient and can be used to solve various kinds of local fractional differential equations. Hence, the introduced method is a powerful tool for solving local fractional linear equations of physical importance.

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