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ON THE HARDY–CARLEMAN INEQUALITY FOR A NEGATIVE EXPONENT

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Abstract. In this paper we settle an open problem raised by B. Yang (2005, *Taiwanese Journal of Mathematics* 9, 469-475), by using Hölder's and Bernoulli's inequalities. We give a strengthened Hardy-Carleman inequality for a negative exponent.

1. Introduction

The following inequality of Hardy's is well known [2, Chap. 9.12]:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k^{\frac{1}{p}}\right)^p<\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n. \tag{1}$$

Here $p>1$, $a_n\geqslant 0$ $(n\in\mathbf{N})$ and $0<\sum_{n=1}^{\infty}a_n<\infty$.

The constant $(\frac{p}{p-1})^p$ in (1) is the best possible. As p tends to infinity the inquality (1) reduces to Carleman's inequality

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}a_n, \tag{2}$$

where the constant e in (2) is still the best possible [2, Chap. 9.12]. The inequalities (1) and (2) are important in analysis and its applications [3].

In [8], we proved the following strengthened version of (2).

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}\left(1+\frac{1}{n+\frac{1}{5}}\right)^{-\frac{1}{2}}a_n. \tag{3}$$

Some other strengthened versions of (2) and related results can be found in [1, 7, 8, 9, 11].

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If we set $p = \frac{1}{r}$ in (1), then we have $0 < r < 1$, and (1) is equivalent to the inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^r\right)^{1/r} < \left(\frac{1}{1-r}\right)^{1/r}\sum_{n=1}^{\infty} a_n, \tag{4}$$

where the constant $(\frac{1}{1-r})^{1/r}$ is the best possible.

Thanh et al. [6] discussed (4) for $r \in (-\infty, 0)$, and proved the following result: If $a_n \geqslant 0$ for $n \in \mathbf{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^r\right)^{1/r} < \left(\frac{1}{1-r}\right)^{1/r}\sum_{n=1}^{\infty} a_n \tag{5}$$

if $-1 \leqslant r < 0$ and

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^r\right)^{1/r} < \frac{r}{r-1}2^{\frac{r-1}{r}}\sum_{n=1}^{\infty} a_n \tag{6}$$

if $r < -1$.

If we replace r by $-r$, in (5) and (6) we obtain

$$\sum_{n=1}^{\infty}\left(\frac{n}{\sum_{k=1}^{n} a_k^{-r}}\right)^{1/r} < (1+r)^{1/r}\sum_{n=1}^{\infty} a_n \tag{7}$$

if $0 < r \leqslant 1$ and

$$\sum_{n=1}^{\infty}\left(\frac{n}{\sum_{k=1}^{n} a_k^{-r}}\right)^{1/r} < \frac{r}{1+r}2^{\frac{1+r}{r}}\sum_{n=1}^{\infty} a_n \tag{8}$$

if $1 < r < \infty$.

Recently, Yang [10] proved that the constant $(1+r)^{1/r}$ in (7) is the best possible for $0 < r \leqslant 1$. At the end of paper [10], Yang posed the question:

Is the constant factor $\frac{r}{1+r}2^{\frac{1+r}{r}}$ in (8) the best possible or not for $1 < r < \infty$?

In this paper we solve this problem. We give a strengthened Hardy-Carleman inequality for a negative exponent.

2. Main results

In this section, we prove the following theorem.

THEOREM 2.1. *Let* $1 < r < \infty$, $a_n \geqslant 0$ $(n \in \mathbf{N})$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty}\left(\frac{n}{\sum_{k=1}^{n} a_k^{-r}}\right)^{1/r} < \frac{1}{r}(1+r)^{\frac{1+r}{r}}\sum_{n=1}^{\infty} a_n. \tag{9}$$

To prove Theorem 2.1, we use Hölder's inequality (with negative exponent p) (see [4, page 29]) and Bernoulli's inequality. For the convenience of the reader we start by recalling these results.

LEMMA 2.1. (Hölder's inequality) *Suppose that* $p<0$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)$, $g(x)\geqslant 0$ *for* $x\in[a,b]$, *and* $f\in L^p[a,b]$, $g\in L^q[a,b]$. *Then*

$$\int_a^b f(t)g(t)dt \geqslant \left(\int_a^b f^p(t)dt\right)^{1/p}\left(\int_a^b g^q(t)dt\right)^{1/q},$$

where equality holds only if there exist real numbers α *and* β, *such that* $\alpha^2+\beta^2>0$, *and* $\alpha f^p(x)=\beta g^q(x)$, *a.e. in* $[a,b]$.

LEMMA 2.2. (Bernoulli inequality) *Suppose that* $x\geqslant -1$ *and* $0<\alpha<1$. *Then*

$$(1+x)^\alpha \leqslant 1+\alpha x,$$

where equality holds if and only if $x=0$.

We also need the following lemmas.

LEMMA 2.3. *Suppose that* $0<\alpha<1$ *and* $x>0$. *Then*

$$1+\frac{\alpha x}{1+(1-\alpha)x}<(1+x)^\alpha.$$

Proof. We rewrite this inequality as

$$1+x+\alpha x(1+x)^\alpha<(1+x)^{1+\alpha}.$$

We define

$$\varphi(x)=(1+x)^{1+\alpha}-\alpha x(1+x)^\alpha-x-1 \quad \text{for} \quad x\geqslant 0.$$

Simple computation yields

$$\varphi'(x)=(1+x)^\alpha-\alpha^2x(1+x)^{\alpha-1}-1,$$
$$\varphi''(x)=(\alpha-\alpha^2)(1+x)^{\alpha-1}-\alpha^2(\alpha-1)x(1+x)^{\alpha-2}.$$

It follows that $\varphi''(x)>0$ for $x>0$ and $0<\alpha<1$, $\varphi'(0)=0$ and $\varphi(0)=0$. Thus, $\varphi(x)$ is strictly increasing and $\varphi(x)>0$ for $x>0$. This completes the proof of Lemma 2.3. $\square$

LEMMA 2.4. *Suppose that* $r>1$ *and* $x\geqslant 1$. *Then*

$$(1+x)^{\frac{1+r}{r}}-x^{\frac{1+r}{r}}>\frac{1+r}{r}x^{\frac{1}{r}}.$$

Proof. We rewrite this inequality as

$$(1+x)\left(1+\frac{1}{x}\right)^{\frac{1}{r}}-x>\frac{1+r}{r}.$$

This is true. Since by Lemma 2.3, we have

$$\left(1+\frac{1}{x}\right)^{\frac{1}{r}}>1+\frac{\frac{1}{r}x}{1+(1-\frac{1}{r})x}>1+\frac{1}{r(1+x)}.$$

This completes the proof of Lemma 2.4. $\square$

LEMMA 2.5. *We have*
(i) $2^{\frac{1}{r}}<\frac{1+r}{r}$ *for* $r>1$.
(ii) $\frac{r}{1+r}2^{\frac{1+r}{r}}>\frac{1}{r}(1+r)^{\frac{1+r}{r}}$ *for* $r>\frac{26}{5}$.

Proof. (i) By Bernoulli's inequality we have

$$2^{\frac{1}{r}}=(1+1)^{\frac{1}{r}}<\frac{1+r}{r}.$$

(ii) We rewrite this inequality as

$$\frac{r^2}{1+r}>\left(\frac{1+r}{2}\right)^{\frac{1+r}{r}}.$$

By Bernoulli's inequality, it follows

$$\begin{aligned}\left(\frac{1+r}{2}\right)^{\frac{1+r}{r}}&=\frac{1+r}{2}\left(1+\frac{r-1}{2}\right)^{\frac{1}{r}}\\&<\frac{1+r}{2}\left(1+\frac{r-1}{2r}\right)=-\frac{r^2(r-5)-r+1}{4r(1+r)}+\frac{r^2}{1+r}\\&<\frac{r^2}{1+r}\end{aligned}$$

for $r>\frac{26}{5}$.

This completes the proof of Lemma 2.5. $\square$

Proof of Theorem 2.1. Let $r>1$ and set $p=-\frac{1}{r}$, $a=1$, $b=x>1$, $f(x)=a_n$, $g_n(x)=(x-1)^{\frac{1}{(1+r)r}}$ for $x\in[n,n+1]$ and $n\in\mathbf{N}$. Hölder's inequality then yields

$$\left(\int_1^x f(t)g(t)dt\right)^{-\frac{1}{r}}\leqslant\left(\int_1^x f^{-\frac{1}{r}}(t)dt\right)\left(\int_1^x g^{\frac{1}{1+r}}(t)dt\right)^{-\frac{1+r}{r}}.$$

It follows that

$$\begin{aligned}\Big(\int_1^x f^{-r}(t)dt\Big)^{-\frac{1}{r}} &= \Big(\int_1^x ((t-1)^{1+r}f(t))^{-r}((t-1)^{(1+r)r})dt\Big)^{-\frac{1}{r}}\\ &< \Big(\int_1^x (t-1)^{1+r}f(t)dt\Big)\Big(\int_1^x (t-1)^r dt\Big)^{-\frac{1+r}{r}}\\ &< (1+r)^{\frac{1+r}{r}}(x-1)^{-\frac{(1+r)^2}{r}}\int_1^x (t-1)^{1+r}f(t)dt.\end{aligned}$$

Then we have

$$\Big(\frac{x-1}{\int_1^x f^{-r}(t)dt}\Big)^{1/r} < (1+r)^{\frac{1+r}{r}}(x-1)^{-r-2}\int_1^x (t-1)^{1+r}f(t)dt.$$

Then we obtain

$$\begin{aligned}\int_1^\infty \Big(\frac{x-1}{\int_1^x f^{-r}(t)dt}\Big)^{1/r}dx &< (1+r)^{\frac{1+r}{r}}\int_1^\infty (x-1)^{-r-2}\int_1^x (t-1)^{1+r}f(t)dtdx\\ &= (1+r)^{\frac{1+r}{r}}\int_1^\infty \Big(\int_t^\infty (x-1)^{-r-2}dx\Big)(t-1)^{1+r}f(t)dt\\ &= (1+r)^{1/r}\int_1^\infty f(t)dt\\ &= (1+r)^{1/r}\sum_{n=1}^\infty a_n.\end{aligned}$$

By the definition of $f(x)$, Lemmas 2.3, 2.4 and 2.5, we have

$$\begin{aligned}\int_1^\infty \Big(\frac{x-1}{\int_1^x f^{-r}(t)dt}\Big)^{1/r}dx &> \int_1^2 \Big(\frac{x-1}{\int_1^x f^{-r}(t)dt}\Big)^{1/r}dx + \int_2^\infty \Big(\frac{x-1}{\int_1^x f^{-r}(t)dt}\Big)^{1/r}dx\\ &> \frac{\int_1^2 (x-1)^{1/r}dx}{a_1^{-1}} + \sum_{n=2}^\infty \frac{\int_n^{n+1}(x-1)^{1/r}dx}{(\sum_{k=1}^n a_k^{-r})^{1/r}}\\ &> \frac{\frac{r}{1+r}}{a_1^{-1}} + \sum_{n=2}^\infty \frac{\frac{r}{1+r}(n^{\frac{1+r}{r}}-(n-1)^{\frac{1+r}{r}})}{(\sum_{k=1}^n a_k^{-r})^{1/r}}\\ &> \frac{\frac{r}{1+r}}{a_1^{-1}} + \sum_{n=2}^\infty \frac{(n-1)^{1/r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}}\\ &> \frac{\frac{r}{1+r}}{a_1^{-1}} + \Big(\frac{1}{2}\Big)^{1/r}\sum_{n=2}^\infty \frac{n^{1/r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}}\\ &> \frac{r}{1+r}\sum_{n=1}^\infty \Big(\frac{n}{\sum_{k=1}^n a_k^{-r}}\Big)^{1/r}.\end{aligned}$$

This completes the proof of Theorem 2.1. □

REMARK 1. By Theorem 2.1 and Lemma 2.5 (ii), we know that the constant factor $\frac{r}{1+r}2^{\frac{1+r}{r}}$ in (8) is not the best possible for $r > \frac{26}{5}$. We give a strengthened Hardy-Carleman inequality (9) for a negative exponent.

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