

ASYMPTOTIC DIRICHLET PROBLEM FOR \mathcal{A} -HARMONIC FUNCTIONS ON MANIFOLDS WITH PINCHED CURVATURE

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ABSTRACT. We study the asymptotic Dirichlet problem for \mathcal{A} -harmonic functions on a Cartan-Hadamard manifold whose radial sectional curvatures outside a compact set satisfy an upper bound

$$K(P) \leq -\frac{1 + \varepsilon}{r(x)^2 \log r(x)}$$

and a pointwise pinching condition

$$|K(P)| \leq C_K |K(P')|$$

for some constants $\varepsilon > 0$ and $C_K \geq 1$, where P and P' are any 2-dimensional subspaces of $T_x M$ containing the (radial) vector $\nabla r(x)$ and $r(x) = d(o, x)$ is the distance to a fixed point $o \in M$. We solve the asymptotic Dirichlet problem with any continuous boundary data $f \in C(\partial_\infty M)$. The results apply also to the Laplacian and p -Laplacian, $1 < p < \infty$, as special cases.

1. INTRODUCTION

In this paper we are interested in the asymptotic Dirichlet problem for \mathcal{A} -harmonic functions on a Cartan-Hadamard manifold M of dimension $n \geq 2$. We recall that a Cartan-Hadamard manifold is a simply connected complete Riemannian manifold with non-positive sectional curvature. Since the exponential map $\exp_o: T_o M \rightarrow M$ is a diffeomorphism for every point $o \in M$, it follows that M is diffeomorphic to \mathbb{R}^n . One can define an asymptotic boundary $\partial_\infty M$ of M as the set of all equivalence classes of unit speed geodesic rays on M . Then the compactification of M is given by $\bar{M} = M \cup \partial_\infty M$ equipped with the *cone topology*. We also notice that \bar{M} is homeomorphic to the closed Euclidean unit ball; for details, see Section 2 and [8].

The *asymptotic Dirichlet problem* on M for some operator \mathcal{Q} is the following: Given a function $f \in C(\partial_\infty M)$ does there exist a (unique) function $u \in C(\bar{M})$ such that $\mathcal{Q}[u] = 0$ on M and $u|_{\partial_\infty M} = f$? We will consider this problem for the \mathcal{A} -harmonic operator (of type p)

$$\mathcal{Q}[u] = -\operatorname{div} \mathcal{A}_x(\nabla u), \tag{1.1}$$

where $\mathcal{A}: TM \rightarrow TM$ is subject to certain conditions; for instance $\langle \mathcal{A}(V), V \rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2} \mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ (see Section 2.3 for the precise definition). A function u is said to be *\mathcal{A} -harmonic* if it satisfies the equation

$$-\operatorname{div} \mathcal{A}_x(\nabla u) = 0. \tag{1.2}$$

The asymptotic Dirichlet problem on Cartan-Hadamard manifolds has been solved for various operators and under various assumptions on the manifold. The first result for this problem was due to Choi [6] when he solved the asymptotic Dirichlet problem for the Laplacian assuming that the sectional curvature has a negative upper bound $K_M \leq -a^2 < 0$, and that any two points at infinity can be

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separated by convex neighborhoods. Anderson [1] showed that such convex sets exist provided the sectional curvature of the manifold satisfies $-b^2 \leq K_M \leq -a^2 < 0$. We point out that Sullivan [13] solved independently the asymptotic Dirichlet problem for the Laplacian under the same curvature assumptions but using probabilistic arguments. Cheng [5] was the first to solve the problem for the Laplacian under the same type of pointwise pinching assumption for the sectional curvatures as we consider in this paper. Later the asymptotic Dirichlet problem has been generalized for p -harmonic and \mathcal{A} -harmonic functions and for minimal graph equation under various curvature assumptions, see [2], [3], [10], [11], [14], [15].

In [14] Vähäkangas had exactly the same pinching condition but with weaker upper bound for the sectional curvatures. Namely, he solved the asymptotic Dirichlet problem assuming the pointwise pinching condition and

$$K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2},$$

where $\phi > 1$ is constant. In [2] the authors showed that, with these stronger assumptions, the solvability result holds also for the minimal graph equation.

In this paper we will use similar techniques as in [2], [3] and [15]. Our main theorem is the following.

Theorem 1.3. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Assume that*

$$K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad (1.4)$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_x M$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \leq C_K |K(P')| \quad (1.5)$$

whenever $x \in M \setminus B(o, R_0)$ and $P, P' \subset T_x M$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Then the asymptotic Dirichlet problem for the \mathcal{A} -harmonic equation (1.2) is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$ provided that $1 < p < n\alpha/\beta$.

In the case of usual Laplacian we have $\alpha = \beta = 1$ and $p = 2$. Hence we obtain the following special case.

Corollary 1.6. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 3$ and assume that the assumptions (1.4) and (1.5) are satisfied. Then the asymptotic Dirichlet problem for the Laplace operator is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.*

We close this introduction by commenting that our upper bound (1.4) is in a sense optimal, since assuming

$$K(P) \geq -\frac{1}{r(x)^2 \log r(x)}$$

and considering \mathcal{A} -harmonic operator of type $p \geq n$, implies that M is p -parabolic i.e. every bounded \mathcal{A} -harmonic function (of type p) is constant. For more detailed discussion, see e.g. [3].

2. PRELIMINARIES

2.1. Cartan-Hadamard manifolds. Recall that a Cartan-Hadamard manifold is a complete and simply connected Riemannian manifold with non-positive sectional curvature. Let M be a Cartan-Hadamard manifold and $\partial_\infty M$ the sphere at infinity, then we denote $\bar{M} = M \cup \partial_\infty M$. The sphere at infinity is defined as the set of all

equivalence classes of unit speed geodesic rays in M ; two such rays γ_1 and γ_2 are equivalent if

$$\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

The equivalence class of γ is denoted by $\gamma(\infty)$. For each $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y}: \mathbb{R} \rightarrow M$ such that $\gamma^{x,y}(0) = x$ and $\gamma^{x,y}(t) = y$ for some $t \in (0, \infty]$. For $x \in M$ and $y, z \in \bar{M} \setminus \{x\}$ we denote by

$$\sphericalangle_x(y, z) = \sphericalangle(\dot{\gamma}_0^{x,y}, \dot{\gamma}_0^{x,z})$$

the angle between vectors $\dot{\gamma}_0^{x,y}$ and $\dot{\gamma}_0^{x,z}$ in $T_x M$. If $v \in T_x M \setminus \{0\}$, $\alpha > 0$, and $R > 0$, we define a cone

$$C(v, \alpha) = \{y \in \bar{M} \setminus \{x\} : \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \alpha\}$$

and a truncated cone

$$T(v, \alpha, R) = C(v, \alpha) \setminus \bar{B}(x, R).$$

All cones and open balls in M form a basis for the cone topology in \bar{M} . With this topology \bar{M} is homeomorphic to the closed unit ball $\bar{B}^n \subset \mathbb{R}^n$ and $\partial_\infty M$ to the unit sphere $\mathbb{S}^{n-1} = \partial B^n$. For detailed study on the cone topology, see [8].

Let us recall that the local Sobolev inequality holds on any Cartan-Hadamard manifold M . More precisely, there exist constants $r_S > 0$ and $C_S < \infty$ such that

$$\left(\int_B |\eta|^{n/(n-1)} \right)^{(n-1)/n} \leq C_S \int_B |\nabla \eta| \quad (2.1)$$

holds for every ball $B = B(x, r_S) \subset M$ and every function $\eta \in C_0^\infty(B)$. This inequality can be obtained e.g. from Croke's estimate of the isoperimetric constant, see [4] and [7].

2.2. Jacobi equation. If $k: [0, \infty) \rightarrow [0, \infty)$ is a smooth function, we denote by $f_k \in C^\infty([0, \infty))$ the solution to the initial value problem

$$\begin{cases} f_k'' = k^2 f_k \\ f_k(0) = 0, \\ f_k'(0) = 1. \end{cases} \quad (2.2)$$

The solution is a non-negative smooth function. Concerning the curvature upper bound (1.4), we have the following estimate by Choi.

Proposition 2.3. [6, Prop. 3.4] *Suppose that $f: [R_0, \infty) \rightarrow \mathbb{R}$, $R_0 > 0$, is a positive strictly increasing function satisfying the equation $f''(r) = a^2(r)f(r)$, where*

$$a^2(r) \geq \frac{1 + \varepsilon}{r^2 \log r},$$

for some $\varepsilon > 0$ on $[R_0, \infty)$. Then for any $0 < \tilde{\varepsilon} < \varepsilon$, there exists $R_1 > R_0$ such that, for all $r \geq R_1$,

$$f(r) \geq r(\log r)^{1+\tilde{\varepsilon}}, \quad \frac{f'(r)}{f(r)} \geq \frac{1}{r} + \frac{1 + \tilde{\varepsilon}}{r \log r}.$$

The pinching condition for the sectional curvatures gives a relation between the maximal and minimal moduli of Jacobi fields along a given geodesic that contains the radial vector:

Lemma 2.4. [5, Lemma 3.2][14, Lemma 3] *Let $v \in T_o M$ be a unit vector and $\gamma = \gamma^v$. Suppose that $r_0 > 0$ and $k < 0$ are constants such that $K_M(P) \geq k$ for every two-dimensional subspace $P \subset T_x M$, $x \in B(o, r_0)$. Suppose that there exists a constant $C_K < \infty$ such that*

$$|K_M(P)| \leq C_K |K_M(P')|$$

whenever $t \geq r_0$ and $P, P' \subset T_{\gamma(t)}M$ are two-dimensional subspaces containing the radial vector $\dot{\gamma}_t$. Let V and \bar{V} be two Jacobi fields along γ such that $V_0 = 0 = \bar{V}_0$, $V'_0 \perp \dot{\gamma}_0 \perp \bar{V}'_0$, and $|V'_0| = 1 = |\bar{V}'_0|$. Then there exists a constant $C_0 = C_0(C_K, r_0, k) > 0$ such that

$$|V_r|^{C_K} \geq C_0 |\bar{V}_r|$$

for every $r \geq r_0$.

To prove the solvability of the \mathcal{A} -harmonic equation, we will need an estimate for the gradient of a certain angular function. This estimate can be obtained in terms of Jacobi fields:

Lemma 2.5. [14, Lemma 2] *Let $x_0 \in M \setminus \{o\}$, $U = M \setminus \gamma^{o, x_0}(\mathbb{R})$, and define $\theta: U \rightarrow [0, \pi]$, $\theta(x) = \angle_o(x_0, x) := \arccos \langle \dot{\gamma}_0^{o, x_0}, \dot{\gamma}_0^{o, x} \rangle$. Let $x \in U$ and $\gamma = \gamma^{o, x}$. Then there exists a Jacobi field W along γ with $W(0) = 0$, $W'_0 \perp \dot{\gamma}_0$, and $|W'_0| = 1$ such that*

$$|\nabla \theta(x)| \leq \frac{1}{|W(r(x))|}.$$

2.3. \mathcal{A} -harmonic functions. Let M be a Riemannian manifold and $1 < p < \infty$. Suppose that $\mathcal{A}: TM \rightarrow TM$ is an operator that satisfies the following assumptions for some constants $0 < \alpha \leq \beta < \infty$: the mapping $\mathcal{A}_x = \mathcal{A}|_{T_x M}: T_x M \rightarrow T_x M$ is continuous for almost every $x \in M$ and the mapping $x \mapsto \mathcal{A}_x(V_x)$ is measurable for all measurable vector fields V on M ; for almost every $x \in M$ and every $v \in T_x M$ we have

$$\begin{aligned} \langle \mathcal{A}_x(v), v \rangle &\geq \alpha |v|^p, \\ |\mathcal{A}_x(v)| &\leq \beta |v|^{p-1}, \\ \langle \mathcal{A}_x(v) - \mathcal{A}_x(w), v - w \rangle &> 0, \end{aligned}$$

whenever $w \in T_x M \setminus \{v\}$, and

$$\mathcal{A}_x(\lambda v) = \lambda |\lambda|^{p-2} \mathcal{A}_x(v)$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. The set of all such operators is denoted by $\mathcal{A}^p(M)$ and we say that \mathcal{A} is of type p . The constants α and β are called the *structure constants* of \mathcal{A} .

Let $\Omega \subset M$ be an open set and $\mathcal{A} \in \mathcal{A}^p(M)$. A function $u \in C(\Omega) \cap W_{\text{loc}}^{1,p}(\Omega)$ is \mathcal{A} -harmonic in Ω if it is a weak solution of the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0. \quad (2.6)$$

In other words, if

$$\int_{\Omega} \langle \mathcal{A}(\nabla u), \nabla \varphi \rangle = 0 \quad (2.7)$$

for every test function $\varphi \in C_0^\infty(\Omega)$. If $|\nabla u| \in L^p(\Omega)$, then it is equivalent to require (2.7) for all $\varphi \in W_0^{1,p}(\Omega)$ by approximation.

In the special case $\mathcal{A}(v) = |v|^{p-2}v$, \mathcal{A} -harmonic functions are called p -harmonic and, in particular, if $p = 2$, we obtain the usual harmonic functions.

A lower semicontinuous function $u: \Omega \rightarrow (-\infty, \infty]$ is called \mathcal{A} -superharmonic if $u \not\equiv \infty$ in each component of Ω , and for each open $D \subset\subset \Omega$ and for every $h \in C(\bar{D})$, \mathcal{A} -harmonic in D , $h \leq u$ on ∂D implies $h \leq u$ in D .

The *asymptotic Dirichlet problem* (for \mathcal{A} -harmonic functions) is the following: for given function $f \in C(\partial_\infty M)$, find a function $u \in C(\bar{M})$ such that $\mathcal{A}(u) = 0$ in M and $u|_{\partial_\infty M} = f$. The asymptotic Dirichlet problem can be solved using the so called Perron's method which we will recall next. The definitions follow [9].

Fix $p \in (1, \infty)$ and let $\mathcal{A} \in \mathcal{A}^p(M)$.

Definition 1. A function $u: M \rightarrow (-\infty, \infty]$ belongs to the upper class \mathcal{U}_f of $f: \partial_\infty M \rightarrow [-\infty, \infty]$ if

- (1) u is \mathcal{A} -superharmonic in M ,
- (2) u is bounded from below, and
- (3) $\liminf_{x \rightarrow x_0} u(x) \geq f(x_0)$ for all $x_0 \in \partial_\infty M$.

The function

$$\overline{H}_f = \inf\{u: u \in \mathcal{U}_f\}$$

is called the *upper Perron solution* and $\underline{H}_f = -\overline{H}_{-f}$ the *lower Perron solution*.

Theorem 2.8. *One of the following is true:*

- (1) \overline{H}_f is \mathcal{A} -harmonic in M ,
- (2) $\overline{H}_f \equiv \infty$ in M ,
- (3) $\overline{H}_f \equiv -\infty$ in M .

We define \mathcal{A} -regular points as follows.

Definition 2. A point $x_0 \in \partial_\infty M$ is called \mathcal{A} -regular if

$$\lim_{x \rightarrow x_0} \overline{H}_f(x) = f(x_0)$$

for all $f \in C(\partial_\infty M)$.

Regularity and solvability of the Dirichlet problem are related. Namely, the asymptotic Dirichlet problem for \mathcal{A} -harmonic functions is uniquely solvable if and only if every point at infinity is \mathcal{A} -regular.

2.4. Young functions. Let $\phi: [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism and let $\psi = \phi^{-1}$. Define *Young functions* Φ and Ψ by setting

$$\Phi(t) = \int_0^t \phi(s) ds$$

and

$$\Psi(t) = \int_0^t \psi(s) ds$$

for each $t \in [0, \infty)$. Then we have the following *Young's inequality*

$$ab \leq \Phi(a) + \Psi(b)$$

for all $a, b \in [0, \infty)$. The functions Φ and Ψ are said to form a *complementary Young pair*. Furthermore, Φ (and similarly Ψ) is a continuous, strictly increasing, and convex function satisfying

$$\lim_{t \rightarrow 0^+} \frac{\Phi(t)}{t} = 0$$

and

$$\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \infty.$$

For a more general definition of Young functions see e.g. [12].

As in [15], we consider complementary Young pairs of a special type. For that, suppose that a homeomorphism $G: [0, \infty) \rightarrow [0, \infty)$ is a Young function that is a diffeomorphism on $(0, \infty)$ and satisfies

$$\int_0^1 \frac{dt}{G^{-1}(t)} < \infty \quad (2.9)$$

and

$$\lim_{t \rightarrow 0} \frac{tG'(t)}{G(t)} = 1. \quad (2.10)$$

Then $G^{(\cdot^{1/p})^p}$, $p > 1$, is also a Young function and we define $F: [0, \infty) \rightarrow [0, \infty)$ so that $G^{(\cdot^{1/p})^p}$ and $F^{(\cdot^{1/p})}$ form a complementary Young pair. The space of such functions F will be denoted by \mathcal{F}_p . Note that if $F \in \mathcal{F}_p$, then also $\lambda F \in \mathcal{F}_p$ and

$F(\lambda \cdot) \in \mathcal{F}_p$ for every $\lambda > 0$. In [15] it is proved that for fixed $\varepsilon_0 \in (0, 1)$ there exists $F \in \mathcal{F}_p$ such that

$$F(t) \leq t^{p+\varepsilon_0} \exp\left(-\frac{1}{t} \left(\log\left(e + \frac{1}{t}\right)\right)^{-1-\varepsilon_0}\right) \quad (2.11)$$

for all $t \in [0, \infty)$.

3. SOLVING THE ASYMPTOTIC DIRICHLET PROBLEM

In order to solve the asymptotic Dirichlet problem for the \mathcal{A} -harmonic equation, we need the following two lemmas, which we state without proofs. Their proofs can be found from the original papers. The first lemma allows us to estimate the supremum of a function in a ball by the integral over a bigger ball. The second lemma shows that we can estimate the previous integral up to another integral, which will be uniformly bounded provided the sectional curvatures of M satisfy (1.4) and (1.5).

Lemma 3.1. [15, Lemma 2.20] *Suppose that $\|\theta\|_{L^\infty} \leq 1$. Suppose that $s \in (0, r_S)$ is a constant and $x \in M$. Assume also that $u \in W_{loc}^{1,p}(M)$ is a function that is \mathcal{A} -harmonic in the open set $\Omega \cap B(x, s)$, satisfies $u - \theta \in W_0^{1,p}(\Omega)$, $\inf_M \theta \leq u \leq \sup_M \theta$, and $u = \theta$ a.e. in $M \setminus \Omega$. Then*

$$\operatorname{ess\,sup}_{B(x,s/2)} \varphi(|u - \theta|)^{p(n-1)} \leq c \int_{B(x,s)} \varphi(|u - \theta|)^p,$$

where the constant c is independent of x .

Lemma 3.2. [3, Lemma 16] *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Suppose that*

$$K(P) \leq -\frac{1 + \varepsilon}{r(x)^2 \log r(x)},$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any plane $P \subset T_x M$ that contains the radial vector $\nabla r(x)$ and x is any point in $M \setminus B(o, R_0)$. Suppose that $U \subset M$ is an open relatively compact set and that u is an \mathcal{A} -harmonic function in U , with $u - \theta \in W_0^{1,p}(U)$, where $\mathcal{A} \in \mathcal{A}^p(M)$ with

$$1 < p < \frac{n\alpha}{\beta},$$

and $\theta \in W^{1,\infty}(M)$ is a continuous function with $\|\theta\|_\infty \leq 1$. Then there exists a bounded C^1 -function $\mathcal{C}: [0, \infty) \rightarrow [0, \infty)$ and a constant $c_0 \geq 1$, that is independent of θ, U and u , such that

$$\begin{aligned} & \int_U \varphi(|u - \theta|/c_0)^p (\log(1+r) + \mathcal{C}(r)) \\ & \leq c_0 + c_0 \int_U F\left(\frac{c_0 |\nabla \theta| r \log(1+r)}{\log(1+r) + \mathcal{C}(r)}\right) (\log(1+r) + \mathcal{C}(r)). \end{aligned} \quad (3.3)$$

In what follows, we will denote by $j(x)$ the infimum, and by $J(x)$ the supremum, of $|V(r(x))|$ over Jacobi fields V along the geodesic $\gamma^{o,x}$ that satisfy $V_0 = 0$, $|V'_0| = 1$ and $V'_0 \perp \dot{\gamma}_0^{o,x}$.

Next we show that the integral appearing in Lemma 3.2 is finite provided the upper bound (1.4) and the pinching condition (1.5) for the sectional curvatures.

Lemma 3.4. *Let M be a Cartan-Hadamard manifold satisfying*

$$K(P) \leq -\frac{1 + \varepsilon}{r(x)^2 \log r(x)},$$

where $K(P)$ is the sectional curvature of any plane $P \subset T_x M$ that contains the radial vector field $\nabla r(x)$ and x is any point in $M \setminus B(o, R_0)$. Then there exists $F \in \mathcal{F}_p$ such that

$$F\left(\frac{r(x)}{c_1 j(x)}\right) \left(\log(1+r) + \mathcal{C}(r)\right) j(x)^{C(n-1)} \leq r(x)^{-2}$$

for any positive constants C and c_1 , and for every $x \in M$ outside a compact set.

Proof. Fix $\varepsilon_0 \in (0, 1)$ and denote $\lambda := 1 + \varepsilon_0$. Then by (2.11) there exists $F \in \mathcal{F}_p$ such that

$$F(t) \leq \exp\left(-\frac{1}{t} \left(\log\left(e + \frac{1}{t}\right)\right)^{-\lambda}\right)$$

for all small t . Hence the claim follows if we show that

$$\exp\left(-\frac{c_1 j(x)}{r(x)} \left(\log\left(e + \frac{c_1 j(x)}{r(x)}\right)\right)^{-\lambda}\right) \left(\log(1+r(x)) + \mathcal{C}(r)\right) j(x)^{C(n-1)} \leq r(x)^{-2},$$

which, by taking logarithms, is equivalent with

$$\begin{aligned} & \frac{c_1 j(x)}{r(x)} \left(\log\left(e + \frac{c_1 j(x)}{r(x)}\right)\right)^{-\lambda} - \log\left(\log(1+r(x)) + \mathcal{C}(r)\right) \\ & - C(n-1) \log j(x) - 2 \log r(x) \geq 0. \end{aligned}$$

Let $\tilde{\varepsilon} \in (0, \varepsilon)$. Then the curvature upper bound and Proposition 2.3 implies that $j(x) \geq r(x)(\log r(x))^{1+\tilde{\varepsilon}}$ for $r(x) \geq R_1 > R_0$, so it is enough to show that

$$f(t) := \frac{c_1 t}{a} \left(\log\left(e + \frac{c_1 t}{a}\right)\right)^{-\lambda} - \log\left(\log(1+a) + \mathcal{C}(a)\right) - C(n-1) \log t - 2 \log a \geq 0$$

for all $t \geq a(\log a)^{1+\tilde{\varepsilon}}$ when a is big enough. By straight computation we get

$$\begin{aligned} f'(t) &= \left(\log\left(e + \frac{c_1 t}{a}\right)\right)^{-\lambda} \left(\frac{-\lambda c_1^2 t}{a^2 \log(e + c_1 t/a)(e + c_1 t/a)} + \frac{c_1}{a}\right) - \frac{C(n-1)}{t} \\ &= \left[\frac{c_1}{a} \left(1 - \frac{\lambda}{\log(e + c_1 t/a)(\frac{ea}{c_1 t} + 1)}\right) / \left(\log\left(e + \frac{c_1 t}{a}\right)\right)^\lambda\right] - \frac{C(n-1)}{t}. \end{aligned}$$

Then we notice that $c_1 t/a \geq c_1(\log a)^{1+\tilde{\varepsilon}}$, which can be made big by increasing a , and $(\log(e + c_1 t/a))^\lambda \leq k(t/a)^\nu$, where k is some constant and $\nu > 0$ can be made as small as we wish. Hence we obtain

$$f'(t) \geq \frac{k_1}{a^{1-\nu} t^\nu} - \frac{C(n-1)}{t} \geq 0$$

for all $t \geq a(\log a)^{1+\tilde{\varepsilon}}$ and some constant k_1 when a is big enough.

Finally we have to check that f is positive at least when t is big enough. To see this, we notice that

$$\begin{aligned} f(a(\log a)^{1+\tilde{\varepsilon}}) &= c_1(\log a)^{1+\tilde{\varepsilon}} \left(\log\left(e + c_1(\log a)^{1+\tilde{\varepsilon}}\right)\right)^{-\lambda} \\ & - \log\left(\log(1+a) + \mathcal{C}(a)\right) - C(n-1) \log\left((a \log a)^{1+\tilde{\varepsilon}}\right) - 2 \log a, \end{aligned}$$

and this being positive is equivalent to

$$\begin{aligned} c_1(\log a)^{1+\tilde{\varepsilon}} &\geq \left(\log\left(e + c_1(\log a)^{1+\tilde{\varepsilon}}\right)\right)^\lambda \left((C(n-1) + 2) \log a\right. \\ & \left. + \log\left(\log(1+a) + \mathcal{C}(a)\right) + C(n-1) \log(\log a)^{1+\tilde{\varepsilon}}\right), \end{aligned}$$

which holds true for a big enough, since $(\log a)^{1+\tilde{\varepsilon}}$ increases faster than $(\log a)(\log(e + c_1(\log a)^{1+\tilde{\varepsilon}}))^\lambda$. \square

To prove the Theorem 1.3, we give a proof for the following localized version that shows the \mathcal{A} -regularity of a point $x_0 \in \partial_\infty M$. That, in turn, implies Theorem 1.3 since the uniqueness follows from the comparison principle.

The proof of the following theorem is the same as the proof of [3, Theorem 17] except that to prove

$$\int_{\Omega} F \left(\frac{c_0 |\nabla \theta| r \log(1+r)}{L(r)} \right) L(r) < \infty,$$

where $L(r) = \log(1+r) + \mathcal{C}(r)$, we use Lemma 3.4 instead of some estimates involving the curvature lower bound. For convenience, we will also write down the proof.

Theorem 3.5. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$ and let $x_0 \in \partial_\infty M$. Assume that x_0 has a cone neighborhood U such that*

$$K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad (3.6)$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_x M$ containing the radial vector $\nabla r(x)$, with $x \in U \cap M$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \leq C_K |K(P')| \quad (3.7)$$

whenever $x \in U \cap M$ and $P, P' \subset T_x M$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Then x_0 is \mathcal{A} -regular for every $\mathcal{A} \in \mathcal{A}^p(M)$ with $1 < p < n\alpha/\beta$.

Proof. Let $f: \partial_\infty M \rightarrow \mathbb{R}$ be a continuous function. To prove the \mathcal{A} -regularity of x_0 , we need to show that

$$\lim_{x \rightarrow x_0} \overline{H}_f(x) = f(x_0).$$

Fix $\varepsilon' > 0$ and let $v_0 = \dot{\gamma}_0^{o, x_0}$ be the initial vector of the geodesic ray from o to x_0 . Furthermore, let $\delta \in (0, \pi)$ and $R_0 > 0$ be such that $T(v_0, \delta, R_0) \subset U$ and that $|f(x_1) - f(x_0)| < \varepsilon'$ for all $x_1 \in C(v_0, \delta) \cap \partial_\infty M$. Fix also $\tilde{\varepsilon} \in (0, \varepsilon)$, where ε is the constant in (3.6), and let $r_1 > \max(2, R_1)$, where $R_1 \geq R_0$ is given by Proposition 2.3.

We denote $\Omega = T(v_0, \delta, r_1) \cap M$ and define $\theta \in C(\bar{M})$ by setting

$$\theta(x) = \min \left(1, \max \left(r_1 + 1 - r(x), \delta^{-1} \angle_o(x_0, x) \right) \right).$$

Let $\Omega_j = \Omega \cap B(o, j)$ for integers $j > r_1$ and let u_j be the unique \mathcal{A} -harmonic function in Ω_j with $u_j - \theta \in W_0^{1,p}(\Omega_j)$. Each $y \in \partial\Omega_j$ is \mathcal{A} -regular and hence functions u_j can be continuously extended to $\partial\Omega_j$ by setting $u_j = \theta$ on $\partial\Omega_j$. Next we notice that $0 \leq u_j \leq 1$, so the sequence (u_j) is equicontinuous, and hence, by Arzelá-Ascoli, we obtain a subsequence (still denoted by (u_j)) that converges locally uniformly to a continuous function $u: \bar{\Omega} \rightarrow [0, 1]$. It follows that u is \mathcal{A} -harmonic in Ω ; see e.g. [9, Chapter 6].

Next we aim to prove that

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} u(x) = 0, \quad (3.8)$$

and for that we use geodesic polar coordinates (r, v) for points $x \in \Omega$. Here $r = r(x) \in [r_1, \infty)$ and $v = \dot{\gamma}_0^{o, x}$, and we denote by $\lambda(r, v)$ the Jacobian of these polar coordinates. Denote $\tilde{\theta} = \theta/c_0$, $\tilde{u}_j = u_j/c_0$ and $\tilde{u} = u/c_0$, where $c_0 \geq 1$ is a constant

given by Lemma 3.2. Then applying Fatou's lemma and Lemma 3.2 to Ω_j we obtain

$$\begin{aligned}
\int_{\Omega} \varphi(|\tilde{u} - \tilde{\theta}|)^p &= \int_{\Omega} \varphi(|u - \theta|/c_0)^p \leq \liminf_{j \rightarrow \infty} \int_{\Omega_j} \varphi(|u_j - \theta|/c_0)^p \\
&\leq \liminf_{j \rightarrow \infty} \int_{\Omega_j} \varphi(|u_j - \theta|/c_0)^p L(r) \\
&\leq c_0 + c_0 \int_{\Omega} F\left(\frac{c_0 |\nabla \theta| r \log(1+r)}{L(r)}\right) L(r) \\
&= c_0 + c_0 \int_{r_1}^{\infty} \int_{S_{oM}} F\left(\frac{c_0 |\nabla \theta(r, v)| r \log(1+r)}{\log(1+r) + \mathcal{C}(r)}\right) (\log(1+r) \\
&\quad + \mathcal{C}(r)) \lambda(r, v) dv dr \\
&\leq c_0 + c_0 \int_{r_1}^{\infty} \int_{S_{oM}} F\left(\frac{r}{c_1 j(r, v)}\right) (\log(1+r) + \mathcal{C}(r)) \frac{1}{C_0} j(r, v)^{C(n-1)} dv dr \\
&< \infty.
\end{aligned} \tag{3.9}$$

At the end we applied also Lemmas 2.4, 2.5, and 3.4.

Next, we extend each u_j to a function $u_j \in W_{\text{loc}}^{1,p}(M) \cap C(M)$ by setting $u_j(y) = \theta(y)$ for every $y \in M \setminus \Omega_j$. Let $x \in \Omega$ and fix $s \in (0, r_S)$. For j large enough, we obtain by Lemma 3.1

$$\sup_{B(x, s/2)} \varphi(|\tilde{u}_j - \tilde{\theta}|)^{p(n+1)} \leq c \int_{B(x, s)} \varphi(|\tilde{u}_j - \tilde{\theta}|)^p.$$

Applying this with dominated convergence theorem, we get

$$\begin{aligned}
\sup_{B(x, s/2)} \varphi(|\tilde{u} - \tilde{\theta}|)^{p(n+1)} &= \sup_{B(x, s/2)} \lim_{j \rightarrow \infty} \varphi(|\tilde{u}_j - \tilde{\theta}|)^{p(n+1)} \\
&\leq \limsup_{j \rightarrow \infty} \sup_{B(x, s/2)} \varphi(|\tilde{u}_j - \tilde{\theta}|)^{p(n+1)} \\
&\leq c \limsup_{j \rightarrow \infty} \int_{B(x, s)} \varphi(|\tilde{u}_j - \tilde{\theta}|)^p = c \int_{B(x, s)} \varphi(|\tilde{u} - \tilde{\theta}|)^p.
\end{aligned} \tag{3.10}$$

Let $(x_k) \subset \Omega$ be a sequence such that $x_k \rightarrow x_0$ as $k \rightarrow \infty$. We apply the estimate (3.10) with $x = x_k$ and a fixed $s \in (0, r_S)$, together with (3.9), to obtain

$$\lim_{k \rightarrow \infty} \sup_{B(x_k, s/2)} \varphi(|\tilde{u} - \tilde{\theta}|)^{p(n+1)} \leq c \lim_{k \rightarrow \infty} \int_{B(x_k, s)} \varphi(|\tilde{u} - \tilde{\theta}|)^p = 0.$$

It follows that

$$\lim_{k \rightarrow \infty} |\tilde{u}(x_k) - \tilde{\theta}(x_k)| = 0,$$

which, in turn, implies (3.8).

Define a function $w: M \rightarrow \mathbb{R}$ by

$$w(x) = \begin{cases} \min(1, 2u(x)) & \text{if } x \in \Omega; \\ 1, & \text{if } x \in M \setminus \Omega. \end{cases}$$

The minimum of two \mathcal{A} -superharmonic functions is \mathcal{A} -superharmonic and hence w is \mathcal{A} -superharmonic. The definition of \overline{H}_f implies that

$$\overline{H}_f \leq f(x_0) + \varepsilon' + 2(\sup |f|)w,$$

and therefore, by (3.8), we have

$$\limsup_{x \rightarrow x_0} \overline{H}_f(x) \leq f(x_0) + \varepsilon'.$$

Similarly one can prove that

$$\liminf_{x \rightarrow x_0} \underline{H}_f(x) \geq f(x_0) - \varepsilon',$$

and because $\overline{H}_f \geq \underline{H}_f$ and ε' was arbitrary, we conclude that

$$\lim_{x \rightarrow x_0} \overline{H}_f(x) = f(x_0).$$

Therefore x_0 is \mathcal{A} -regular point. \square

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