DEPENDENCE LOGIC WITH GENERALIZED QUANTIFIERS: AXIOMATIZATIONS

FREDRIK ENGSTRÖM, JUHA KONTINEN, AND JOUKO VÄÄNÄNEN

ABSTRACT. We prove two completeness results, one for the extension of dependence logic by a monotone generalized quantifier Q with weak interpretation, weak in the meaning that the interpretation of Q varies with the structures. The second result considers the extension of dependence logic where Q is interpreted as "there exists uncountable many." Both of the axiomatizations are shown to be sound and complete for $FO(Q)$ consequences.

1. INTRODUCTION

Generalized quantifiers constitute a well-studied method of extending the expressive power of first order logic. A more recent extension of first order logic is obtained by adding dependence atoms, permitting the expression of partially ordered quantification. In this paper we study the combination of the two methods, adding to first order logic both generalized quantifiers and dependence atoms as defined in [4]. It was shown in [5] that the resulting extension properly extends both the respective extension by generalized quantifiers and the extension by dependence atoms. We analyse further the expressive power and give natural axioms for the new logic. There are theoretical limits to the extent that the axioms can be complete but we give partial completeness results in the sense that completeness is shown with respect to $FO(Q)$ consequences.

Generalized quantifiers were introduced by Mostowski [17]. The most important of them is perhaps the quantifier

$$\mathbb{M} \vDash Q_1 x \phi(x, \bar{b}) \iff \mathbb{M} \vDash \phi(a, \bar{b}) \text{ for uncountably many } a \in M$$

owing to the beautiful axiomatization of it by Keisler [11]. On the other hand, generalized quantifiers have made an entrance to both linguistics [18] and computer science [12]. In natural language we can use generalized quantifiers to analyse constructs such as

Most boys run,

where we think of "most" as a generalized quantifier. Other natural language quantifiers are "two thirds", "quite a few", "many", etc. There are various so-called polyadic lifts of such quantifiers such as branching, Ramseyfication, and resumption:

A binary relation R satisfies the branching of a quantifier Q with itself if there are two sets satisfying Q whose cartesian product is contained in R. A natural interpretation of

A few boys in my class and a few girls in your class have dated each other. [2]

uses the branching of "few" with itself. On the other hand, a relation R satisfies the k-ary Ramseyfication, or the Ramsey lift, of Q if there is a set satisfying Q such that each tuple of k distinct elements from the set satisfies R. The following sentence can be interpreted using the binary Ramseyfication of "at least two thirds".

> At least two thirds of the boys in your class like each other. [6]

Finally, the k-ary resumption transfers the meaning from elements to tuples, for example a relation R satisfies the binary resumption of "most" if R contains most of the pairs. An example of this would be the following sentence:

> Most neighbours like each other. [6]

In computer science, or more exactly finite model theory, we have the *counting quantifiers*

$$\mathbb{M} \vDash \exists_{\geq k} x \phi(x, \bar{b}) \iff \mathbb{M} \vDash \phi(a, \bar{b}) \text{ for at least } k \text{ elements } a \in M$$

which, although first order definable, have turn out relevant, but also the non-first order

$$\mathbb{M} \vDash \exists_{\mathrm{even}} x \phi(x, \bar{b}) \iff \mathbb{M} \vDash \phi(a, \bar{b}) \text{ for an even number of } a \in M$$

and

$$\mathbb{M} \vDash \exists_{n/2} x \phi(x, \bar{b}) \iff \mathbb{M} \vDash \phi(a, \bar{b}) \text{ for at least } 50\% \text{ of the elements } a \in M$$

and other similar ones. The lifts, also called vectorizations, are important in finite model theory, too. For example, the resumption lift sequence of the transitive closure quantifiers characterises in ordered models NLOGSPACE [9], and the resumption lift sequence of the so-called alternating transitive closure quantifier characterises, even in unordered models, least fixpoint logic [3].

Dependence logic was introduced in [19] by adding an explicit way to express functional dependence between variables to first-order logic. This formalism, unlike previous ones, makes it possible to express complex dependence properties of teams, i.e., sets of assignments, drawing full advantage of Hodges' compositional semantics [8]. It also enables a compositional analysis of the partially ordered prefixes of Henkin [7]:

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \phi(x, y, u, v) \iff \exists f \, \exists g \, \forall x \, \forall u \, \phi(x, f(x), u, g(u)).$$

The partially ordered prefix can be broken down into a linear prefix:

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \phi(x, y, u, v) \iff \forall x \, \exists y \, \forall u \, \exists v \, (=(u, v) \wedge \phi(x, y, u, v)),$$

where $=(u, v)$ is a so-called *dependence atom*.

Dependence logic has the same expressive power as existential second order logic [13]. Thus dependence logic alone cannot express, for example, uncountability, in fact not even finiteness.

The idea of combining partially ordered quantifiers and generalized quantifiers was first suggested by Barwise [2]. He used this combination to analyse polyadic lifts such as the above mentioned Ramsey lift and the branching lift. It was proved in [6] that the polyadic lifts of monotone (unbounded) generalized quantifiers lead to a strong hierarchy, giving immediately the result that there is no finite number of generalized quantifiers, including partially ordered quantifiers, which would be

able to express all Ramsey lifts of a given monotone (unbounded) quantifier. The same is true of branching lifts, and to a lesser extent of resumption lifts [6].

The situation is quite different with the extension of *dependence* logic (rather than *first order* logic) by a monotone generalized quantifier. All the mentioned polyadic lifts (and vectorizations) can be readily defined (in all arities). Let us see how this is done for the Ramsey lift of a monotone quantifier Q.

$$\exists A \in Q \,\forall x \in A \,\forall y \in A \,\phi(x, y, \bar{z})$$

$$\Longleftrightarrow$$

$$\exists w (=(\bar{z}, w) \wedge Qx \exists y (\quad y = w \,\wedge$$
$$\forall u \exists v (=(\bar{z}, u, v) \wedge (x = u \to v = w) \wedge$$
$$\forall u' \exists v' (=(\bar{z}, u', v') \wedge (u = u' \to v' = w) \wedge$$
$$((v = w \wedge v' = w) \to \phi(u, u', \bar{z})))))))$$

Respectively, the branching lift can be expressed as follows:

$$\exists A \in Q \,\exists B \in Q \,\forall x \in A \,\forall y \in B \,\phi(x, y, \bar{z})$$

$$\Longleftrightarrow$$

$$\exists w, w' (\quad =(\bar{z}, w) \wedge =(\bar{z}, w') \wedge$$
$$Qx \exists y (y = w \wedge =(\bar{z}, x, y) \,\wedge$$
$$Qx' \exists y' (y' = w' \wedge =(\bar{z}, x', y') \,\wedge$$
$$\forall u \exists v (=(\bar{z}, u, v) \wedge (x = u \to v = w) \wedge$$
$$\forall u' \exists v' (=(\bar{z}, u', v') \wedge (x' = u' \to v' = w') \wedge$$
$$((v = w \wedge v' = w') \to \phi(u, u', \bar{z}))))))$$

Resumption can be handled similarly.

Thus putting generalized quantifiers and dependence atoms together results in a powerful combination extending far beyond either generalized quantifiers alone or dependence atoms alone.

This paper is organised as follows. In Section 2 we review the basics on dependence logic and generalized quantifiers in the dependence logic context. In Section 3 we present a system of natural deduction for the extension $D(Q, \check{Q})$ of dependence logic by a monotone generalized quantifier Q and its dual $\check{Q}$, and show that these rules are sound.

Finally in Section 4 two completeness results for $FO(Q)$ consequences are shown for $D(Q, \check{Q})$. In the first result Q has the so-called weak interpretation, where the interpretation of the quantifier symbol Q is allowed to vary with the structure, and in the second Q is interpreted as Q_1, that is, "there exists uncountable many."

The main ideas behind the natural deduction system are the following: First, we add rules making it possible to deduce a certain normal form from a sentence, which is introduced in Section 4.1. Second, we add a rule that enables us to replace sentences in normal form with a first-order approximation. These rules differ in the two cases; in the case of weak interpretations we approximate sentences in normal form by an infinite set of first-order sentences; and in the case of Q_1 we use Skolemization rather than approximations.

2. Preliminaries

2.1. **Dependence Logic.** In this section we give a brief introduction to dependence logic. For a detailed account see [19].

The syntax of dependence logic extends the syntax of first order logic with new atomic formulas, the dependence atoms. There is one dependence atom for each arity. We write the atom expressing that the term t_n is uniquely determined by the values of the terms $t_1, \ldots, t_{n-1}$ as $=(t_1, \ldots, t_n)$. We consider formulas where negation can only appear in front of formulas without dependence atoms. For a vocabulary τ, $\mathrm{D}[\tau]$ denotes the set of τ-formulas of dependence logic. The set $\mathrm{FV}(\phi)$ of free variables of a formula ϕ is defined as in first order logic except that

$$\mathrm{FV}(=(t_1, \ldots, t_n)) = \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n).$$

To define a compositional semantics for dependence logic we use *sets of assignments*, called *teams*, instead of single assignments as in first order logic. An assignment is a function $s : V \to M$ where V is a finite set of variables and M is the universe under consideration. A team on M is a set of assignments for some fixed finite set of variables V. If $V = \emptyset$ there is only one assignment, the empty function $\emptyset$. Observe that the team of the empty assignment $\{\emptyset\}$ is different from the empty team $\emptyset$.

- Given an assignment $s : V \to M$ and $a \in M$ let $s[a/x] : V \cup \{x\} \to M$ be the assignment:

$$s[a/x] : y \mapsto \begin{cases} s(y) & \text{if } y \in V \setminus \{x\}, \text{ and} \\ a & \text{if } x = y. \end{cases}$$

- Let $X[M/y]$ be the team

$$\{s[a/y] \mid s \in X, a \in M\},$$

- and whenever $f : X \to M$, let $X[f/y]$ denote the team

$$\{s[f(s)/y] \mid s \in X\}.$$

The domain of a non-empty team X, denoted $\mathrm{dom}(X)$, is the set of variables V. The interpretation of the term t in the model $\mathbb{M}$ under the assignment s is denoted by $t^{\mathbb{M},s}$. We write $s(\bar{x})$ for the tuple obtained by pointwise application of s to the finite sequence $\bar{x}$ of variables.

The satisfaction relation for dependence logic $\mathbb{M}, X \vDash \phi$ is defined as follows. Below, the notation $\mathbb{M}, s \vDash \phi$ refers to the ordinary satisfaction relation of first order logic. We also assume that $\mathrm{FV}(\phi) \subseteq \mathrm{dom}(X)$.

(1) For formulas ψ without dependence atoms: $\mathbb{M}, X \vDash \psi$ iff $\forall s \in X : \mathbb{M}, s \vDash \psi$.
(2) $\mathbb{M}, X \vDash =(t_1, \ldots, t_{n+1})$ iff $\forall s, s' \in X : \bigwedge_{1 \leq i \leq n} t_i^{\mathbb{M},s} = t_i^{\mathbb{M},s'} \to t_{n+1}^{\mathbb{M},s} = t_{n+1}^{\mathbb{M},s'}$
(3) $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\mathbb{M}, X \vDash \phi$ and $\mathbb{M}, X \vDash \psi$
(4) $\mathbb{M}, X \vDash \phi \vee \psi$ iff there are Y and Z such that $X = Y \cup Z$, and $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
(5) $\mathbb{M}, X \vDash \exists y \phi$ iff there is $f : X \to M$, such that $\mathbb{M}, X[f/y] \vDash \phi$
(6) $\mathbb{M}, X \vDash \forall y \phi$ iff $\mathbb{M}, X[M/y] \vDash \phi$.

We define $\mathbb{M} \vDash \sigma$ for a sentence σ to hold if $\mathbb{M}, \{\emptyset\} \vDash \sigma$. Let us make some easy remarks.

- Every formula is satisfied by the empty team.

- The satisfaction relation is downwards closed: If $\mathbb{M}, X \vDash \phi$ and $Y \subseteq X$ then $\mathbb{M}, Y \vDash \phi$.
- The satisfaction relation is local: $\mathbb{M}, X \vDash \phi$ iff $\mathbb{M}, Y \vDash \phi$ where

$$Y = \{\, s \restriction \mathrm{FV}(\phi) \mid s \in X \,\}.$$

The expressive power for sentences of dependence logic is the same as that of existential second order logic.

2.2. **D(Q).** The notion of a generalized quantifier goes back to Mostowski [17] and Lindström [15]. In [4] semantics for generalized quantifiers in the framework of dependence logic was introduced. We will review the definitions below.

Let Q be a quantifier of type $\langle k \rangle$, meaning that Q is a class of τ-structures, where the signature τ has a single k-ary relation symbol. Also, assume that Q is monotone increasing, i.e., for every M and every $A \subseteq B \subseteq M^k$, if $A \in Q_M$ then also $B \in Q_M$, where $Q_M = \{\, R \subseteq M^k \mid (M, R) \in Q \,\}$.

The formulas of dependence logic extended with a quantifier Q, $\mathrm{D}(Q)$, is built up from $\mathrm{FO}(Q)$-formulas and dependence atoms using the connectives $\wedge$ and $\vee$, and the quantifier expressions $\exists x$, $\forall x$ and Qx in the usual way. We write $\phi \to \psi$ as a shorthand for $\neg \phi \vee \psi$, where ϕ is a formula without dependence atoms.

An assignment s satisfies a formula $Q\bar{x}\,\phi$ in a structure $\mathbb{M}$,

$$\mathbb{M}, s \vDash Q\bar{x}\,\phi, \text{ if the set } \{\, \bar{a} \in M^k \mid \mathbb{M}, s[\bar{a}/\bar{x}] \vDash \phi \,\} \text{ is in } Q_M.$$

In the context of teams we say that a team X satisfies a formula $Q\bar{x}\,\phi$,

(1) $\mathbb{M}, X \vDash Q\bar{x}\,\phi$, if there exists $F : X \to Q_M$ such that $\mathbb{M}, X[F/\bar{x}] \vDash \phi$,

where $X[F/\bar{x}] = \{\, s[\bar{a}/\bar{x}] \mid \bar{a} \in F(s) \,\}$. This definition works well only with monotone (increasing) quantifiers, see [4] for details.

The following easy proposition suggests that we indeed have the right truth condition for monotone quantifiers:

Proposition 2.1 ([4]).   *(i)* $\mathrm{D}(Q)$ *is downwards closed.*
 (ii) $\mathrm{D}(Q)$ *is local, in the sense that* $\mathbb{M}, X \vDash \phi$ *iff* $\mathbb{M}, (X \restriction FV(\phi)) \vDash \phi$.
(iii) *Viewing* $\exists$ *and* $\forall$ *as generalized quantifiers of type* $\langle 1 \rangle$, *the truth conditions in* (1) *are equivalent to the truth conditions of dependence logic.*
 (iv) *For every* $\mathrm{D}(Q)$ *formula* ϕ *we have* $\mathbb{M}, \emptyset \vDash \phi$.

As proved in [5], the expressive power of $\mathrm{D}(Q)$ sentences corresponds to that of a certain natural extension of existential second order logic by Q.

In order to get a prenex normal form for all formulas we will focus on the logics $\mathrm{D}(Q, \check{Q})$, where $\check{Q}$ is the dual of Q, i.e,

$$\check{Q} = \{\, (M, M^k \setminus R) \mid R \subseteq M^k, (M, R) \notin Q \,\},$$

instead of $\mathrm{D}(Q)$. Note that, according to our definition of $\mathrm{D}(Q)$, a formula $Qx\phi$ may be negated only if ϕ is a $\mathrm{FO}(Q)$ formula.

We will consider monotone increasing quantifiers Q satisfying two non-triviality assumptions: $(M, \emptyset) \notin Q$ and $(M, M^k) \in Q$ for all M. In [5] the following normal form for sentences of $\mathrm{D}(Q)$ was shown for such non-trivial quantifiers.

Theorem 2.2. *Every* $\mathrm{D}(Q)$ *sentence in negation normal form, where* Q *is non-trivial, can be written as*

$$\mathcal{H}^1 \bar{x}_1 \ldots \mathcal{H}^m \bar{x}_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \le j \le n} =(\bar{z}^i, y_i) \wedge \theta \big),$$

where $\mathcal{H}^i$ is either Q or $\forall$ and θ is a quantifier-free FO-formula.

In the present paper a similar normal form for all $D(Q, \check{Q})$ formulas is obtained in Proposition 4.1.

A *weak semantics* can be given for $D(Q)$ (and $FO(Q)$, etc) by regarding Q as an interpreted symbol rather than a logical constant in the following way (see [11] and [10] for more on this). A weak model is a structure together with an interpretation of Q, often denoted by q. We define $T \vDash_w \sigma$ to hold if every weak model (M, q) of T satisfies σ. In this paper we require the interpretation q of Q to be monotone increasing and non-trivial (In essence this is the *monotone logic* of [16]). In the weak semantics for $D(Q, \check{Q})$ we require that the interpretation $\check{q}$ of $\check{Q}$ is the dual of the interpretation q of Q. Thus, if $T \cup \{\sigma\}$ consists of $D(Q, \check{Q})$ sentences, then $T \vDash_w \sigma$ if every model $(M, q, \check{q})$ of T satisfies σ.

3. Natural deduction for $D(Q, \check{Q})$

In this section we present a set of natural deduction rules for the logic $D(Q, \check{Q})$, and prove its soundness.

3.1. A system of natural deduction.
In this section we present a set of natural deduction rules for the logic $D(Q, \check{Q})$, where Q is monotone and satisfies the non-triviality conditions: $(M, \emptyset) \notin Q$ and $(M, M^k) \in Q$ for all M. Observe that then also the dual quantifier $\check{Q}$ satisfies these conditions. To simplify notation, we will restrict attention to type $\langle 1 \rangle$ quantifiers.

We use an abbreviation $\bar{x} = \bar{y}$ for the formula $\bigwedge_{1 \le i \le \text{len}(\bar{x})} x_i = y_i$, assuming of course that $\bar{x}$ and $\bar{y}$ are tuples of the same length $\text{len}(\bar{x})$. The substitution of a term t to the free occurrences of x in ψ is denoted by $\psi[t/x]$. Analogously to first order logic, no variable of t may become bound in such a substitution. For tuples $\bar{t} = (t_1, \ldots, t_n)$ and $\bar{x} = (x_1, \ldots, x_n)$ we write $\psi[\bar{t}/\bar{x}]$ to denote the simultaneous substitution of x_i by t_i for $1 \le i \le n$.

Here is a list of all the rules:

(1) Conjunction:

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge\text{I} \qquad \frac{\phi \wedge \psi}{\phi} \wedge\text{E} \qquad \frac{\phi \wedge \psi}{\psi} \wedge\text{E}$$

(2) Disjunction:

$$\frac{\phi}{\phi \vee \psi} \vee\text{I} \qquad \frac{\psi}{\phi \vee \psi} \vee\text{I} \qquad \frac{\phi \vee \psi \quad \overset{[\phi]}{\underset{\gamma}{\vdots}} \quad \overset{[\psi]}{\underset{\gamma}{\vdots}}}{\gamma} \vee\text{E}$$

where γ is a $FO(Q, \check{Q})$ formula.

(3) Negation and duality:

$$\frac{\overset{[\phi]}{\underset{\bot}{\vdots}}}{\neg\phi} \neg\text{I} \qquad \frac{\overset{[\neg\phi]}{\underset{\bot}{\vdots}}}{\phi} \text{RAA} \qquad \frac{\phi \quad \neg\phi}{\bot} \bot\text{I} \qquad \frac{\check{Q}x\phi}{\neg Qx\neg\phi}$$

where ϕ is a $FO(Q, \check{Q})$ formula.

(4) Universal quantifier:

$$\frac{\phi}{\forall x_i \phi} \ \forall\text{I} \qquad \frac{\forall x_i \phi}{\phi[t/x_i]} \ \forall\text{E}$$

In $\forall\text{I}$ the variable x_i cannot appear free in any non-discharged assumption used in the derivation of ϕ.

(5) Existential quantifier:

$$\frac{\phi(t/x_i)}{\exists x_i \phi} \ \exists\text{I} \qquad \frac{\exists x_i \phi \quad \overset{\displaystyle [\phi] \atop \displaystyle \vdots}{\psi}}{\psi} \ \exists\text{E}$$

In $\exists\text{E}$ the variable x_i cannot appear free in ψ and in any non-discharged assumption used in the derivation of ψ, except in ϕ.

(6) Disjunction substitution:

$$\frac{\phi \vee \psi \quad \overset{\displaystyle [\psi] \atop \displaystyle \vdots}{\gamma}}{\phi \vee \gamma}$$

(7) Commutation and associativity of disjunction:

$$\frac{\psi \vee \phi}{\phi \vee \psi} \qquad\qquad \frac{(\phi \vee \psi) \vee \gamma}{\phi \vee (\psi \vee \gamma)}$$

(8) Extending scope:

$$\frac{\mathcal{H}x\phi \vee \psi}{\mathcal{H}x(\phi \vee \psi)} \qquad\qquad \frac{Qx\phi \wedge \psi}{Qx(\phi \wedge \psi)}$$

where $\mathcal{H} \in \{Q, \check{Q}, \exists, \forall\}$, and the prerequisite for applying these rules is that x does not appear free in ψ. The rule on the right is also assumed for $\check{Q}$.

(9) Unnesting:

$$\frac{=(t_1, ..., t_n)}{\exists z(=(t_1, ..., z, ..., t_n) \wedge z = t_i)}$$

where z is a new variable.

(10) Dependence distribution: let

$$\begin{aligned}
\phi &= \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq j \leq n} =(\bar{z}^j, y_j) \wedge \phi_0 \big), \\
\psi &= \exists y_{n+1} \ldots \exists y_m \big(\bigwedge_{n+1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge \psi_0 \big).
\end{aligned}$$

where ϕ_0 and ψ_0 are quantifier-free formulas without dependence atoms, and y_i, for $1 \leq i \leq n$, does not appear in ψ and y_i, for $n+1 \leq i \leq m$, does not appear in ϕ. Then,

$$\frac{\phi \vee \psi}{\exists y_1 \ldots \exists y_m (\bigwedge_{1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge (\phi_0 \vee \psi_0))}$$

(11) Dependence introduction:

$$\frac{\exists x \forall y \phi}{\forall y \exists x (=(\bar{z}, x) \wedge \phi)} \qquad \frac{\exists x Q y \phi}{Q y \exists x (=(\bar{z}, x) \wedge \phi)}$$

where $\bar{z}$ lists the variables in $\mathrm{FV}(\phi) - \{x, y\}$. Similar for $\check{Q}$.

(12) Monotonicity of Q and $\check{Q}$:

$$\frac{Qx\phi \quad \begin{array}{c} [\phi] \\ \vdots \\ \psi \end{array}}{Qx\psi}$$

where the prerequisite for applying this rule is that the variable x cannot appear free in any non-discharged assumption used in the derivation of ψ, except for ϕ. Similar for $\check{Q}$.

(13) Bound variables:

$$\frac{Qx\phi}{Qy\phi[y/x]},$$

where y does not appear in ϕ. Similar for $\check{Q}$.

(14) Identity rules:

$$\frac{}{t=t} \qquad \frac{\phi[r/x] \quad t = r}{\phi[t/x]}$$

where ϕ is an $\mathrm{FO}(Q, \check{Q})$ formula.

Observe that $\mathrm{FO}(Q, \check{Q}) \equiv \mathrm{FO}(Q)$, but syntactically $\mathrm{FO}(Q, \check{Q})$ includes more formulas.

3.2. **Soundness of the rules.** In this section we show the soundness of the rules introduced in the previous section under any monotone and non-trivial interpretation of Q. Clearly this is the same as soundness in the weak semantics for Q.

The following lemmas will be needed in the proof.

Lemma 3.1. *Let $\phi(x)$ be a $\mathrm{D}(Q, \check{Q})$ formula, and t a term such that in the substitution $\phi[t/x]$ no variable of t becomes bound. Then for all $\mathbb{M}$ and teams X, where $(FV(\phi) - \{x\}) \cup \mathrm{Var}(t) \subseteq \mathrm{dom}(X)$*

$$\mathbb{M}, X \vDash \phi[t/x] \Leftrightarrow \mathbb{M}, X[F/x] \vDash \phi(x),$$

where $F \colon X \to A$ is defined by $F(s) = t^{\mathbb{M}, s}$.

Proof. Analogous to Lemma 8 in [14]. □

It is easy to verify that Lemma 3.1 gives the following familiar property concerning changing free variables.

Lemma 3.2 (Change of free variables). *Let the free variables of $\phi \in \mathrm{D}(Q, \check{Q})$ be $x_1, \ldots, x_n$ and let $y_1, \ldots, y_n$ be distinct variables. Then for all structures $\mathbb{M}$ and teams X with domain $\{x_1, \ldots, x_n\}$ it holds that*

$$\mathbb{M}, X \vDash \phi \Leftrightarrow \mathbb{M}, X' \vDash \phi[\bar{y}/\bar{x}],$$

where X' is the team with domain $\{y_1, \ldots, y_n\}$ containing the assignments $s' \colon y_i \mapsto s(x_i)$ for $s \in X$.

Proposition 3.3. *Assume that Q is monotone and non-trivial. Let $T \cup \{\phi\}$ be a set of sentences of $\mathrm{D}(Q, \check{Q})$. If $T \vdash \phi$, then $T \vDash \phi$.*

Proof. We prove the statement that if $T \vdash \phi$, where $T \cup \{\phi\}$ is a set of *formulas*, then for any $\mathbb{M}$ and X where $\mathrm{dom}(X) \supseteq \mathrm{FV}(T) \cup \mathrm{FV}(\phi)$, if $\mathbb{M}, X \vDash T$ then $\mathbb{M}, X \vDash \phi$. This is done by using induction on the length of derivation.

It suffices to consider the rules 3 (only duality), 8, 11, 12, and 13 since the soundness of the other rules can be proved analogously to [14] using the fact that $\mathrm{D}(Q, \check{Q})$ is local and has downwards closure (see (ii) and (i) of Proposition 2.1). In particular, Lemma 3.1 is used in the soundness proofs of the rules $\exists$ I and $\forall$ E.

(3) Assume $\mathbb{M}, X \vDash \check{Q}x\phi$ then, since $\check{Q}x\phi$ is a $\mathrm{FO}(Q, \check{Q})$ formula we have $\mathbb{M}, s \vDash \check{Q}x\phi$ for all $s \in X$. This clearly implies that $\mathbb{M}, s \vDash \neg Qx\neg\phi$ for all $s \in X$, which is equivalent to $\mathbb{M}, X \vDash \neg Qx\neg\phi$.

(8) These rules preserve logical equivalence analogously to Lemma 3.2 in [5].

(11) The soundness of this rule follows from the logical equivalence

$$Qy\exists x(=(\bar{z}, x) \wedge \phi) \equiv \exists x Qy\phi$$

the proof of which is analogous to the case where Q is replaced by $\forall$ (see [14]).

(12) Assume that we have a natural deduction proof of $Qx\psi$ from the assumptions

$$\{\gamma_1, \ldots, \gamma_k\}$$

with the last rule 13. Let $\mathbb{M}$ and X be such that $\mathbb{M}, X \vDash \phi_i$, for $1 \leq i \leq k$. By the assumption, we have a shorter deduction of $Qx\phi$ from the assumptions $\{\gamma_{n_1}, \ldots, \gamma_{n_l}\}$ and a deduction of ψ from the assumptions $\{\phi, \gamma_{n_{l+1}}, \ldots, \gamma_{n_m}\}$. Hence by the induction assumption it holds that $\mathbb{M}, X \vDash Qx\phi$. Therefore, there is $F \colon X \to Q_M$ such that $\mathbb{M}, X[F/x] \vDash \phi$. Since the variable x cannot appear free in the formulas $\gamma_{n_{l+1}}, \ldots, \gamma_{n_m}$ it follows that $\mathbb{M}, X[F/x] \vDash \gamma_i$, for $i \in \{n_{l+1}, \ldots, n_m\}$. Now by the induction assumption we get that $\mathbb{M}, X[F/x] \vDash \psi$ and $\mathbb{M}, X \vDash Qx\psi$.

(13) This rule preserves logical equivalence by Lemma 3.2. $\qquad\square$

Note that since Proposition 3.3 holds for every monotone non-trivial quantifier Q we get also soundness for weak semantics: If $T \vdash \phi$ then $T \vDash_w \phi$.

4. COMPLETENESS RESULTS FOR $\mathrm{FO}(Q, \check{Q})$ CONSEQUENCES

4.1. **Deriving a normal form for** $\mathrm{D}(Q, \check{Q})$**.** In this section we show that from each formula $\phi \in \mathrm{D}(Q, \check{Q})$ we can derive a logically equivalent formula in the following normal form:

$$(2) \qquad \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq j \leq n} =(\bar{x}^i, y_i) \wedge \theta \big),$$

where $\mathcal{H}^i$ is either Q, $\check{Q}$ or $\forall$, and θ is a quantifier-free FO-formula.

Proposition 4.1. *Let ϕ be a formula of $\mathrm{D}(Q, \check{Q})$. Then $\phi \vdash \phi'$, where ϕ' is of the form (2), and ϕ' is logically equivalent to ϕ.*

Proof. The proof of this Proposition is analogous to the proof of the corresponding result for dependence logic formulas in [14]. We will indicate how the proof of [14] can be extended for the formulas of $\mathrm{D}(Q, \check{Q})$.

We will establish the claim in several steps. Without loss of generality, we assume that in ϕ each variable is quantified only once and that, in the dependence atoms of ϕ, only variables (i.e. no complex terms) occur.

Step 1. We derive from ϕ an equivalent sentence in prenex normal form:

$$(3) \qquad \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \chi,$$

where $\mathcal{H}^i \in \{\exists, \forall, Q, \check{Q}\}$ and χ is a quantifier-free formula.

We will prove the claim for every formula ϕ satisfying the assumptions made in the beginning of the proof and the assumption (if ϕ has free variables) that no variable appears both free and bound in ϕ. It suffices to consider the case $\phi := \psi \vee \theta$, since the case of conjunction is analogous and the other cases are trivial.

By the induction assumption, we have derivations $\psi \vdash \psi^*$ and $\theta \vdash \theta^*$, where

$$\begin{aligned} \psi^* &= \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \psi_0, \\ \theta^* &= \mathcal{H}^{m+1} x_{m+1} \ldots \mathcal{H}^{m+n} x_{m+n} \theta_0, \end{aligned}$$

and $\psi \equiv \psi^*$ and $\theta \equiv \theta^*$. Now $\phi \vdash \psi^* \vee \theta^*$, using two applications of the rule 6. Next we prove using induction on m that, from $\psi^* \vee \theta^*$, we can derive

$$(4) \qquad \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \mathcal{H}^{m+1} x_{m+1} \ldots \mathcal{H}^{m+n} x_{m+n} (\psi_0 \vee \theta_0).$$

Let $m = 0$. We prove this case again by induction; for $n = 0$ the claim holds. Suppose that $n = l + 1$. We assume that $\mathcal{H}^1 = Q$. The case $\mathcal{H}^1 = \check{Q}$ is analogous, and the cases $\mathcal{H}^1 \in \{\exists, \forall\}$ are handled exactly as in [14]. The following deduction now shows the claim:

(1) $\psi_0 \vee Q x_1 \ldots \mathcal{H}^n x_n \theta_0$
(2) $Q x_1 \ldots \mathcal{H}^n x_n \theta_0 \vee \psi_0$ (rule 7)
(3) $Q x_1 (\mathcal{H}^2 x_2 \ldots \mathcal{H}^n x_n \theta_0 \vee \psi_0)$ (rule 8)
(4) $Q x_1 \ldots \mathcal{H}^n x_n (\psi_0 \vee \theta_0)$ (rule 12 and D1),

where D1 is the derivation

(1) $\mathcal{H}^2 x_2 \ldots \mathcal{H}^n x_n \theta_0 \vee \psi_0$
(2)   .
(3)   .
(4)   .
(5) $\mathcal{H}^2 x_2 \ldots \mathcal{H}^n x_n (\theta_0 \vee \psi_0)$ (induction assumption)
(6)   .
(7)   .
(8)   .
(9) $\mathcal{H}^2 x_2 \ldots \mathcal{H}^n x_n (\psi_0 \vee \theta_0)$ (D2)

where D2 is a derivation that swaps the disjuncts. This concludes the proof for the case $m = 0$.

Assume then that $m = k + 1$ and that the claim holds for k. Now the following derivation shows the claim. Again we consider only the case $\mathcal{H}^1 = Q$.

(1) $Q x_1 \mathcal{H}^2 x_2 \ldots \mathcal{H}^m x_m \psi_0 \vee \mathcal{H}^{m+1} x_{m+1} \ldots \mathcal{H}^{m+n} x_{m+n} \theta_0$
(2) $Q x_1 (\mathcal{H}^2 x_2 \ldots \mathcal{H}^m x_m \psi_0 \vee \mathcal{H}^{m+1} x_{m+1} \ldots \mathcal{H}^{m+n} x_{m+n} \theta_0)$ (rule 8)
(3) $Q x_1 \ldots \mathcal{H}^m x_m \mathcal{H}^{m+1} x_{m+1} \ldots \mathcal{H}^{m+n} x_{m+n} (\psi_0 \vee \theta_0)$ (rule 12 and D3)

where D3 is the following derivation given by the induction assumption:

(1) $\mathcal{H}^2 x_2 \ldots \mathcal{H}^m x_m \psi_0 \vee \mathcal{H}^{m+1} x_{m+1} \ldots \mathcal{H}^{m+n} x_{m+n} \theta_0$
(2)   .
(3)   .
(4)   .

(5) $\mathcal{H}^2 x_2 \ldots \mathcal{H}^m x_m \mathcal{H}^{m+1} x_{m+1} \ldots \mathcal{H}^{m+n} x_{m+n} (\psi_0 \vee \theta_0)$

This concludes the proof.

Step 2. The next step is to show that from a quantifier-free formula θ it is possible to derive an equivalent formula of the form:

$$(5) \qquad \exists z_1 \ldots \exists z_n (\bigwedge_{1 \leq j \leq n} = (\bar{x}^j, z_j) \wedge \theta^*),$$

where θ^* is a quantifier-free formula without dependence atoms. Again the claim is proved using induction on θ using in particular rule 10. Note that the quantifier Q does not play any role in this step, hence the claim can be proved exactly as in [14].

Step 3. The deductions in Step 1 and 2 can be combined (from ϕ to (3), and then from θ to (5)) to show that

$$(6) \qquad \phi \vdash \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists z_1 \ldots \exists z_n (\bigwedge_{1 \leq j \leq n} = (\bar{x}^j, z_j) \wedge \theta^*).$$

Note that for $\mathcal{H}_i = Q$, rule 12 is needed in this deduction.

Step 4. We transform the $\mathcal{H}^i$-quantifier prefix in (6) to the required form (see (2)) by using rule 11 and pushing the new dependence atoms as new conjuncts to

$$(7) \qquad \bigwedge_{1 \leq j \leq n} = (\bar{x}^j, z_j).$$

We prove the claim using induction on the length m of the $\mathcal{H}$-quantifier block in (6). For $m = 0$ the claim holds. Suppose that the claim holds for k and $m = k + 1$. We consider first the case $\mathcal{H}^1 = Q$. The following derivation now shows the claim:

(1) $Q x_1 \mathcal{H}^2 x_2 \ldots \mathcal{H}^m x_m \exists z_1 \ldots \exists z_n (\bigwedge_{1 \leq j \leq n} = (\bar{x}^j, z_j) \wedge \theta^*)$

(2) $Q x_1 \mathcal{H}^{i_1} x_{i_1} \cdots \mathcal{H}^{i_h} x_{i_h} \exists \bar{x}' \exists \bar{z} (\bigwedge_{1 \leq j \leq n'} = (\bar{x}^j, w_j) \wedge \theta^*)$ (rule 12 and D4)

where $\mathcal{H}_{i_j}$, for $1 \leq j \leq h$, is either Q, $\check{Q}$ or $\forall$, and D4 is the following derivation that exists by the induction assumption:

(1) $\mathcal{H}^2 x_2 \ldots \mathcal{H}^m x_m \exists z_1 \ldots \exists z_n (\bigwedge_{1 \leq j \leq n} = (\bar{x}^j, z_j) \wedge \theta^*)$

(2)　　　　.

(3)　　　　.

(4)　　　　.

(5) $\mathcal{H}^{i_1} x_{i_1} \cdots \mathcal{H}^{i_h} x_{i_h} \exists \bar{x}' \exists \bar{z} (\bigwedge_{1 \leq j \leq n'} = (\bar{x}^j, w_j) \wedge \theta^*)$

The case $\mathcal{H}^1 = \forall$ can be proved analogously. Next we consider the case $\mathcal{H}^1 = \exists$ and $\mathcal{H}^i = \exists$ for all $2 \leq i \leq m$. In this case the quantifier $\mathcal{H}^1$ is already in the right place in the quantifier prefix. We will record the variables determining $\mathcal{H}^1$ by a new dependence atom and then move it to the quantifier free part of the formula. This is done because each existentially quantified variable is determined by one and only one dependence atom in the normal form (2). We will use the following auxiliary derivation D5:

(1) $\exists x \chi(\bar{z})$

(2) $\exists x \forall y \chi(\bar{z})$ ($\exists$ E and D6)

(3) $\forall y \exists x (= (\bar{z}, x) \wedge \chi(\bar{z}))$ (rule 10)

(4) $\exists x (= (\bar{z}, x) \wedge \chi(\bar{z}))$ ($\forall$ E),

where D6 refers to the following derivation

(1) $\chi(\bar{z}, x)$

(2) $\forall y \chi(\bar{z}, x)$ ($\forall$ I with y a fresh variable)

(3) $\exists x \forall y \chi(\bar{z})$ ($\exists$ I)

Let us now prove the case $\mathcal{H}^1 = \exists$ with $\mathcal{H}^i = \exists$ for all $2 \leq i \leq m$.

(1) $\exists x_1 \exists x_2 \dots \exists x_m \exists z_1 \dots \exists z_n (\bigwedge_{1 \leq j \leq n} = (\bar{x}^j, z_j) \wedge \theta^*)$

(2) $\exists x_1 \exists x_2 \dots \exists x_m \exists z_1 \dots \exists z_n (\bigwedge_{1 \leq j \leq n+m-1} = (\bar{x}^j, w_j) \wedge \theta^*)$ ($\exists$ E and D7)

(3) $\exists x_1 (=(\bar{x}^{n+m}, x_1) \wedge \exists x_2 \dots \exists x_m \exists \bar{z} (\bigwedge_{1 \leq j \leq n+m-1} = (\bar{x}^j, w_j) \wedge \theta^*))$ (D5)

(4) $\exists x_1 \exists x_2 \dots \exists x_m \exists z_1 \dots \exists z_n (\bigwedge_{1 \leq j \leq n+m} = (\bar{x}^j, w_j) \wedge \theta^*)$ (D8)

where D7 is a derivation that exists by the induction assumption, and D8 is a derivation that pushes $=(\bar{x}^{n+m}, x_1)$ into the quantifier free part of the formula. The case $\mathcal{H}^1 = \exists$, where $\mathcal{H}^i = \forall$ for some $2 \leq i \leq m$ can be proved similarly to [14] adding one additional trasformation in which the redundant dependence atoms (created not by the first swap of $\mathcal{H}^1$ with $\forall$, Q, or $\check{Q}$) are deleted from the formula using essentially the rule $\wedge$ E.

Steps 1-4 show that from a formula ϕ a formula of the form can be deduced

$$(8) \qquad \mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m \exists y_1 \dots \exists y_n \big(\bigwedge_{1 \leq j \leq n} = (\bar{x}^i, y_i) \wedge \theta \big),$$

where $\mathcal{H}^i$ is either Q, $\check{Q}$ or $\forall$ and θ is a quantifier-free FO-formula. Furthermore, ϕ and the formula in (8) are logically equivalent since logical equivalence is preserved in each of the Steps 1-4. □

4.2. Completeness for $D(Q, \check{Q})$.

In this section we prove a completeness result for $D(Q, \check{Q})$ with respect to $FO(Q, \check{Q})$ consequences of $D(Q, \check{Q})$-sentences, with weak semantics. Analogously to [14], we approximate $D(Q, \check{Q})$-sentences in the normal form (2) by an infinite set of $FO(Q, \check{Q})$ sentences. We use an extra predicate R to encode a team witnessing the satisfiability of the quantifier prefix $\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m$.

Let σ be

$$\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m \exists y_1 \dots \exists y_n \big(\bigwedge_{1 \leq i \leq n} = (\bar{x}^i, y_i) \wedge \theta(x_1, \dots, x_m, y_1, \dots, y_n) \big),$$

where each $\mathcal{H}^i$ is either Q, $\check{Q}$ or $\forall$.

We define finite approximations $A^i \sigma$ of σ as follows. The first approximation, $A^1 \sigma$, is

$$\forall x_1 \dots \forall x_m \exists y_1 \dots \exists y_n \big(R(x_1, \dots, x_m) \rightarrow \theta(x_1, \dots, x_m, y_1, \dots, y_n) \big),$$

or in compressed form:

$$\forall \bar{x} \exists \bar{y} \big(R(\bar{x}) \rightarrow \theta(\bar{x}, \bar{y}) \big).$$

The second approximation $A^2 \sigma$ is

$$\forall \bar{x}_1 \exists \bar{y}_1 \forall \bar{x}_2 \exists \bar{y}_2 \big(R(\bar{x}_1) \wedge R(\bar{x}_2) \rightarrow \theta(\bar{x}_1, \bar{y}_1) \wedge \theta(\bar{x}_2, \bar{y}_2) \wedge$$
$$\bigwedge_{1 < i < n} (\bar{x}_1^i = \bar{x}_2^i \rightarrow y_{i,1} = y_{i,2}) \big)$$

With the notational convention that $(x_{i_1}, \ldots, x_{i_k})_j$ is the sequence $(x_{i_1,j}, \ldots, x_{i_k,j})$. By generalizing this construction we get the k:th approximation:

$$\forall \bar{x}_1 \exists \bar{y}_1 \ldots \forall \bar{x}_k \exists \bar{y}_k \Big(\bigwedge_{1 \leq j \leq k} R(\bar{x}_j) \rightarrow \bigwedge_{1 \leq j \leq k} \theta(\bar{x}_j, \bar{y}_j) \wedge$$

$$\bigwedge_{\substack{1 \leq i \leq n \\ 1 \leq j, j' \leq k}} (\bar{x}_j^i = \bar{x}_{j'}^i \rightarrow y_{i,j} = y_{i,j'}))$$

Also we need a sentence saying that R is of the right kind, witnessing the quantifier prefix: Let $B\sigma$ be

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m R(x_1, \ldots, x_m).$$

We will adopt the following approximation rule in our deduction system:

$$
\begin{array}{cc}
[B\sigma] & [A^n\sigma] \\
\ddots & \ddots \\
\cfrac{\sigma \qquad\qquad \psi}{\psi} & \text{(Approx)}
\end{array}
$$

where σ is a sentence in normal form, and R does not appear in ψ nor in any uncancelled assumptions in the derivation of ψ, except for $B\sigma$ and $A^n\sigma$.

Lemma 4.2. *Adding the approximation rule to the inference system results in a sound system for* $\mathrm{D}(Q, \check{Q})$ *with regard to weak semantics.*

Proof. We plug in the following induction step to the proof of Proposition 3.3:

Assume that there is a derivation of ψ from Γ ending with the approximation rule. Then there are shorter derivations from Γ of σ and from $\Gamma', B\sigma, A^n\sigma$ of ψ, where $\Gamma' \subseteq \Gamma$ is such that R does not occur in Γ'. By the induction hypothesis we get $\Gamma \vDash \sigma$ and $\Gamma', B\sigma, A^n\sigma \vDash \psi$. We will prove that $\Gamma \vDash \psi$, by assuming $\mathbb{M}, X \vDash \Gamma$ for some non-empty X and proving that $\mathbb{M}, X \vDash \psi$.

Assume σ is of the form

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \Big(\bigwedge_{1 \leq i \leq n} = (\bar{x}^i, y_i) \wedge \theta \Big).$$

where θ is a quantifier free first order formula.

From the fact that $\mathbb{M}, X \vDash \sigma$ we get $\mathbb{M} \vDash \sigma$ and thus there is a (non-empty) team Y such that

$$\mathbb{M}, Y \vDash \exists y_1 \ldots \exists y_n \Big(\bigwedge_{1 \leq i \leq n} = (\bar{x}^i, y_i) \wedge \theta \Big).$$

Let $r \subseteq M^m$ be the relation $Y(\bar{x})$ corresponding to Y. Then $(\mathbb{M}, r) \vDash B\sigma$ and, it should also be clear that $(\mathbb{M}, r) \vDash A^n\sigma$. Since R does not occur in Γ' we have $(\mathbb{M}, r), X \vDash \Gamma', B\sigma, A^n\sigma$. By the induction hypothesis $(\mathbb{M}, r), X \vDash \psi$, and since R does not occur in ψ we have $\mathbb{M}, X \vDash \psi$. $\qquad \square$

The main result of this section can now be stated as follows.

Theorem 4.3. *Let T be a set of sentences of* $\mathrm{D}(Q, \check{Q})$ *and $\phi \in \mathrm{FO}(Q, \check{Q})$ a sentence. Then the following are equivalent:*

    (I): $T \vDash_w \phi$
    (II): $T \vdash \phi$

The following lemmas are needed in the proof.

Lemma 4.4. *If T is a set of* $\mathrm{FO}(Q, \check{Q})$*-sentences consistent in the deduction system described above then there are a countable recursively saturated model $\mathbb{M}$ and an interpretation q of Q such that* $(\mathbb{M}, q, \check{q}) \vDash T$.

Proof. First translate T to $T^{\neg}$ in which each $\check{Q}x\phi$ is replaced by $\neg Qx\neg\phi$. By using the same argument as in [11, 10] we may reduce $\mathrm{FO}(Q)$ to FO by replacing subformulas of the form $Qx\phi$ with new relation symbols $R_\phi(\bar{y})$, $\bar{y}$ being the free variables of $Qx\phi$. This will reduce the set $T^{\neg}$ to a set T^*. Let T' be T^* together with the translations of the universal closures of

- $(\phi \to \psi) \to (Qx\phi \to Qx\psi)$, for all ϕ and ψ; and
- $Qx\phi \to Qy(\phi[y/x])$, for all ϕ such that the substitution is legal.

Now T' is consistent by the same argument as in [10]. Let $\mathbb{M}^*$ be a countable recursively saturated model of T', and M its reduct to the original signature. Now we may define q to be

$$\{\, A \subseteq M \mid A \supseteq \{\, a \in M \mid \mathbb{M}^*, s[a/x] \vDash \phi^* \,\} \text{ for some } \phi \text{ s.t. } M^*, s \vDash Qx\phi^* \,\}.$$

Proposition 2.3.4 in [10] shows that $(\mathbb{M}, q) \vDash T^{\neg}$, and thus $(\mathbb{M}, q, \check{q}) \vDash T$. □

Lemma 4.5. *In a countable recursively saturated weak model $(\mathbb{M}, q, \check{q})$ in which $B\sigma$ and $A^n\sigma$ holds for all n, σ holds.*

Proof. Suppose σ is

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big).$$

Note that the sentences $A^n\sigma$ can be viewed as the finite approximations as defined in [14] (and see also [1]) of the D sentence σ':

$$\forall \bar{x} \exists \bar{y} \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge (R(\bar{x}) \to \theta) \big).$$

Thus by Theorem 2.4 in [1] (see also [14]), we know that $\mathbb{M} \vDash \sigma'$.

Let X be the team $\{\, s : \{\, x_1, \ldots, x_k \,\} \to M \mid (s(x_1), \ldots, s(x_m)) \in R^{\mathbb{M}} \,\}$. To prove that $(\mathbb{M}, q, \check{q}) \vDash \sigma$ we find $F_1, \ldots, F_m$ so that $\{\, \emptyset \,\}[F_1/x_1] \ldots [F_m/x_m] = X$.

$$F_1(\emptyset) = X \restriction \{\, x_1 \,\},$$
$$F_{i+1} : \{\, \emptyset \,\}[F_1/x_1] \ldots [F_i/x_i] \to M$$
$$F_{i+1}(s) = \{\, a \in M \mid \exists s' \in X : (s'(x_1), \ldots, s'(x_{i+1})) = (s(x_1), \ldots, s(x_i), a) \,\}.$$

By the assumption $(\mathbb{M}, q, \check{q}) \vDash B\sigma$ it follows that $F_i(s) \in \mathcal{H}_M^i$. Furthermore, since $\mathbb{M} \vDash \sigma'$ we get that

$$\mathbb{M}, X \vDash \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big).$$

Therefore

$$(\mathbb{M}, q, \check{q}) \vDash \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big)$$

as wanted. □

Proof of Theorem 4.3. (I) $\Rightarrow$ (II): This is just a special case (for sentences) of soundness.

(II) $\Rightarrow$ (I): Suppose $T \nvdash \phi$, where ϕ is a FO$(Q, \check{Q})$-sentence. We will construct a weak model of $T \cup \{\neg\phi\}$ showing that $T \nvDash_w \phi$. Replacing T with the set $T' = \{B\sigma, A^n\sigma \mid \sigma \in T, n \in \mathbb{N}\}$ we can conclude that $T' \cup \{\neg\phi\} \nvdash \bot$. By applying Lemma 4.4 we get a weak countable recursively saturated model $(M, q, \check{q})$ of T'. Lemma 4.5 implies that $(M, q, \check{q}) \vDash T$. Now since $(M, q, \check{q}) \nvDash \phi$, we get $T \nvDash_w \phi$ as wanted. □

4.3. **Completeness for** $\mathrm{D}(Q_1, \check{Q}_1)$**.** We will now prove a completeness result similar to Theorem 4.3 for the logic $\mathrm{D}(Q, \check{Q})$ where Q is interpretated as Q_1, the quantifier "there exists uncountably many." In this section we consider only structures over uncountable universes.

We add the following two rules from [11] to the system presented in Section 3. Note that the approximation rule of section 4.2 is not included.

$$\overline{\neg Qx(x = y \vee x = z)}$$

$$\frac{Qx\exists y\phi}{\exists yQx\phi \vee Qy\exists x\phi}$$

The intuitive meaning of the second rule is that a countable union of countable sets is countable. The first is needed to avoid Q being interpreted as the quantifier "the exists at least two."

For each $\mathrm{D}(Q, \check{Q})$ sentence σ

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} = (\bar{x}^i, y_i) \wedge \theta\big)$$

in normal form we define the Skolem translation $S\sigma$ of σ to be:

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \theta(f_i(\bar{x}^i)/y_i),$$

where the f_i's are new function symbols of the right arity. If σ is a sentence in the signature τ then $S\sigma$ will be in the extended signature $\tau \cup \{f_1, \ldots, f_n\}$.

The last rule of the deduction system is the following:

$$[S\sigma]$$
$$\vdots$$
$$\frac{\sigma \qquad \psi}{\psi} \text{ (Skolem)}$$

Here σ is a $\mathrm{D}(Q, \check{Q})$ sentence in normal form, and the function symbols $f_1, \ldots, f_n$ do not occur in ψ nor in any uncancelled assumption of the derivation of ψ, except for $S\sigma$.

Proposition 4.6. *If $T \vdash \phi$ in the deduction system for $\mathrm{D}(Q_1, \check{Q}_1)$ then $T \vDash \phi$.*

Proof. We extend the proof of Proposition 3.3 to also cover the three new rules:

(1) The soundness of the first rule is easily seen by observing that the formula $Q_1x(x = y \vee x = z)$ is a FO(Q_1) formula and thus a team satisfies it iff every assignment in the team satisfies the formula.

(2) For the second rule we need to prove that if $\mathbb{M}, X \vDash \Gamma, Q_1x\exists y\phi$ then $\mathbb{M}, X \vDash \Gamma, \exists yQ_1x\phi \vee Q_1y\exists x\phi$. By the assumption we get functions $F : X \to Q_M$ and $f :$

$X[F/x] \to M$ such that $\mathbb{M}, X[F/x][f/y] \vDash \phi$. Thus, for each $s \in X$ there is a binary relation $R_s = \{\, (a, f(s[a/y])) \mid a \in F(s) \,\}$ such that $(M, R_s) \vDash Q_1 x \exists y R(x, y)$. Let

$$Y = \{\, s \in X \mid (M, R_s) \vDash \exists y Q_1 x R(x, y) \,\}$$

and

$$Z = \{\, s \in X \mid (M, R_s) \vDash Q_1 y \exists x R(x, y) \,\}.$$

By the validity of the rule for $\mathrm{FO}(Q_1)$ we see that $X = Y \cup Z$.

It should be clear that $Y \vDash \exists y Q_1 x \phi$ since by letting $g(s)$ be such that $(M, R_s) \vDash Q_1 x R(x, g(s))$ and

$$G(s[g(s)/y]) = \{\, a \in M \mid (M, R_s) \vDash R(a, g(s)) \,\},$$

we have that $Y[g/y][G/x] \subseteq X[F/x][f/y]$ and thus by downward closure

$$\mathbb{M}, Y[g/y][G/x] \vDash \phi.$$

Similarly we can prove that $\mathbb{M}, Z \vDash Q_1 y \exists x \phi$, and thus that $\mathbb{M}, X \vDash \Gamma, \exists y Q_1 x \phi \vee Q_1 y \exists x \phi$.

(3) For the Skolem rule assume that there is a derivation of ψ from Γ ending with the Skolem rule. Then there are shorter derivations from Γ of σ and from $\Gamma, S\sigma$ of ψ. By the induction hypothesis we get $\Gamma \vDash \sigma$ and $\Gamma, S\sigma \vDash \psi$. We will prove that $\Gamma \vDash \psi$, by assuming $\mathbb{M}, X \vDash \Gamma$ for some non-empty X and proving that $\mathbb{M}, X \vDash \psi$.

From the proof of Theorem 3.5 in [5] we see that $\mathbb{M} \vDash \sigma$ iff $\mathbb{M} \vDash \exists f_1 \ldots \exists f_k S\sigma$. From $\mathbb{M}, X \vDash \Gamma$ and $\Gamma \vDash \sigma$ we get that $\mathbb{M} \vDash \sigma$ and thus there are $f_1, \ldots, f_k$ such that $(M, f_1, \ldots, f_k) \vDash S\sigma$. Now since the f_i's do not occur in formulas $\Gamma' \subseteq \Gamma$ used in the derivation of ψ, $\mathbb{M}, X \vDash \Gamma$ implies that $(M, f_1, \ldots, f_k), X \vDash \Gamma'$. By locality, we also have $(M, f_1, \ldots, f_k), X \vDash S\sigma$. Therefore, by the induction hypothesis, we get that $(M, f_1, \ldots, f_k), X \vDash \psi$, and, since the f_i's do not occur in ψ, $\mathbb{M}, X \vDash \psi$ follows. $\qquad\square$

Theorem 4.7. *If T is a set of $\mathrm{D}(Q_1, \check{Q}_1)$ sentences and ϕ is a $\mathrm{FO}(Q_1, \check{Q}_1)$ sentence then $T \vdash \phi$ iff $T \vDash \phi$.*

Proof. Assume $T \nvdash \phi$. We build a model of $T' = T \cup \{\, \neg\phi \,\} \nvdash \bot$ by translating sentences σ of T into normal form σ_{nf} and considering the $\mathrm{FO}(Q, \check{Q})$ theory $T_S = \{\, S\sigma_{\mathrm{nf}} \mid \sigma \in T \,\} \cup \{\, \neg\phi \,\}$. This theory is consistent, since otherwise the Skolem rule and Proposition 4.1 would allow us to derive a contradiction from T'.

Since the deduction system for $\mathrm{D}(Q_1, \check{Q}_1)$ contains Keisler's system [11] we may apply the completeness theorem for $\mathrm{FO}(Q_1)$ and get a model $\mathbb{M}$ of $T_S \cup \{\, \neg\phi \,\}$. By the remark made in the proof of Proposition 4.6 and Proposition 4.1 $\mathbb{M}$ is also a model of $T \cup \{\, \neg\phi \,\}$. Thus $T \nvDash \phi$. $\qquad\square$

5. Conclusion

In this article we have presented inference rules and axioms for extensions of dependence logic by monotone generalized quantifiers. We also proved two completeness results for $\mathrm{FO}(Q)$ consequences in the cases where Q either has a weak interpretation or Q it is interpreted as "there exists uncountable many." In the first completeness theorem, an important feature of the proof is the approximation of a $\mathrm{D}(Q_1, \check{Q}_1)$ sentence by an infinite set of $\mathrm{FO}(Q)$ sentences. In the second completeness theorem the approximations were replaced by the Skolem rule which however is slightly unsatisfactory due to the extra function symbols f_i used in its formulation. In future work our plan is to further analyze the completeness theorem of

$D(Q_1, \check{Q}_1)$, and replace the Skolem rule with rules that do not rely on the explicit use of the Skolem functions f_i.

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Fredrik Engström
Department of Philosophy, Linguistics and Theory of Science
University of Gothenburg, Sweden
fredrik.engstrom@gu.se

Juha Kontinen
Department of Mathematics and Statistics
University of Helsinki, Finland
juha.kontinen@helsinki.fi

Jouko Väänänen
Department of Mathematics and Statistics
University of Helsinki, Finland
and
Insitute for Logic, Language and Computation
University of Amsterdam, The Netherlands
jouko.vaananen@helsinki.fi

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