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# Adaptive estimation of autoregressive models with time-varying variances

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## Abstract

Stable autoregressive models are considered with martingale differences errors scaled by an unknown nonparametric time-varying function generating heterogeneity. An important special case involves structural change in the error variance, but in most practical cases the pattern of variance change over time is unknown and may involve shifts at unknown discrete points in time, continuous evolution or combinations of the two. This paper develops kernel-based estimators of the residual variances and associated adaptive least squares (ALS) estimators of the autoregressive coefficients. Simulations show that efficiency gains are achieved by the adaptive procedure.

*JEL classification:* C14; C22

*Keywords:* Adaptive estimation; Autoregression; Heterogeneity; Nonstationary volatility; Weighted regression

## 1. Introduction

The failure of the assumption of homogenous innovations in many time series models has been well documented in the macroeconomics and empirical finance literatures. Ignoring this problem leads to inefficient estimation and unreliable inference on the conditional mean function. To account for conditional heteroskedasticity, it is common practice to assume that the innovations follow some parametric ARCH or GARCH models based on those proposed by Engle (1982) and Bollerslev (1986). Efficient estimation of the mean function in this case is achieved by quasi-maximum likelihood-based or other adaptive procedures, and recent developments on this topic have been surveyed by Li et al. (2002).

Although the GARCH-type model is successful in capturing many important features in macroeconomic or financial time series such as volatility clustering and persistent autocorrelation, a crucial weakness is its nonrobustness to the stationarity assumption. In typical GARCH-type models, the time-varying volatility is exclusively attributed to the conditional variance or covariance structure, while the unconditional variance is

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assumed to be constant over time. When this condition fails, ARCH or GARCH-based approaches may lead to serious model mis-specification. For instance, artificial IGARCH effects may be observed due to nonstationary changes in the unconditional volatility (Diebold, 1986; Mikosch and Stărică, 2004). This problem is particularly relevant in view of the strong evidence against constancy of unconditional second moments shown in the empirical literatures, e.g., in time series of exchange rates, interest rates, GDP and other macroeconomic variables (*inter alia*, Loretan and Phillips, 1994; Watson, 1999; McConnell and Perez Quiros, 2000; van Dijk et al., 2002). Recently, more complicated GARCH-type models have been proposed to allow for unconditional heteroskedasticity, e.g., varying coefficients GARCH models (Polzehl and Spokoiny, 2006) and spline GARCH models (Engle and Rangel, 2004).

An alternative approach to modeling time-varying volatility is to use a smooth deterministic nonparametric framework, assuming that the unconditional variance is the main time-changing feature to be captured (see, e.g., Hsu et al., 1974; Officer, 1976; Merton, 1980; French et al., 1987). Compared to stochastic heteroskedasticity modeling like GARCH-type models, this deterministic framework is technically easier to handle and allows for nonstationarity. Recently, Drees and Stărică (2002) and Stărică (2003) used a deterministic nonstationary framework to analyze time series of S&P 500 returns, and found that this approach outperforms the GARCH-type models in both fitting the data and forecasting the next-day volatility. However, in the typical setting of this framework, the volatility is specified as a smooth function of time thereby ruling out important practical features like structural breaks in the underlying series. Meanwhile, there are other contributions focusing particularly on modeling structural changes in volatility. For instance, Wichern et al. (1976) investigated the AR(1) model when there are a finite number of step changes at unknown time points in the error variance. These authors used iterative maximum likelihood methods to locate the change points and then estimated the error variances in each block by averaging the squared least squares residuals. The resulting feasible weighted least squares (WLS) estimator was shown to be efficient for the specific model considered. Alternative methods to detect step changes in the variances of time series models have been studied by Abraham and Wei (1984), Baufays and Rassin (1985); Tsay (1988); Park et al. (2000); Lee and Park (2001); de Pooter and van Dijk (2004) and Galeano and Peña (2004).<sup>1</sup>

However, in practice the pattern of variance changes over time, which may be discrete or continuous, is unknown to the econometrician and it seems desirable to use methods that can adapt for a wide range of possibilities. Accordingly, this paper combines two strands of the literatures mentioned above by providing a general framework to modeling nonparametric deterministic volatility in a stable linear AR( $p$ ) model, and develops an efficient estimation procedure that adapts for the presence of different and unknown forms of variance dynamics. Specifically, the model errors are assumed to be martingale differences multiplied by a time-varying scale factor which is a continuous or discontinuous function of time, thereby permitting a spectrum of variance dynamics that include step changes and smooth transitions.

Efficient estimation of linear models under heteroskedasticity with *iid* predictors was earlier investigated by Carroll (1982) and Robinson (1987), and more recently by Kitamura et al. (2004) using empirical likelihood methods in a general conditional moment restriction setting. In the time series context, Kuersteiner (2002) developed efficient instrumental variables estimators for autoregressive models under conditional heteroskedasticity but assuming constancy of the unconditional variances over time. Harvey and Robinson (1988) focused on a regression model with deterministically trending regressors only, whose error is an AR( $p$ ) process scaled by a continuous function of time, thereby allowing for both serial correlation and nonstationarity but ruling out jump behavior in the innovations. In a closely related paper, Hansen (1995) considered the linear regression model, nesting autoregressive models as special cases, when the conditional variance of the model error is a function of a covariate that has the form of a nearly integrated stochastic process with no deterministic drift. Using a kernel-weighted technique similar to ours, he also obtained the adaptive estimation results. There are some important differences between Hansen's paper and ours. The first is model formulation. Instead of focusing on stochastic trends in volatility as in Hansen (1995), we consider deterministic trends in volatility allowing particularly for single or multiple abrupt structural breaks. By doing

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<sup>1</sup>Related literature also includes testing and estimation of structural change points of the *mean* function in parametric (Bai, 1994; Bai and Perron, 1998 and references therein) and nonparametric (Yin, 1988; Müller, 1992; Wu and Chu, 1993; Delgado and Hidalgo, 2000) frameworks.

so, a different scale parameter is employed to obtain sensible limit theory. Second, in constructing the adaptive least squares (ALS) estimator, we consider two-sided kernel estimates of the residual variances, which are more accurate than Hansen's one-sided kernel estimates when variances are discontinuous over time. For this reason his proof of adaptiveness cannot be extended here. Third, we allow for multiple covariates in the mean function by studying  $p$ th order autoregressive processes. Fourth, we analyze how specific nonstationary variance patterns, such as shifts and monotone trends in variance, affect the inefficiency of the ordinary least squares (OLS) estimator relative to the generalized least squares (GLS) estimator. Finally, we also mention that regression models in which the conditional variance of the error is an unscaled function of an integrated time series were recently investigated by [Chung and Park \(2007\)](#) using Brownian local time limit methods developed in [Park and Phillips \(1999, 2001\)](#).

The remainder of the paper proceeds as follows. Section 2 describes the autoregressive model with general nonstationary deterministic volatility. Several assumptions are introduced and discussed. A limit theory is developed in Section 3 for a class of WLS estimators, including efficient (infeasible) (GLS). A range of examples show that OLS can be extremely inefficient asymptotically in some cases while nearly optimal in others. Section 4 proposes a kernel-based estimator of the residual variance and shows the associated ALS estimator to be asymptotically efficient, in the sense of having the same limit distribution as the infeasible GLS estimator. Simulation experiments are conducted in Section 5 to assess the finite sample performance of the adaptive estimator. Section 6 concludes. Proofs of the main results are collected in two appendices.

## 2. The model and assumptions

Suppose the sample  $\{Y_{-p+1}, \dots, Y_0, Y_1, \dots, Y_T\}$  from the following data generating process for the time series  $Y_t$  is observed:

$$A(L)Y_t = u_t, \quad (1)$$

$$u_t = \sigma_t \varepsilon_t, \quad (2)$$

where  $L$  is the lag operator,  $A(L) = 1 - \beta_1 L - \beta_2 L^2 - \dots - \beta_p L^p$ ,  $\beta_p \neq 0$ , is assumed to have all roots outside the unit circle and the lag order  $p$  is finite and known. We assume  $\{\sigma_t\}$  is a deterministic sequence and  $\{\varepsilon_t\}$  is a martingale difference sequence with respect to  $\{\mathcal{F}_t\}$ , where  $\mathcal{F}_t = \sigma(\varepsilon_s, s \leq t)$  is the  $\sigma$ -field generated by  $\{\varepsilon_s, s \leq t\}$ , with unit conditional variance, i.e.,  $\mathbb{E}(\varepsilon_t^2 | \mathcal{F}_{t-1}) = 1$ , a.s., for all  $t$ . The conditional variance of  $\{u_t\}$  is characterized fully by the multiplicative factor  $\sigma_t$ , i.e.,  $\mathbb{E}(u_t^2 | \mathcal{F}_{t-1}) = \sigma_t^2$ , a.s. This paper focuses on unconditional heteroskedasticity and  $\sigma_t^2$  is assumed to be modeled as a general deterministic function, which rules out conditional dependence of  $\sigma_t$  on the past events of  $Y_t$ . The autoregressive coefficient vector  $\beta = (\beta_1, \beta_2, \dots, \beta_p)'$  is the parameter of interest. OLS estimation gives  $\hat{\beta} = (\sum_{t=1}^T X_{t-1} X'_{t-1})^{-1} (\sum_{t=1}^T X_{t-1} Y_t)$ , where  $X_{t-1} = (Y_{t-1}, Y_{t-2}, \dots, Y_{t-p})'$ . Throughout the rest of the paper we impose the following conditions.

**Assumption.** (i)  $\{\sigma_t\}$  satisfies  $\sigma_t = g(t/T)$ , where  $g(\cdot)$  is a measurable and strictly positive function on the interval  $(0, 1]$  such that  $0 < \inf_{r \in (0, 1]} g(r) \leq \sup_{r \in (0, 1]} g(r) < \infty$ , and  $g(r)$  satisfies a Lipschitz condition except at a finite number of points of discontinuity;

(ii)  $\{\varepsilon_t\}$  satisfies  $\mathbb{E}(\varepsilon_t | \mathcal{F}_{t-1}) = 0$ ,  $\mathbb{E}(\varepsilon_t^2 | \mathcal{F}_{t-1}) = 1$ , a.s., for all  $t$ .

(iii)  $\sup_t \mathbb{E}|\varepsilon_t|^{4\mu} < \infty$  for some  $\mu > 1$ .

Under Assumption (i) the function  $g$  is integrable on the interval  $(0, 1]$  to any finite order. For brevity, we write  $\int_0^1 g^m(r) dr$  as  $\int g^m$  for any finite positive integer  $m$ . Formally, of course, the assumption induces a triangular array structure to the processes  $u_t$  and  $Y_t$ , but we dispense with the additional affix  $T$  in the arguments that follow. Assumption (ii) stipulates  $\{\varepsilon_t\}$  is a martingale difference (m.d.) sequence and therefore uncorrelated, but may be dependent via higher moments.

In contrast to modeling  $\sigma_t$  in a setting with finitely many parameters, Assumption (i) is nonparametric and  $\sigma_t$  depends only on the relative position of the error in the sample. It allows for a wide range of nonstationary variance dynamics including single or multiple step changes and smooth transitions (e.g., trending or periodic variances. See Examples 1 and 2). Assumption (i) excludes the dependence of  $\mathbb{E}(u_t^2 | \mathcal{F}_{t-1})$  on past events.

A more flexible formulation is to assume  $\sigma_t$  as a function of scaled ( $T^{-1}$ ) integrated time series with a time trend (see the discussion in the next paragraph).

Our model of nonstationary volatility is related to that of [Hansen \(1995\)](#). In his paper, the volatility process is specified as a function of a first-order nearly integrated process, viz.  $E(u_t^2|F_{t-1}) = g^2(c_1 + c_2 S_t/\sqrt{T})$ , where  $S_t = (1 - c_3/T)S_{t-1} + z_t$  with martingale differences  $z_t$  and constants  $c_i$ ,  $i = 1, 2, 3$ . Without accounting for structural breaks explicitly, his model focuses on stochastic volatility, which asymptotically reduces to ours in Assumption (i) by a simple extension. To illustrate, suppose a time trend (or drift)  $c_4 t$  is added to the nearly unit root process  $S_t$ . Since a stochastic trend is dominated by a deterministic trend in the long run at least for a scalar process, Hansen's model in this case is no longer applicable and the normalization factor needs to be adjusted to  $1/T$  rather than  $1/\sqrt{T}$ , as in Hansen's formulation, to achieve a non-degenerate asymptotic theory.

Combining (1) with (2) is particularly useful in accounting for nonstationary volatility that may be present in macroeconomic and financial data. [Watson \(1999\)](#) and [McConnell and Perez Quiros \(2000\)](#) found evidence of monotone trending behavior in variability (corresponding to a monotone version of the function  $g(\cdot)$  in Assumption (i)) for US short and long term interest rates and GDP series over specified periods. The volatility structure in (2) was also used by [Stărică et al. \(2005\)](#) in the analysis of the dynamics of stock indexes—see also [Stărică and Granger \(2005\)](#).

We conclude this section by mentioning that much attention has recently been paid to potential structural error variance changes in integrated process models. The effects of step breaks in the innovation variance on unit root tests and stationarity tests were studied by [Hamori and Tokihisa \(1997\)](#); [Kim et al. \(2002\)](#); [Busetti and Taylor \(2003\)](#) and [Cavaliere \(2004a\)](#). A general framework to analyze the effect of time-varying variances on unit root tests was given in [Cavaliere \(2004b\)](#) and [Cavaliere and Taylor \(2004\)](#). By contrast, little work of this general nature (as in Assumption (i), which is attributed to [Cavaliere, 2004b](#)) has been done on autoregressions with coefficients satisfying the stable condition, most of the attention in the literature being concerned with the case of step changes or smooth transitions in the error variance, as discussed above. The present paper therefore contributes by focusing on efficient estimation of the  $AR(p)$  model with time-varying variances of a general form that includes step changes as a special case.

### 3. Limit theory

Under the stated assumptions, the process  $Y_t$  has the following representation:

$$Y_t = \sum_{i=0}^{\infty} \alpha_i u_{t-i}, \quad (3)$$

where the coefficients  $\{\alpha_i\}$  satisfy

$$\sum_{i=0}^{\infty} |\alpha_i| < \infty. \quad (4)$$

Under Assumptions (i)–(iii),  $\hat{\beta}$  is asymptotically normal with limit distribution ([Phillips and Xu, 2006a](#)):<sup>2</sup>

$$\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, A), \quad (5)$$

where

$$A = \frac{\int g^4}{(\int g^2)^2} \Gamma^{-1}$$

and  $\Gamma$  is the  $p \times p$  positive definite matrix with the  $(i, j)$ th element  $\gamma_{|i-j|}$ , and  $\gamma_k = \sum_{i=0}^{\infty} \alpha_i \alpha_{i+k}$  with  $|\gamma_k| < \infty$ , for  $0 \leq k \leq p-1$ . The matrix  $\Gamma^{-1}$  can be consistently estimated by

$$\hat{\Gamma}^{-1} = (\hat{\gamma}_{|i-j|})_{i,j}^{-1}, \quad (6)$$

<sup>2</sup>In a more general framework allowing for both stochastic and deterministic nonstationary volatility, this limit distribution assumes a general form involving stochastic integrals ([Xu, 2006](#), see also [Hansen, 1995](#)).

where  $\widehat{\gamma}_0, \widehat{\gamma}_1, \dots, \widehat{\gamma}_{p-1}$  are the first  $p$  elements in the first column of the  $(p^2 \times p^2)$  matrix  $[I_{p^2} - F \otimes F]^{-1}$ , where  $\otimes$  indicates the Kronecker product and

$$F = \begin{pmatrix} \widehat{\beta}_1 & \widehat{\beta}_2 & \cdots & \widehat{\beta}_p \\ & & & 0 \\ & I_{p-1} & & \vdots \\ & & & 0 \end{pmatrix}.$$

Result (5) is a consequence of the following more general theorem. In what follows, let  $C > 0$  be a generic positive constant.

**Theorem 1.** *Suppose  $\omega_t^2$  is nonstochastic and satisfies (i)  $0 < \omega_t^2 < C < \infty$  for all  $t$ ; (ii) there exists a function  $\omega(\cdot)$  on  $(0, 1]$ , continuous except for a finite number of discontinuities, such that  $\omega_{[T\tau]}^2 \rightarrow \omega^2(r)$  for any  $r \in (0, 1]$  at which  $\omega(\cdot)$  is continuous; (iii)  $\int \omega^2 > 0$ . Then, under Assumptions (i)–(iii), the WLS estimator*

$$\widehat{\beta}_{\text{WLS}} = \left( \sum_{t=1}^T \omega_t^2 X_{t-1} X'_{t-1} \right)^{-1} \left( \sum_{t=1}^T \omega_t^2 X_{t-1} Y_t \right) \quad (7)$$

satisfies

$$\sqrt{T}(\widehat{\beta}_{\text{WLS}} - \beta) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\int \omega^4 g^4}{(\int \omega^2 g^2)^2} \Gamma^{-1} \right), \quad (8)$$

as  $T \rightarrow \infty$ .

Naturally, the estimator with the smallest asymptotic variance matrix in the class (7) is achieved by GLS

$$\beta^* = \left( \sum_{t=1}^T X_{t-1} X'_{t-1} \sigma_t^{-2} \right)^{-1} \left( \sum_{t=1}^T X_{t-1} Y_t \sigma_t^{-2} \right), \quad (9)$$

with weights  $\omega_t^2 = \sigma_t^{-2}$  in which case<sup>3</sup>

$$\sqrt{T}(\beta^* - \beta) \xrightarrow{d} \mathcal{N}(0, \Gamma^{-1}), \quad (10)$$

as  $T \rightarrow \infty$ .

**Remarks.** Clearly, the asymptotic variance matrix of  $\widehat{\beta}$  differs from that of  $\beta^*$  by the factor  $\int g^4 / (\int g^2)^2$ , and since  $\Gamma^{-1}$  is invariant to the function  $g(\cdot)$  the inefficiency of the OLS estimator  $\widehat{\beta}$  depends crucially on this factor. The following examples<sup>4</sup> show that the factor can be large and OLS can be very inefficient in some cases, whereas in others, the factor is close to unity and OLS is close to optimal.

**Example 1** (*A single abrupt shift in the innovation variance*). Let  $\tau \in [0, 1]$  and  $g(r)$  be the step function

$$g(r)^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2) \mathbf{1}_{\{r \geq \tau\}}, \quad r \in [0, 1],$$

giving error variance  $\sigma_0^2$  before the break point  $[T\tau]$ , and  $\sigma_1^2$  afterwards. The steepness of the variance shift is measured by the ratio  $\delta := \sigma_1 / \sigma_0$  of the post-break and pre-break standard deviation. By (5) the asymptotic variance matrix of OLS is

$$A = \frac{\tau + (1 - \tau)\delta^4}{(\tau + (1 - \tau)\delta^2)^2} \Gamma^{-1} := f_1^2(\tau, \delta) \Gamma^{-1},$$

<sup>3</sup>The optimality of  $\beta^*$  can also be justified by the theory of unbiased linear estimating equations, as in Godambe (1960) and Durbin (1960).

<sup>4</sup>We follow the formulation of the variance function in Cavaliere (2004a,b, Section 5, pp. 271–283), who investigates heteroskedastic unit root testing.

where  $f_1^2(\tau, \delta) = (\tau + (1 - \tau)\delta^2)^{-2}(\tau + (1 - \tau)\delta^4)$ , which is a function of the break date  $\tau$  and the shift magnitude  $\delta$ .

Fig. 1 plots the value of  $f_1(\tau, \delta)$  across  $\delta \in [0.01, 100]$  for different values of  $\tau$ . The variance of the OLS estimator largely depends on where the break in the innovation variance occurs. For the negative ( $\delta < 1$ ) shift,  $f_1(\tau, \delta)$  increases steeply as  $\delta$  decreases when  $\tau = 0.1$ , and is relatively steady and nearly unity when  $\tau = 0.9$ . The graph shows that OLS has large variance when the break occurs at the beginning ( $\tau = 0.1$ ) but much smaller variance, and in fact close to that of infeasible GLS, when the break is at the end ( $\tau = 0.9$ ) of the sample. This difference is explained by the fact that when the break in variance occurs early in the sample, the large innovation variance in the early part of the sample affects all later observations via the autoregressive mechanism. By contrast, when the break occurs near the end of the sample, only later observations are directly affected, so the impact of a negative shift is small. This argument applies when there is a negative shift—a shift to a smaller variance at the end of the sample—and a reverse argument applies in the case of a positive shift.

In fact, under a positive ( $\delta > 1$ ) shift, OLS has large variance when the shift occurs late ( $\tau = 0.9$ ) but small variance and more closely approximates infeasible GLS when it is early ( $\tau = 0.1$ ) in the sample. These phenomena are confirmed in the simulation experiment of Gaussian AR(1) case, reported in Section 5.

**Example 2** (*Trending variances in the innovations*). Let  $m$  be a positive integer and  $g(r)$  be

$$g(r)^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)r^m, \quad r \in [0, 1],$$

giving error variance changing from  $\sigma_0^2$  to  $\sigma_1^2$  continuously according to an  $m$ th order power function. Then

$$A = \frac{1 + 2(\delta^2 - 1)/(m + 1) + (\delta^2 - 1)^2/(2m + 1)}{[1 + (\delta^2 - 1)/(m + 1)]^2} \Gamma^{-1} := f_2^2(m, \delta) \Gamma^{-1},$$

where  $f_2^2(m, \delta) = (1 + (\delta^2 - 1)/(m + 1))^{-2}(1 + (2(\delta^2 - 1)/(m + 1) + ((\delta^2 - 1)^2/2m + 1))$  and  $\delta = \sigma_1/\sigma_0$ .

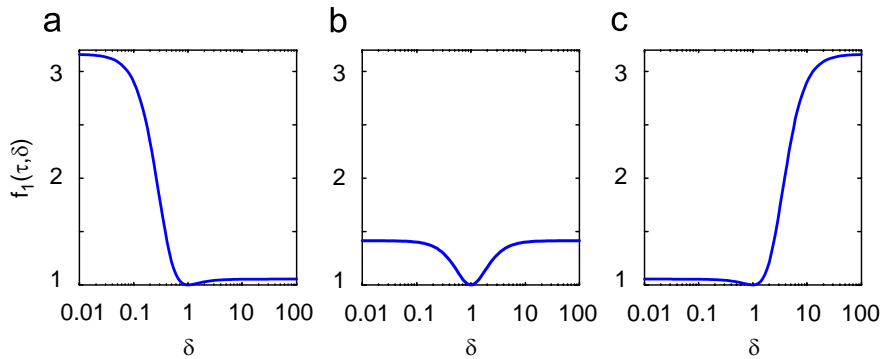


Fig. 1. The values of  $f_1(\tau, \delta)$  (y-axis) in Example 1 across  $\delta$  (x-axis) for different values of  $\tau$ : (a)  $\tau = 0.1$ ; (b)  $\tau = 0.5$ ; (c)  $\tau = 0.9$ .

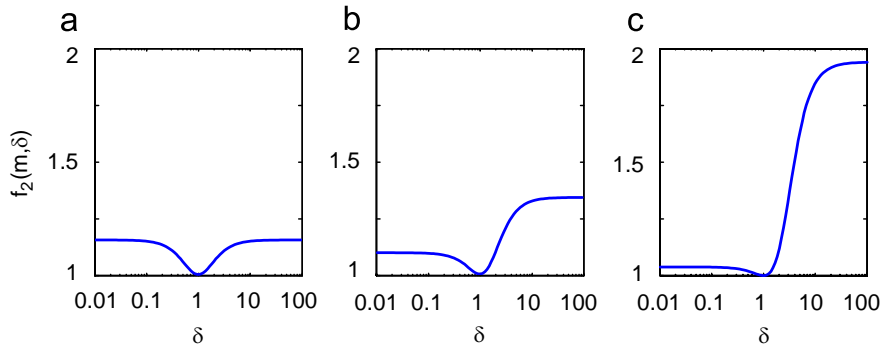


Fig. 2. The values of  $f_2(m, \delta)$  (y-axis) in Example 2 across  $\delta$  (x-axis) for different values of  $m$ : (a)  $m = 1$ ; (b)  $m = 2$ ; (c)  $m = 6$ .

Fig. 2 plots the value of  $f_2(m, \delta)$  across  $\delta \in [0.01, 100]$  for different values of  $m$ , so that both positive ( $\delta > 1$ ) and negative ( $\delta < 1$ ) trending heteroskedasticity is allowed. Compared with the case of a single abrupt shift in the innovation variance (Example 1), the multiplicative factor  $f_2(m, \delta)$  changes more steadily for a given value of  $m$ , especially when  $m$  is small (say,  $m = 1$ ). In the case of large  $m$  (say,  $m = 6$ ), much inefficiency in OLS is sustained when there is positive trending heteroskedasticity ( $\delta > 1$ ).

#### 4. Adaptive estimation

The GLS estimator  $\beta^*$  in (9) is infeasible, since the true values of  $\sigma_t$  are unknown. To produce a feasible procedure, we propose a kernel-based estimator  $\tilde{\beta}$  that have the same asymptotic distribution as  $\beta^*$ . Let  $\hat{u}_t = Y_t - X'_{t-1}\beta$  be the OLS residuals and  $K(z)$  be a bounded nonnegative continuous kernel function defined on the real line such that  $\int_{-\infty}^{\infty} K(z) dz = 1$ . Define

$$\tilde{\beta} = \left( \sum_{t=1}^T X_{t-1} X'_{t-1} \hat{\sigma}_t^{-2} \right)^{-1} \left( \sum_{t=1}^T X_{t-1} Y_t \hat{\sigma}_t^{-2} \right), \quad (11)$$

where

$$\hat{\sigma}_t^2 = \sum_{i=1}^T w_{ti} \hat{u}_i^2 \quad (12)$$

and  $w_{ti} = \left( \sum_{i=1}^T K_{ii} \right)^{-1} K_{ti}$  with

$$K_{ti} = \begin{cases} K\left(\frac{t-i}{Tb}\right) & \text{if } t \neq i, \\ 0 & \text{if } t = i. \end{cases} \quad (13)$$

Here  $b$  is a bandwidth parameter, dependent on  $T$ .  $\tilde{\beta}$  is called the ALS estimator of  $\beta$ . The rationale for this is, of course, that  $\hat{\sigma}_t^2$  deputizes for  $\sigma_t^2$ .<sup>5</sup> For technical reasons in (12), we use the leave-one-out procedure and omit the observation  $\hat{u}_t^2$ .

The implementation of the estimator  $\hat{\sigma}_t^2$  depends on the choice of kernel function  $K$  and the bandwidth  $b$ . Commonly used kernels such as the uniform, Epanechnikov, biweight and Gaussian functions can be applied. Bandwidth selection is more crucial. As usual, too small a bandwidth produces less bias for the true residual variance but has higher variability. A simple data-driven method to choose the parameter  $b$  is cross-validation (CV) on the average squared error—see Wong (1983). The cross-validatory choice of  $b$  is the value  $b^*$  which minimizes

$$\widehat{CV}(b) = \frac{1}{T} \sum_{t=1}^T (\hat{u}_t^2 - \hat{\sigma}_t^2)^2.$$

We use the following assumptions that modify and extend the earlier assumptions to facilitate the development of an asymptotic theory for  $\tilde{\beta}$ .

**Assumption.** (iii')  $\sup_t \mathbb{E}(\varepsilon_t^8) < \infty$ ;  
(iv) As  $T \rightarrow \infty$ ,  $b + 1/Tb^2 \rightarrow 0$ .

We replace Assumption (iii) by the stronger Assumption (iii'), which requires the existence of eighth moments of  $\varepsilon_t$  for all  $t$ . This moment condition simplifies the proof of the main theorem and is, no doubt, stronger than necessary. Assumption (iv) is a rate condition that requires  $b \rightarrow 0$  at a slower rate than  $T^{-1/2}$ .

The main result is as follows.

<sup>5</sup>There is a vast literature on kernel-based conditional variance estimation, e.g., see Fan and Yao (1998); Yu and Jones (2004) and Phillips and Xu (2006b) for recent contributions.



**Theorem 2.** Let  $g^2(r-) = \lim_{\bar{r} \uparrow r} g^2(\bar{r})$  and  $g^2(r+) = \lim_{\bar{r} \downarrow r} g^2(\bar{r})$ , for  $r \in (0, 1]$ . Under Assumptions (i)–(iv) with (iii') instead of (iii), as  $T \rightarrow \infty$ ,

$$\hat{\sigma}_{[Tr]}^2 \xrightarrow{p} g^2(r-) \int_{-\infty}^0 K(z) dz + g^2(r+) \int_0^{\infty} K(z) dz, \quad (14)$$

for  $r \in (0, 1)$ , and  $\hat{\sigma}_T^2 \xrightarrow{p} g^2(1-) \int_{-\infty}^0 K(z) dz$ . Let  $\beta^*$  and  $\tilde{\beta}$  be defined in (9) and (11), respectively, then

$$\sqrt{T}(\tilde{\beta} - \beta) = \sqrt{T}(\beta^* - \beta) + o_p(1) \xrightarrow{d} \mathcal{N}(0, \Gamma^{-1}), \quad (15)$$

where  $\Gamma^{-1}$  is estimated by (6).

Result (14) shows that  $\hat{\sigma}_{[Tr]}^2$  converges in probability to  $g^2(r)$  for interior points  $r$  when the function  $g$  is continuous, but in general to a point between  $g^2(r-)$  and  $g^2(r+)$  depending on the shape of the kernel. The inconsistency of the error variance function estimator at points of discontinuities has a diminishing effect on the behavior of adaptive estimators of the autoregressive coefficients when the sample size is large, as is clear from (15). A one-sided kernel estimator of the residual variance at time  $t$ , as proposed by Hansen (1995), can be also constructed by using information up to time  $t - 1$ . But this estimator has larger bias in small samples at discontinuous points since it always converges in probability to  $g^2(r-)$ , although the difference on adaptive estimation diminishes as the sample size increases.

Another adaptive estimator is suggested by Harvey and Robinson (1988), who dealt with time series regression in the presence of trending regressors. Rather than estimating each  $\sigma_t^2$  separately, they split the data into  $K$  blocks and estimated  $\sigma_t^2$  in one block by the average of  $\hat{u}_t^2$  in this block. So only  $K$  distinct estimators are used. It can be shown under the regularity assumptions, the resulting WLS estimator of  $\beta$  also has the same asymptotic distribution as  $\tilde{\beta}$  if  $1/T_1 + T/T_1^2 + T_2/T \rightarrow 0$ , as  $T \rightarrow \infty$ , where  $T_1$  and  $T_2$  are the minimum and maximum lengths of the  $K$  blocks. Compared to our estimator, this estimator is faster to compute but it does not integrate in an efficient way the information of  $\hat{u}_s^2$  where  $s$  is close to  $t$  when estimating  $\sigma_t^2$ , especially when  $t$  is close to the boundary of the block.

## 5. Simulations

This section examines the finite sample performance of the ALS efficient procedure proposed in Section 4 using simulations of the heteroskedastic AR(1) model

$$Y_t = \beta Y_{t-1} + u_t, \quad u_t = \sigma_t \varepsilon_t,$$

where  $\sigma_t = g(t/T)$ . We use  $\beta \in \{0.1, 0.9\}$  and  $\varepsilon_t \sim iid \mathcal{N}(0, 1)$ .

Our simulation design basically follows Cavaliere (2004a, b) and Cavaliere and Taylor (2004). The  $g$  function generating heteroskedasticity is taken as the step function used in Example 1, viz.,

$$g(r)^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2) \mathbf{1}_{\{r \geq \tau\}}, \quad r \in [0, 1].$$

The break date is chosen from  $\{0.1, 0.9\}$  and the ratio of post-break and pre-break standard deviations  $\delta = \sigma_1/\sigma_0$  is set to the values  $\{0.2, 5\}$ . Without loss of generality, we let  $\sigma_0 = 1$ . The estimates of  $\beta$  are obtained with sample size  $T = 50$  and 200, and the number of replications is set to 10,000. Other models (say the trending variance in Example 2) are also considered in our experiments, although not reported here, and they yield the results similar to those obtained below.

We report estimates for  $\beta$  obtained by OLS, infeasible GLS and ALS. For the ALS estimator (11), we use the Gaussian kernel function,  $K(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ , for  $-\infty < z < \infty$ . When a different kernel (such as Epanechnikov kernel) is used, the results do not change much. Five bandwidths are considered, i.e., four fixed bandwidths  $h_i = c_i T^{-0.4}$ ,  $i = 1, \dots, 4$ , where  $\{c_1, c_2, c_3, c_4\} = \{0.25, 0.4, 0.6, 0.75\}$  as well as a data-driven bandwidth chosen by the CV procedure described in Section 4.

Table 1 reports the ratios of the root mean squared errors (RMSE) of estimators considered relative to the RMSE of GLS. The levels (rather than the ratios) of RMSE are reported for GLS in brackets. Clearly, OLS is inefficient and the ALS estimator works reasonably well in all cases considered. The largest inefficiency in OLS is observed when an early shift in the innovation variance is negative, for instance,  $(\tau, \delta) = (0.1, 0.2)$ , and when

Table 1

The ratios of the RMSEs of OLS estimator and ALS estimators using four fixed bandwidths and cross-validated bandwidth, relative to that of GLS (the levels of RMSE are reported for GLS in brackets)

$\tau$	$\delta$	$T$	OLS	ALS					CV	GLS		
				$h_1$	$h_2$	$h_3$	$h_4$					
$\beta = 0.1$												
0.1	0.2	50	1.9749	1.5029	1.5278	1.6169	1.6865	1.5612	[.1236]			
		200	2.4751	1.1501	1.1501	1.1830	1.2182	1.1538	[.0636]			
	1	50	1.0000	1.1586	1.0745	1.0375	1.0241	1.0329	[.0885]			
		200	1.0000	1.0466	1.0280	1.0187	1.0151	1.0155	[.0374]			
		5	50	1.0333	1.1220	1.0754	1.0561	1.0498	1.0612	[.1471]		
			200	1.0351	1.0780	1.0676	1.0631	1.0600	1.0594	[.0667]		
0.9	0.2	50	1.1801	1.3196	1.2625	1.2339	1.2199	1.2359	[.1170]			
		200	1.1100	1.1253	1.1172	1.1164	1.1151	1.1198	[.0691]			
	5	50	1.9576	1.1925	1.1958	1.2583	1.3177	1.2555	[.1433]			
		200	2.2333	1.0859	1.0784	1.0952	1.1208	1.0795	[.0701]			
		$\beta = 0.9$										
		0.1	0.2	50	2.0748	1.4599	1.4968	1.5742	1.6417	1.5380	[.0633]	
200	2.3822			1.1994	1.2020	1.2270	1.2450	1.1995	[.0283]			
1	50		1.0000	1.0931	1.0374	1.0172	1.0110	1.0191	[.0851]			
	200		1.0000	1.0398	1.0213	1.0115	1.0080	1.0103	[.0346]			
	5		50	1.0427	1.1260	1.0749	1.0628	1.0592	1.0754	[.0885]		
			200	1.0225	1.0571	1.0425	1.0380	1.0354	1.0362	[.0374]		
0.9	0.2	50	1.2853	1.2581	1.2763	1.2875	1.2904	1.2838	[.0664]			
		200	1.1856	1.1315	1.1540	1.1781	1.1866	1.1844	[.0291]			
	5	50	2.0607	1.2049	1.1773	1.2188	1.2769	1.2068	[.0887]			
		200	2.2663	1.0903	1.0748	1.0825	1.0983	1.0823	[.0346]			

Error distribution: normal. Parameter values:  $\beta \in \{0.1, 0.9\}$ ,  $\tau \in \{0.1, 0.9\}$ ,  $\delta \in \{0.2, 5\}$  and the sample size  $T = \{50, 200\}$ .

a late shift is positive, for instance,  $(\tau, \delta) = (0.9, 5)$ . The former is explained by the fact that the large variance early in the sample affects all later observations and the latter is explained by the fact that the large variance in the last part of the sample means that the OLS estimator is more closely approximated by the terms involving the last few observations, thereby effectively reducing the sample size. In both these cases, substantial efficiency gains are achieved by the ALS estimator. In contrast, when there is a positive early shift or a negative late shift in the innovation variance, for instance,  $(\tau, \delta) = (0.1, 5)$  or  $(0.9, 0.2)$ , OLS works nearly as well as GLS, especially when the sample size is large. The ALS estimator performs comparably well with OLS in those cases. When the sample size is increased from  $T = 50$  to 200, the ALS estimators have the smaller ratio of RSME, while no improvement (or even larger inefficiency) is observed for OLS.

We also note that the CV procedure to choose the bandwidth of the ALS estimator works satisfactorily. Sometime the ALS estimator with the cross-validated bandwidth is outperformed by certain specified fixed bandwidth in certain cases (in most case by  $h_2$ ), but is not uniformly dominated by a single fixed bandwidth from the four we considered. In practice we recommend using the cross-validated bandwidth or the fixed bandwidth  $h_2$ .

Simulations results, along with those not reported here, also show that, in both models the improvement of the ALS procedure relative to OLS is insensitive to the location of the true value of the autoregressive parameter  $\beta$ , as long as  $|\beta| < 1$ .

We also check the homoskedastic case when  $\delta = 1$  and show results in Table 1. OLS is equivalent to GLS when the errors are homoskedastic, so the ratio of RMSE of OLS relative to GLS is unity. We observe that in this case the ALS estimator is also close to one, so that ALS may be used satisfactorily even when the errors are homoskedastic.

Furthermore, to check the robustness of our ALS procedure to skewed or heavy-tailed error distributions, we let  $\varepsilon_t$  be subject to a  $\chi^2(5)$  or a  $t(5)$  distribution each with degree of freedom five, normalized so that it has

Table 2

The ratios of the RMSEs of OLS estimator and ALS estimators using four fixed bandwidths and cross-validated bandwidth, relative to that of GLS (the levels of RMSE are reported for GLS in brackets)

Error dist.	$T$	OLS	ALS					CV	GLS
			$h_1$	$h_2$	$h_3$	$h_4$			
$\beta = 0.1$ $\frac{\chi^2(5) - 5}{\sqrt{10}}$	50	2.0441	1.3597	1.3298	1.3983	1.4721	1.4277	[.1375]	
	200	2.1478	1.1170	1.1022	1.1148	1.1364	1.1157	[.0701]	
$\sqrt{0.6}t_5$	50	1.9072	1.4207	1.3863	1.4259	1.4834	1.4405	[.1394]	
	200	2.1648	1.1687	1.1477	1.1583	1.1767	1.1545	[.0704]	
$\beta = 0.9$ $\frac{\chi^2(5) - 5}{\sqrt{10}}$	50	2.0241	1.3419	1.3424	1.3853	1.4286	1.4208	[.0902]	
	200	2.2665	1.1729	1.1345	1.1278	1.1364	1.1457	[.0327]	
$\sqrt{0.6}t_5$	50	2.0371	1.3108	1.3060	1.3605	1.4243	1.3851	[.0850]	
	200	2.1579	1.1515	1.1233	1.1216	1.1337	1.1321	[.0364]	

Error distribution:  $\chi^2(5)$  or  $t_5$ . Parameter values:  $\beta \in \{0.1, 0.9\}$ ,  $\tau = 0.9$ ,  $\delta = 5$  and the sample size  $T = \{50, 200\}$ .

zero mean and unit variance. Apparently when  $\varepsilon_t \sim t(5)$ , the technical Assumption (iii') is violated. This model is incorporated to illustrate that the conclusion of Theorem 2 extends to more general error distributions. The corresponding results are reported only for the case of a positive late shift (i.e.,  $\tau = 0.9$ ,  $\delta = 5$ ) in Table 2. Again, we can see that major efficiency gains are achieved by the ALS estimator compared to the OLS procedure. Just as the cases with Gaussian errors we consider above, ALS is almost as efficient as the infeasible GLS estimator when  $T$  is increased from 50 to 200.

In summary, our kernel-based ALS estimator and CV procedure both appear to perform reasonably well, at least within the simulation design considered. The advantages are clear—they are convenient for practical use and have uniformly good performance over the parameter space.

## 6. Further remarks

This paper considers efficient estimation of finite order autoregressive models under unconditional heteroskedasticity of unknown form. Several extensions of the approach taken in the paper are possible. One of these is to consider efficient estimation of unconditionally heteroskedastic stable autoregressions of possible infinite order. The issue here is whether the nonparametric feasible GLS estimator considered here is still asymptotically efficient when the order of autoregression,  $p$ , increases with the sample size,  $T$ . We leave this and other extensions for future research.

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## Appendix A. Proofs of the theorems

This section gives the proofs of Theorems 1 and 2. We use  $|\cdot|$  to denote the Euclidean norm  $|X| = (X_1^2 + \dots + X_n^2)^{1/2}$  for  $X = (X_1, \dots, X_n)'$ , and  $\|\cdot\|_K$  to denote the  $L^K$ -norm, so that  $\|\xi\|_K = (E|\xi|^K)^{1/K}$  for a random vector  $\xi$ .

**Proof of Theorem 1.** The WLS estimator  $\widehat{\beta}_{\text{WLS}}$  satisfies

$$\sqrt{T}(\widehat{\beta}_{\text{WLS}} - \beta) = \left( \frac{1}{T} \sum_{t=1}^T \omega_t^2 X_{t-1} X'_{t-1} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \omega_t^2 X_{t-1} u_t \right). \quad (16)$$

Consider the first term of the right-hand side of (16) first. We want to show

$$A_n = \frac{1}{T} \sum_{t=1}^T (\omega_t^2 Y_{t-h} Y_{t-h-k} - \omega_t^2 \mathbb{E}(Y_{t-h} Y_{t-h-k})) \xrightarrow{P} 0 \quad (17)$$

for  $1 \leq h \leq p$ ,  $0 \leq k \leq p - h$ . Define  $Y_{t,m} = \sum_{i=0}^m \alpha_i u_{t-i}$  and

$$A_n^m = \frac{1}{T} \sum_{t=1}^T (\omega_t^2 Y_{t-h,m} Y_{t-h-k,m} - \omega_t^2 \mathbb{E}(Y_{t-h,m} Y_{t-h-k,m})).$$

To prove (17), we need to show: (a)  $A_n^m \xrightarrow{P} 0$ , as  $n \rightarrow \infty$ , for each fixed  $m$ ; (b)  $\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(|A_n^m - A_n| \geq \delta) = 0$  for all  $\delta > 0$ . Then (17) follows from Proposition 6.3.9 of [Brockwell and Davis \(1991\)](#) (see also [Billingsley, 1968](#), Theorem 4.2). It is straightforward to show that (a) follows from the LLN for uniformly integrable  $L^1$ -mixingales ([Andrews, 1988](#)) and (b) from the Markov inequality and Assumptions (i)–(iii). Thus (17) holds. Lemma A(ii) of [Phillips and Xu \(2006a\)](#) shows that for every continuous point  $r$  of  $g(\cdot)$ ,  $\lim_{T \rightarrow \infty} \mathbb{E} Y_{[Tr]-h} \cdot Y_{[Tr]-h-k} = g^2(r) \gamma_k$ , where  $[\cdot]$  refers to the integer part. Let  $r_1 < r_2 < \dots < r_Q$  be the discontinuous points of  $g(\cdot)$  and  $w(\cdot)$ , where  $Q$  is a finite number (independent of  $T$ ). So by (17), for sufficiently large  $T$ ,  $T^{-1} \sum_{t=1}^T \omega_t^2 Y_{t-h} Y_{t-h-k} = T^{-1} \sum_{t=1}^T \omega_t^2 \mathbb{E}(Y_{t-h} Y_{t-h-k}) + o_p(1) = \sum_{t=1}^T \int_{t/T}^{(t+1)/T} \omega_{[Tr]}^2 \mathbb{E} Y_{[Tr]-h} Y_{[Tr]-h-k} dr + o_p(1) = \int_{1/T}^{r_1} \omega_{[Tr]}^2 \mathbb{E} Y_{[Tr]-h} Y_{[Tr]-h-k} dr + \sum_{j=1}^{Q-1} \int_{r_j}^{r_{j+1}} \omega_{[Tr]}^2 \cdot \mathbb{E} Y_{[Tr]-h} Y_{[Tr]-h-k} dr + \int_{r_Q}^{(T+1)/T} \omega_{[Tr]}^2 \mathbb{E} Y_{[Tr]-h} Y_{[Tr]-h-k} dr + o_p(1) \xrightarrow{P} (\int \omega^2 g^2) \gamma_k$ . So we have  $T^{-1} \sum_{t=1}^T \omega_t^2 \cdot X_{t-1} X'_{t-1} \xrightarrow{P} (\int \omega^2 g^2) \Gamma$ . Next we show that  $T^{-1} \sum_{t=1}^T \omega_t^4 \cdot X_{t-1} X'_{t-1} u_t^2 \xrightarrow{P} (\int \omega^4 g^4) \Gamma$ , which holds if  $T^{-1} \sum_{t=1}^T \omega_t^4 Y_{t-h} Y_{t-h-k} u_t^2 \xrightarrow{P} \gamma_k$  for  $1 \leq h \leq p$ ,  $0 \leq k \leq p - h$ . Indeed, since  $\{\omega_t^4 Y_{t-h} Y_{t-h-k} u_t^2 - \omega_t^4 \sigma_t^2 \mathbb{E} Y_{t-h} Y_{t-h-k}, \mathcal{F}_t\}$  are martingale differences, so  $T^{-1} \sum_{t=1}^T \omega_t^4 Y_{t-h} Y_{t-h-k} u_t^2 = T^{-1} \sum_{t=1}^T \omega_t^4 \cdot \sigma_t^2 \mathbb{E} Y_{t-h} Y_{t-h-k} + o_p(1) \xrightarrow{P} (\int \omega^4 g^4) \gamma_k$  by [Andrews \(1988\)](#)'s LLN for uniformly integrable  $L^1$ -mixingales. Furthermore,  $\mathbb{E}|\omega_t^2 X_{t-1} u_t|^4 < \infty$  by Lemma A(b) with  $\mu = 2$ . By the central limit theorem for vector martingale differences,  $T^{-1/2} \sum_{t=1}^T \omega_t^2 X_{t-1} u_t \xrightarrow{d} \mathcal{N}(0, (\int \omega^4 g^4) \Gamma)$ . Then Theorem 1 follows from (16).  $\square$

**Proof of Theorem 2.** First we prove (14). Recall that  $\widehat{u}_t$ 's are the OLS residuals. Let  $\overline{\sigma}_t^2 = \sum_{i=1}^T w_{it} \sigma_i^2$ , and it is easy to see that

$$\left| \left( \frac{1}{Tb} \sum_{i=1}^T K_{ii} \right) (\widehat{\sigma}_t^2 - \overline{\sigma}_t^2) \right| \leq \left| \frac{1}{Tb} \sum_{i=1}^T K_{ii} (u_i^2 - \sigma_i^2) \right| + o_p(1) = o_p(1). \quad (18)$$

Actually, if we let  $a_i = u_i^2 - \sigma_i^2$ , then  $\{a_i\}$  is an m.d. sequence and  $\mathbb{E}((1/Tb)\sum_{i=1}^T K_{[Tr]i} a_i)^2 = (1/(Tb)^2)\sum_{i=1}^T K_{[Tr]i}^2 \mathbb{E} a_i^2 \leq (1/Tb)(\sup_i K_{[Tr]i})(\sup_i \mathbb{E} a_i^2)((1/Tb)\sum_{i=1}^T K_{[Tr]i}) = O(1/Tb) \rightarrow 0$ , in view of Lemma A(c). On the other hand, we have

$$\begin{aligned} \frac{1}{Tb} \sum_{i=1}^T K_{[Tr]i} \sigma_i^2 &= \frac{1}{b} \int_{1/T}^{(T+1)/T} K\left(\frac{[Ts] - [Tr]}{Tb}\right) g^2\left(\frac{[Ts]}{T}\right) ds + o(1) \\ &\stackrel{z=(s-r)/b}{=} \int_{(1-Tr)/Tb}^{(T+1-Tr)/Tb} K\left(\frac{[T(r+bz)] - [Tr]}{Tb}\right) g^2\left(\frac{[T(r+bz)]}{T}\right) dz + o(1) \\ &\rightarrow g^2(r-) \int_{-\infty}^0 K(z) dz + g^2(r+) \int_0^{\infty} K(z) dz, \end{aligned} \quad (19)$$

for interior points  $r \in (0, 1)$ . Combining (18) and (19) gives  $\hat{\sigma}_{[Tr]}^2 = ((1/Tb)\sum_{i=1}^T K_{[Tr]i} \sigma_i^2) + o_p(1) = (1/Tb)\sum_{i=1}^T K_{[Tr]i} \sigma_i^2 + o_p(1) \xrightarrow{p} g^2(r-) \int_{-\infty}^0 K(z) dz + g^2(r+) \int_0^{\infty} K(z) dz$  as claimed. Similarly, for  $r = 1$ ,  $\hat{\sigma}_T^2 \xrightarrow{p} g^2(1-) \int_{-\infty}^0 K(z) dz$ .

Now we prove (15). We follow closely the proof of the theorem in Robinson (1987) using some of his notation. First, note that  $\tilde{\beta}$  satisfies

$$\sqrt{T}(\tilde{\beta} - \beta) = \left( \frac{1}{T} \sum_{t=1}^T X_{t-1} X'_{t-1} \hat{\sigma}_t^{-2} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T X_{t-1} u_t \hat{\sigma}_t^{-2} \right).$$

Define  $a(f) = (1/\sqrt{T})\sum_{t=1}^T X_{t-1} u_t f_t^{-2}$  and  $A(f) = (1/T)\sum_{t=1}^T X_{t-1} X'_{t-1} f_t^{-2}$ , then we have  $\sqrt{T}(\beta^* - \beta) = A(\sigma)^{-1} a(\sigma)$  and  $\sqrt{T}(\tilde{\beta} - \beta) = A(\hat{\sigma})^{-1} a(\hat{\sigma}) = A(\sigma)^{-1} a(\sigma) + A(\hat{\sigma})^{-1} (a(\hat{\sigma}) - a(\sigma)) - A(\sigma)^{-1} (A(\hat{\sigma}) - A(\sigma)) A(\hat{\sigma})^{-1} a(\sigma)$ . We have  $A(\sigma) \xrightarrow{p} \Gamma$  which is positive definite, and  $a(\sigma) = O_p(1)$ , which follows from Markov's inequality and  $\mathbb{E}((1/\sqrt{T})\sum_{t=1}^T Y_{t-h} u_t \sigma_t^{-2})^2 = (1/T)\sum_{t=1}^T \sigma_t^{-4} \mathbb{E} Y_{t-h}^2 u_t^2 \leq C(1/T)\sum_{t=1}^T \mathbb{E} Y_{t-h}^2 u_t^2 < \infty$ , by Lemma A(b). Hence (15) follows if we prove

$$A(\hat{\sigma}) - A(\sigma) \xrightarrow{p} 0, \quad a(\hat{\sigma}) - a(\sigma) \xrightarrow{p} 0. \quad (20)$$

Define  $\tilde{\sigma}_t^2 = \sum_{i=1}^T w_{it} u_i^2$  and  $\bar{\sigma}_t^2 = \sum_{i=1}^T w_{it} \sigma_i^2$ , and (20) follows from the following six results as in Robinson (1987):

(a)  $a(\hat{\sigma}) - a(\tilde{\sigma}) \xrightarrow{p} 0$ ; (b)  $a(\tilde{\sigma}) - a(\bar{\sigma}) \xrightarrow{p} 0$ ; (c)  $a(\bar{\sigma}) - a(\sigma) \xrightarrow{p} 0$ ; (d)  $A(\hat{\sigma}) - A(\tilde{\sigma}) \xrightarrow{p} 0$ ; (e)  $A(\tilde{\sigma}) - A(\bar{\sigma}) \xrightarrow{p} 0$ ; (f)  $A(\bar{\sigma}) - A(\sigma) \xrightarrow{p} 0$ . These will be shown as follows:

(a) Since  $a(\hat{\sigma}) - a(\tilde{\sigma}) = (1/\sqrt{T})\sum_t X_{t-1} u_t (\hat{\sigma}_t^{-2} - \tilde{\sigma}_t^{-2}) / \hat{\sigma}_t^2 \tilde{\sigma}_t^2$ , we have  $|a(\hat{\sigma}) - a(\tilde{\sigma})| \leq (\min_t \tilde{\sigma}_t^2)^{-1} (\min_t \hat{\sigma}_t^2)^{-1} \cdot \sum_{t=1}^T \frac{|X_{t-1} u_t|}{\sqrt{T}} |\tilde{\sigma}_t^2 - \hat{\sigma}_t^2| \leq (\min_t \tilde{\sigma}_t^2)^{-1} (\min_t \hat{\sigma}_t^2)^{-1} ((1/T)\sum_{t=1}^T |X_{t-1} u_t|^2)^{1/2} (\sum_{t=1}^T |\tilde{\sigma}_t^2 - \hat{\sigma}_t^2|^2)^{1/2} = O_p(1/Tb) \xrightarrow{p} 0$ , by Lemma A (b, h, j, k).

(b) We write

$$\begin{aligned} a(\tilde{\sigma}) - a(\bar{\sigma}) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T X_{t-1} u_t (\tilde{\sigma}_t^{-2} - \bar{\sigma}_t^{-2}) \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^T X_{t-1} u_t (\bar{\sigma}_t^2 - \tilde{\sigma}_t^2) \bar{\sigma}_t^{-4} + \frac{1}{\sqrt{T}} \sum_{t=1}^T X_{t-1} u_t (\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)^2 \tilde{\sigma}_t^{-2} \bar{\sigma}_t^{-4}, \end{aligned} \quad (21)$$

which holds since for two any nonzero real numbers  $p$  and  $q$  we have the following equality  $p^{-1} - q^{-1} = (q - p)q^{-2} + (q - p)^2 p^{-1} q^{-2}$ . We will show the two terms of (21) vanishes in probability. For the first term, we

note that  $\{X_{t-1}u_t(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)\bar{\sigma}_t^{-4}, \mathcal{F}_t\}$  is an m.d. sequence. Indeed, we have

$$\begin{aligned} & \mathbb{E}(X_{t-1}u_t(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)\bar{\sigma}_t^{-4} | \mathcal{F}_{t-1}) \\ &= \bar{\sigma}_t^{-2} \mathbb{E}(X_{t-1}u_t | \mathcal{F}_{t-1}) - \bar{\sigma}_t^{-4} \mathbb{E}\left(X_{t-1}u_t \sum_{i<t} w_{ii}u_i^2 | \mathcal{F}_{t-1}\right) \\ & \quad - \bar{\sigma}_t^{-4} \mathbb{E}\left(X_{t-1}u_t \sum_{i>t} w_{ii}u_i^2 | \mathcal{F}_{t-1}\right). \end{aligned} \quad (22)$$

Both the last two terms are zero, since for the term  $i>t$ ,  $\mathbb{E}(X_{t-1}u_t u_i^2 | \mathcal{F}_{t-1}) = X_{t-1} \mathbb{E}(u_t u_i^2 | \mathcal{F}_{t-1}) = X_{t-1} \mathbb{E}(u_t \mathbb{E}(u_i^2 | \mathcal{F}_{i-1}) | \mathcal{F}_{t-1}) = X_{t-1} \mathbb{E}(u_t | \mathcal{F}_{t-1}) = 0$ , and for the term  $i<t$ ,  $\mathbb{E}(X_{t-1}u_t u_i^2 | \mathcal{F}_{t-1}) = X_{t-1} u_i^2 \cdot \mathbb{E}(u_t | \mathcal{F}_{t-1}) = 0$ . Thus, by (22)  $\mathbb{E}(X_{t-1}u_t(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)\bar{\sigma}_t^{-4} | \mathcal{F}_{t-1}) = 0$ . So the first term of (21) converges to zero in probability by the Markov inequality and  $\mathbb{E}|(1/\sqrt{T})\sum_{t=1}^T X_{t-1}u_t(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)\bar{\sigma}_t^{-4}|^2 \leq (C/T)\sum_{t=1}^T \mathbb{E}|X_{t-1}u_t|^2(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)^2 \leq (C/T)\sum_{t=1}^T (\mathbb{E}|X_{t-1}u_t|^4)^{1/2} \cdot (\mathbb{E}(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)^4)^{1/2} \leq (\max_t \mathbb{E}(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)^4)^{1/2} \cdot (C/T)\sum_{t=1}^T (\mathbb{E}|X_{t-1}u_t|^4)^{1/2} = O_p(1/Tb) \rightarrow^p 0$ , by Lemma A(a,f). For the second term of (21),  $|\sum_{t=1}^T (X_{t-1}u_t/\sqrt{T})(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)\bar{\sigma}_t^{-2}\bar{\sigma}_t^{-4}| \leq C((1/T)\sum_{t=1}^T |X_{t-1}u_t|^2)^{1/2}(\sum_{t=1}^T (\bar{\sigma}_t^2 - \tilde{\sigma}_t^2)^4)^{1/2} = O_p(1/T^{1/2}b) \xrightarrow{p} 0$ , by Lemma A(a,f). This completes the proof of (b).

(c) First we note

$$\sigma_t^2(\bar{\sigma}_t^{-2} - \sigma_t^{-2})^2 \leq \bar{\sigma}_t^{-4} \sigma_t^{-2} |\bar{\sigma}_t^2 + \sigma_t^2| \cdot |\bar{\sigma}_t^2 - \sigma_t^2| \leq C|\bar{\sigma}_t^2 - \sigma_t^2|. \quad (23)$$

Since  $\{X_{t-1}u_t\}$  is an m.d. sequence, we get  $\mathbb{E}|a(\bar{\sigma}) - a(\sigma)|^2 = (1/T)\sum_{t=1}^T \mathbb{E}(|X_{t-1}|^2 u_t^2)(\bar{\sigma}_t^{-2} - \sigma_t^{-2})^2 = (1/T)\sum_{t=1}^T \mathbb{E}(|X_{t-1}|^2 \mathbb{E}(u_t^2 | \mathcal{F}_{t-1}))(\bar{\sigma}_t^{-2} - \sigma_t^{-2})^2 = (1/T)\sum_{t=1}^T \mathbb{E}|X_{t-1}|^2 \sigma_t^2 |\bar{\sigma}_t^{-2} - \sigma_t^{-2}|^2 \leq (C/T)\sum_{t=1}^T \mathbb{E}|X_{t-1}|^2 \cdot |\bar{\sigma}_t^2 - \sigma_t^2| \leq C \max_t \mathbb{E}|X_{t-1}|^2 \cdot (1/T)\sum_{t=1}^T |\bar{\sigma}_t^2 - \sigma_t^2| = o_p(1)$ , by Lemma A(a, 1).

(d) It follows from  $|A(\hat{\sigma}) - A(\tilde{\sigma})| \leq (\min_t \hat{\sigma}_t^2)^{-1} (\min_t \tilde{\sigma}_t^2)^{-1} (1/T)\sum_{t=1}^T |X_{t-1}|^2 |\hat{\sigma}_t^2 - \tilde{\sigma}_t^2| \leq C \cdot \max_t |\hat{\sigma}_t^2 - \tilde{\sigma}_t^2| \cdot (1/T)\sum_{t=1}^T |X_{t-1}|^2 = O_p(1/\sqrt{Tb})$ , by Lemma A(a, h, i, j).

(e) This can be proved in the same way as (d) by employing Lemma A(g).

(f) It follows from  $|A(\bar{\sigma}) - A(\sigma)| \leq (\min_t \bar{\sigma}_t^2)^{-1} (\min_t \sigma_t^2)^{-1} (1/T)\sum_{t=1}^T |X_{t-1}|^2 |\bar{\sigma}_t^2 - \sigma_t^2| \leq (\min_t \bar{\sigma}_t^2)^{-1} \cdot (\min_t \sigma_t^2)^{-1} \cdot \max_t |X_{t-1}|^2 \cdot (1/T)\sum_{t=1}^T |\bar{\sigma}_t^2 - \sigma_t^2| = o_p(1)$ , by Lemma A(a, e, l).  $\square$

## Appendix B. Supplementary results and proofs

This section states and proves some results (Lemma A) used in the proofs of the theorems.

**Lemma A.** (a) If  $\sup_{1 \leq t \leq T} \mathbb{E}|\varepsilon_t|^{2\mu} < \infty$ ,  $1 \leq \mu < \infty$ , then  $\sup_{1 \leq t \leq T} \mathbb{E}|Y_{t-h}|^{2\mu} < \infty$  holds for  $1 \leq h \leq p$ ;

(b) if  $\sup_{1 \leq t \leq T} \mathbb{E}|\varepsilon_t|^{4\mu} < \infty$ ,  $1 \leq \mu < \infty$ , then  $\sup_{1 \leq t \leq T} \mathbb{E}|Y_{t-h}u_t|^{2\mu} < \infty$  holds for  $1 \leq h \leq p$ ;

(c) let  $t = [Tr]$  for any fixed  $r \in (0, 1]$ , then  $(1/Tb)\sum_{i=1}^T K_{ti} \rightarrow \int_{-\infty}^{\infty} K(z) dz = 1$ , where  $K_{ti}$  is defined in (13);

(d)  $\max_{t,i} w_{ii} = O(1/Tb)$ ;

(e)  $\min_{1 \leq t \leq T} \bar{\sigma}_t^2 \geq C > 0$ ;

(f)  $\max_{1 \leq t \leq T} \mathbb{E}|\bar{\sigma}_t^2 - \tilde{\sigma}_t^2|^4 = O(1/(Tb)^2)$ ;

(g)  $\max_t |\bar{\sigma}_t^2 - \tilde{\sigma}_t^2|^\delta = O_p(T^{-\delta/4} b^{-\delta/2})$ , for  $\delta = 1, 2$ ;

(h)  $(\min_{1 \leq t \leq T} \tilde{\sigma}_t^2)^{-1} = O_p(1)$ , as  $T \rightarrow \infty$ ;

(i)  $\max_{1 \leq t \leq T} |\bar{\sigma}_t^2 - \tilde{\sigma}_t^2| = O_p(1/\sqrt{Tb})$ ;

(j)  $(\min_{1 \leq t \leq T} \hat{\sigma}_t^2)^{-1} = O_p(1)$ , as  $T \rightarrow \infty$ ;

(k)  $\sum_{t=1}^T (\hat{\sigma}_t^2 - \tilde{\sigma}_t^2)^2 = O_p(1/(Tb)^2)$ ;

(l)  $(1/T)\sum_{t=1}^T |\bar{\sigma}_t^2 - \sigma_t^2| = o(1)$ .

**Proof of Lemma A.** (a) Note that  $Y_{t-h}^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \alpha_k \alpha_l u_{t-h-k} u_{t-h-l}$  and  $\mathbb{E}|u_{t-h-k} \cdot u_{t-h-l}|^\mu \leq (\mathbb{E}|u_{t-h-k}|^{2\mu} \mathbb{E}|u_{t-h-l}|^{2\mu})^{1/2} < \infty$ . So we have  $\mathbb{E}|Y_{t-h}|^{2\mu} = \|Y_{t-h}\|_\mu^{2\mu} \leq (\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\alpha_k \alpha_l| \cdot \|u_{t-h-k} u_{t-h-l}\|_\mu)^\mu \leq C (\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\alpha_k \alpha_l|)^\mu = C (\sum_{k=0}^{\infty} |\alpha_k|)^{2\mu} < \infty$ .

(b) Since  $Y_{t-h}^2 u_t^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \alpha_k \alpha_l u_{t-h-k} u_{t-h-l} u_t^2$  and  $\mathbb{E}|u_{t-h-k} u_{t-h-l} u_t^2|^\mu \leq (\mathbb{E}|u_{t-h-k}|^{4\mu} \cdot \mathbb{E}|u_{t-h-l}|^{4\mu})^{1/4} \cdot (\mathbb{E}|u_t|^{4\mu})^{1/2} < \infty$ , so  $\mathbb{E}|Y_{t-h} u_t|^{2\mu} = \|Y_{t-h} u_t\|_\mu^{2\mu} \leq (\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\alpha_k \alpha_l| \cdot \|u_{t-h-k} u_{t-h-l} u_t^2\|_\mu)^\mu \leq C \cdot (\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\alpha_k \alpha_l|)^\mu < \infty$ .

(c) Let  $t-i = [Tx]$ , where  $x$  is a real number,  $|x| < 1$ . Then  $(1/Tb) \sum_{i=1}^T K_{ti} = (1/Tb) \sum_{i=1}^T K((t-i)/Tb) + o(1) = \sum_{i=1}^T \int_{(t-i)/T}^{(t-i+1)/T} K([Tx]/Tb) d(x/b) + o(1) \stackrel{z=x/b}{=} \sum_{i=1}^T \int_{(t-i)/Tb}^{(t-i+1)/Tb} K([Tbz]/Tb) dz + o(1) = \int_{(t-T)/Tb}^{t/Tb} K([Tbz]/Tb) dz + o(1) \rightarrow \int_{-\infty}^{\infty} K(z) dz = 1$ .

(d) It follows from  $w_{ti} = ((1/Tb) \sum_{i=1}^T K_{ti})^{-1} K_{ti}/Tb$  and (c).

(e) It follows from  $\min_{1 \leq t \leq T} \bar{\sigma}_t^2 \geq \min_{1 \leq i \leq T} \sigma_i^2 \cdot (\sum_{i=1}^T w_{ti}) \geq \inf_{s \in [0,1]} g^2(s) \geq C > 0$ .

(f) We make use of the Burkholder's inequality (BI) (c.f. Shiryaev, 1995, p. 499): for the m.d. sequence  $\xi_1, \dots, \xi_T$  and  $p > 1$ , there exists constant  $A_p$  and  $B_p$ , such that

$$A_p \left\| \left( \sum_{t=1}^T \xi_t^2 \right)^{1/2} \right\|_p \leq \left\| \sum_{t=1}^T \xi_t \right\|_p \leq B_p \left\| \left( \sum_{t=1}^T \xi_t^2 \right)^{1/2} \right\|_p.$$

Let  $a_i = u_i^2 - \sigma_i^2$ , then  $a_i$  is a m.d. sequence and  $\mathbb{E}a_i^4 < \infty$ . Then  $\mathbb{E}(\tilde{\sigma}_t^2 - \bar{\sigma}_t^2)^4 = \mathbb{E}(\sum_{i=1}^T w_{ti} a_i)^4 \stackrel{BI(p=4)}{\leq} \mathbb{E}(\sum_{i=1}^T w_{ti}^2 a_i^2)^2 \stackrel{(d)}{\leq} (1/(Tb)^2) \mathbb{E}(\sum_{i=1}^T w_{ti} a_i^2)^2 \stackrel{\text{Jensen}}{\leq} 1/(Tb)^2 \sum_{i=1}^T w_{ti} \mathbb{E}a_i^4 = O(1/(Tb)^2)$ , where the last inequality is by Jensen's  $f(\sum_{i=1}^T w_{ti} a_i^2) \leq \sum_{i=1}^T w_{ti} f(a_i^2)$  with convex function  $f(x) = x^2$ .

(g) It holds since for arbitrary  $C > 0$ ,  $\mathbb{P}(\max_t |\tilde{\sigma}_t^2 - \bar{\sigma}_t^2|^\delta > CT^{-\delta/4} b^{-\delta/2}) \leq \sum_{t=1}^T \mathbb{P}(|\tilde{\sigma}_t^2 - \bar{\sigma}_t^2|^\delta > CT^{-\delta/4} b^{-\delta/2})$

$$\stackrel{\text{Markov}}{\leq} \begin{cases} C^{-4} T b^2 \sum_{t=1}^T \mathbb{E}|\tilde{\sigma}_t^2 - \bar{\sigma}_t^2|^4 \stackrel{(f)}{=} O(C^{-4}), & \delta = 1, \\ C^{-2} T b^2 \sum_{t=1}^T \mathbb{E}|\tilde{\sigma}_t^2 - \bar{\sigma}_t^2|^4 \stackrel{(f)}{=} O(C^{-2}), & \delta = 2. \end{cases}$$

(h) It follows from  $0 < C \stackrel{(e)}{\leq} \min_{1 \leq t \leq T} \bar{\sigma}_t^2 \leq \min_{1 \leq t \leq T} \tilde{\sigma}_t^2 + \max_t |\tilde{\sigma}_t^2 - \bar{\sigma}_t^2| = \min_{1 \leq t \leq T} \tilde{\sigma}_t^2 + o_p(1)$ .

(i) Note that  $\hat{\sigma}_t^2 - \tilde{\sigma}_t^2 = \sum_{i=1}^T w_{ti} (\hat{u}_i^2 - u_i^2) = \sum_{i=1}^T w_{ti} ((\hat{\beta} - \beta)' X_{i-1} X_{i-1}' (\hat{\beta} - \beta) - 2u_i X_{i-1}' (\hat{\beta} - \beta))$ , and  $\max_{t,i} \sum_{i=1}^T w_{ti}^2 \leq \max_{t,i} w_{ti} \cdot \sum_{i=1}^T w_{ti} = O(1/Tb)$ . We also have  $\hat{\beta} - \beta = O(T^{-1/2})$  by (5). Thus  $\max_{1 \leq t \leq T} |\hat{\sigma}_t^2 - \tilde{\sigma}_t^2| \leq \max_{1 \leq t \leq T} \sum_{i=1}^T w_{ti} |(\hat{\beta} - \beta)' X_{i-1} X_{i-1}' (\hat{\beta} - \beta) - 2u_i X_{i-1}' (\hat{\beta} - \beta)|$

$$\leq \max_{1 \leq t \leq T} \sum_{i=1}^T w_{ti} |\hat{\beta} - \beta|^2 |X_{i-1}|^2 + 2 \max_{1 \leq t \leq T} \sum_{i=1}^T w_{ti} |u_i X_{i-1}'| \cdot |\hat{\beta} - \beta|$$

$$\leq \max_{t,i} w_{ti} \cdot |\hat{\beta} - \beta|^2 \sum_{i=1}^T |X_{i-1}|^2 + 2|\hat{\beta} - \beta| \cdot \left( \max_{t,i} \sum_{i=1}^T w_{ti}^2 \right)^{1/2} \cdot \left( \sum_{i=1}^T |u_i X_{i-1}'|^2 \right)^{1/2}$$

$$= O_p\left(\frac{1}{Tb}\right) + O_p\left(\frac{1}{\sqrt{Tb}}\right) = O_p\left(\frac{1}{\sqrt{Tb}}\right).$$

(j) It follows from  $0 < C \stackrel{(h)}{\leq} \min_{1 \leq t \leq T} \tilde{\sigma}_t^2 \leq \min_{1 \leq t \leq T} \hat{\sigma}_t^2 + \max_t |\hat{\sigma}_t^2 - \tilde{\sigma}_t^2| = \min_{1 \leq t \leq T} \hat{\sigma}_t^2 + o_p(1)$ .

(k) Since  $\hat{\sigma}_i^2 - \tilde{\sigma}_i^2 = \sum_{i=1}^T w_{ii}(\hat{u}_i^2 - u_i^2) = (\hat{\beta} - \beta)'(\sum_{i=1}^T w_{ii}^2 X_{i-1} X'_{i-1})(\hat{\beta} - \beta) - 2(\sum_{i=1}^T w_{ii}^2 u_i X'_{i-1}) \cdot (\hat{\beta} - \beta)$ , then  $\sum_{i=1}^T (\hat{\sigma}_i^2 - \tilde{\sigma}_i^2)^2$  is bounded by

$$\begin{aligned} & \sum_{i=1}^T C \left( |\hat{\beta} - \beta|^4 \left| \sum_{i=1}^T w_{ii}^2 X_{i-1} X'_{i-1} \right|^2 + \left| \sum_{i=1}^T w_{ii}^2 u_i X'_{i-1} \right|^2 |\hat{\beta} - \beta|^2 \right) \\ & \leq |\hat{\beta} - \beta|^4 \sum_{i=1}^T C \left( \sum_{i=1}^T w_{ii}^2 |X_{i-1}|^2 \right)^2 + |\hat{\beta} - \beta|^2 \sum_{i=1}^T C \left( \sum_{i=1}^T w_{ii}^2 |u_i X'_{i-1}| \right)^2. \end{aligned} \quad (24)$$

The first term of (24) is bounded by

$$|\hat{\beta} - \beta|^4 \sum_{i=1}^T C \left( \sup_i |X_{i-1}|^2 \cdot \max_{t,i} w_{ti} \cdot \sum_{i=1}^T w_{ti} \right)^2 = O_p \left( \frac{1}{T^3 b^2} \right),$$

by (a) and (d), and similarly the second term of (24) is  $O_p(1/T^2 b^2)$ . So (k) follows.

(l) Let  $r_1 < r_2 < \dots < r_D$  be the discontinuous points of  $g(\cdot)$ , where  $D$  is finite. Then for sufficiently large  $T$ ,  $(1/T) \sum_{i=1}^T |\hat{\sigma}_i^2 - \sigma_i^2| = \sum_{i=1}^T \int_{i/T}^{(i+1)/T} |\hat{\sigma}_{[nr]}^2 - \sigma_{[nr]}^2| dr = \int_{1/T}^{r_1} |\hat{\sigma}_{[nr]}^2 - \sigma_{[nr]}^2| dr + \sum_{j=1}^{D-1} \int_{r_j}^{r_{j+1}} |\hat{\sigma}_{[nr]}^2 - \sigma_{[nr]}^2| dr + \int_{r_D}^{(T+1)/T} |\hat{\sigma}_{[nr]}^2 - \sigma_{[nr]}^2| dr \rightarrow 0$ , provided that

$$\hat{\sigma}_{[nr]}^2 \rightarrow g^2(r) \quad (25)$$

when  $g$  is continuous at  $r$ . Indeed, following the proof of (c) we can similarly have  $(1/Tb) \sum_{i=1}^T K_{ti} \sigma_i^2 \rightarrow g^2(r)$  when  $g$  is continuous at  $r$ . Thus (25) holds by (c).  $\square$

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