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Lottery Rather than Waiting-Line Auction

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Abstract This paper investigates the allocative efficiency of two non-price allocation mechanisms – the lottery (random allocation) and the waiting-line auction (queue system) – for the cases where consumers possess identical time costs (the homogeneous case), and where time costs are correlated with time valuations (the heterogeneous case). We show that the relative efficiency of the two mechanisms depends critically on a scarcity factor (measured by the ratio of the number of objects available for allocation over the number of participants) and on the shape of the distribution of valuations. We show that the lottery dominates the waiting-line auction for a wide range of situations, and that while consumer heterogeneity may improve the relative allocative efficiency of the waiting-line auction, the ranking on relative efficiency is not reversed.

1 Introduction

Governments often play a key role in the allocation of goods and services when prices are set below market clearing levels. Two commonly-used mechanisms are

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the lottery and the waiting-line auction (i.e. first-served queue system).[1] Lotteries have been used widely to allocate hunting permits, fishing berths, oil drill leases, or even admission to universities, while waiting line auctions have been used to allocate publicly-provided goods and services such as medical care services or subsidized public housing.

In selecting an allocation mechanism, one must consider its *equity* and *efficiency*. The *equity* of a mechanism is measured by the welfare impact of the allocation. The case for choosing the lottery to allocate goods and burdens (e.g. military draft) is frequently made on the grounds of horizontal equity, i.e. individuals who possess the same relevant characteristics should be treated equally [see Eckhoff (1989); Elster (1991); Goodwin (1992); Boyce (1994)]. The *efficiency* of a mechanism is measured by the degree of rent dissipation, due to *resource misallocation* and the incurrence of *rent seeking costs*. In a waiting-line auction, individuals who queued up earlier may be the ones with lower opportunity cost of time rather than the ones with higher valuations, while in the case of the lottery, individuals who value the objects most may not receive an allocation.[2] There are no rent-seeking costs in a lottery, but waiting in line creates both disutility and potential loss of income.

In this paper, we study the allocative efficiency of the lottery versus the waiting-line auction, when individuals possess identical time costs (the homogeneous case), and when time costs are correlated with time valuations (the heterogeneous case). For the homogeneous case, our results generalize those in Taylor et al. (2003).[3] There are two key findings in this paper. Firstly, relative efficiency is critically dependent on the shape of the distribution of time valuations and a scarcity factor (measured by the ratio of number of objects over number of participants); secondly, the lottery is almost always more efficient than the waiting-line auction unless there are very few objects to be allocated (i.e. high scarcity) and there are only a few participants possessing high values (i.e. the distribution of time valuations is L-shaped). For the heterogeneous case, we study a model where time costs and time valuations are correlated. When there is a positive (negative) correlation, the relative efficiency of the waiting-line auction improves (declines). However, the the ranking of allocative efficiency vis-a-vis the lottery is unchanged.[4]

The rest of the paper is organized as follows. Section 2 presents the basic model. Section 3 analyzes the relative allocative efficiency for the case of homogenous consumers under different distributional assumptions for time valuations. Section 4 extends the analysis to the case of heterogenous consumers. Section 5 concludes the paper.

2 The model

There are m identical and objects to be distributed freely to $n(>m)$ individuals, at most one object per person, using either a lottery or a waiting-line auction. The

[1] Another allocation mechanism is by merit, according to a set of pre-determined criteria.

[2] Studies on the economics of rationing and the queue system have been carried out by Tobin (1952); Nichols et al. (1971); Barzel (1974); Suen (1989).

[3] Using numerical analysis, they considered the local impact of mean-preserving dispersions in valuations on relative allocative efficiency.

[4] These results are related to the analysis in Sah (1987), which concluded that the sufficiently poor would always prefer non-convertible rations (i.e. a lottery) to queuing.

opportunity costs of time of the n individuals (measured by their wage rates) are denoted by $w_1, w_2, \ldots, w_n$, and their monetary valuations (measured in dollars) are denoted by $v_1, v_2, \ldots, v_n$. Thus, the ratio $y_i = v_i/w_i$ describes an individual's valuation of an object measured in time units. We refer to v_i as *monetary valuation* and y_i as *time valuation*. In our analysis, it is often more convenient to work with time valuations.

Individuals are risk neutral; they know their own valuations and time costs, but not those of others. Each individual has identical subjective beliefs about the possible monetary valuations and time costs of other individuals. Specifically, each individual believes that the monetary valuations and time costs for the $n-1$ rival claimants are independent realizations of a pair of continuous random variables $\{V, W\}$ having a joint distribution function $F(v, w)$ with support $[\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$, for some finite non-negative $\underline{v}$ and positive $\underline{w}$. The marginal distributions of V and W are denoted by $F_V(v)$ and $F_W(w)$, respectively. Similarly, the marginal distribution of Y is denoted by $F_Y(y)$.

The efficiency of an allocation mechanism is measured by the *expected social surplus*, defined as the sum of the expected payoffs for all n consumers.

2.1 Lottery

At a pre-specified time, m individuals are randomly chosen and allocated an object. The probability that the ith individual obtains an object is

$$H^R = \frac{m}{n}.$$

If the ith individual has a monetary valuation of v_i, his monetary payoff is

$$\pi^R(v_i) = v_i H^R = \frac{m v_i}{n}.$$

Given the symmetric treatment of all individuals, the expected social surplus is

$$S^R = nE[\pi^R(V)] = mE(V). \qquad (1)$$

Hence, the expected social surplus generated by a lottery depends only on the number of objects and the mean value of the distribution of monetary valuations.

2.2 Waiting-line auction

In a waiting-line auction, objects are allocated at a pre-specified time and location, on a first-come-first-served basis . Following Holt and Sherman (1982), we consider an individual's decision whether to join the queue, conditional on an expected waiting time. Each individual occupies only one position in the queue. Individuals who arrive after the mth person will be notified so that no unsuccessful persons will spend time in the queue.[5] The time taken to reach the queue is negligible compared with the waiting time.

[5] Holt and Sherman (1982) shows that if unsuccessful individuals also have to wait, individuals will optimally reduce their waiting time. In equilibrium, expected waiting time as well as payoff remains unchanged.

The equilibrium queuing time. For each individual i, there is an optimal queuing time $\tau(y_i)$, which is a strictly increasing function of time valuation y_i. Under the assumption that $\tau(y)$ is differentiable, $\tau(y)$ can be written as

$$\tau(y) = \frac{1}{H_Y^Q(y)} \int_{\underline{y}}^{y} x h_Y^Q(x)\mathrm{d}x = y - \frac{1}{H_Y^Q(y)} \int_{\underline{y}}^{y} H_Y^Q(x)\mathrm{d}x \qquad (2)$$

where $h_Y^Q(y)$ and $H_Y^Q(y)$ are, respectively, the density function and the distribution function of the mth largest order-statistic among the $n-1$ independent draws from the distribution of time valuations.[6] Denoting the marginal distribution of Y by $F_Y(y)$, we can verify that

$$H_Y^Q(y) = \sum_{k=n-m}^{n-1} \binom{n-1}{k} [F_Y(y)]^k [1 - F_Y(y)]^{n-k-1}.$$

It is straightforward to verify that the optimal waiting time $\tau(v_i)$ is a decreasing function in m and an increasing function in n. As shown in Holt and Sherman (1982), if the arrival time at the queue is chosen according to $\tau(y)$, expected payoff will be globally maximized. The probability that individual i will receive an object is simply $H_Y^Q(y_i)$.

The equilibrium expected payoff. The expected payoff, in time units is

$$\pi^Q(y_i) = (y_i - \tau(y_i))H^Q(y_i) = \int_{\underline{y}}^{y_i} H_Y^Q(x)\mathrm{d}x$$

for individual i. Multiplying $\pi^Q(y_i)$ by the time cost w_i yields the expected monetary payoff. The expected social surplus generated is

$$S^Q = nE\left(W \int_{\underline{y}}^{Y} H_Y^Q(x)\mathrm{d}x \right). \qquad (3)$$

Note that S^Q depends on the joint distribution of time valuation Y and time cost W. To compare allocative efficiency, we must specify the joint distribution of Y and W, in order to derive closed-form expressions for S^R and S^Q, and compute the ratio S^R/S^Q.

3 Efficiency comparison: homogeneous consumers

When individuals have identical time costs (i.e. $w_i = w_c, i = 1, \cdots, n$), the expected social surplus of a waiting-line auction can be expressed in terms of

[6] The second part of the equation follows from integration by parts.

$V = Yw_c$,

$$S^Q = nE\left(\int_{\underline{v}}^{V} H_V^Q(x)\mathrm{d}x\right),$$

where,

$$H_V^Q(v) = \sum_{k=n-k}^{n-1} \binom{n-1}{k} [F_V(v)]^k [1 - F_V(v)]^{n-k-1}$$

By switching the order of integrations and then the order of integration and summation,

$$\begin{aligned} S^Q &= n\int_{\underline{v}}^{\overline{v}} \left(\int_{\underline{v}}^{v} H_V^Q(x)\mathrm{d}x\right) f_V(v)\mathrm{d}v \\ &= n\int_{\underline{v}}^{\overline{v}} \left(\int_{x}^{\overline{v}} f_V(v)\mathrm{d}v\right) H_V^Q(x)\mathrm{d}x \\ &= n\int_{\underline{v}}^{\overline{v}} [1 - F_V(v)]H_V^Q(v)\mathrm{d}v \\ &= n\sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_{\underline{v}}^{\overline{v}} F_V(v)^k[1 - F_V(v)]^{n-k}\mathrm{d}v \end{aligned} \tag{4}$$

Thus, S^Q depends on the number of objects to be allocated m, the number of individuals n, and the distribution of monetary valuations $F_V(v)$.

When time costs are homogeneous, resource misallocation does not occur in a waiting-line auction; inefficiency results from the rent-seeking costs of waiting in line. As noted earlier, rent dissipation in a lottery is due solely to resource misallocation. Hence, the lottery is more (less) efficient than the waiting-line auction if resource misallocation in the lottery is smaller (larger) than the rent-seeking costs incurred in a waiting-line auction.

To analyze the relative efficiency of the two allocation mechanisms, four classes of distributions for V are considered: power function, Weibull, logistic and beta.[7] We summarize the technical results in the lemmas (with proofs provided in the Appendix) and focus our discussion on the corresponding Propositions.

[7] These four classes cover the broad range of distributional forms: L-shaped (the majority have very low valuations and a few have very high valuations), U-shaped (the majority have either very high or very low valuations), J-shaped (the majority have high valuations and a few have low valuations), flat and unimodal, (the majority have valuations in the middle, with a few having low or high valuations), etc.

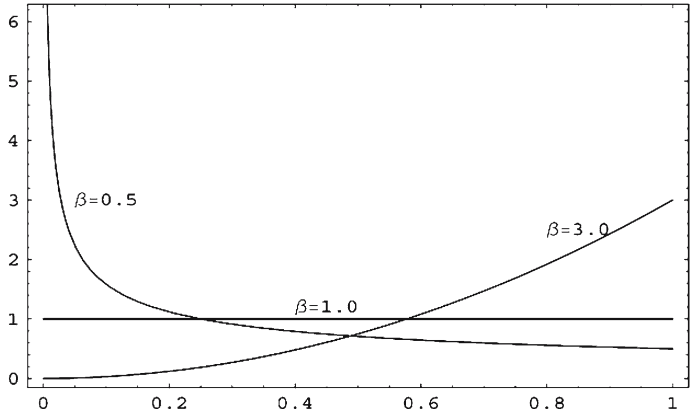

Fig. 1 Plots of pdf of power function distribution: $\theta = 1.0$

3.1 Monetary valuation with the power function distribution

The cumulative distribution function (cdf) takes the following form[8]:

$$F(v;\theta,\beta) = \left(\frac{v}{\theta}\right)^{\beta}, \quad 0 < v < \theta;\ \ \theta > 0,\ \ \beta > 0, \tag{5}$$

where β determines the *shape* of the distribution and θ controls the range or *scale* of the V values. The probability density function (pdf) is decreasing (L-shaped) when $\beta < 1$, constant (uniform) when $\beta = 1$, and increasing (J-shaped) when $\beta > 1$. Figure 1 illustrates these cases. The mean and variance of the distribution are $E(V) = \frac{\theta\beta}{\beta+1}$ and $Var(V) = \frac{\theta^2\beta}{(\beta+2)(\beta+1)^2}$, respectively.

Lemma 1 *If monetary valuations are drawn from the power function distribution, the expected social surplus functions are* $S^R = m\theta\beta/(\beta+1)$ *and* $S^Q = S^R h(\beta, n, m)$, *where*

$$h(\beta, n, m) = \frac{n}{m} - \frac{n!\,(\beta n + m + 1)\Gamma(n - m + \frac{1}{\beta})}{\beta m(n-m-1)!\Gamma(n+1+\frac{1}{\beta})}, \tag{6}$$

with $\Gamma(\cdot)$ *being the gamma function. Furthermore,* $h(\beta, n, m)$ *satisfies: (i) it is strictly increasing in m, decreasing in n, and decreasing in* β*; (ii)* $h(1, n, m) = \frac{m+1}{n+1}$*; (iii)* $h(\frac{1}{2}, n, m) = \frac{3mn+3n+2-2m^2}{(n+1)(n+2)}$*; (iv)* $h(\beta, n, 1) = \frac{n(1+\beta)}{(1+n\beta)(1+n\beta-\beta)}$*; and (v)* $\lim_{\beta\to\infty} h(\beta, n, m) = 0$.

Proposition 1 *Suppose monetary valuations are drawn from the power function distribution. For any given* θ*, m and n, the lottery is more efficient that the waiting-line auction when* $\beta \geq 1$*. The degree of relative efficiency, as measured by*

[8] Taylor et al. (2003) considered the power function distribution with $m = 1$, and the beta distribution with $\beta = 1$. As noted in Sect. 3.4, the latter is really a special case of the power function distribution.

$h(\beta, n, m)$ increases as β increases. When $\beta < 1$, the lottery is still more efficient than the waiting-line auction provided the ratio m/n is sufficiently small (For instance, we can show that $h(\frac{1}{2}, n, m) \leq 1$ if $m/n \leq \frac{1}{2}$).

Two factors influence $h(\beta, n, m)$: a *scarcity factor* and a *distributional shape factor*. This scarcity factor is measured by the ratio m/n. Since $h(\beta, n, m)$ and $\tau(v)$ have opposite signs with respect to changes in m/n, relative efficiency of the waiting-line auction improves (worsens) when the m/n ratio rises (falls). The distributional shape factor is measured by the parameter β in the power function distribution. When $\beta < 1$, the pdf is L-shaped, so that an individual in the waiting-line auction is more likely to face mostly competitors with low valuations. As β increases, the distribution of V shifts to the right; the likelihood of facing competitors with higher valuations increases, leading to an increase in optimal waiting time $\tau(v)$. This causes $h(\beta, n, m)$ to fall in value, so that the relative efficiency of the lottery over the waiting-line auction improves.

Proposition 1 indicates that when $\beta \geq 1$, the lottery is always more efficient than the waiting-line auction, regardless of the m/n ratio. For the case where $\beta < 1$, when the expected social surplus is smaller and the rent-seeking costs (i.e. waiting times) are lower, the waiting-line auction may be more efficient, if resource misallocation incurred under the lottery is greater. Thus, when $\beta < 1$, the scarcity factor m/n is critical in determining the relative efficiency of the two mechanisms. As stated in Proposition 1, when β drops to $\frac{1}{2}$, the waiting-line auction will dominate the lottery if m/n is greater than $\frac{1}{2}$. Otherwise, the lottery remains the more efficient allocation mechanism. In general, if the change in the rent-seeking costs is larger than the change in the expected surplus, the allocative efficiency of the waiting-line auction deteriorates relative to the lottery.

3.2 Monetary valuation with the Weibull distribution

The Weibull distribution models valuation distributions that are (i) extremely positively-skewed, (ii) unimodal and positively-skewed, and (iii) nearly symmetric. The cumulative distribution function of the Weibull distribution takes the form

$$F(v; \theta, \beta) = 1 - \exp\left(-\left(\frac{v}{\theta}\right)^{\beta}\right), \quad v > 0;\ \theta > 0,\ \beta > 0, \tag{7}$$

where β is the shape parameter and θ is the scale parameter. When $\beta < 1$, the density function is a decreasing function in v. When $\beta > 1$, the density function is unimodal with a longer tail to the right.[9] When $\beta = 1$, the Weibull distribution is an exponential distribution. The mean and variance of a Weibull random variable are, respectively, $E(V) = \theta\Gamma(1+\frac{1}{\beta})$ and $\mathrm{Var}(V) = \theta^2[\Gamma(1+\frac{2}{\beta}) - [\Gamma(1+\frac{1}{\beta})]^2]$. Figure 2 illustrates the density function of the Weibull distribution.

Lemma 2 *If monetary valuations are drawn from the Weibull distribution, the expected social surplus functions are given by $S^R = m\theta\Gamma(1 + 1/\beta)$ and $S^Q =$*

[9] It can be shown that when $\beta = 3.768$, the density function of the Weibull distribution is very similar to that of a normal distribution [See Hernandez and Johnson (1980)].

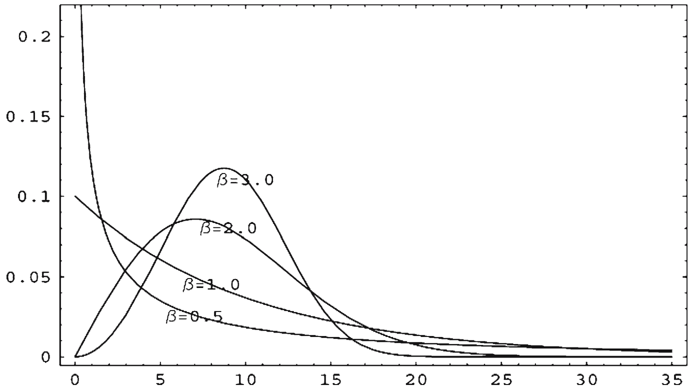

Fig. 2 Plots of pdf of Weibull distribution: $\theta = 10.0$

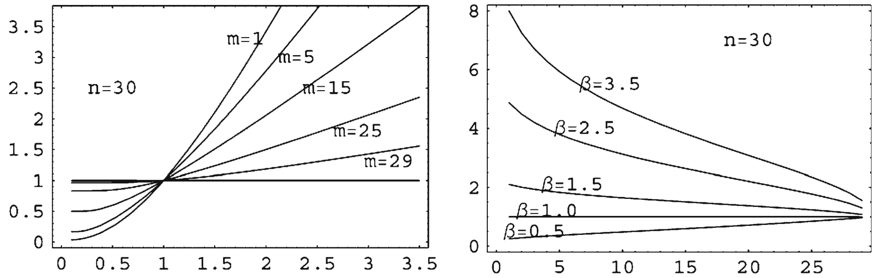

Fig. 3 Plots of S^R/S^Q versus β (left), and versus m for Weibull distribution

$S^R h(\beta, n, m)$, where,

$$h(\beta, n, m) = \frac{n}{m} \sum_{n-m}^{n-1} \sum_{j=0}^{k} \binom{n-1}{k} \binom{k}{j} (-1)^j \left(\frac{1}{n-k+j} \right)^{1/\beta}. \tag{8}$$

Furthermore, $h(\beta, n, m)$ is decreasing in β and $h(1, n, m) = 1$.

Proposition 2 *Suppose monetary valuations are drawn from the Weibull distribution. The lottery is more efficient than the the waiting-line auction if $\beta > 1$; the waiting-line dominates the lottery if $\beta < 1$; the two allocation mechanisms are equally efficient when $\beta = 1$, which represents the case where monetary valuations follow an exponential distribution.*

The above results indicate that when valuations are drawn from a Weibull distribution, the relative efficiency of the lottery over the waiting-line auction depends only on the shape parameter β. The magnitude of relative efficiency still depends on the scarcity factor m/n. We plot the relative efficiency of the lottery over the waiting-line auction (i.e., $S^R/S^Q = 1/h(\beta, n, m)$) in Fig. 3. The numerical results show that when the scarcity factor $m/n \approx 1$, the difference in allocative efficiency is small, so that it does not really matter which allocation mechanism is chosen.

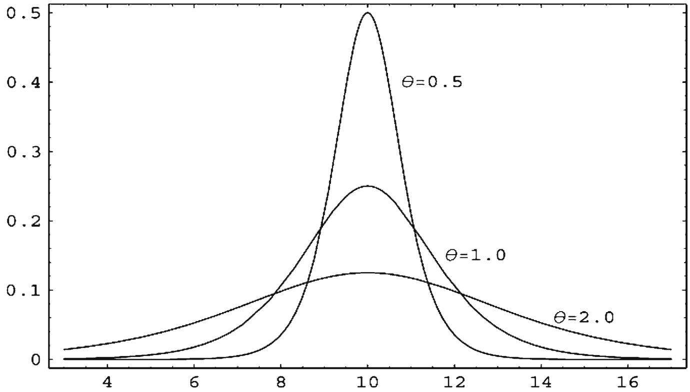

Fig. 4 Plots of pdf of logistic distribution: $\mu = 10.0$

3.3 Monetary valuation with the logistic distribution

In a variety of economic settings, the distribution of valuations are best modelled as symmetric and unimodal. The logistic distribution function serves this purpose.[10] The cumulative distribution function of the logistic distribution has the following form

$$F(v; \mu, \theta) = 1 - \frac{1}{1 + \exp[(v - \mu)/\theta]},$$
$$-\infty < v < \infty; \quad -\infty < \mu < \infty, \ \theta > 0. \tag{9}$$

Note that the logistic distribution is symmetric around the mean $E(V) = \mu$ with variance $\text{Var}(V) = \frac{1}{3}\pi^2\theta^2$. Note that μ is a location parameter and θ is the scale parameter. The larger the value of θ, the flatter is the pdf. A few plots of the logistic pdf are provided in Fig. 4 for illustrative purpose.[11]

Lemma 3 *If monetary valuations are drawn from the logistic distribution, the expected social surplus functions are*

$$S^R = m\mu \ \text{ and } S^Q = n\theta[\Psi(n) - \Psi(n - m)]$$

where $\Psi(\cdot)$ is the digamma function defined as $\Psi(z) = \mathrm{d}\log\Gamma(z)/\mathrm{d}z$. Taking $\mu/\theta = 10$,

$$\frac{S^Q}{S^R} = \frac{n}{10m}[\Psi(n) - \Psi(n - m)],$$

[10] We chose the class of logistic distributions over the class normal distributions for our analysis, as the latter class of distributions does not allow us to derive closed-form expressions for the expected surplus functions. With suitable choice of parameters, the logistic distribution may approximate a normal distribution.

[11] Note that the logistic distribution may assume negative values, which has no economic meaning for the problem at hand, since an individual with a negative monetary valuation will not choose to participate in either the lottery or the waiting-line auction. However, we can make the probability of negative values negligible by having a large mean-to-scale ratio, i.e., $\mu/\theta \geq 10$. The probability of negative values is $F(0, \mu, \theta) = 1 - 1/[1 + \exp(-\mu/\theta)]$. When $\mu/\theta \geq 10$, we have $F(0, \mu\theta) \leq 4.5 \times 10^{-5}$.

which is an increasing function of m. We can show that $\max_m(S^Q/S^R) < 1$ *for* $n < 10,000$.

Proposition 3 *If monetary valuations are drawn from the logistic distribution with a negligible probability of negative valuations (*$\mu/\theta \geq 10$*), the lottery almost always dominates the waiting-line auction.*

The above result is particularly striking as it indicates that when time costs are homogeneous and valuations can be modeled as a symmetric distribution, the optimal allocation mechanism is almost always a lottery, regardless of the number of objects to be allocated or the number of participants.

3.4 Monetary valuation with the beta distribution

The probability density function (pdf) of the beta distribution has the following form

$$F(v;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1}, \quad 0 \leq v \leq 1, \;\; \alpha > 0, \;\; \beta > 0. \quad (10)$$

In terms of the potential shapes of the density function, the beta distribution is the richest family of distributions. It is U-shaped if $\alpha < 1$ and $\beta < 1$, uniform if $\alpha = 1$ and $\beta = 1$, L-shaped if $\alpha < 1$ and $\beta > 1$, J-shaped if $\alpha > 1$ and $\beta < 1$, and unimodal, otherwise. Furthermore, when $\beta = 1$, the beta distribution becomes a special case of the power function distribution. The mean and variance of the beta distribution are $E(V) = \frac{\alpha}{\alpha+\beta}$ and $\text{Var}(V) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, respectively. We provide plots of the beta pdf in Fig. 5.

There are no closed-form functions of S^R and S^Q, since the cdf of the beta distribution does not have a closed-form expression. We consider two special cases - when $\alpha = 1$ and when $\beta = 1$ - which allow us to obtain closed-form expression of the ratio S^R/S^Q, as well as compute the values of the ratio S^R/S^Q [using the general expression given in (4)] for a range of parameter configurations of β, n and m. These results are presented in Fig. 6.

When $\beta = 1$, the cdf is a power function distribution; hence, the Lemma 1 applies. We can infer that when $\beta = 1$, the lottery is more efficient than the waiting-line auction if $\alpha \geq 1$, and if $\alpha < 1$ it continues to dominate the waiting-line auction provided that m/n is sufficiently small. Similarly, for $\beta > 1$, the lottery remains the more efficient allocation mechanism. For the second special case when $\alpha = 1$, we first prove the following lemma.

Lemma 4 *If monetary valuations are drawn from the beta distribution and* $\alpha = 1$*, the expected social surplus functions are* $S^R = m/(1+\beta)$ *and* $S^Q = S^R h(\beta,n,m)$*, where*

$$h(\beta,n,m) = \frac{n!\Gamma(m+1+1/\beta)}{m!\Gamma(n+1+1/\beta)}. \quad (11)$$

Furthermore, $h(\beta,n,m)$ *is increasing in* β*, and* $\lim_{\beta\to\infty} h(\beta,n,m) = 1$.

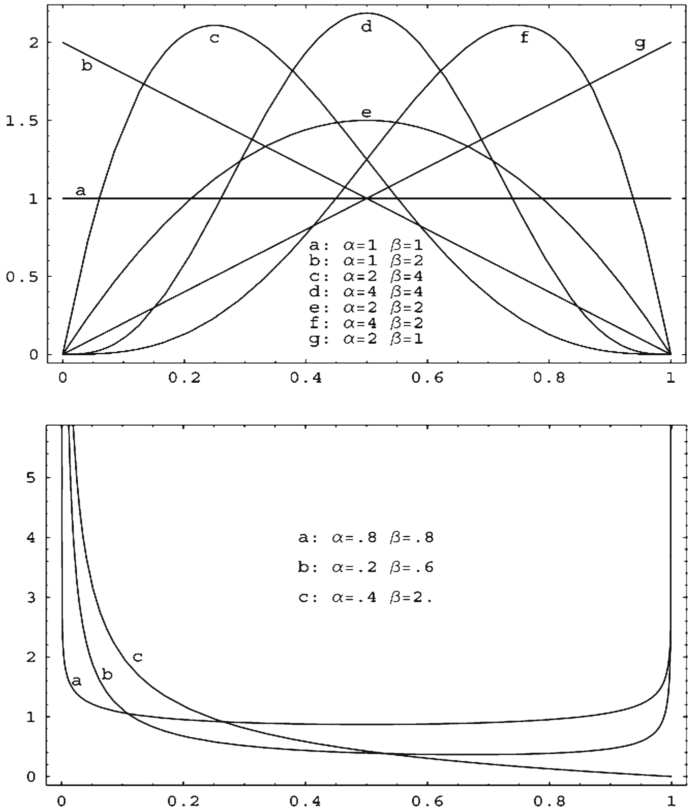

Fig. 5 Plots of pdf of beta distribution

Hence, when $\alpha = 1$,the lottery is more efficient than the the waiting-line auction regardless of the value of β or m/n. This conclusion can be generalized to the case when $\alpha > 1$. The intuition here is that, as α increases above 1, the weight of the pdf shifts to the right. In turn, this means a greater likelihood that individuals participating in the waiting-line auction possess higher valuations. Thus, competition intensifies and each individual will raise the optimal waiting time. Rent-seeking costs are higher; as a result, the $h(\beta, n, m)$ function decreases in value as α increases (see Fig. 6). Combining Lemmas 1 and 4, we can conclude that

Proposition 4 *If monetary valuations are drawn from the beta distribution, the lottery is more efficient than the waiting-line auction, except possibly when* $\alpha < 1$, $\beta \geq 1$, *i.e., when the beta distribution is L-shaped, and* m/n *is sufficiently large.*

4 Efficiency comparison: heterogeneous consumers

In Sect. 3, we analyzed the case where time costs are identical. When time costs vary significantly, it is more appropriate to consider a joint distribution for time

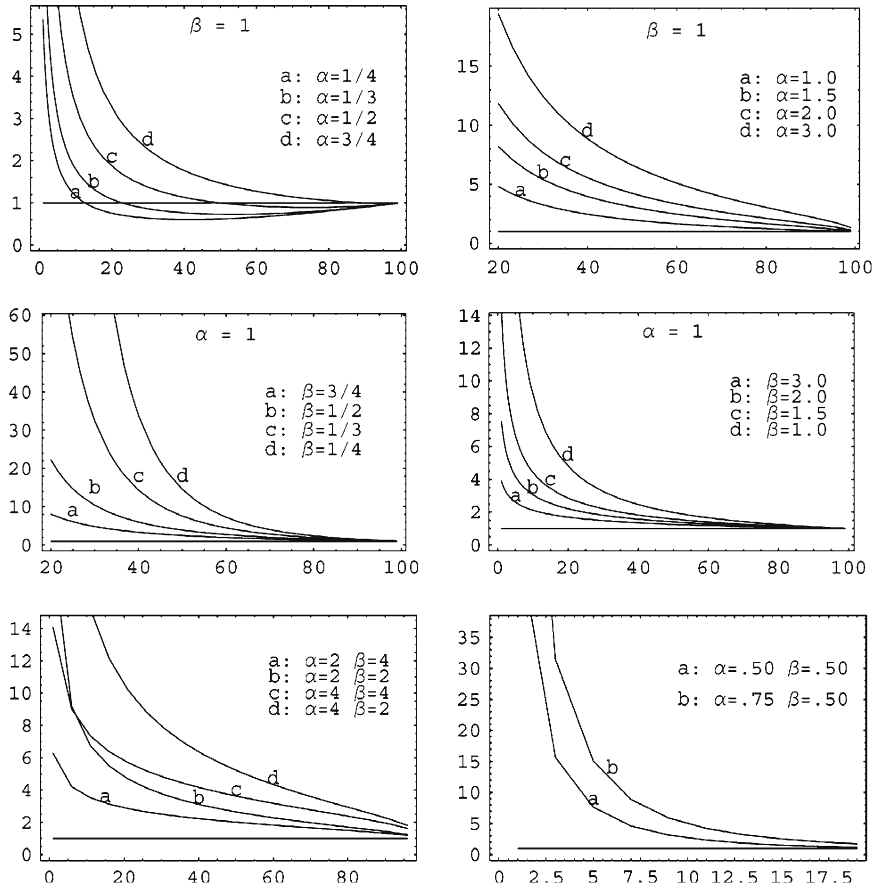

Fig. 6 Plots of S^R/S^Q versus m for beta distribution, $n = 100$ for the first five plots and $n = 20$ for the last one. The last two plots are based on a few selected points

valuations Y and time costs W (or, equivalently, a joint distribution for monetary valuation V and time cost W). When time costs and time valuations are correlated, it is clear that rent dissipation in the waiting-line auction includes potential resource misallocation as well. The results for the following special cases are straightforward: (i) V and W are independent and (ii) Y and W are independent. Using the general expression derived in (3), some simple conditioning arguments show that when W is independent of V, all the results of Lemmas 1–4 go through. Again, using (3), a direct manipulation shows that the S^R/S^Q function remains the same if W is no longer constant but is still independent of Y. Therefore, the heterogeneity in time costs does not affect our results on the relative efficiency of the lottery over the waiting-line auction if time costs are independent of monetary valuations or time valuations.

For the general case, we present a model of joint distribution between time valuations and time costs to analyze the impact of heterogeneity on the relative allocative efficiency.[12] Our analysis suggests that the relative efficiency of the waiting-line auction in fact improves when there is positive correlation in time costs and monetary valuations, and deteriorates when the correlation is negative. When the correlation is zero, the relative efficiency is not affected by heterogeneity in time costs.

4.1 Positively correlated time valuation and time cost

Consider the case where Y and W are uniformly distributed on the area $A = \{(y, w) : [0 \leq y \leq 1], [0 \leq w \leq \beta y^{\beta-1}]\}$. The joint pdf of Y and W is

$$f(y, w) = 1, \quad 0 \leq y \leq 1, \; 0 \leq w \leq \beta y^{\beta-1}; \; \beta \geq 1. \tag{12}$$

It is easy to verify that $f_Y(y) = \beta y^{\beta-1}, 0 \leq y \leq 1$, i.e., the marginal distribution of Y is the power function distribution with the scale parameter equal to 1, and $f_W(w) = 1 - (w/\beta)^{\frac{1}{\beta-1}}, 0 \leq w \leq \beta$. The correlation coefficient between Y and W is

$$\rho(Y, W) = \frac{\beta - 1}{2\beta}\sqrt{\frac{(\beta+2)(9\beta-6)}{7\beta^2 - 2\beta + 4}}.$$

For $\beta = 1, 2, 4, \infty$, we have $\rho(Y, W) = 0, 0.32, 0.48$, and 0.57, respectively. This shows that Y and W are uncorrelated when $\beta = 1$, and that the correlation increases as β increases (with an upper limit of 0.57).

Lemma 5 *When time valuation Y and time cost W are drawn jointly from the distribution specified in* (12), *the expected social surplus functions are $S^R = \frac{1}{4}m\beta$, and $S^Q = S^R h(\beta, n, m)$, where,*

$$h(\beta, n, m) = \frac{2\beta}{2\beta - 1}\left(\frac{n}{m} - \frac{1}{\beta} + \frac{m+1}{2\beta(n+1)} - \frac{n!\Gamma(n - m + \frac{1}{\beta})}{m(n-m-1)!\Gamma(n + \frac{1}{\beta})}\right). \tag{13}$$

Furthermore, $h(\beta, n, m)$ is decreasing in β, $h(1, n, m) = \frac{m+1}{n+1}$ and $\lim_{\beta\to\infty} h(\beta, n, m) = 0$.

A comparison of the h functions, defined in Lemmas 1 and 5 allows us to determine the effect of positive correlation of time costs and valuations on allocative efficiency. When $\beta = 1$, both h functions have the same value. This implies that when time valuations and time costs are uncorrelated, relative efficiency is unchanged. Therefore, the results of Proposition 1 continues to hold even if consumers are heterogeneous in their time costs. When β tends to ∞, both h functions are trivially identical, as both h functions tend to zero.

[12] From a modeling point of view, we could consider either a joint distribution of time valuations and time costs or a joint distribution of monetary valuations and time costs. As our motivation here is to demonstrate the impact of heterogeneity on relative efficiency, we have chosen the current specification for its tractability.

More importantly, as β increases – causing the marginal distribution of time valuations $f_Y(y)$ to become more negatively skewed – the waiting-line auction is less efficient, regardless of the degree of correlation between Y and W. This result follows directly from Lemma 5, which states that $h(\beta, n, m) \leq 1$ when $\beta = 1$ and is decreasing in β. Furthermore, it is straightforward to verify that the relative efficiency of the lottery increases with β, and decreases with m/n. These results indicate that the allocative efficiency of the lottery improves when time costs are positively correlated with time valuations.

4.2 Negatively correlated time valuation and time cost

To analyze the impact of negative correlation on relative efficiency, consider the following specification of time cost: $W^* = \beta - W$.[13] Then,

$$\rho(Y, W*) = -\rho(Y, W) = -\frac{\beta - 1}{2\beta}\sqrt{\frac{(\beta + 2)(9\beta - 6)}{7\beta^2 - 2\beta + 4}},$$

i.e., the time valuation Y and time cost W^* are negatively correlated.

Lemma 6 *If Y and W follow the joint distribution specified in* (12), *the expected social surplus functions under the time valuation Y and time cost* $W^* = \beta - W$ *are* $S^R = \frac{1}{4}m\beta$, *and* $S^Q = S^R h(\beta, n, m)$, *where,*

$$h(\beta, n, m) = \frac{4\beta}{3\beta - 1} h_1(\beta, n, m) - \frac{\beta + 1}{3\beta - 1} h_2(\beta, n, m) \tag{14}$$

with h_1 *being the h function defined in Lemma 1 and* h_2 *the h function defined in Lemma 5. Furthermore,* $h(\beta, n, m)$ *is decreasing in* β, $h(1, n, m) = \frac{m+1}{n+1}$ *and* $\lim_{\beta\to\infty} h(\beta, n, m) = 0$.

Together, Lemmas 5 and 6 allow us to compare allocative efficiency under the alternative scenarios of positive and negative correlations of time costs and time valuations. We provide graphical analysis, in Fig. 7.

In Fig. 7, we have plotted the three h functions (defined in Lemmas 1, 5 and 6), against m, for a given value of β and for $n = 50$. The plots in Fig. 7 show that when Y and W are positively correlated, the relative efficiency of the waiting-line auction over the lottery is higher than in the case when Y and W are un-correlated. The converse is true when Y and W are negatively correlated.[14]

---

[13] This specification is chosen for its tractability and does not affect the generality of the results presented here.

[14] Note that since monetary valuation is a product of time valuation and time cost, $V = YW$, a positive correlation between Y and W implies a positive correlation between V and W. Similarly, it is straightforward to show that V and W^* are negatively correlated.

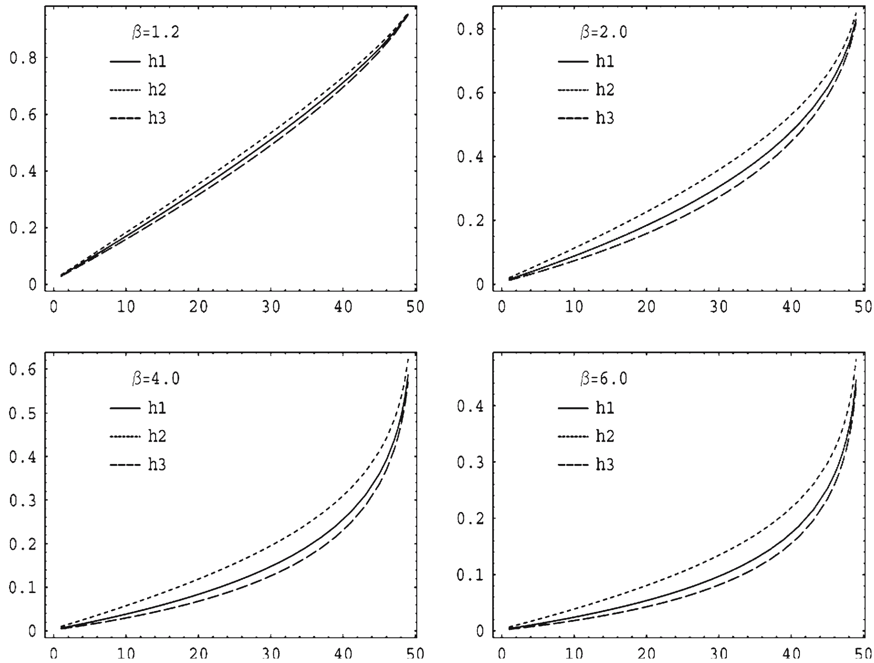

Fig. 7 Plots of the three h functions defined in Lemmas 1, 5 and 6: $n = 50$

Note that even though the allocative efficiency of the waiting-line auction improves in the case of positive correlation between time costs and time valuations, the lottery remains the more efficient mechanism (as noted in the discussion following Lemma 5) in the model that we present here. We summarize our findings below.

Proposition 5 *If time valuations Y and time costs W follow the joint distribution specified in* (12), *the allocative efficiency of the waiting-line auction improves (deteriorates) if Y and W are more positively (negatively) correlated, compared with the case when they are uncorrelated.*

A common perception and argument against the use of the waiting-line auction is that less wealthy with lower monetary valuations are also likely to have lower time-costs. The assertion is that these individuals are likely to join the queue earlier in a waiting-line auction, so that the objects are not necessarily allocated to those who might possess higher monetary valuations. Hence, when time valuations Y and time costs W are positively correlated, the allocative efficiency of the waiting-line auction is lower than when time costs are uncorrelated (or negatively correlated) with time valuations.

Proposition 5 indicates that, contrary to conventional wisdom, the efficiency of the waiting-line auction in fact improves when time cost is positively correlated with monetary valuation. In this case, the gain in efficiency here more than offsets the costs of waiting in line. In the case of negative correlation, the decline in the efficiency of the waiting line auction is particularly easy to understand in the following example. Suppose all consumers have the same monetary valuations of the good. Then time costs and time valuations are negatively correlated. In this case, since everyone values the good identically, so that the queuing costs have no offsetting benefit in terms of allocative efficiency.

5 Conclusion

By comparing the expected social surplus functions of the two allocation mechanisms, we are able to delineate the circumstances under which a lottery is more efficient than the waiting-line auction, and vice versa. Our analysis suggests that when time costs are homogeneous, random allocation is the optimal mechanism in a wide range of circumstances (Propositions 1 to 4). We show that the relative efficiency of the waiting-line auction improves when there is positive correlation between time costs and time valuations, but deteriorates when the correlation is negative (Proposition 5).

Our results indicate that besides its *equity* appeal, the lottery is also the *efficient* non-price allocation mechanism in a wide variety of situations. Although heterogeneity in time costs may improve the relative efficiency of the waiting-line auction when there there is positive correlation with time valuations, our study shows that the improvement is not likely to be significant enough to reverse the efficiency ranking in most situations, unless the marginal distribution of time valuation is extremely positively skewed (i.e. L-shaped).

While the analysis in this paper is *positive* and does not address the issue of equity, we hope the results presented here will contribute to a better understanding among policy-makers on the choice of the appropriate non-price allocation mechanism. Clearly, if more weight is assigned to the welfare (expected payoffs) of a particular group of individuals, the relative desirability of two allocation schemes may not follow the ranking based on allocative efficiency. Specifically, it is conceivable that if it is desirable that the allocation favors, say, the lower-income group, and time costs and positively correlated with time valuation, then a waiting-line auction may be the preferred allocation scheme. However, a potentially better allocation scheme may be to segregate the n participants into two or more groups, based on income, and conduct separate lotteries for the different groups. Relatively more of the m objects could be alloted to the lower-income group. This meets the objective of favoring the lower-income group, without incurring rent-seeking costs of waiting in line.

Appendix: Proofs of the Lemmas

The proofs of Lemmas 1–4 are basically the derivations of S^Q give in (4) in terms of money valuation V. That is,

$$S^Q = n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_{\underline{v}}^{\overline{v}} F_V(v)^k [1 - F_V(v)]^{n-k} \mathrm{d}v.$$

Proof of Lemma 1 With the power function distribution, we have $F_V(v) = (v/\theta)^\beta$, and

$$S^R = mE(V) = \frac{m\theta\beta}{1+\beta}$$

$$S^Q = n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\int_0^\theta [(v/\theta)^\beta)]^k[1-(v/\theta)^\beta]^{n-k}\mathrm{d}v$$

$$= n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\frac{\theta}{\beta}\int_0^1 u^{k+\frac{1}{\beta}-1}(1-u)^{n-k}\mathrm{d}u,\ \text{(by letting } u = (v/\theta)^\beta)$$

$$= n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\frac{\theta}{\beta}\frac{\Gamma(k+\frac{1}{\beta})\Gamma(n-k+1)}{\Gamma(n+1+\frac{1}{\beta})}$$

$$= \frac{\theta}{\beta}\frac{\Gamma(n+1)}{\Gamma(n+1+\frac{1}{\beta})}\sum_{k=n-m}^{n-1}\frac{\Gamma(k+\frac{1}{\beta})}{k!}(n-k)$$

$$= \frac{\theta}{\beta}\frac{\Gamma(n+1)}{\Gamma(n+1+\frac{1}{\beta})}\left(\frac{\beta^2\Gamma(n+1+\frac{1}{\beta})}{(1+\beta)\Gamma(n)} - \frac{\beta(\beta n+m+1)\Gamma(n-m+\frac{1}{\beta})}{(1+\beta)\Gamma(n-m)}\right)$$

$$= \frac{m\theta\beta}{1+\beta}\left(\frac{n}{m} - \frac{n!(\beta n+m+1)\Gamma(n-m+\frac{1}{\beta})}{\beta m(n-m-1)!\Gamma(n+1+\frac{1}{\beta})}\right).$$

It is straightforward to verify the properties of the h function stated in the Lemma.

Proof of Lemma 2 With the Weibull distribution, we have $F_V(v) = 1 - \exp[-(v/\theta)^\beta]$, and

$$S^R = mE(V) = m\theta\Gamma\left(1+\frac{1}{\beta}\right)$$

$$S^Q = n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\int_0^\infty [1-\exp(-(v/\theta)^\beta)]^k[\exp(-v/\theta)^\beta]^{n-k}\mathrm{d}v.$$

Making a change of variable $u = (v/\theta)^\beta$, and then applying the binominal expansion to $[1-\exp(-u)]^k$, the integral in the summation for S^Q becomes

$$\int_0^\infty [1-\exp(-(v/\theta)^\beta)]^k[\exp(-v/\theta)^\beta]^{n-k}\mathrm{d}v$$

$$= \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}[1-\exp(-u)]^k[\exp(-u)]^{n-k}\mathrm{d}u$$

$$= \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}\left\{\sum_{j=0}^{k}\binom{k}{j}(-1)^j\exp(-ju)\right\}[\exp(-(n-k)u)]\mathrm{d}u$$

$$= \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}\left\{\sum_{j=0}^{k}\binom{k}{j}(-1)^j\exp(-(n-k+j)u)\right\}\mathrm{d}u$$

$$= \frac{\theta}{\beta}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\int_0^\infty u^{1/\beta-1}\exp[-(n-k+j)u]\mathrm{d}u$$

$$= \frac{\theta}{\beta}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\Gamma(1/\beta)\left(\frac{1}{n-k+j}\right)^{1/\beta}.$$

Substituting this back into the expression for S^Q, we have

$$S^Q = n\frac{\theta}{\beta}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\Gamma(1/\beta)\left(\frac{1}{n-k+j}\right)^{1/\beta}$$

$$= n\theta\Gamma(1+1/\beta)\sum_{k=n-m}^{n-1}\sum_{j=0}^{k}\binom{n-1}{k}\binom{k}{j}(-1)^j\left(\frac{1}{n-k+j}\right)^{1/\beta}$$

$$= m\theta\Gamma(1+1/\beta)h(\beta,n,m).$$

Since $1/(n-k+j)\leq 1$ with equality occurring only when $k = n-1$, and $j = 0$, the terms in the summation of $h(\beta,n,m)$ are thus either constant or increasing in β. Hence h is an increasing function of β. Finally,

$$h(1,n,m) = \frac{n}{m}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{1}{n-k+j}\right)$$

$$= \frac{n}{m}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\frac{k!}{(n-k)(n-k+1)\cdots(n-1)n}$$

$$= \frac{n}{m}\sum_{k=n-m}^{n-1}\frac{1}{n} = 1.$$

Note that the first summation is handled by a combinatory formula

$$\sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{a+j} = \frac{k!}{a(a+1)\cdots(a+k)}, \quad \text{for } a \neq 0, -1, -2, \ldots, -k.$$

Proof of Lemma 3 With the logistic distribution, we have

$$S^R = mE(V) = m\mu$$

$$\int_{-\infty}^{\infty} F_V(v)^k [1 - F_V(v)]^{n-k} \mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \left[1 - \frac{1}{1+\exp((v-\mu)/\theta)}\right]^k \left[\frac{1}{1+\exp((v-\mu)/\theta)}\right]^{n-k} \mathrm{d}v.$$

Letting $w = \{1 + \exp[(v-\mu)/\theta]\}^{-1}$, the above integral becomes

$$\int_0^1 (1-w)^k w^{n-k} \frac{\theta}{w(1-w)} \mathrm{d}w$$

$$= \theta \int_0^1 (1-w)^{k-1} w^{n-k-1} \mathrm{d}w$$

$$= \theta \frac{\Gamma(k)\Gamma(n-k)}{\Gamma(n)}$$

This gives

$$S^Q = n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \frac{\theta\Gamma(k)\Gamma(n-k)}{\Gamma(n)}$$

$$= n\theta \sum_{k=n-m}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} \frac{(k-1)!(n-k-1)!}{(n-)!}$$

$$= n\theta \sum_{k=n-m}^{n-1} \frac{1}{k} = n\theta[\Psi(n) - \Psi(n-m)].$$

Proof of Lemma 4 When $\alpha = 1$, the beta distribution becomes $F_V(v) = 1 - (1-v)^\beta$. This gives

$$
\begin{aligned}
S^R &= mE(V) = \frac{m}{1+\beta} \\
S^Q &= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_0^1 [1-(1-v)^\beta]^k (1-v)^{\beta(n-k)} \mathrm{d}v \\
&= n \sum_{k=n-m}^{n-1} \binom{n-1}{k} \frac{1}{\beta} \frac{\Gamma(n-k+\frac{1}{\beta})}{\Gamma(n+1+\frac{1}{\beta})} \\
&= \frac{n!}{\beta\Gamma(n+1+\frac{1}{\beta})} \sum_{k=n-m}^{n-1} \frac{\Gamma(n-k+\frac{1}{\beta})}{\Gamma(n-k)} \\
&= \frac{n!}{\beta\Gamma(n+1+\frac{1}{\beta})} \frac{\beta\Gamma(m+1+\frac{1}{\beta})}{(1+\beta)\Gamma(m)}, \quad \text{(by Mahthematica)} \\
&= \frac{m}{1+\beta} \frac{n!\Gamma(m+1+\frac{1}{\beta})}{m!\Gamma(n+1+\frac{1}{\beta})}
\end{aligned}
$$

The rest of the proof is straightforward.

Proof of Lemma 5

$$
\begin{aligned}
S^Q &= nE\left(W \int_0^Y H_Y^Q(x)\mathrm{d}x \right) \\
&= n \int_0^1 \int_0^{\beta y^{\beta-1}} w \left(\int_0^y H_Y^Q(x)\mathrm{d}x \right) f(y,w)\mathrm{d}w\mathrm{d}y \\
&= n \int_0^1 \left(\int_0^y H_Y^Q(x)\mathrm{d}x \right) \frac{1}{2}\beta^2 y^{2(\beta-1)}\mathrm{d}y \\
&= \frac{n\beta^2}{2} \int_0^1 \left(\int_x^1 y^{2(\beta-1)}\mathrm{d}y \right) H_Y^Q(x)\mathrm{d}x \\
&= \frac{n\beta^2}{2(2\beta-1)} \int_0^1 (1-x^{2\beta-1}) H_Y^Q(x)\mathrm{d}x \\
&= \frac{n\beta^2}{2(2\beta-1)} \sum_{k=n-m}^{n-1} \binom{n-1}{k}
\end{aligned}
$$

$$\times \int_0^1 (1 - x^{2\beta-1})(x^\beta)^k (1 - x^\beta)^{n-k-1} \mathrm{d}x, \quad (\text{letting } u = x^\beta)$$

$$= \frac{n\beta}{2(2\beta - 1)} \sum_{k=n-m}^{n-1} \binom{n-1}{k}$$

$$\times \left(\int_0^1 u^{k+\frac{1}{\beta}-1(1-u)^{n-k-1}} - \int_0^1 u^{k+1}(1-u)^{n-k-1} \right) \mathrm{d}u$$

$$= \frac{n\beta}{2(2\beta - 1)} \sum_{k=n-m}^{n-1} \binom{n-1}{k} \left(\frac{\Gamma(k+\frac{1}{\beta})\Gamma(n-k)}{\Gamma(n+\frac{1}{\beta})} - \frac{\Gamma(k+2)\Gamma(n-k)}{\Gamma(n+2)} \right)$$

$$= \frac{\beta}{2(2\beta - 1)} \sum_{k=n-m}^{n-1} \left(\frac{\Gamma(n+1)\Gamma(k+\frac{1}{\beta})}{\Gamma(n+\frac{1}{\beta})\Gamma(k+1)} - \frac{k+1}{n+1} \right)$$

$$= \frac{\beta}{2(2\beta - 1)} \left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{\beta})} \left(\frac{\beta\Gamma(n+\frac{1}{\beta})}{\Gamma(n)} - \frac{\beta\Gamma(n-m+\frac{1}{\beta})}{\Gamma(n-m)} \right) \right.$$

$$\left. - \frac{m(n+1) - \frac{1}{2}m(m+1)}{n+1} \right)$$

$$= \left(\frac{m\beta}{4} \right) \frac{2\beta}{2\beta - 1} \left(\frac{n}{m} - \frac{1}{\beta} + \frac{m+1}{2\beta(n+1)} - \frac{n!\Gamma(n-m+\frac{1}{\beta})}{m(n-m-1)!\Gamma(n+\frac{1}{\beta})} \right) \#$$

It is then straightforward to verify the properties of the h function stated in the Lemma.

Proof of Lemma 6

$$S^Q = nE \left(W^* \int_0^Y H_Y^Q(x)\mathrm{d}x \right) = nE \left((\beta - W) \int_0^Y H_Y^Q(x)\mathrm{d}x \right)$$

$$= n\beta E \left(\int_0^Y H_Y^Q(x)\mathrm{d}x \right) - nE \left(W \int_0^Y H_Y^Q(x)\mathrm{d}x \right)$$

The first part can be obtained from Lemma 1 and the second part can be obtained from Lemma 5.

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