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Extreme Events and the Copula Pricing of Commercial Mortgage-Backed Securities

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Abstract

Commercial mortgage-backed securities (CMBS), as a portfolio-based financial product, have gained great popularity in financial markets. This paper extends Childs, Ott and Riddiough's (1996, JFQA) model by proposing a copula-based methodology for pricing CMBS bonds. Default on underlying commercial mortgages within a pool is a crucial risk associated with CMBS transactions. Two important issues associated with such default—extreme events and default dependencies among the mortgages—have been identified to play crucial roles in determining credit risk in the pooled commercial mortgage portfolios. This article pays particular attention to these two issues in pricing CMBS bonds. Our results show the usefulness and potential of copula-based models in pricing CMBS bonds, and the ability of such models to correctly price CMBS tranches of different seniorities. It is also important to sufficiently consider complex default dependence structure and the likelihood of extreme events occurring in pricing various CMBS bonds.

Key words: CMBS, copula model, extreme events, heavy tail, tail dependence

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1. Introduction

Commercial mortgage backed securities (CMBS) have become a major financing and investment instrument in the U.S. The U.S. CMBS market had grown to about $500 billion outstanding bonds by 2005. CMBS are typical portfolio-based financial instruments, whose payoffs are contingent on the default realization of the pooled underlying commercial mortgages. As a result, default on the underlying mortgages is a critical risk exposure to CMBS investors and a major consideration in the pricing of CMBS in real estate literature. Two important issues associated with such default—extreme events and default dependencies among pooled commercial mortgages—have been identified to play crucial roles in determining the pooled mortgage portfolios' credit risk exposure, especially for the CMBS transactions backed by a large number of mortgage loans. [1] In this paper, our contribution is to examine the pricing of CMBS bonds with particular focus on these two issues. Specifically, we introduce and employ a copula-based model framework, which will be shown to be a powerful method of analyzing simultaneous defaults of underlying mortgages and the joint default dependencies among them.

Previous studies on CMBS pricing in the real estate literature usually utilized a contingent-claims approach to the valuation of CMBS bonds.[2] In this approach, the log-normal distribution of underlying asset value or equivalently normal distribution of its continuously compounded returns is a crucial assumption, as in the seminal papers of

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[1] Childs, Ott and Riddiough (1996) examine the effect of default dependencies among pooled commercial mortgages on the pricing of multiclass CMBS bonds by considering the correlation structure among underlying commercial properties in the pool. Their numerical results demonstrate that the correlation structure is an important determinant of required yield spreads for multiclass CMBS bonds. Fan, Sing, and Ong (2008) allow for a default contagion function in examining the impact of default clustering of pooled commercial mortgages on CMBS prices. Their findings support the critical role of default dependence structure among pooled commercial mortgages in the pricing of CMBS bonds. On the other hand, the effect of extreme events on the credit quality of CMBS bonds has also attracted considerable attention as a result of Hurricane Katrina in 2005 (Bach, *et al.*, 2006).

[2] An exception is Fan, Sing, and Ong (2008), who use an intensity-based model to price CMBS bonds.

Black and Scholes's option pricing theory (1973) and Markowitz's portfolio optimization theory (1952). Under the assumption of geometric Brownian property value processes, the default on commercial mortgages is determined based on the stochastic evolution of the property value relative to the default threshold level [see Titman and Torous (1989) and Childs, Ott and Riddiough (1996)]. However, empirical results have challenged the assumption of lognormal diffusion property price processes. Based on the method of McCulloch (1986) and employing the U.S. Russell-NCREIF data base, Young and Graff (1995) demonstrated that the annual returns of individual properties are not normally distributed during the period from 1978 to 1992. In particular, they found that the sample data exhibited a "heavy-tailed" distribution [See also similar evidence in Graff, Harrington and Young (1997, 1999) and Young (2007)]. These results have important implications for the specification of CMBS pricing models.

First, the lognormal distribution of underlying commercial property value, though analytically convenient, may be misspecified, since the returns of real estate, in effect, deviate from normality. The assumption that real estate returns are normally distributed actually neglects the probable effect of their higher moments such as kurtosis on default dependency of pooled commercial mortgages, while the empirical results above implies the importance of examining these effects.

Secondly, specification of the linear correlation between any two underlying commercial properties is insufficient to capture default dependence structure of the pooled mortgages, whose adequate modeling has been identified as one of the most important and pressing issues in the pricing of credit risk in CMBS bonds [see Fan, Sing, and Ong (2008)]. Childs, Ott and Riddiough (1996, henceforth COR) provides a good attempt to look at the default dependence structure among pooled commercial mortgages by considering the linear correlation coefficients of the underlying commercial properties. However, recent studies such as Embrechts, McNeil and Straumann (1999, 2002) have demonstrated that the linear correlation coefficient is not an adequate measure of the dependency between any two assets and can only capture their linear dependence, and a sufficient consideration of the full joint multivariate distribution of asset values is indispensable in modeling default dependency of pooled loans.

Thirdly, Hurricane Katrina in 2005 has drawn considerable attention to the effect of extreme events on the credit quality of CMBS bonds, and several independent credit rating agencies like Moody's and Fitch Ratings have discussed the likely effect of the particular event on CMBS transactions [see Bach, *et al*. (2006)]. Using the standard contingent claim approach, however, can lead to an underestimation of the effects of extreme events on CMBS credit quality, because the multivariate normal distribution does not exhibit tail dependence. Tail dependence in more recent research has become known as a powerful measure of the dependency between the occurrences of extreme observations of underlying random variables [see, e.g., Malevergne and Sornette (2006, Chapter 4)]. Given that extreme events are likely to result in significant illiquidity in real estate markets and even a sharp decline in real estate values, it is important to allow for the likelihood of extreme events occurring via property value tail dependence in modeling the default dependence structure of pooled commercial mortgages.

In order to overcome the above difficulties in pricing CMBS bonds, we introduce a novel copula-based framework to the CMBS pricing issue. Copulas are becoming popular as a promising tool in the research and practice of credit risk valuation associated with multi-dimensional distributions of asset portfolios, because they can provide a complete description of the dependence structure among the pooled assets rather than use the simple normalized covariation. The copula-based methodology has been utilized recently for pricing Collateral Debts Obligations (CDOs).[3]

Copula functions are a category of appealing mathematical tools to describe the complex dependence structure among multiple random variables, which can therefore be used to model default dependency among the underlying loans in a portfolio and its credit risk in extreme events (Li, 2000). CMBS are an important class of portfolio financial

[3] For a good review on copula models for pricing CDOs, see Elizalde (2005).

instruments with pooled commercial mortgages as collateral. This article will show the usefulness and potential of copula models in pricing CMBS bonds, in particular allowing for complex default dependence structure among pooled commercial mortgages and the credit risk of CMBS bonds in extreme events. Two recent developments in copula models are the incorporation of the conception of both factor models and conditional independence into the pricing of credit risk in portfolio-based financial instruments [see Saunders, et al. (2007)]. Under the conception of factor models, we assume that the value of a commercial property is driven by a set of fundamental random variables common to all other commercial properties in addition to a property-specific factor. CMBS transactions are often backed by a mortgage pool with a large number of loans, each of whose values is likely to be very small relative to the whole pool. The specification of a factor model can reduce significantly the dimensional problem in the analysis of default risk associated with the large portfolio, and therefore make the pricing of CMBS bonds backed by a large mortgage pool tractable mathematically. On the other hand, given the fact that default dependency of pooled mortgage loans are to a large extent driven by some common fundamental factors, the concept of conditional independence, in effect, implies that since the default times of pooled mortgage loans are usually determined by these factors, conditional on the factors these default times can be treated to be independent. Such specification further allows the modeling of the joint default of pooled commercial mortgages to be tractable in utilizing a semi-analytical approach or Monte Carlo simulation.

Extending the contingent-claims model of Childs, Ott and Riddiough (1996), this article explores the CMBS pricing issue using the copula-based methodology. Our numerical results show that the default dependence structure among pooled commercial mortgages significantly affects required yield spreads for various CMBS bonds. Given that the real default dependence structure of pooled commercial mortgages is usually unknown, it is therefore important to allow for copula-based pricing models, which can provide a complete description of the dependence structure. Moreover, the effect of possible extreme events on CMBS yield spreads is also noticeable and implies the importance of considering property value tail dependence in modeling the joint default of multiple commercial mortgages. The remainder of this paper is organized as follows. Section 2 introduces the basic model framework for pricing CMBS bonds. Section 3 briefly presents several important copula functions. Section 4 extends the copula-based model to price multiclass CMBS bonds. Section 5 applies a semi-analytic approach to pricing CMBS bonds. Section 6 draws relevant conclusions.

2. Default Models

Since the seminal work of Merton (1974), there has been a large body of literature that prices credit risk associated with defaultable bonds based on the classical option pricing theory developed by Black and Scholes (1973). Such category of credit risk models are usually known as contingent-claims or structural-based models, where a firm's default is determined by the stochastic evolution of its asset value relative to a pre-specified default threshold. [4] Childs, Ott and Riddiough (1996) provided an excellent framework for evaluating credit risk associated with CMBS bonds using the contingent-claims approach. Their model takes account of an underlying mortgage pool consisting of $N \geq 1$ commercial mortgages, each of which is guaranteed by a commercial property. The value of any commercial property i is assumed to evolve following a log-normal diffusion process

$$dV_i = \mu_i V_i dt + \sigma_i V_i dZ_i \tag{1}$$

where μ_i is the expected return on property i, σ_i is the volatility of property i returns, and Z_i is a Wiener process. To further highlight the key feature of the COR model, we neglect the stochastic evolution of interest rates and solve this stochastic differential equation (SDE) under the risk neutral probability measure as

$$V_i(t) = V_i(0)\exp\left[\left(r - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W\sqrt{t}\right] \tag{2}$$

where r is the constant risk-free interest rate, W is a normally distributed random variable with zero mean and unit standard deviation, and $V_i(t)$ and $V_i(0)$ are the values of property i at times t and 0, respectively.

[4] See Sing, Ong, Fan and Lim (2005) for a good survey on structural-based pricing models of credit risks.

Let V_i^* be the default threshold value,[5] and suppose that default on commercial mortgage *i* occurs when its underlying property value falls below this threshold level. This default condition can be expressed as

$$V_i\left(0\right)\exp\left[\left(r-\frac{1}{2}\sigma_i^2\right)t+\sigma_i W\sqrt{t}\right]<V_i^*. \tag{3}$$

As a result, the default probability of mortgage *i* at time *t* is

$$\begin{aligned}
&\Pr\left[V_i\left(0\right)\exp\left[\left(r-\frac{1}{2}\sigma_i^2\right)t+\sigma_i W\sqrt{t}\right]<V_i^*\right]\\
&=\Pr\left[W<-\frac{\ln\left(V_i\left(0\right)/V_i^*\right)+\left(r-\frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}\right]\\
&=G\left(-\frac{\ln\left(V_i\left(0\right)/V_i^*\right)+\left(r-\frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}\right),
\end{aligned} \tag{4}$$

where $G\left(\cdot\right)$ is the cumulative normal distribution function. Expression (4) provides a foundation implied in the COR model for determining the default probability of individual commercial mortgages.

Extending the contingent-claims model by incorporating the conception of factor models, we specify that the value of underlying commercial property *i* evolves according to a one-factor model

$$V_i\left(t\right)=\sqrt{\rho}M+\sqrt{1-\rho}\varepsilon_i, \tag{5}$$

[5] In structural-based models, default occurs if the firm's asset value drops below a default threshold, which can be determined both exogenously and endogenously. These models were first introduced by Black and Cox (1976), in which the default threshold is exogenously specified. Similarly, Longstaff and Schwartz (1995) incorporate an exogenously determined default threshold into their structural-based model for pricing risky debt. Following their wisdom, copula-based models usually assume that the default threshold values are given exogenously, because a more general specification for the default threshold does not provide additional insight into the pricing of portfolio-based bonds in our model. This is slightly different from Childs, Ott and Riddiough (1996), where default boundary values can be determined endogenously and found using the traditional backward pricing equation approach.

where ρ is a nonnegative constant with $-1 \le \rho \le 1$, M is a random variable representing a common market factor, and for $i = 1,...,N$, ε_i represent the idiosyncratic factors that are independent and identically distributed (i.i.d.) and independent of M, and both M and ε_i have a zero mean and unit variance[see Vasicek (1991), Schönbucher (2000) and Saunders, et al. (2007)]. Such a specification implies that any two underlying commercial properties are correlated with a correlation coefficient ρ.[6] [7]

Let F_{V_i}, F_{ε_i} and F_M represent the cumulative distribution functions of the random variables V_i, ε_i and M , respectively. Suppose that default on commercial mortgage i occurs if $V_i(t) < V_i^*$, where V_i^* still represents the default threshold level for mortgage i. Thus, given the cumulative distribution function F_{V_i} of commercial property i and its mortgage default probability $p_i(t)$ over the time period (0, t), we may obtain

$$V_i^* = F_{V_i}^{-1}\left(p_i(t)\right) \tag{6}$$

where $F_{V_i}^{-1}$ is the inverse of distribution function F_{V_i}. Conditional on the realization of the common market factor $M = m$, substituting (5) into the default condition of mortgage i gives

$$\varepsilon_i < \frac{\left(V_i^* - \sqrt{\rho}m\right)}{\sqrt{1-\rho}}. \tag{7}$$

Accordingly, the conditional default probability of this mortgage can be expressed as

$$\Pr\left(\tau_i < t \middle| M = m\right) = F_{\varepsilon_i}\left(\frac{V_i^* - \sqrt{\rho}m}{\sqrt{1-\rho}}\right), \tag{8}$$

where τ_i is the time until default for mortgage i. Substituting (6) into (8) produces

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[6] Alternatively, we can specify the one-factor model as $V_i(t) = \rho_i M + \sqrt{1-\rho_i^2}\varepsilon_i$, where ρ_i represents the sensitivity of V_i to M. This specification implies that the correlation between any two underlying assets V_i and V_j is $\rho_i\rho_j$ instead of ρ in our model specified above [see, e.g., Guegan and Houdain (2007)].

[7] This study uses property correlation coefficients as dependence parameters to examine the effect of default dependencies among pooled commercial mortgages on the probability of their joint default in a large portfolio.

$$\Pr\left(\tau_i < t \middle| M = m\right) = F_{\varepsilon_i}\left(\frac{F_{V_i}^{-1}\left(p_i\left(t\right)\right) - \sqrt{\rho} m}{\sqrt{1-\rho}}\right). \tag{9}$$

More generally, equation (5) can be extended as a multi-factor model:

$$V_i\left(t\right) = \sqrt{\rho_1} M_1 + \sqrt{\rho_2} M_2 + \cdots \sqrt{\rho_K} M_K + \sqrt{1-\rho_1-\rho_2-\cdots-\rho_K}\,\varepsilon_i \tag{10}$$

where M_k, $k = 1,...,K$, represent K independent common factors, ε_i is the idiosyncratic factor of commercial mortgage i, ρ_k are the corresponding nonnegative constants, and M_k and ε_i are independent of each other.[8] It can be shown that the correlation between any two commercial properties *i* and *j* are

$$\mathrm{Corr}\left(V_i, V_j\right) = \begin{cases} 1, \text{ for } i = j \\ \rho_1 + \rho_2 + \cdots + \rho_K, \text{ for } i \neq j. \end{cases}$$

Given the realizations of $M_1 = m_1, M_2 = m_2, \cdots, M_K = m_K$, the conditional default probability of underlying mortgage *i* can be written as [9]

$$\Pr\left(\tau_i < t \middle| M_1 = m_1, M_2 = m_2, \cdots, M_K = m_K\right) = F_{Z_i}\left(\frac{V_i^* - \sqrt{\rho_1} m_1 - \sqrt{\rho_2} m_2 - \cdots - \sqrt{\rho_K} m_K}{\sqrt{1-\rho_1-\rho_2-\cdots-\rho_K}}\right).$$

3. Copulas

Copulas are functions that reveal the relationship of the marginal (individual) distribution functions of individual random variables with their multivariate distribution function. Gaussian and Student-*t* copulas are two of the most widely used copula models in the

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[8] Although all the relevant results below are developed using equation (5), similar results can be easily found using the more general equation (10). We keep to the parsimonious case to highlight the contributions in this paper without causing unnecessary complications that throw no additional insights.

[9] Alternatively, the multi-factor model can be specified as $V_i\left(t\right) = \rho_{i1} M_1 + \rho_{i2} M_2 + ... + \rho_{iK} M_K + \omega_i \varepsilon_i$ or $V_i\left(t\right) = \rho_{i1} M_1 + \rho_{i2} M_2 + ... + \rho_{iK} M_K + \sqrt{1-\rho_{i1}^2-\rho_{i2}^2-...-\rho_{iK}^2}\,\varepsilon_i$, where ρ_{ik} represents the sensitivity of V_i to the *k*th factor *M* and ω_i represents the sensitivity of V_i to ε_i. This implies that the correlation between any two underlying properties V_i and V_j is $Corr(V_i, V_j) = \rho_{i1}\rho_{j1} + \rho_{i2}\rho_{j2} + ... + \rho_{iK}\rho_{jK}$ instead of $\rho_1 + \rho_2 + ... \rho_K$ in our model specified above [see, e.g., Hull and White (2004) and Guegan and Houdain (2007)].

relevant literature [see Hull (2004, 2005) and Burtschell, Gregory and Laurent (2005)].[10] The correlation coefficients in Gaussian and Student-*t* copulas have apparent economic implication, and can be interpreted as dependence on common market or sectoral factors. Among them, the Student-*t* copula has attracted more recent attention in modeling the dependent structure implicit in multivariate financial data. In particular, Coleman and Mansour (2005) showed that the distribution characteristics of real estate data, such as significant heavy tail, can be better captured using the Student-*t* distribution.[11]

3.1 Elliptical Models

In a Gaussian model, the value of any underlying commercial property is assumed to follow a normal distribution. Suppose that both M and ε_i are independent Gaussian random variables. Consequently, conditional on the realization of the common factor M, the default probability of mortgage i can be obtained from equation (9)

$$\Pr\left(\tau_i < t \middle| M = m\right) = G\left(\frac{G^{-1}\left(p_i\left(t\right)\right) - \sqrt{\rho}m}{\sqrt{1-\rho}}\right), \tag{11}$$

where $G(\cdot)$ is the cumulative normal distribution function.

In contrast, we can also assume that the value of any underlying commercial property evolves in a Student-*t* distribution. The Student-*t* distribution is the quotient of a normal random variable and the square root of a Chi-Square random variable scaled by its degree of freedom. Let $X_i = \sqrt{\rho}M + \sqrt{1-\rho}\varepsilon_i$, where both M and ε_i are independent Gaussian random variables. Consequently, we have the following Student-*t* distribution with v degrees of freedom

$$V_i = \frac{X_i}{\sqrt{z/v}} \sim T_v, \tag{12}$$

[10] See Appendix 1 for a brief review on the theory of copula function. For a comprehensive review on this theory, see Nelsen, R. (1999) and Cherubini, Luciano and Vecchiato (2004).

[11] Coleman and Mansour (2005) also apply the noncentral Student-*t* distribution for considering significant skewness. The standard Student-*t* is a symmetric distribution with heavy tails, but it can be generalized as the noncentral Student-*t* distribution for measuring asymmetric behavior by introducing a noncentrality parameter of controlling the degree of skewness.

where X_i is a normally distributed random variable with zero mean and unit variance, and z is a Chi-square random variable with v degrees of freedom. This is a symmetric Student-*t* distribution, which has a bell shape curve similar to that of the normal distribution, but with a heavier tail. Since the idiosyncratic factors satisfy the normal distribution, the conditional default probability can be expressed as

$$\Pr\left(\tau_i < t \middle| M = m\right) = G\left(\frac{T_v^{-1}\left(p_i\left(t\right)\right)\sqrt{z/v} - \sqrt{\rho}m}{\sqrt{1-\rho}}\right), \tag{13}$$

where $T_v(\cdot)$ is the cumulative Student-*t* distribution function. [12]

3.2 Copula Functions

To examine the default dependency of pooled commercial mortgages, consider a CMBS backed by N equally weighted underlying mortgages, and let the random variables τ_i, $i = 1...N$, represent their default times. For simplicity of analysis, suppose that conditional on the common market factor, the default probabilities of the individual underlying mortgages are independent of one another. We also make the assumption that recovery rate is constant. This may be further relaxed, but does not provide additional insights into the models. The default probability for the *i*th underlying mortgage is then defined as:

$$p_i\left(t_i\right) = \Pr\left(\tau_i \le t_i\right) = 1 - \exp\left(-\int_0^{t_i} h_i\left(u\right)du\right),$$

where 0 denotes the issue date of the CMBS bonds, t_i represents the maturity date of underlying mortgage i and $h_i\left(u\right)$ is the hazard rate of default at any time u [see, e.g., Li(2000) and Fan, Sing, and Ong (2008)] .

The joint distribution of the default times can be expressed in the following form even if the mortgages are not equally weighted

[12] Alternatively, Hull and White (2004) also suggest a double Student-*t* copula, while this model is not stable under convolution and does not provide additional insight into the pricing of CMBS bonds [see also Burtschell, *et al*. (2005)].

$$\Pr(\tau_1 \le t_1, \tau_2 \le t_2, ..., \tau_N \le t_N) = \Phi_N\left(\phi^{-1}\left(p_1\left(t_1\right)\right), \phi^{-1}\left(p_2\left(t_2\right)\right), \cdots, \phi^{-1}\left(p_N\left(t_N\right)\right)\right), \qquad (14)$$

where Φ_N is a *N*-dimensional cumulative distribution function (cdf) with correlation matrix Σ, and ϕ^{-1} is the inverse of the marginal distribution function [see, e.g., Li (2000)]. For an *N*-dimensional Gaussian or Student *t* copula function, there is an explicit expression for its probability density function (pdf). By differentiating, the copula function will uniquely determine the *N*-dimension pdf as shown in Appendix A1.

A Gaussian copula is the copula of a multivariate normal distribution. Based on Sklar's theorem, the Gaussian copula can produce the joint normal distribution function. According to the definition of the Gaussian copula, we can easily obtain its density function from equation (A4)

$$c\left(t_1, ..., t_N\right) = \frac{1}{\left|\Sigma\right|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\varsigma^T\left(\Sigma^{-1} - I\right)\varsigma\right) \qquad (15)$$

where $\varsigma = \left(G^{-1}\left(t_1\right), ..., G^{-1}\left(t_N\right)\right)'$, and *I* denotes the identity matrix. Both upper and lower tail dependence are absent in the Gaussian copula, therefore implying no likelihood of extreme default observations jointly occurring in this copula.

A Student-*t* copula can be viewed as an extension of the Gaussian copula. For a multivariate Student-*t* copula with degree of freedom *v*, its default copula density can be written as [13]

$$c(t_1, ..., t_N) = \left|\Sigma\right|^{-\frac{1}{2}} \frac{\Gamma\left(\frac{v+N}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \left(\frac{\Gamma\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v+1}{2}\right)}\right)^N \frac{\left(1 + \frac{1}{v}\varsigma'\Sigma^{-1}\varsigma\right)^{-\frac{v+N}{2}}}{\prod_{j=1}^N \left(1 + \frac{\varsigma_j^2}{v}\right)^{-\frac{v+1}{2}}}, \qquad (16)$$

where $\varsigma_j = T_v^{-1}\left(t_j\right)$ is the inverse of the univariate cdf of Student-*t* with *v* degrees of freedom, $\Gamma\left(x\right)$ is a Gamma function, and $\varsigma = \left(T_v^{-1}\left(t_1\right), ..., T_v^{-1}\left(t_N\right)\right)'$. Upper and lower tail dependence in the Student-*t* copula can be expressed as

[13] See Andersen, Sidenius and Basu (2003).

$$\lambda_U = \lambda_L = 2T_{v+1}\left(-\sqrt{v+1}\sqrt{\frac{1-\rho}{1+\rho}}\right), \tag{17}$$

where ρ is the correlation coefficient, and T_{v+1} represents a Student-*t* distribution with *v*+1 degrees of freedom [see Meneguzzo and Vecchiato (2004) and Demarta and McNeil (2005)]. The factor conditional independent model can reduce the dimension of correlation matrix Σ in the Student-*t* copula, and therefore facilities the simplification of default dependence computation associated with CMBS pricing.

4. CMBS Pricing

In this section we discuss the pricing of the pooled commercial mortgages and their multi-class CMBS bonds using the copula models defined above. We first specify the remaining value of each mortgage as

$$R_i = \frac{1-L_i 1_{\{\tau_i \le t\}}}{N}, \tag{18}$$

where L_i is the deterministic loss rate of the *i*th pooled mortgage, and $1_{\{\tau_i \le t\}}$ is the indicator function of mortgage *i* default. We express the cumulative default loss of the underlying mortgage portfolio at time *t* in the percentage form

$$L(t) = \sum_{i=1}^{N} \frac{L_i 1_{\{\tau_i \le t\}}}{N}. \tag{19}$$

Then we can obtain the cumulative remaining value in percentage form

$$R(t) = \sum_{i=1}^{N} \frac{1-L_i 1_{\{\tau_i \le t\}}}{N}. \tag{20}$$

In a CMBS transaction, due to the use of the credit enhancement of senior/subordinated structure, two or more classes or tranches of debt securities are usually issued. The CMBS tranches are redeemed sequentially from senior tranche to the first loss tranche, and default risk is therefore shifted from the senior tranche to more subordinated tranches. Suppose that the CMBS is structured with three tranches, namely senior (*S*), mezzanine (*M*) and junior/first loss piece (*J*) tranches, $\varphi \in \{S, M, J\}$. We also assume that there exist two default thresholds—α and β, which satisfy $0 \le \alpha \le \beta \le 1$. Let $MR(t)$ represent the cumulative remaining value on the mezzanine tranche. We specify that

$$MR(t) = \begin{cases} \beta - \alpha & \text{if } L(t) \le \alpha \\ \beta - L(t) & \text{if } \alpha < L(t) < \beta \\ 0 & \text{if } L(t) \ge \beta. \end{cases}$$

That is,

$$MR(t) = (\beta - L(t)) 1_{\{L(t) \in (\alpha, \beta)\}} + (\beta - \alpha) 1_{\{L(t) \in [0, \alpha]\}}. \tag{21}$$

Similarly, for the cumulative remaining value on the junior tranche $JR(t)$, we have

$$JR(t) = (\alpha - L(t)) 1_{\{L(t) \in [0, \alpha]\}}, \tag{22}$$

while for the senior tranche's remaining value $SR(t)$, we define

$$SR(t) = (1 - L(t)) 1_{\{L(t) \in [\beta, 1]\}} + (1 - \beta) 1_{\{L(t) \in [0, \beta)\}}. \tag{23}$$

Let $K(t) \in \{SR(t), MR(t), JR(t)\}$, and $B(0,t)$ denotes the discount factor for maturity *t*. Then under the risk-neutral probability measure, the cumulative remaining value of a given tranche can be expressed as

$$\mathrm{E}^{Q}\left[\int_0^t B(0,s)\,dK(s)\right]. \tag{24}$$

Since $K(t)$ is a pure jump process, (24) can therefore be defined under the Riemann-Stieltjes integration. Given that $d(B(0,s))/ds = -f(0,s)B(0,s)$, where $f(0,s)$ is the instantaneous forward rate, then (24) can be integrated by parts formula [14]

$$\mathrm{E}^{Q}\left[\int_0^t B(0,s)\,dK(s)\right] = B(0,t)\mathrm{E}^{Q}(K(t)) + \int_0^t \mathrm{E}^{Q}(K(s)) f(0,s) B(0,s)\,ds. \tag{25}$$

For simplicity of analysis, suppose that the interest rate and instantaneous forward rate are constant, and that the hazard rate is flat. Expression (24) can therefore be rewritten as

$$\mathrm{E}^{Q}\left[\int_0^t B(0,s)\,dK(s)\right] = \exp(-rt)\mathrm{E}^{Q}(K(t)) + r\int_0^t \exp(-rs)\mathrm{E}^{Q}(K(s))\,ds. \tag{26}$$

[14] See Meneguzzo and Vecchiato (2004) for a similar derivation, while they only consider the cumulative default loss.

This implies that we only need allow for the first moment of the cumulative remaining value on this tranche in the numerical analysis.

On the other hand, the present value of a given tranche may be computed as follows:

$$V(0)=\sum_{j=1}^{t}\exp\left(-\left(y+r\right)j\right)C_j+\exp\left(-\left(y+r\right)t\right)F_{\varphi}, \qquad (27)$$

where j represents a coupon payment date, y is the required yield spread, r is the risk-free interest rate, C_j represents the coupon payment of the CMBS tranche at time j, and F_{φ} is its face value. Then we find that the yield spread y can be calculated by setting (26) and (27) to be equal.

4.2 Semi-analytic Approximation

The true loss distribution (equivalently, remaining value distribution) of an underlying commercial mortgage pool can be approximated using that of a homogeneous reference portfolio if this pool consists of a large number of mortgage loans that are not too inhomogeneous in credit quality. As a consequence, we can obtain a semi-analytical solution for pricing CMBS bonds. In this approximation, defaults on commercial mortgages are treated to be independent of one another, as they are conditional on the realization of the common factor(s). Given the common factor(s), the remaining value for mortgage i in the underlying portfolio is written as

$$R_i\left(t\right)=\begin{cases}1-L_i, & \text{with probability } p_i\left(t|m\right)\\ 1, & \text{with probability } 1-p_i\left(t|m\right).\end{cases} \qquad (28)$$

Suppose that the underlying mortgage pool is homogeneous, and omit the mortgage subscripts for $L_i=\bar{L}$. Since the conditional defaults are independent, the conditional probability of having $n\leq N$ mortgage defaults may be written as

$$\Pr\left(R\left(t\right)=1-\frac{n\bar{L}}{N}|m\right)=\binom{N}{n}\left(p\left(t|m\right)\right)^{n}\left(1-p\left(t|m\right)\right)^{N-n}. \qquad (29)$$

Correspondingly, the unconditional probability of n mortgage defaults is given by

$$\Pr\left(R(t)=1-\frac{n\bar{L}}{N}\right)=\int_{-\infty}^{\infty}\binom{N}{n}\left(p(t|m)\right)^{n}\left(1-p(t|m)\right)^{N-n}dF_M(m) \tag{30}$$

where $F_M(m)$ is the cdf for the common factor *M*.

As shown in (25), there are two parts associated with the expectation computation of CMBS bonds. Since

$$\mathrm{E}^Q\left(K(t)\right)=\int_{-\infty}^{\infty}\left[\sum_{n=0}^{N}\Pr\left(R(t)=1-\frac{n\bar{L}}{N}|m\right)K(t)\right]dF_M(m), \tag{31}$$

substituting equation (31) into equation (25), we have

$$B(0,t)\mathrm{E}^Q\left(K(t)\right)=B(0,t)\int_{-\infty}^{\infty}\left[\sum_{n=0}^{N}\Pr\left(R(t)=1-\frac{n\bar{L}}{N}|m\right)K(t)\right]dF_M(m), \tag{32}$$

and

$$\begin{aligned}
&\int_0^t \mathrm{E}^Q\left(K(s)\right)f(0,s)B(0,s)ds\\
&=\int_0^t\left[\int_{-\infty}^{\infty}\sum_{n=0}^{N}\Pr\left(R(s)=1-\frac{n\bar{L}}{N}|m\right)K(s)dF_M(m)\right]f(0,s)B(0,s)ds \qquad (33)\\
&=\int_{-\infty}^{\infty}\left[\int_0^t\sum_{n=0}^{N}\Pr\left(R(s)=1-\frac{n\bar{L}}{N}|m\right)K(s)f(0,s)B(0,s)ds\right]dF_M(m).
\end{aligned}$$

This exchange of integration in equation (33) is for convenience of computation. That is, we calculate conditional expectation of $K(s)$ first, and then compute the unconditional expectation about $F_M(m)$. More specifically, based on equations (21) and (31) the expected time-*t* value of the mezzanine tranche $\mathrm{E}^Q\left(MR(t)\right)$ can be expressed as

$$\begin{aligned}
&\mathrm{E}^Q\left(MR(t)\right)=\int_{-\infty}^{\infty}\left[\sum_{n=0}^{N}\Pr\left(R(t)=1-\frac{n\bar{L}}{N}|m\right)MR(t)\right]dF_M(m)\\
&=\int_{-\infty}^{\infty}\left[\sum_{n=A+1}^{B}\left(\beta-L(t)\right)\Pr\left(R(t)=1-\frac{n\bar{L}}{N}\right)+(\beta-\alpha)\sum_{n=0}^{A}\Pr\left(R(t)=1-\frac{n\bar{L}}{N}\right)\right]dF_M(m).
\end{aligned} \tag{34}$$

where $A=\left[\alpha N/\bar{L}\right], B=\left[\beta N/\bar{L}\right]$, and $[X]$ is the maximum integer less or equal to *X*. Similarly, we can derive the expected expressions for the senior and junior tranches based on equations (22), (23) and (31). Based on the resulting formulas above, the cumulative remaining value on a given tranche can therefore be computed using the numerical integration.

5. Numerical Results for Pricing CMBS

In order to illustrate the usefulness of the copula-based model in pricing CMBS bonds, we employ the semi-analytic approach for pricing the CMBS based on the results derived in the previous section.[15] We price a hypothetical CMBS transaction backed by 100 commercial mortgages of the same type and size. Three different tranches, namely senior, mezzanine and junior bonds, are issued for financing the purchase of the commercial mortgages. The security structure of the CMBS transaction is 70:20:10.[16] Suppose that those underlying mortgages are fixed-rate balloon loans with an average loan-to-value (LTV) ratio of 75%. For simplicity, their notional value is standardized as 1/100, while their remaining maturity terms are seven years coinciding with the scheduled redemption time of the CMBS tranches. We shall ignore the prepayment or early redemption of the underlying mortgages due to the usual imposition of lockout and prepayment penalties on them.[17]

We investigate the sensitivity of required yield spreads on CMBS bonds to the varying correlation coefficient of the underlying commercial properties from 0 to 1. Table 1 reports the input parameter values in the basic case scenario. Figure 1 shows that for the Gaussian copula, required yield spreads for the senior tranche are an increasing function in the correlation between underlying commercial properties. More specifically, our results show that the yield spreads decrease from about 17 to 0 basis points, when the correlation coefficient changes from 1 to 0. This implies that better diversification due to $\rho \to 0$ helps further reduce the default risk of the senior tranche with a 70-percent pool share in that the overwhelming majority of the default risk in the underlying pool is shifted to the more subordinated tranches occupying 30 percent of the pool through the

[15] See Bluhm and Overbeck (2004) for more technical details about this approach. Alternatively, we can use Monte Carlo simulation for the pricing purpose.

[16] In a CMBS transaction, senior tranches typically occupy not less than 70 percent of the issued bond size [see, e.g., Childs, Ott and Riddiough (1996)].

[17] In this numerical analysis, we determine the default threshold level according to equation (6).

senior/subordinated security structure. For the Student-*t* copula, a similar function relationship of required yield spreads with property correlation can also be identified from Figure 1. The yield spreads are found to be positively associated with the correlation but have slightly greater values than those produced by the Gaussian copula. This is because the Student-*t* copula allows for tail dependence, that is, the likelihood of joint occurrence of extreme observations, while the Gaussian copula does not exhibit any such dependence.

[Insert Table 1]

[Insert Figure 1]

Figure 2 displays the impact of varying property correlation on required yield spreads for the mezzanine tranche. Similarly to the case of the senior bond, the yield spreads are found to be positively related to property correlation for both the copulas but are more sensitive to its changes. For the Gaussian copula, the yield spread increases from 0 to 177 basis points as the correlation coefficient increases from 0 to 1. This is because that both the mezzanine and junior tranches bear the majority of the default risk of the pool. Also, we find that since the junior tranche with a 10-percent pool share absorbs most of the default risk, a well-diversified pool may make the mezzanine tranche immune from default loss risk, while decreased pool diversification causes this tranche to bear a higher default risk exposure. In the case of the Student-*t* copula, the yield spread increases from 50 to 174 basis points when the correlation coefficient changes in the range from 0 to 1. Compared with those results produced by the Gaussian copula, the higher yield spreads are due to the impact of tail dependence of underlying property values. This implies that property value tail dependence plays an important role in determining the default risk exposure to this tranche. Moreover, like the senior tranche, Figure 2 also shows that the larger the correlation coefficient becomes, the closer the yields spreads on the mezzanine tranche in these two cases are to each other. When the correlation coefficient is very close to 1, this implies that the pooled commercial properties can be actually viewed as a single, large commercial property so that in these two cases the default risk exposure to this tranche is not affected by property value tail dependence.

[Insert Figure 2]

Figure 3 plots the relationship between junior tranche yield spreads and property correlation. It is noteworthy that varying property correlation produces an opposing impact on the yield spreads compared with those on yield spreads for the senior and mezzanine tranches, and the yield spreads on this tranche are fairly high when the correlation coefficient is close to zero. For the Gaussian copula, it is shown that the yield spreads decrease from 803 to 177 basis points as the correlation coefficient increases from 0 to 1. The inverse impact of increasing property correlation on junior tranche yield spreads is consistent with the finding in Childs, Ott and Riddiough (1996). For the mezzanine and senior tranches, increasing correlation amongst the commercial properties produces higher probabilities of large mortgage default losses and therefore results in the increase in their default risk exposure. In such scenarios, these two tranches have to absorb some of these default losses. Thus higher spreads are required as compensation for these two tranches as the correlation becomes larger. However, higher property correlation also leads to a higher survival correlation among the underlying commercial mortgages and therefore these mortgages also tend to survive together. As a result, the junior tranche will require less default risk premium due to higher survival correlation.

On the other hand, for the Student-*t* copula, our numerical results also show that required yield spreads for the junior tranche decrease with the increase in the property correlation. Figure 3 shows that if the correlation coefficient changes from 0 to 1, the yield spreads on this tranche decrease from about 627 to 175 basis points. In particular, one can readily find that there are lower yield spreads for the junior tranche under the Student-*t* copula than under the Gaussian copula. This is due to the impact of property value tail dependence on the allocation of the pool's default risk among these three tranches. The Student-*t* copula produces higher joint default or survival probabilities at the tail. Given that the expected default loss of the pool remains unchanged in these two cases, if the more default risk of the pool is shifted to the mezzanine tranche for the case of the Student-*t* copula, then the less default risk loss will be correspondingly borne by the junior tranche in this case.

[Insert Figure 3]

It is noteworthy that our numerical results are comparable with those with the same security structure in Childs, Ott and Riddiough (1996). Their study has demonstrated that mortgage pool diversification plays an important role in determining required yield spreads for various CMBS tranches. An important result contradicting the conventional wisdom in the COR model is the opposing impacts of pool diversification on the yield spreads for the mezzanine and junior tranches.[18] Table 2 further confirms these results. However, this table also shows that mezzanine tranche yield spreads derived from the Student-t copula are higher than those produced in the COR model due to the impact of property value tail dependence. Senior tranche yield spreads in our model are found to be a decreasing function of pool diversification like yield spreads on the mezzanine tranche, while the COR model shows a slightly different relationship. In addition, our results also show that junior tranche yield spreads derived from both the copulas are smaller than those given in the COR model. This is mainly because the COR model also take into account the impact of interest rate risk on commercial mortgage value by specifying the stochastic evolution of interest rates, while for the purpose of this study we focus on the critical role of various property value dependency in pricing CMBS bonds. In particular, for the Student-*t* copula, the lower yield spreads for the junior tranche is also due partly to the impact of property value tail dependence on the allocation of the pool's default risk among these three tranches as discussed above.

[Insert Table 2]

[18] Childs, Ott and Riddiough (1996) further show that better pool diversification increases the value of mezzanine CMBS tranches, but decreases the value of junior CMBS tranches.

6. Conclusions

This paper develops a copula-based model to price CMBS bonds. Given that commercial real estate returns usually deviate from normal distribution and the dependency among underlying properties cannot be sufficiently captured by the linear correlation coefficient, extant pricing models of CMBS that neglect these important features do not offer accurate valuation. The two important issues associated with CMBS bond default—extreme events and default dependence structure among pooled commercial mortgages—play crucial roles in determining the pooled commercial mortgage portfolio's credit risk exposure to CMBS investors. To address these problems, we employ the copula-based method.

We have demonstrated that the proposed copula-based model has great flexibility in sufficiently taking into account the two crucial issues associated with CMBS pricing. Our numerical results show that property value dependence structure plays an important role in determining required yield spreads for various CMBS tranches. Specifically, pool diversification are found to have opposing effects on required yield spreads for junior CMBS tranches and those more senior CMBS tranches. This confirms the findings in Childs, Ott and Riddiough (1996). Moreover, it is shown that property value tail dependence is another important determinant of required yield spreads for CMBS bonds. This suggests that it is important to allow for the effect of possible extreme events in the valuation of CMBS bonds. The neglect of complex mortgage default dependence structure and the likelihood of extreme events occurring in pricing CMBS can lead to an inaccurate valuation of CMBS bonds. Given that CMBS have become major investment instruments for many investment and hedge funds, these results have important implication for their investment decisions associated with CMBS bonds. For the fund portfolios containing CMBS, the copula-based methodology provides a useful tool for the fund managers to identify major risk factors and analyze the sensitivity of CMBS bond yield spreads with respect to the risk sources and enable hedging of the risks properly.

APPENDIX

A1. Basic Definitions and Properties

The theory of copula functions investigates and describes the dependence structure of multiple random variables. On one hand, copulas are functions that connect the marginal (individual) distribution functions of individual random variables to their multivariate distribution function. On the other hand, the copula function provides an analytical tractable way of characterizing the dependence structure of joint random variables.[19]

A copula can be defined as follows:

Definition: *Let* $C:[0,1]^n \rightarrow [0,1]$ *be an n-dimensional distribution function on* $[0,1]^n$. *Then C is called a copula if it has uniform marginal distributions on the interval* [0, 1].

Based on the above definition, we have the following fundamental theorem and corollary for copulas.

Theorem (Sklar, 1959): *Let* F *be an n-dimensional joint distribution function with marginal distributions* $F_1(x_1),...,F_n(x_n)$. *Then there exists a copula function* C, *such that for all* $(x_1,...,x_n) \in R^n$,

$$F(x_1,...,x_n) = C(F_1(x_1),...,F_n(x_n)). \tag{A1}$$

Also, C *is unique if* $F_1(x_1),...,F_n(x_n)$ *are all continuous; if not,* C *is uniquely determined on* $RanF \times \cdots \times RanF_n$, *where* $RanF_i$ *denotes the range of* $F_i(x_i)$ *for* $i = 1,...,n$.

Conversely, if C is an *n*-copula function and $F_1(x_1),...,F_n(x_n)$ are marginal distribution functions, then the function F defined above is an *n*-dimensional joint distribution function with margins $F_1(x_1),...,F_n(x_n)$. Sklar's theorem shows that the copula function

[19] See Nelsen, R. (1999) and Cherubini, Luciano and Vecchiato (2004) for a comprehensive introduction of copulas.

can partition a multivariate distribution into two components, i.e., the marginal distributions of the individual random variables and their dependence structure.

The following corollary shows how to obtain the copula of a multi-dimensional distribution function.

Corollary: *Let F be an n-dimensional continuous distribution function with marginal distributions* $F_1(x_1),...,F_n(x_n)$*. Then the corresponding copula C has representation*

$$C(u_1,...,u_n) = F\left(F_1^{-1}(u_1),...,F_n^{-1}(u_n)\right) \tag{A2}$$

where $F_1^{-1},...,F_n^{-1}$ *denote the generalized inverses of the distribution functions* $F_1(x_1),...,F_n(x_n)$*, i.e. for all* $u_1,...,u_n \in (0,1)$: $F_i^{-1}(u_i) = \inf\left\{x \in R \middle| F_i(x_i) \geq u_i\right\}$*,* $i = 1,...n$*.*

An important property of copula is the invariance property. That is, if one carries out strictly increasing transformations for the underlying random variables, the transformed variables have the same copula as the original variables. When the random variables are independent, their copula can be simply written as

$$C(u_1,...,u_n) = F(x_1,...,x_n) = \prod_{i=1}^{n} F(x_i) = \prod_{i=1}^{n} u_i \ . \tag{A3}$$

On the other hand, the density of the multi-dimensional distribution function *F* can be expressed as follows

$$f(x_1,...,x_n) = c\left(F_1(x_1),...,F_n(x_n)\right)\prod_{i=1}^{n} f_i(x_i) \tag{A4}$$

where $c(.,...,.)$ is the density of the copula *C*

$$c\left(F_1(x_1),...,F_n(x_n)\right) = \frac{\partial^n\left[C\left(F_1(x_1),...,F_n(x_n)\right)\right]}{\partial F_1(x_1)...\partial F_n(x_n)} \tag{A5}$$

and $f_i(\cdot)$ are the densities of the marginal distributions.

Any copula $C(u_1,...,u_n)$ also satisfies the following bounds

$$\max\left(u_1+\ldots+u_n-1,0\right)\le C\left(u_1,\ldots,u_n\right)\le \min\left(u_1,\ldots,u_n\right). \tag{A6}$$

This inequality is known as Fréchet-Hoeffding Bounds, which represent the largest possible positive and negative dependence of the underlying random variables.

An n-copula $C\left(u_1,\ldots,u_n\right)$ is non-decreasing in each argument. In particular, its partial derivative with regard to u_i exists almost everywhere and satisfies

$$0\le \frac{\partial C}{\partial u_i}\left(u_1,\ldots,u_n\right)\le 1;$$

it also has mixed kth-order partial derivatives almost surely, which for $1\le l\le n$, satisfies

$$0\le \frac{\partial^l C\left(u_1,\ldots,u_n\right)}{\partial u_1,\ldots,\partial u_l}\le 1.$$

The properties imply that copulas have nice smoothness conditions.

A2. Tail Dependence

Tail dependence is a powerful measure of the dependency between the occurrences of extreme observations of the underlying random variables, and can therefore be used to model probabilities of highly correlated defaults.

Definition (Tail dependence)

Let $X=\left(X_1,X_2\right)$ be a two-dimensional random vector. Then the upper tail dependence of X is defined as

$$\lambda_U=\lim_{u\to 1}\Pr\left[X_1\ge F_1^{-1}\left(u\right)\middle|X_2\ge F_2^{-1}\left(u\right)\right], \tag{A7}$$

while its lower tail dependence is

$$\lambda_L=\lim_{u\to 0}P\left[X_1\le F_1^{-1}\left(u\right)\middle|X_2\le F_2^{-1}\left(u\right)\right], \tag{A8}$$

where F_1 and F_2 are the marginal distribution functions of X_1 and X_2, respectively.

A positive probability of positive or negative outliers jointly occurring implies the presence of upper or lower tail dependence, respectively. (A7) and (A8) can be rewritten as

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u} \tag{A9}$$

and

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u}. \tag{A10}$$

If λ_U or $\lambda_L > 0$, the two random variables (X_1, X_2) are asymptotically dependent in the upper or lower tail and their extreme observations tend to occur simultaneously with probability λ_U or λ_L. On the other hand, if λ_U or $\lambda_L = 0$, the two random variables are asymptotically independent in the upper or lower tail. That is, the copula has no upper or lower tail dependence [(see, e.g., Meneguzzo and Vecchiato (2004)].

If the two random variables are independent in the upper and lower tails, then

$$C(u,u) = u^2$$

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u} = \lim_{u \to 1} (1-u) = 0$$

and

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u} = \lim_{u \to 0} u = 0.$$

On the other hand, if

$$\lim_{u \to 1} C(u,u) = \lim_{u \to 1} \left(1 - 2(1-u) + o(1-u)\right),$$

then

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u} = \lim_{u \to 1} o(1-u) = 0.$$

If

$$\lim_{u \to 0} C(u,u) = o(u).$$

then

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u} = \lim_{u \to 0} \frac{o(u)}{u} = 0 .$$

The two cases have no tail dependence of the underlying random variables. To analyze their tail dependence structure, the copula functions are usually chosen with these two limits not equal to zero.

If copulas have no closed-form expressions, we can use the approach of Embrechts, Lindskog & Mcneil (1999, 2002) in calculating tail dependence. It is shown that the upper tail dependence λ_U can be expressed using conditional probabilities if the following limit exists:

$$\lambda_U = \lim_{u \to 1} \Pr\left[\left(U_1 > u \middle| U_2 = u \right) + \left(U_2 > u \middle| U_1 = u \right) \right]. \tag{A11}$$

If (U_1, U_2) have the same marginal distribution of normality or Student-*t* and the copula is exchangeable, then:

$$\lambda_U = \lim_{u \to 1} \Pr\left[\left(U_1 > u \middle| U_2 = u \right) + \left(U_2 > u \middle| U_1 = u \right) \right] = 2 \lim_{u \to 1} \Pr\left(U_1 > u \middle| U_2 = u \right) \tag{A12}$$

However, the tail dependent coefficient for the bi-normal distribution is zero, implying that extreme events occur independently in each margin. Thus, the Gaussian or normal copula does not have a useful tail dependence structure for mortgage default risk management.

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Table 1: Assumptions for Base Case Scenario

Input Parameter	**Assumption**
Hazard rate of default	$h = 1.5\%$
Risk-free interest rate	$r = 6\%$
Mortgage rate	$M_r = 8.5\%$
Coupon rate of senior CMBS bond	$c_S = 7.5\%$
Coupon rate of mezzanine CMBS bond	$c_M = 8\%$
Coupon rate of junior CMBS bond	$c_J = 8\%$

Table 2: Comparison of Yield Spread Estimation

Investment Class	Pool Share (%)	Asset Correlation	Estimated Yield Spreads		
			Gaussian Copula	Student *t* Copula	Estimates of Childs et al (1996)
Senior	70%	0	0.0000	0.0000	0.0010
		0.5	0.0000	0.0001	0.0009
		1	0.0017	0.0017	0.0008
Mezzanine	20%	0	0.0000	0.0050	0.0011
		0.5	0.0066	0.0115	0.0059
		1	0.0177	0.0174	0.0166
Junior	10%	0	0.0803	0.0627	0.0892
		0.5	0.0593	0.0439	0.0746
		1	0.0177	0.0175	0.0476

Figure 1: Effect of Correlations on Yield Spreads of Senior Bonds

(Student-*t* Copula vs. Gaussian Copula)

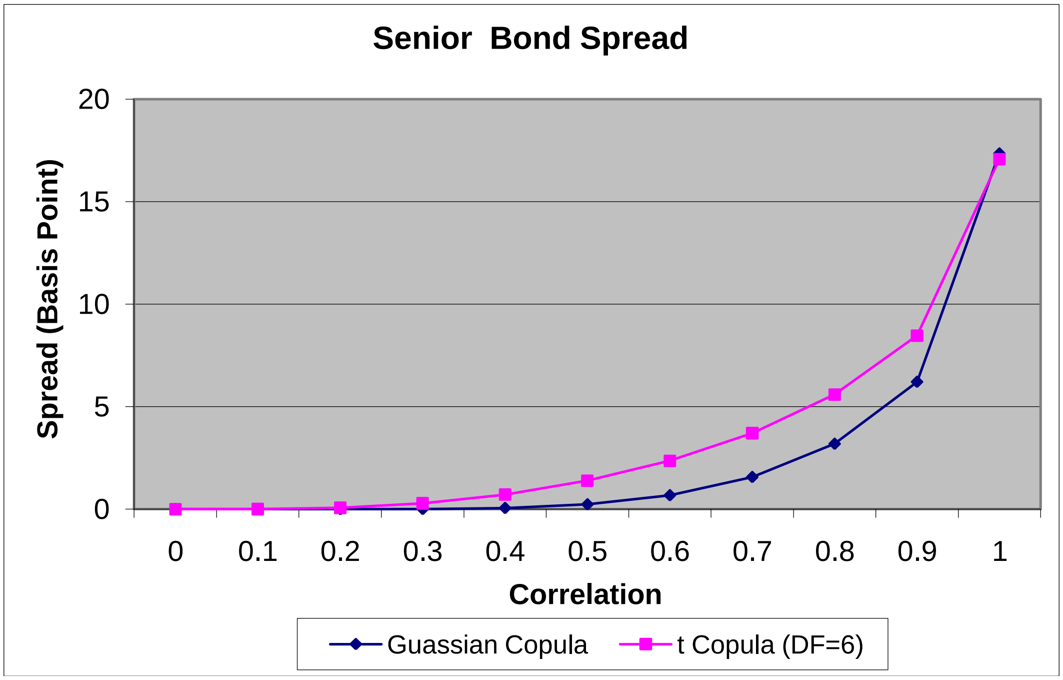

Figure 2: Effect of Correlations on Yield Spreads of Mezzanine Bonds

(Student-*t* Copula vs. Gaussian Copula)

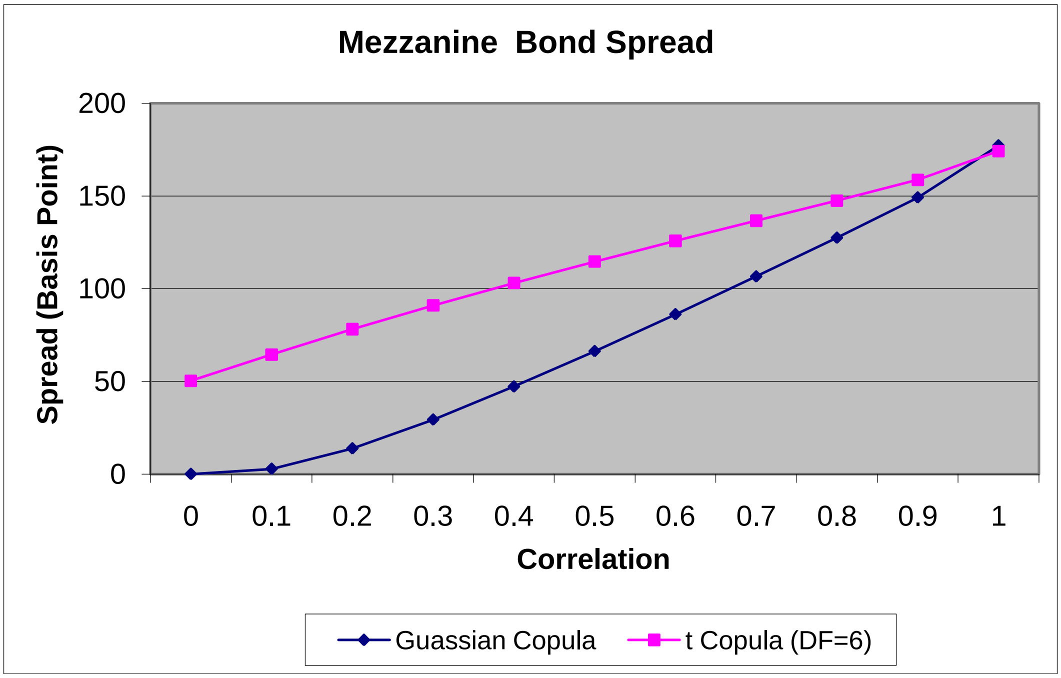

Figure 3: Effect of Correlations on Yield Spreads of Junior Bonds

(Student-*t* Copula vs. Gaussian Copula)