Well-posedness and Dynamics of a Fractional Stochastic Integro-Differential Equation

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Abstract

In this paper we investigate the well-posedness and dynamics of a fractional stochastic integro-differential equation describing a reaction process depending on the temperature itself. Existence and uniqueness of solutions of the integro-differential equation is proved by the Lumer-Phillips theorem. Besides, under appropriate assumptions on the memory kernel and on the magnitude of the nonlinearity, the existence of random attractor is achieved by obtaining first some a priori estimates. Moreover, the random attractor is shown to have finite Hausdorff dimension.

Key words: fractional stochastic integro-differential equation; random attractor; Lumer-Phillips theorem; Hausdorff dimension; a priori estimates

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1 Introduction

This paper focuses on the following fractional stochastic partial integro-differential equations, which is derived in the framework of the well-established theory of heat flows with memory (see [8]) on $O \subset \mathbb{R}^3$, which is a bounded domain with smooth boundary ∂O,

$$\frac{\partial u}{\partial t} + \beta(1-\gamma)(-\Delta)^\alpha u + \int_0^\infty \mu(s)(-\Delta)^\alpha u(t-s)ds + f(u) = k(x) + h(x)\frac{dW}{dt}, \quad x \in O, \ t > 0, \tag{1.1}$$

with boundary condition

$$u(x,t) = 0, \quad x \in \partial O, \quad t > 0, \tag{1.2}$$

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and initial condition

$$u(x,t) = u_0(x,t), \quad x \in \mathcal{O},\ t \le 0. \tag{1.3}$$

Here, $\alpha \in (0,1), \beta \in (0,+\infty)$ and $\gamma \in (0,1)$, $u(x,t)$ is the unknown function, while μ is a decreasing and non-negative memory kernel; f is a nonlinear reaction term (for instance, $f(u) = u^3 - u$), $k(\cdot) \in L^2(\mathcal{O})$ and $h(\cdot) \in H^{2\alpha}(\mathcal{O})$ are given functions. W is a two-sided real-valued Wiener process on a probability space which will be specified later. In the present case, the dynamics of u relies on the past history of the diffusion term, that is, $\int_0^\infty \mu(s)(-\Delta)^\alpha u(t-s)ds$.

Problem (1.1) with $\alpha = 1$ as well as $h(x) = 0$ is well known and has been extensively studied (see [5, 6, 9, 18, 19]), and can be interpreted as a model of heat diffusion with memory which also accounts for a reaction process depending on the temperature itself (see [19] and related references therein). Namely, if $u(t)$ represents the temperature of a material occupying $\mathcal{O}$ for any time t, as in [8], we can consider the following heat flux law

$$\overrightarrow{q}(x,t) = -\beta(1-\gamma)\nabla u(x,t) - \int_0^\infty \mu(s)\nabla u(x,t-s)ds,$$

where $\beta(1-\gamma)$ is the instantaneous heat conductivity and $\mu(s)$ is a memory or relaxation kernel. Then, assuming the total energy is proportional to u (with proportionality constant 1 for simplicity), the standard semilinear heat equation with memory, i.e.,

$$\frac{\partial u}{\partial t} - \beta(1-\gamma)\Delta u - \int_0^\infty \mu(s)\Delta u(t-s)ds + f(u) = k(x), \tag{1.4}$$

could be recovered from the energy balance

$$u_t + \nabla \cdot \overrightarrow{q} = k - f(u),$$

(see [6] for a more detailed explanation and more references on the topic). This kind of equation can also be proposed to describe many different phenomena, such as the evolution of the velocity of certain viscoelastic fluids [14, 32], the thermomechanical behavior of polymers [15, 25, 36], the diffusion of the chemical potential of a penetrant in polymers near the glass transition [26], and some models in population dynamics [17]. Concerning equation (1.4) (which is a deterministic heat equation with memory) existence, uniqueness, and asymptotic behavior results can be found in [9, 20, 21, 23]. In particular, equation (1.4) is shown to have a uniform attractor, which has finite Hausdorff dimension (see [21]), whereas in [20] the existence of absorbing sets in suitable function spaces is achieved.

Observe that the aforementioned literature mainly dealt with versions of Eq. (1.4) in a deterministic context. But, it is sensible to assume that the models of certain phenomena from the real world are more realistic if some kind of uncertainty, for instance, some randomness or environmental noise, is also considered in the formulation. In fact, the random perturbations are intrinsic effects in a variety of settings and spatial scales. They may be most obviously influential at the microscopic and smaller scales but indirectly they play an important role in macroscopic phenomena. We will take into account an additive noise in our model which we interpret as the environmental noisy effect produced on the system, and will exploit the theory of random dynamical systems (see [1, 3]) to obtain information on the dynamics of our model, in particular we will be able to prove the existence of random attractor. When $\alpha = 1$, problem (1.1) reduces to a standard stochastic heat equation with memory. In this case, a similar stochastic equation with additive noise in materials with memory is studied in [6], and the existence of pullback attractors is also established, while in [4], the existence and stability of solutions for stochastic heat equations with multiplicative noise in materials with memory is proved.

Nevertheless, the previously cited references are concerned with equations with standard Laplace operator, namely, $\alpha = 1$ in equation (1.1). However, it is mentioned in [2] that some research on classical diffusion equation may be inadequate to model many real situations, for instance, a particle plume spreads faster than that predicted by the classical model, and may exhibit significant self-organization phenomena or asymmetry, see details in [39]. In this case, these situations are called anomalous diffusion. One popular model for anomalous diffusion is the fractional diffusion equation, where the usual second derivative operator in space, i.e., the Laplacian operator $-\Delta$, is replaced by a fractional derivative operator $(-\Delta)^{\alpha}$ with $0 < \alpha < 1$. Indeed, equations with fractional derivative are becoming a focus of interest since the fractional derivative and fractional integral have a wide range of applications in physics, biology, chemistry, population dynamics, geophysical fluid dynamics, finance and other fields of applied sciences. One meets them in the theory of systems with chaotic dynamics (see [37, 41]); dynamics in a complex or porous medium [16, 38]; random walks with a memory and flights [22, 29, 30, 42] and many other situations. When $\mu = 0$, this is the case of no memory term, (1.1) reduces to a fractional stochastic parabolic equation with noise. In this case, the ergodicity of a stochastic fractional reaction-diffusion equation with additive noise is studied in [24], whereas the existence of random attractor for a fractional stochastic reaction-diffusion equation is proved in [27] under the assumption of $\alpha \in [\frac{1}{2}, 1)$. However, as far as we know, there are no works dealing with fractional stochastic reaction-diffusion equations with both white noise term and memory terms, and this is the reason of the current investigation in this paper.

Inspired by [6, 24], we are devoted to investigating a stochastic fractional integro-differential equation. More precisely, in this work, we analyze the well-posedness and dynamics of a fractional stochastic reaction-diffusion equation with memory term, which is expressed by convolution integrals and represent the past history of one or more variables. The main features of the present paper work are summarized as follows: Both the fractional diffusion term (instead of standard diffusion term, i.e., $-\Delta u$) and the memory term are considered. Besides, the well-posedness is analyzed by a semigroup method (see [35] for more information), which is different from the classical Faedo-Galerkin method (see [40]). Then the existence of random attractor is established by a priori estimates and solutions decomposition. Moreover, by using the method introduced by Debussche in [13], we obtain that the random attractor has finite Hausdorff dimension.

The structure of the paper is as follows. In the next Section, we recall some notations and introduce basic hypotheses. Our main results are also stated in this section. In Section 3, we first transform the stochastic equation (1.1) into a deterministic one only with random parameters and then study the well-posedness of the problem, and prove that it generates a random dynamical system Φ. Then, the existence and uniqueness of a random attractor is proved in Section 4. Finally, we show that the random attractor has finite Hausdorff dimension in Section 5.

2 Preliminaries

In this section, we briefly recall some concepts and the basic theory of random dynamical systems. For a detailed information and related applications the reader is referred to [1, 7, 10, 11, 40].

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with the Borel $\sigma-$algebra $\mathcal{B}(X)$, and $\{\theta_t : \Omega \to \Omega,\ t \in \mathbb{R}\}$ be a family of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$.

Definition 2.1. *$(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F}, \mathcal{F})-$measurable, θ_0 is the identity on Ω, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$. and $\theta_t(P) = P$ for all $t \in \mathbb{R}$.*

Definition 2.2. *A random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping*

$$\Phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \Phi(t, \omega)x,$$

which is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X), \mathcal{B}(X))-$measurable and satisfies for $P-a.e.\ \omega \in \Omega$,

(i) $\Phi(0, \omega) = Id_X$ *on* X;

(ii) $\Phi(t + s, \omega) = \Phi(t, \theta_s\omega) \circ \Phi(s, \omega)$, *for all* $t, s \in \mathbb{R}^+$. *(cocycle property)*

An RDS Φ is said to be continuous if $\Phi(t, \omega) : X \to X$ is continuous for all $t \in \mathbb{R}^+$, $\omega \in \Omega$.

Definition 2.3. *Let X be a metric space with a metric d. A set-valued map $\omega \to B(\omega)$ taking values in the closed/compact subsets of X is said to be a random closed/compact set in X if the mapping $\omega \longmapsto dist(x, B(\omega))$ is measurable for all $x \in X$, where $d(x, D) := \inf_{y\in D} d(x, y)$. A set-valued map $\omega \mapsto U(\omega)$ taking values in the open subsets of X is said to be a random open set if $\omega \mapsto U^c(\omega)$ is a random closed set, where $U^c(\omega)$ denotes the complement of U, i.e., $U^c := X\backslash U$.*

Definition 2.4. *A random set $B : \Omega \to 2^X$ is called a bounded random set if there is a random variable $r(\omega) \in [0, \infty)$, $\omega \in \Omega$, such that*

$$d(B(\omega)) := \sup\Big\{\|x\|_X :\ x \in B(\omega)\Big\} \le r(\omega) \ \text{for all}\ \omega \in \Omega.$$

A bounded random set $B(\omega)$ is said to be tempered with respect to $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ if for $P-a.e.\ \omega \in \Omega$,

$$\lim_{t\to+\infty} e^{-\mu t} \sup_{x\in B(\theta_{-t}\omega)} \|x\|_X = 0 \ \text{ for all } \mu > 0.$$

Definition 2.5. *Let $\mathcal{D}$ be a collection of random sets in X. A random set $B \in \mathcal{D}$ is called a $\mathcal{D}$-random absorbing set for an RDS Φ if for any random set $D \in \mathcal{D}$ and $P-a.e.\ \omega \in \Omega$, there exists $T_D(\omega) > 0$ such that*

$$\Phi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega), \ \ \textit{for all } t \geq T_D(\omega).$$

A collection $\mathcal{D}$ of random sets in X is called inclusion closed if whenever E is a random set, and F is in $\mathcal{D}$ with $E(\omega) \subset F(\omega)$ for all $\omega \in \Omega$, then E must belong to $\mathcal{D}$. A collection $\mathcal{D}$ of random sets in X is said to be universe if it is inclusion-closed.

Definition 2.6. *Let $\mathcal{D}$ be a collection of random subsets of X and Φ be a continuous random dynamical system. Then a random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a random attractor for Φ if*

(i) $\mathcal{A}$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$.

(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant, i.e.,

$$\Phi\,(t, \omega)\,\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega), \ \textit{for all } t \geq 0.$$

(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to+\infty} \mathrm{dist}\Big(\Phi\,(t, \theta_{-t}\omega)\, B(\theta_{-t}\omega), \mathcal{A}\,(\omega)\Big) = 0,$$

where $\mathrm{dist}(\cdot,\cdot)$ *denotes the Hausdorff semi-distance under the norm of X, i.e., for two nonempty sets $A, B \subset X$,*

$$\mathrm{dist}_X(A, B) := \sup_{a\in A} \mathrm{dist}_X(a, B) = \sup_{a\in A} \inf_{b\in B} \parallel a - b \parallel_X .$$

Definition 2.7. *Let A be a linear operator on a Hilbert space X. For any $m \in \mathbb{N}$, the $m-$dimensional trace of A is defined as*

$$Tr_m(A) = \sup_Q \sum_{j=1}^{m} (Au_j, u_j)_X,$$

where the supremum ranges over all possible orthogonal projections Q in X on the $m-$dimensional space QX belonging to the domain of A, and $\{u_1, u_2, \cdots, u_m\}$ is an orthonormal basis of QX.

The following proposition can be found in [6, 7, 10, 11, 13].

Proposition 2.8. *Let Φ be a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. If Φ possesses a compact attracting set K in $\mathcal{D}$, then Φ has a unique random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$ given by*

$$A(\omega) = \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} \Phi(t, \theta_{-t}\omega)K(\theta_{-t}\omega)}, \ \ \textit{for each } \omega \in \Omega.$$

Proposition 2.9. *(See [28]) Let $\mathcal{A}(\omega)$ be a compact measurable set which is invariant under a random map $\Psi(\omega)(\cdot)$, $\omega \in \Omega$, for some ergodic metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Assume that the following conditions are satisfied.*

(i) $\Psi(\omega)(\cdot)$ *is almost surely uniformly differentiable on* $\mathcal{A}(\omega)$*, that is, for every* $u, u+h \in \mathcal{A}(\omega)$*, there exists* $D\Psi(\omega, u)$ *in* $\mathcal{L}(X)$*, the space of the bounded linear operators from* X *to* X*, such that*

$$\|\Psi(\omega)(u+h) - \Psi(\omega)(u) - D\Psi(\omega, u)h\| \leq \bar{k}(\omega)\|h\|^{1+\rho},$$

where $\rho > 0$ *and* $\bar{k}(\omega)$ *is a random variable satisfying* $\bar{k}(\omega) \geq 1$ *and* $E(\ln \bar{k}) < \infty$*.*

(ii) $\omega_d(D\Psi(\omega, u)) \leq \overline{\omega}_d(\omega)$ *holds when* $u \in \mathcal{A}(\omega)$ *and there is some random variable* $\overline{\omega}_d(\omega)$ *satisfying* $E(\ln(\overline{\omega}_d)) < 0$*, where*

$$\omega_d(D\Psi(\omega, u)) = \alpha_1(D\Psi(\omega, u)) \cdots \alpha_d(D\Psi(\omega, u)),$$

$$\alpha_d(D\Psi(\omega, u)) = \sup_{G \subset X, dimG \leq d} \inf_{v \in G, \|v\|_X = 1} \|D\Psi(\omega, u)v\|.$$

(iii) $\alpha_1(D\Psi(\omega, u)) \leq \overline{\alpha_1}(\omega)$ *holds when* $u \in \mathcal{A}(\omega)$ *and there is a random variable* $\overline{\alpha_1}(\omega) \geq 1$ *with* $E(\ln \overline{\alpha_1}) < \infty$*.*

Then the Hausdorff dimension $d_H(\mathcal{A}(\omega))$ *of* $\mathcal{A}(\omega)$ *is less than d almost surely.*

Throughout the work, we denote by $A = (-\Delta)^{\alpha}$ $(0 < \alpha < 1)$ the fractional Laplace operator with domain $D(A) = H^{2\alpha}(\mathcal{O})$. With usual notation, we introduce the space L^p, H^k and H_0^k acting on $\mathcal{O}$. Let $\|\cdot\|$ and $(\cdot, \cdot)$ denote the norm and the inner product on the real Hilbert space $L^2(\mathcal{O})$, respectively, and let $\|\cdot\|_p$ denote the L^p-norm. With abuse of notation, we use $(\cdot, \cdot)$ to denote also duality between L^p and its dual space L^q. The inner products on $H^{\alpha}(\mathcal{O})$, $H^{2\alpha}(\mathcal{O})$ can be defined in the following manner:

$$(u, v)_{H^{\alpha}(\mathcal{O})} = ((-\Delta)^{\frac{\alpha}{2}} u, (-\Delta)^{\frac{\alpha}{2}} v)$$

and

$$(u, v)_{H^{2\alpha}(\mathcal{O})} = ((-\Delta)^{\alpha} u, (-\Delta)^{\alpha} v).$$

Assuming $\mu(\infty) = 0$, set

$$g(s) = -\mu'(s). \tag{2.1}$$

In what follows, we take $\beta = 2, \gamma = \frac{1}{2}$ for simplicity, and the following set of hypotheses are required:

(H1) $g(\cdot) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $g(s) \geq 0$, $g'(s) \leq 0$, $g'(s) + \delta g(s) \leq 0$, $\forall\ s \in \mathbb{R}^+$ and some $\delta > 0$;

(H2) $f(\cdot) \in C^1(\mathbb{R}^+)$, $f(u)u \geq \alpha_1 |u|^p - \alpha_2$, $f'(u) > -\alpha_3$, $|f(u)| \leq \alpha_4(1 + |u|^{p-1})$,

where α_i $(i = 1, 2, 3, 4)$, $p \geq 1$ are positive numbers.

Note that (H1) implies the exponential decay of $g(\cdot)$. Nevertheless, it allows $g(\cdot)$ to have a singularity at $s = 0$, whose order is less than 1, since $g(\cdot)$ is a non-negative L^1-function.

Now, let $L_g^2(\mathbb{R}^+, L^2(\mathcal{O}))$ be the Hilbert space of L^2-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(\eta_1, \eta_2)_{L_g^2(\mathbb{R}^+, L^2(\mathcal{O}))} = \int_0^{\infty} g(s) \int_{\mathcal{O}} \eta_1(s, x) \cdot \eta_2(s, x) dx ds.$$

Similarly on $M := L^2_g(\mathbb{R}^+, H^\alpha(O))$ and $M_1 := L^2_g(\mathbb{R}^+, H^{2\alpha}(O))$, respectively, we have the inner products

$$(\eta_1, \eta_2)_{L^2_g(\mathbb{R}^+, H^\alpha(O))} = \int_0^\infty g(s) \int_O (-\Delta)^{\alpha/2}\eta_1(s,x) \cdot (-\Delta)^{\alpha/2}\eta_2(s,x)dxds$$

and

$$(\eta_1, \eta_2)_{L^2_g(\mathbb{R}^+, H^{2\alpha}(O))} = \int_0^\infty g(s) \int_O (-\Delta)^{\alpha}\eta_1(s,x) \cdot (-\Delta)^{\alpha}\eta_2(s,x)dxds,$$

where operators $(-\Delta)^{\alpha/2}$ and $(-\Delta)^\alpha$ are considered with respect the spatial variable $x \in O$. In the sequel, we will omit the variable x when no confusion is possible.

Finally, we introduce the Hilbert spaces

$$\mathcal{H} = L^2(O) \times L^2_g(\mathbb{R}^+, H^\alpha(O))$$

and

$$\mathcal{V} = H^\alpha(O) \times L^2_g(\mathbb{R}^+, H^{2\alpha}(O)).$$

To this end, along the lines of the procedure suggested by Dafermos in his pioneering work [12], we introduce the new variable

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^t u(x,r)dr, \quad s \geq 0,$$

and

$$u^t(x,s) = u(x,t-s), \quad s \geq 0.$$

Using (2.1), a formal integration by parts transforms Eq. (1.1)-(1.3) into

$$\frac{\partial u}{\partial t} + (-\Delta)^\alpha u + \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s)ds + f(u) = k(x) + h(x)\frac{dW}{dt}, \quad x \in O, \ t > 0, \tag{2.2}$$

$$\partial_t \eta^t = -\partial_s \eta^t + u, \quad x \in O, \ t > 0, \ s > 0, \tag{2.3}$$

with boundary condition

$$u(x,t) = 0, \quad x \in \partial O, \ t > 0, \tag{2.4}$$

and initial condition

$$u(x,t) = u_0(x,t), \ \eta^0(x,s) = \eta_0(x,s), \quad x \in O, \ t \leq 0, \ s > 0. \tag{2.5}$$

And the term

$$\eta^0(x,s) = \int_0^s u^0(x,r)dr = \int_{-s}^0 u(x,r)dr, \ x \in O, \quad s \geq 0,$$

is the prescribed initial integral past history of $u(x,t)$, which does not depend on $u_0(x,t)$, and is assumed to vanish on ∂O, as well as $u(x,t)$. As a consequence it follows that

$$\eta^t(x,s) = 0, \quad x \in \partial O, \quad t > 0 \text{ and } s > 0.$$

Indeed, the above assertion is obvious if $t \geq s$, and if $t < s$ we can write

$$\eta^t(x,s) = \eta_0(x, s-t) + \int_0^t u(x,r)dr.$$

In order to present our results, let us write system (2.2)-(2.5) as a Cauchy problem. Denote $w(t) = (u(t), \eta^t)$, $w_0 = (u_0, \eta_0)$, and set

$$Lw = (-(-\Delta)^\alpha u - \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s)ds, u - \partial_s \eta^t).$$

and

$$F(w, \theta_t\omega) = (k - f(u) + h\frac{dW}{dt}, 0).$$

Problem (2.2)-(2.5) can be written

$$\frac{dw}{dt} = Lw + F(w, \theta_t\omega) \tag{2.6}$$

$$w(x,t) = 0, \;\; x \in \partial\mathcal{O}, \; t > 0, \tag{2.7}$$

$$w(x,t) = w_0(x,t), \; x \in \mathcal{O}, \; t \leq 0. \tag{2.8}$$

Now we present our main results of this paper.

Theorem 3.4 *Assume that hypotheses (H1)-(H2) are satisfied and initial data $(u_0, \eta_0) \in \mathcal{H}$. Then, problem* (2.6)-(2.8) *possesses a unique mild solution in the class*

$$u \in C([0,\infty); L^2(\mathcal{O})), \; \text{and } \eta^t \in C([0,\infty); M). \tag{2.9}$$

If initial data $(u_0, \eta_0) \in D(L)$, then the solution is more regular, i.e., $u \in C([0,\infty); H^\alpha(\mathcal{O}))$, and $\eta^t \in C([0,\infty); M_1)$. In addition, if $w(t) = (u, \eta^t)$ and $\bar{w}(t) = (\bar{u}, \bar{\eta}^t)$ are two mild solutions of (2.6)-(2.8)*, then for any $T > 0$,*

$$\|w(t) - \bar{w}(t)\|^2_{\mathcal{H}} \leq e^{cT}\|w(0) - \bar{w}(0)\|^2_{\mathcal{H}}, \;\; 0 \leq t \leq T, \tag{2.10}$$

where $c > 0$ is a constant independent of the initial data.

The proof of Theorem 3.4 is presented in Section 3 by means of semigroup arguments.

The next main result of our paper concerns the generation of a random dynamical system, the existence of the corresponding random attractor and its finite Hausdorff dimension. These are included in Theorems 4.11 and 5.2 which are the content included in the theorem below (see Sections 4 and 5).

Theorem (See Theorem 4.11 and 5.2) *Assume that $k(\cdot) \in L^2(\mathcal{O})$ and that hypotheses (H1)-(H2) hold with $\alpha \in [\frac{1}{2}, 1)$ and $p \in [2, 1 + \frac{3}{3-2\alpha})$. Then the random dynamical system Φ generated by* (2.6)-(2.8) *possesses a random attractor $\mathcal{A}$ in $\mathcal{H}$. Moreover, if the second derivative of f is bounded, then the random attractor has finite Hausdorff dimension.*

3 Well-posedness

In this section, we show the existence, uniqueness and continuous dependence of mild solutions of the problem (2.6)-(2.8).

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we will identify $W(t)$ with $\omega(t)$, i.e., $\omega(t) = W(t, \omega)$, $t \in \mathbb{R}$.

Define the time shift by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system (see [1]).

To our end, we need to transform our stochastic equation (2.6) into a deterministic one with random parameters but without noise terms.

Writing

$$z^*(\omega) = -\int_{-\infty}^{0} e^s \omega(s) ds, \tag{3.1}$$

it is easy to check that $\bar{z}(t, \omega) = z^*(\theta_t \omega)$ is an Ornstein-Uhlenbeck process which solves the Itô equation

$$d\bar{z} + \bar{z}dt = dW.$$

Then, if we denote $z(\omega)(x) = z^*(\omega)h(x)$, it holds that the real-valued stochastic process $z(\theta_t\omega)(x) = z^*(\theta_t\omega)h(x)$ is solution to

$$dz + zdt = h(x)dW.$$

Now, we recall that (see Proposition 4.3.3 in [1]) that there exists $r_1(\omega) > 0$ tempered s.t.

$$|z^*(\omega)|^2 + |z^*(\omega)|^p + |(-\Delta)^{\frac{\alpha}{2}} z^*(\omega)|^2 + |(-\Delta)^{\alpha} z^*(\omega)|^2 \le r_0(\omega), \text{ where } r_0(\theta_t\omega) \le e^{\frac{\lambda}{2}|t|} r_0(\omega),$$

and λ will be specified later.

Then, it is straightforward to check that

$$|z(\omega)|^2 + |z(\omega)|^p + |(-\Delta)^{\frac{\alpha}{2}} z(\omega)|^2 + |(-\Delta)^{\alpha} z(\omega)|^2 \le r(\omega), \tag{3.2}$$

where $r(\omega)$ satisfies the same as $r_0(\omega)$.

It is well known (see [1, 7]) that there is a θ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full P measure such that for every $\omega \in \tilde{\Omega}$, $t \to z(\theta_t \omega)$ is continuous in t. For convenience, in the following we write $\tilde{\Omega}$ as Ω whenever no confusion is possible.

Then it follows from (3.2) that, for P-a.e. $\omega \in \Omega$,

$$|z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p + |(-\Delta)^{\frac{\alpha}{2}} z(\theta_t\omega)|^2 + |(-\Delta)^{\alpha} z(\theta_t\omega)|^2 \le e^{\frac{\lambda}{2}|t|} r(\theta_t\omega), \ t \in \mathbb{R}. \tag{3.3}$$

Formally, if u solves Eq. (2.2), then the variable $v(t) = u(t) - z(\theta_t\omega)$ should satisfy

$$\frac{\partial v}{\partial t} + (-\Delta)^{\alpha} v + \int_0^{\infty} g(s)(-\Delta)^{\alpha} \eta^t(s) ds + f(v + z) = k(x) + z - (-\Delta)^{\alpha} z.$$

with boundary condition and initial condition:

$$v(x,t)=0,\ \eta^t(x,s)=0,\ \ x\in\partial\mathcal{O},\ t>0,\ \ v(x,t)=v_0(x,t),\ \eta^0(x,s)=\eta_0(x,s), x\in\mathcal{O}, s\geq 0, t\leq 0.$$

Similarly, we can write the above system as a Cauchy problem. To this end, denote $\varphi(t,\omega,\varphi_0)=(v(t,\omega,v_0),\eta^t(\omega,\eta_0(\cdot)))$ with $v_0=u_0-z(\omega)$, $\eta^0=\eta_0(\cdot)$, we have the following compact form

$$\begin{aligned}\frac{d\varphi}{dt}&=L\varphi+F(\varphi,\theta_t\omega)\\ \varphi(0,\omega,\varphi_0)&=(v_0,\eta_0(\cdot)):=\varphi_0,\end{aligned}\tag{3.4}$$

where

$$L\varphi=\left(-(-\Delta)^\alpha v-\int_0^\infty g(s)(-\Delta)^\alpha\eta^t(s)ds,-\partial_s\eta^t+v\right)\tag{3.5}$$

and

$$F(\varphi,\theta_t\omega)=\Big(k-f(u)-(-\Delta)^\alpha z(\theta_t\omega)+z(\theta_t\omega),z(\theta_t\omega)\Big).\tag{3.6}$$

With respect to the variable η^t, it can be shown as in Pata and Zucchi [34] that

$$\partial_t\eta^t=-\partial_s\eta^t+v+z(\theta_t\omega),\ \eta(0)=0,$$

can be considered $\partial_t\eta^t=T\eta^t+v+z(\theta_t\omega)$, where

$$T\eta^t=-\partial_s\eta^t,\ \ \eta^t\in D(T),$$

is the generator of a translation semigroup with domain

$$D(T)=\{\eta^t\in M|\partial_s\eta^t\in M,\ \eta(0)=0\}.$$

Since the domain of L is defined by

$$D(L)=\{\varphi\in\mathcal{H}|L\varphi\in\mathcal{H}\},$$

one has

$$D(L)=\Big\{(v,\eta^t)\in\mathcal{H}\mid v\in L^2(\mathcal{O}),\ \eta^t\in D(T),\ -(-\Delta)^\alpha v-\int_0^\infty g(s)(-\Delta)^\alpha\eta^t(s)ds\in L^2(\mathcal{O})\Big\}.$$

We begin with the following lemma, which is an important step to prove the existence of mild solution of problem (3.4).

Lemma 3.1. *Operator L is the infinitesimal generator of a C^0−semigroup of contractions e^{Lt} in $\mathcal{H}$.*

Proof. We show that L is m-dissipative in $\mathcal{H}$. By (H1) and the definition of $L\varphi$, we infer that

$$(L\varphi,\varphi)=-\|(-\Delta)^{\frac{\alpha}{2}}v\|^2+\frac{1}{2}\int_0^\infty g'(s)\|(-\Delta)^{\frac{\alpha}{2}}\eta^t(s)\|^2ds\leq 0,$$

for all $\varphi = (v, \eta^t) \in D(L)$. This proves that L is dissipative in $\mathcal{H}$.

Next we show that L is maximal, that is, for each $F \in \mathcal{H}$, there exists a solution $\varphi \in D(L)$ of

$$(I - L)\varphi = F.$$

Equivalently, for each $F = (f_1, f_2) \in \mathcal{H}$, there exists $\varphi = (v, \eta^t) \in D(L)$ such that

$$v + (-\Delta)^\alpha v + \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s)ds = f_1, \tag{3.7}$$

$$\eta^t - v + \partial_s \eta^t = f_2. \tag{3.8}$$

To solve system (3.7)-(3.8), we first multiply (3.8) by e^s and integrate over $(0, s)$. Then,

$$\eta^t = v(1 - e^{-s}) + \int_0^s e^{\tau - s} f_2(\tau)d\tau. \tag{3.9}$$

Including (3.9) into (3.7) we obtain, by denoting $k_1 = \int_0^\infty g(s)(1 - e^{-s})ds$,

$$v + (-\Delta)^\alpha v + k_1(-\Delta)^\alpha v = f_1 - \int_0^\infty g(s) \int_0^s e^{\tau - s}(-\Delta)^\alpha f_2(\tau)d\tau ds. \tag{3.10}$$

In order to solve equation (3.10) we define the bilinear form

$$a(w_1, w_2) = \int_O w_1 w_2 dx + \int_O (-\Delta)^{\frac{\alpha}{2}} w_1 \cdot (-\Delta)^{\frac{\alpha}{2}} w_2 dx + k_1 \int_O (-\Delta)^{\frac{\alpha}{2}} w_1 \cdot (-\Delta)^{\frac{\alpha}{2}} w_2 dx,\ w_1, w_2 \in H^\alpha(O).$$

It is easy to check that $a(w_1, w_2)$ is continuous and coercive in $H^\alpha(O)$. And we have

$$H^\alpha(O) \hookrightarrow L^2(O) \hookrightarrow H^{-\alpha}(O).$$

We now aim at applying the Lax-Milgram theorem. It suffices to prove that the right hand side of (3.10) is an element of $H^{-\alpha}(O)$. Obviously,

$$f_1 \in H^{-\alpha}(O).$$

Let f^* denote the last term in (3.10), and we only need to show that $f^* \in H^{-\alpha}(O)$. We apply arguments similar to those used by Giorgi [19]. For $w \in H^\alpha(O)$ with $\|(-\Delta)^{\frac{\alpha}{2}} w\| \le 1$,

$$\begin{aligned}
\left|(f^*, w)_{H^{-\alpha}, H^\alpha}\right| &= \left|\int_0^\infty g(s) \int_0^s e^{\tau - s}\left(\int_O (-\Delta)^{\frac{\alpha}{2}} f_2(\tau)(-\Delta)^{\frac{\alpha}{2}} w dx\right) d\tau ds\right| \\
&\le \int_0^\infty e^\tau \|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\| \int_\tau^\infty g(s)e^{-s} ds d\tau \\
&\le \int_0^\infty e^\tau g(\tau)\|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\| \int_\tau^\infty e^{-s} ds d\tau \\
&= \int_0^\infty g(\tau)\|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\| d\tau < \infty,
\end{aligned}$$

which implies that $f^* \in H^{-\alpha}(\mathcal{O})$. Then, thanks to Lax-Milgram's theorem, Eq. (3.10) has a weak solution

$$\tilde{v} \in H^{\alpha}(\mathcal{O}).$$

Now, in view of (3.8), it follows

$$\tilde{\eta}^t(s) = \tilde{v}(1 - e^{-s}) + \int_0^s f_2(\tau)e^{\tau-s}d\tau.$$

Let us show that $\tilde{\eta}^t \in M$. From (3.9), taking into account that $\tilde{v} \in H^{\alpha}(\mathcal{O})$, we obtain

$$\|(-\Delta)^{\frac{\alpha}{2}}\tilde{\eta}^t(s)\|^2 \leq \|(-\Delta)^{\frac{\alpha}{2}}\tilde{v}\|^2 + \int_0^s e^{\tau-s}\|(-\Delta)^{\frac{\alpha}{2}}f_2(\tau)\|^2 d\tau.$$

Then, as above in the proof of f^*,

$$\begin{aligned}\int_0^\infty g(s)\|(-\Delta)^{\frac{\alpha}{2}}\tilde{\eta}^t(s)\|^2 ds &\leq k_0\|(-\Delta)^{\frac{\alpha}{2}}\tilde{v}\|^2 + \int_0^\infty g(s)\int_0^s e^{\tau-s}\|(-\Delta)^{\frac{\alpha}{2}}f_2(\tau)\|^2 d\tau ds \\ &\leq k_0\|(-\Delta)^{\frac{\alpha}{2}}\tilde{v}\|^2 + \int_0^\infty g(\tau)\|(-\Delta)^{\frac{\alpha}{2}}f_2(\tau)\|^2 d\tau \\ &\leq k_0\|(-\Delta)^{\frac{\alpha}{2}}\tilde{v}\|^2 + \|f_2(\tau)\|_M^2 < \infty,\end{aligned}$$

and hence $\tilde{\eta}^t \in M$. It follows that

$$\tilde{\varphi} = (\tilde{v}, \tilde{\eta}^t) \in \mathcal{H}$$

is a weak solution of (3.7)-(3.8).

To complete the proof of maximality of L we prove that $\tilde{\varphi} \in D(L)$. Indeed, from (3.8) we see that

$$\partial_s \tilde{\eta}^t = f_2 + \tilde{v} - \tilde{\eta}^t \in M.$$

Obviously $\tilde{\eta}(0) = 0$, we conclude that $\tilde{\eta}^t \in D(T)$. By inspection of (3.7) we find that

$$(-\Delta)^{\alpha}\tilde{v} + \int_0^\infty g(s)(-\Delta)^{\alpha}\tilde{\eta}^t(s)ds = -\tilde{v} + f_1 \in L^2(\mathcal{O}).$$

Therefore $(\tilde{v}, \tilde{\eta}) \in D(L)$. □

Lemma 3.2. *The operator $F : \mathcal{H} \to \mathcal{H}$ defined in* (3.6) *is locally Lipschitz continuous.*

Proof. Let B be a bounded set in $\mathcal{H}$ and $\varphi, \bar{\varphi} \in B$. Writing $\varphi = (v, \eta^t)$, $\bar{\varphi} = (\bar{v}, \bar{\eta}^t)$ and using (H2), one obtains

$$\begin{aligned}\|F(\varphi, \theta_t\omega) - F(\bar{\varphi}, \theta_t\omega)\|_{\mathcal{H}}^2 &= \|f(\bar{\varphi}) - f(\varphi)\|_2^2 \\ &= \int_{\mathcal{O}} | f(\bar{v} + z) - f(v + z) |^2 dx \\ &= \int_{\mathcal{O}} | -f' \cdot (v - \bar{v}) |^2 dx \\ &\leq \int_{\mathcal{O}} | \alpha_3(v - \bar{v}) |^2 dx \\ &\leq \alpha_3^2\|v - \bar{v}\|^2 \\ &\leq \alpha_3^2\|\varphi - \bar{\varphi}\|_{\mathcal{H}}^2.\end{aligned}$$

□

To complete the existence of solution, we still need the following lemma.

Lemma 3.3. *Assume that (H1)-(H2) hold. Then for any fixed $T > 0$, the solution φ of problem* (3.4) *satisfies the following inequality:*

$$\|\varphi(t,\omega,\varphi_0)\|_{\mathcal{H}}^2 \leq \|\varphi_0\|_{\mathcal{H}}^2 + c\int_0^T e^{\lambda s}(\|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}} z(\theta_s\omega)\|^2)ds$$
$$+ c(e^{\lambda T} - 1), \quad \forall t \in [0, T].$$

Proof. Taking the inner product of (3.4) with φ in $\mathcal{H}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_{\mathcal{H}}^2 = (L\varphi, \varphi)_{\mathcal{H}} + (F(\varphi, \theta_t\omega), \varphi)_{\mathcal{H}}, \tag{3.11}$$

where

$$(L\varphi, \varphi)_{\mathcal{H}} = -\|(-\Delta)^{\frac{\alpha}{2}} v\|^2 - \int_0^\infty g(s)\int_{\mathcal{O}} (-\Delta)^{\frac{\alpha}{2}}\eta^t \cdot (-\Delta)^{\frac{\alpha}{2}} v dxds + (-\partial_s \eta^t + v, \eta^t)_M. \tag{3.12}$$

From (H1), we have

$$\begin{aligned}(-\partial_s\eta^t + v, \eta^t)_M &= \frac{1}{2}\int_0^\infty g'(s)\|(-\Delta)^{\frac{\alpha}{2}}\eta^t\|^2 ds + \int_0^\infty g(s)\int_{\mathcal{O}}(-\Delta)^{\frac{\alpha}{2}} v \cdot (-\Delta)^{\frac{\alpha}{2}}\eta^t dxds \\ &\leq -\frac{\delta}{2}\int_0^\infty g(s)\|(-\Delta)^{\frac{\alpha}{2}}\eta^t\|^2 ds + \int_0^\infty g(s)\int_{\mathcal{O}}(-\Delta)^{\frac{\alpha}{2}} v \cdot (-\Delta)^{\frac{\alpha}{2}}\eta^t dxds.\end{aligned} \tag{3.13}$$

On the other hand,

$$(F(\varphi, \theta_t\omega), \varphi)_{\mathcal{H}} = \int_{\mathcal{O}}\Big(k - f(u) - (-\Delta)^\alpha z + z\Big)v dx + (z, \eta^t)_M. \tag{3.14}$$

By Hölder's inequality and Young's inequality, we obtain

$$(z, \eta^t)_M = \int_0^\infty g(s)\int_{\mathcal{O}}(-\Delta)^{\frac{\alpha}{2}} z \cdot (-\Delta)^{\frac{\alpha}{2}}\eta^t dxds \leq \frac{\delta}{4}\|\eta^t\|_M^2 + c\|(-\Delta)^{\frac{\alpha}{2}} z\|^2. \tag{3.15}$$

From (H2), and Young's inequality,

$$\begin{aligned} -\int_{\mathcal{O}} f(u)v dx &\leq -\frac{\alpha_1}{2}\|u\|_p^p + c(1 + \|z\|^2 + \|z\|_p^p), \\ \int_{\mathcal{O}} kv dx &\leq \frac{\lambda_1}{8}\|v\|^2 + \frac{2\|k\|^2}{\lambda_1}, \\ (-(-\Delta)^\alpha z, v) &\leq \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}} v\|^2 + \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}} z\|^2, \\ (z, v) &\leq \frac{\lambda_1}{8}\|v\|^2 + \frac{2}{\lambda_1}\|z\|^2. \end{aligned} \tag{3.16}$$

It follows from (3.11)-(3.16) that

$$\frac{d}{dt}\|\varphi\|_{\mathcal{H}}^2 + \|(-\Delta)^{\frac{\alpha}{2}} v\|^2 + \frac{\delta}{2}\|\eta^t\|_M^2 + \alpha_1\|u\|_p^p \leq \frac{\lambda_1}{2}\|v\|^2 + c(1 + \|z\|^2 + \|z\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}} z\|^2).$$

By Young's inequality with $\frac{1}{p/2} + \frac{1}{p/(p-2)} = 1$, we obtain

$$\lambda_1 \|v\|^2 \le \frac{\alpha_1}{2^p} \|v\|_p^p + c|O| \le \alpha_1 \|u\|_p^p + c(1 + \|z\|_p^p).$$

Take $\lambda = \min\{\frac{\lambda_1}{2}, \frac{\delta}{2}\}$, then

$$\frac{d}{dt}\|\varphi\|_{\mathcal{H}}^2 + \lambda\|\varphi\|_{\mathcal{H}}^2 + \|(-\Delta)^{\frac{\alpha}{2}} v\|^2 \le c(1 + \|z\|^2 + \|z\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}} z\|^2). \tag{3.17}$$

By the Gronwall lemma,

$$\begin{aligned} \|\varphi(t, \omega, \varphi_0(\omega))\|_{\mathcal{H}}^2 \le e^{-\lambda t}\|\varphi_0(\omega)\|_{\mathcal{H}}^2 + c\int_0^t e^{\lambda(s-t)}\Big(1 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_p^p \\ + \|(-\Delta)^{\frac{\alpha}{2}} z(\theta_s\omega)\|^2\Big) ds. \end{aligned} \tag{3.18}$$

Notice that $z(\theta_t\omega)$ is continuous in t, for any fixed $T > 0$ and $t \in [0, T]$. Then, we obtain

$$\begin{aligned} \|\varphi(t, \omega, \varphi_0(\omega))\|_{\mathcal{H}}^2 \le \|\varphi_0(\omega)\|_{\mathcal{H}}^2 + c\int_0^T e^{\lambda s}\Big(\|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_p^p \\ + \|(-\Delta)^{\frac{\alpha}{2}} z(\theta_s\omega)\|^2\Big) ds + c(e^{\lambda T} - 1) < \infty. \end{aligned}$$

The proof is completed. □

Theorem 3.4. *(Well-posedness) Assume that hypotheses (H1)-(H2) are satisfied and initial data* $(u_0, \eta_0) \in \mathcal{H}$. *Then, problem* (2.6)-(2.8) *possesses a unique mild solution in the class*

$$u \in C([0, \infty); L^2(O)), \ \textit{and}\ \eta^t \in C([0, \infty); M). \tag{3.19}$$

If initial data $(u_0, \eta_0) \in D(L)$*, then the solution is more regular, i.e.,* $u \in C([0, \infty); H^\alpha(O))$*, and* $\eta^t \in C([0, \infty); M_1)$*. In addition, if* $w(t) = (u, \eta^t)$ *and* $\bar{w}(t) = (\bar{u}, \bar{\eta}^t)$ *are two mild solutions of* (2.6)-(2.8)*, then for any* $T > 0$*,*

$$\|w(t) - \bar{w}(t)\|_{\mathcal{H}}^2 \le e^{cT}\|w(0) - \bar{w}(0)\|_{\mathcal{H}}^2, \ \ 0 \le t \le T, \tag{3.20}$$

where $c > 0$ *is a constant independent of the initial data.*

Proof. From Lemma 3.1 and 3.2, and Lumer-Phillip's theorem (see for instance Pazy [35], Theorem 6.1.4 and 6.1.5), problem (3.4) has a unique local mild solution

$$\varphi(t, \omega, \varphi_0) = e^{Lt}\varphi_0(\omega) + \int_0^t e^{L(t-r)} F(\varphi(r, \omega, \varphi_0), \theta_r\omega) dr \tag{3.21}$$

defined in $[0, T]$.

Let us prove that $T = \infty$. Indeed, Lemma 3.3 implies that the local solution (v, η^t) cannot blow-up in finite time and thus $T = \infty$. Hence, problem (3.4) has a global solution $\varphi(\cdot, \omega, \varphi_0) \in C([0, \infty), \mathcal{H})$ with $\varphi(0, \omega, \varphi_0) = \varphi_0(\omega)$ for all $t \ge 0$. Then, (3.19) holds. Moreover, the continuity with respect to initial data, i.e. Eq. (3.20), follows from the representation formula (3.21) and the Lipschitz property of F. □

Note that $u(t,\omega,u_0) = v(t,\omega,u_0 - z(\omega)) + z(\theta_t\omega)$. Then the process $\phi = (u, \eta^t)$ is the solution of problem (1.1)-(1.3). We now define a mapping $\Phi : \mathbb{R}^+ \times \Omega \times \mathcal{H} \to \mathcal{H}$ by

$$
\begin{aligned}
\Phi(t,\omega)\phi_0 &= \phi(t,\omega,\phi_0) \\
&= (u(t,\omega,u_0), \eta^t(\omega,\eta_0)) \\
&= (v(t,\omega,u_0 - z(\omega)) + z(\theta_t\omega), \eta^t(\omega,\eta_0)), \quad \text{for all } (t,\omega,\phi_0) \in \mathbb{R}^+ \times \Omega \times \mathcal{H}.
\end{aligned}
\tag{3.22}
$$

Then Φ satisfies conditions (i)-(iii) in Definition 2.2. Therefore, Φ is a continuous random dynamical system associated with the fractional stochastic reaction-diffusion equation with memory on $\mathcal{O}$. In the next section, we establish uniform estimates for the solutions of problem (1.1)-(1.3) and prove the existence of a random attractor for Φ.

4 Existence of random attractor

In this section we prove the existence of random attractor for our problem. First we will recall some technical lemmas that will be necessary for our analysis.

4.1 Auxiliary technical lemmas

To describe the asymptotic behavior of the solutions to our system we need to recall the following Gagliardo-Nirenberg inequality.

Lemma 4.1. *(Gagliado-Nirenberg)(see [31]) Suppose that $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $u \in L^q(\mathcal{O})$ and its derivatives of order m, $D^m u$ belong to $L^r(\mathcal{O})$, where $1 \le q, r \le \infty$. Then for the derivatives $D^j u, 0 \le j < m$, there holds*

$$
\|D^j u\|_{L^p} \le c\|u\|^{\sigma}_{W^{m,r}} \|u\|^{1-\sigma}_{L^q}, \tag{4.1}
$$

where

$$
\frac{1}{p} = \frac{j}{n} + \sigma\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\sigma)\frac{1}{q},
$$

for all σ in the interval

$$
\frac{j}{m} \le \sigma < 1.
$$

Here the constant c depends only on n, m, j, q, r and σ.

Also we need to introduce the space $\mathcal{T}^p_b(\mathbb{R}, X)$ of $L^p_{loc}-$translation bounded functions with values in a Banach space X, namely

$$
\mathcal{T}^p_b(\mathbb{R}, X) = \left\{ f \in L^p_{loc}(\mathbb{R}, X) : \|f\|_{\mathcal{T}^p_b(\mathbb{R},X)} = \sup \left(\int_{\xi}^{\xi+1} \|f(y)\|^p_X dy \right)^{\frac{1}{p}} \right\}.
$$

In a similar way, given $\tau \in \mathbb{R}$, we define the space $\mathcal{T}^p_b([\tau, +\infty), X)$.

The following lemmas will be useful in this paper, and readers are referred to [20, 33] for more details.

Lemma 4.2. *Let ϕ be a non-negative, absolutely continuous function on $\mathbb{R}_\tau$, $\tau \in \mathbb{R}$, which satisfies for some $\epsilon > 0$ and $0 \le \sigma < 1$ the differential inequality*

$$\frac{d}{dt}\phi + \epsilon\phi \le \Lambda + m_1(t)\phi(t)^\sigma + m_2(t) \quad t \in \mathbb{R}_\tau,$$

where $\Lambda \ge 0$, and m_1 and m_2 are non-negative locally summable functions on $\mathbb{R}_\tau$. Then

$$\phi(t) \le \frac{1}{1-\sigma}\left[\phi(\tau)e^{-\epsilon(t-\tau)} + \frac{\Lambda}{\epsilon}\right] + \left[\int_\tau^t m_1(y)e^{-\epsilon(1-\sigma)(t-y)}dy\right]^{\frac{1}{1-\sigma}} + \frac{1}{1-\sigma}\int_\tau^t m_2(y)e^{-\epsilon(t-y)}dy,$$

for any $t \in \mathbb{R}_\tau$.

Lemma 4.3. *Let $m \in \mathcal{T}_b^p(\mathbb{R}, X)$ for some $\tau \in \mathbb{R}$. Then, for every $\epsilon > 0$,*

$$\int_\tau^t m(y)e^{-\epsilon(t-\tau)}dy \le c(\epsilon)\|m\|_{\mathcal{T}_b^p(\mathbb{R},X)},$$

where $c(\epsilon) = \frac{e^\epsilon}{1-e^{-\epsilon}}$.

Lemma 4.4. *([34]) Let $g \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ be a non-negative function, such that $g(s) = 0$ whenever $g(s_0) = 0$ and $s > s_0$, for some $s_0 \in \mathbb{R}^+$. Let B_0, B, B_1 be three Banach spaces, such that B_0 and B_1 are reflexive and*

$$B_0 \hookrightarrow B \hookrightarrow B_1$$

the first injection being compact. Let $\mathcal{N} \subset L_g^2(\mathbb{R}^+, B)$ satisfy the following hypotheses:

(i) $\mathcal{N}$ *is bounded in* $L_g^2(\mathbb{R}^+, B_0) \cap H_g^1(\mathbb{R}^+, B_1)$

(ii) $\sup_{\eta^t \in \mathcal{N}} \|\eta^t(\theta_{-t}\omega, \eta_0)\|_B^2 \le h(s), \quad \forall\, s \in \mathbb{R}^+$ *for some* $h \in L_g^1(\mathbb{R}^+)$.

Then $\mathcal{N}$ is relatively compact in $L_g^2(\mathbb{R}^+, B)$.

4.2 A priori estimates

Now, we first prove the existence of random absorbing sets for the RDS Φ, which is necessary to establish the existence of random attractors. From now on, we always assume that $\mathcal{D}$ is the collection of all tempered subsets of $\mathcal{H}$ with respect to $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. The next lemma shows that Φ has a random absorbing set in $\mathcal{H}$.

Lemma 4.5. *Assume that $(H1)-(H2)$ hold. Then there exists a random absorbing set $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ for Φ in $\mathcal{H}$, i.e., for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $P-a.e.\ \omega \in \Omega$, there is $T_{1B}(\omega) > 0$ such that*

$$\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega), \quad \forall\, t \ge T_{1B}(\omega).$$

Proof. The process is similar to that of Lemma 3.3 with slight modifications. We only sketch it. We first derive uniform estimates on $\varphi(t,\omega,\varphi_0) = (v(t,\omega,v_0),\eta^t(\omega,\eta_0)) = (u(t,\omega,u_0)-z(\theta_t\omega),\eta^t(\omega,\eta_0))$, from which the uniform estimates on $\phi = (u(t,\omega,u_0),\eta^t(\omega,\eta_0))$ follow immediately.

Multiply (3.17) by $e^{\lambda t}$ and integrate over $[0,t]$ to obtain

$$\begin{aligned}&\|\varphi(t,\omega,\varphi_0)\|_{\mathcal{H}}^2 + \int_0^t e^{\lambda(s-t)}\|(-\Delta)^{\frac{\alpha}{2}}v(s,\omega,v_0(\omega))\|^2 ds\\ &\le e^{-\lambda t}\|\varphi_0(\omega)\|_{\mathcal{H}}^2 + c\int_0^t e^{\lambda(s-t)}(1+\beta(\theta_s\omega))ds,\end{aligned}\tag{4.2}$$

where

$$\beta(\theta_t\omega) := \|z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}}z(\theta_t\omega)\|^2 \le r(\theta_t\omega),$$

and $r(\theta_t\omega)$ satisfies

$$r(\theta_t\omega) \le e^{\frac{\lambda}{2}|t|}r(\omega),\ \ t\in\mathbb{R}.$$

Replacing ω by $\theta_{-t}\omega$ in (4.2) yields

$$\begin{aligned}&\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_{\mathcal{H}}^2 + \int_0^t e^{\lambda(s-t)}\|(-\Delta)^{\frac{\alpha}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds\\ &\le e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + c\int_0^t e^{\lambda(s-t)}(1+\beta(\theta_{s-t}\omega))ds\\ &\le e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + c\int_{-t}^0 e^{\lambda s}e^{-\frac{\lambda}{2}s}r(\omega)dr + c(1-e^{-\lambda t})\\ &\le e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + \frac{cr(\omega)}{\lambda}(1-e^{-\frac{\lambda}{2}t}) + c.\end{aligned}\tag{4.3}$$

Note that $\Phi(t,\omega)\phi_0(\omega) = \phi(t,\omega,\phi_0(\omega)) = (v(t,\omega,u_0-z(\omega))+z(\theta_t\omega),\eta^t(\omega,\eta_0))$. Consequently, from (4.3), we have, for all $t\ge 0$,

$$\begin{aligned}&\|\Phi(t,\theta_{-t}\omega)\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2\\ &= \|v(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)-z(\theta_{-t}\omega))+z(\omega)\|^2 + \|\eta^t(\theta_{-t}\omega,\eta_0(\theta_{-t}\omega))\|_M^2\\ &\le 2\|v(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)-z(\theta_{-t}\omega))\|^2 + 2\|z(\omega)\|^2 + \|\eta^t(\theta_{-t}\omega,\eta_0(\theta_{-t}\omega))\|_M^2\\ &\le 2e^{-\lambda t}(\|u_0(\theta_{-t}\omega)-z(\theta_{-t}\omega)\|^2 + \|\eta_0(\theta_{-t}\omega)\|_M^2) + cr(\omega) + c + 2\|z(\omega)\|^2\\ &\le 4e^{-\lambda t}(\|u_0(\theta_{-t}\omega)\|^2 + \|\eta_0(\theta_{-t}\omega)\|_M^2 + \|z(\theta_{-t}\omega)\|^2) + cr(\omega) + c + 2\|z(\omega)\|^2\\ &= 4e^{-\lambda t}(\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + \|z(\theta_{-t}\omega)\|^2) + cr(\omega) + c + 2\|z(\omega)\|^2.\end{aligned}\tag{4.4}$$

Since $\phi_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)(\in\mathcal{D})$ and $\|z(\omega)\|^2$ is tempered, there exists $T_{1B}(\omega)>0$, such that for all $t\ge T_{1B}(\omega)$,

$$4e^{-\lambda t}(\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + \|z(\theta_{-t}\omega)\|^2) \le cr(\omega) + c,$$

which along with (4.4) shows that, for all $t\ge T_{1B}(\omega)$,

$$\|\Phi(t,\theta_{-t}\omega)\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 \le c(1+r(\omega)+\|z(\omega)\|^2) := R_0(\omega).\tag{4.5}$$

Given $\omega \in \Omega$, denote by

$$K(\omega) = \left\{ \phi \in \mathcal{H} : \ \|\phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega))\|_{\mathcal{H}}^2 \le R_0(\omega) \right\}.$$

It is obviously that $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Further, (4.5) indicates that $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for Φ in $\mathcal{H}$, which completes the proof. □

We next derive uniform estimates for u in $H^\alpha(O)$.

Lemma 4.6. *Assume that* $(H1) - (H2)$ *hold. Let* $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. *Then there exists* $T_{2B}(\omega) > T_{1B}(\omega)$, *such that for all* $t \ge T_{2B}(\omega)$ *and* $P - a.e.\ \omega \in \Omega$, *it follows*

$$\int_t^{t+1} e^{\lambda(s-t)} \|(-\Delta)^{\frac{\alpha}{2}} u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 ds \le c R_0(\omega),$$

where $R_0(\omega)$ *is defined as in Lemma 4.5.*

Proof. By a similar procedure as it was done in Lemma 4.3 in [3], we can obtain

$$\int_t^{t+1} \|(-\Delta)^{\frac{\alpha}{2}} v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 ds \le c(1 + r(\omega)).$$

On the other hand,

$$\begin{aligned} &\|(-\Delta)^{\frac{\alpha}{2}} u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 = \|(-\Delta)^{\frac{\alpha}{2}} v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) + (-\Delta)^{\frac{\alpha}{2}} z(\theta_{s-t-1}\omega)\|^2 \\ &\le 2\|(-\Delta)^{\frac{\alpha}{2}} v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 + 2\|(-\Delta)^{\frac{\alpha}{2}} z(\theta_{s-t-1}\omega)\|^2 \text{ for all } t \ge 0. \end{aligned} \tag{4.6}$$

Integrating inequality (4.6) with respect to s over $[0, t]$, one can check that there exists $T_{2B}(\omega) > T_{1B}(\omega)$, such that for all $t > T_{2B}(\omega)$ we have

$$\int_t^{t+1} \|(-\Delta)^{\frac{\alpha}{2}} u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 ds \le c(1 + r(\omega) + \|z(\omega)\|^2).$$

The proof is therefore completed. □

In order to show the existence of random attractor for Φ associated with the problem (1.1)-(1.3), we need to prove the existence of compact measurable attracting set of Φ.

4.3 Asymptotic compactness

In this subsection, our main purpose is to obtain a random compact attracting set of Φ. To this end, we decompose the solution of (3.4) into a sum of two parts: one decays exponentially and the other is bounded in a "higher regular" space by using the method in [23], and obtain some a priori estimates for the solutions, which are the basis to construct a compact measurable attracting set for Φ. More precisely, we split the solution φ to (3.4) as the sum $\varphi = \varphi_L + \varphi_N$, where $\varphi_L = \varphi_L(t, \omega, \varphi_0) = (v_L, \eta_L^t)$ and $\varphi_N = \varphi_N(t, \omega, \varphi_0) = (v_N, \eta_N^t)$ satisfy, respectively,

$$\begin{cases} \partial_t \varphi_L = L\varphi_L, \\ \varphi_L(t, \omega, \varphi_0) = \varphi_{0L}(\omega) = (v_0, \eta_0), \quad s \ge 0, \end{cases} \tag{4.7}$$

and

$$\begin{cases} \partial_t \varphi_N = L\varphi_N + F(\varphi, \theta_t\omega), \\ \varphi_N(t, \omega, \varphi_0) = (0, 0), \quad s \geq 0. \end{cases} \tag{4.8}$$

First we have to show that φ_L has an exponential decay, that is,

$$\|\varphi_L(t, \theta_{-t}\omega, \varphi_{0L}(\theta_{-t}\omega))\|_{\mathcal{H}}^2 \leq e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2, \ \ \forall\ \varphi_0(\theta_{-t}\omega) \in \mathcal{H}. \tag{4.9}$$

It is apparent that the solution φ_L to (4.7) fulfills the estimates (4.3) with $c = 0$, namely,

$$\|\varphi_L(t, \theta_{-t}\omega, \varphi_{0L}(\theta_{-t}\omega))\|_{\mathcal{H}}^2 \leq e^{-\lambda t}\|\varphi_{0L}(\theta_{-t}\omega)\|_{\mathcal{H}}^2 = e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2. \tag{4.10}$$

Note that $\phi_L = \varphi_L$, we have

$$\|\phi_L(t, \theta_{-t}\omega, \phi_{0L}(\theta_{-t}\omega))\|_{\mathcal{H}}^2 \leq e^{-\lambda t}\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2. \tag{4.11}$$

Since

$$\|\phi_N(t, \theta_{-t}\omega, 0)\|_{\mathcal{H}}^2 \leq 2\|\phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega))\|_{\mathcal{H}}^2 + 2\|\phi_L(t, \theta_{-t}\omega, \phi_{0L}(\theta_{-t}\omega))\|_{\mathcal{H}}^2,$$

we also have

$$\|\phi_N(t, \theta_{-t}\omega, 0)\|_{\mathcal{H}}^2 \leq 10e^{-\lambda t}\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + c(1 + r(\omega)). \tag{4.12}$$

For further reference, we denote by $\eta_N^t(\omega, \eta_0)$ the second component of the solution ϕ_N to (4.7) at time t with initial time 0 and initial value $\phi(0, \omega, \phi_0) = \phi_0(\omega)$. Observe that η_N^t can be computed explicitly from the second component of (4.8) and the zero boundary data as follows:

$$\eta_N^t(\omega, \eta_0) = \begin{cases} \int_0^s u_N(t - r)dr, & 0 < s \leq t, \\ \int_0^t u_N(t - r)dr, & s > t. \end{cases} \tag{4.13}$$

Our goal is to build a compact attracting set for the random dynamical system Φ.

Lemma 4.7. *Assume that (H1)-(H2) hold,* $\alpha \in [\frac{1}{2}, 1)$ *and* $p \in [2, 1 + \frac{3}{3-2\alpha})$*. Then there exists* $T_{3B}(\omega) > T_{2B}(\omega)$*, such that for all* $t \geq T_{3B}(\omega)$ *and P-a.e.* $\omega \in \Omega$*, it follows*

$$\|\phi_N(t, \theta_{-t}\omega, 0)\|_{\mathcal{V}}^2 + \frac{1}{2}\int_0^t e^{\lambda(s-t)}\|(-\Delta)^\alpha v_N(s, \theta_{-t-1}\omega, 0)\|^2 ds \leq R_1(\omega),$$

where $R_1(\omega) := c\left(1 + C(\lambda)\left(R_0^2(\omega) + R_0(\omega) + r(\omega) + 1\right)\right)$.

Proof. First we take the inner product of the first part of (4.8) with $(-\Delta)^\alpha v_N$ in $L^2(O)$ to deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 = -\|(-\Delta)^\alpha v_N\|^2 - \int_0^\infty g(s)\int_O (-\Delta)^\alpha \eta_N^t \cdot (-\Delta)^\alpha v_N dxds \\ + \int_O (k - f(u) - (-\Delta)^\alpha z + z) \cdot (-\Delta)^\alpha v_N dx. \end{aligned} \tag{4.14}$$

Using (H2), Lemma 4.1 and Young's inequality, we obtain

$$
\begin{aligned}
-\int_{\mathcal{O}} f(u)(-\Delta)^{\alpha} v_N dx &\le \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c\|1+|u|^{p-1}\|^2 \\
&\le \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c + c\|u\|_{2p-2}^{2p-2} \\
&\le \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c + c\|(-\Delta)^{\frac{\alpha}{2}} u\|^{\zeta}\|u\|^{1-\zeta} \\
&\le \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c + c(1+\|(-\Delta)^{\frac{\alpha}{2}} u\|^2)(1+\|u\|^2),
\end{aligned}
\tag{4.15}
$$

where $\zeta = \frac{3}{2\alpha}(\frac{p-2}{p-1})$.

On the other hand, by Young's inequality, we have

$$
\begin{aligned}
\int_{\mathcal{O}} (k(x) - (-\Delta)^{\alpha} z)\cdot(-\Delta)^{\alpha} v_N dx &\le \frac{3}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c(1+\|(-\Delta)^{\alpha} z\|^2), \\
\int_{\mathcal{O}} z\cdot(-\Delta)^{\alpha} v_N dx &\le \frac{\lambda_1}{4}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 + \frac{1}{\lambda_1}\|(-\Delta)^{\frac{\alpha}{2}} z\|^2.
\end{aligned}
$$

By the precedent inequalities,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 &= \frac{\lambda_1}{4}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 - \frac{1}{2}\|(-\Delta)^{\alpha} v_N\|^2 - \int_0^{\infty} g(s)\int_{\mathcal{O}} (-\Delta)^{\alpha}\eta_N^t\cdot(-\Delta)^{\alpha} v_N dxds \\
&\quad + c(1+\|(-\Delta)^{\frac{\alpha}{2}} u\|^2)(1+\|u\|^2) + c(1+\|(-\Delta)^{\frac{\alpha}{2}} z\|^2 + \|(-\Delta)^{\alpha} z\|^2).
\end{aligned}
\tag{4.16}
$$

Taking now the inner product of the second part of (4.8) with $(-\Delta)^{2\alpha}\eta_N^t$, and thanks to similar computations as above,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{2\alpha}\eta_N^t\|_{M_1}^2 &= -\int_0^{\infty} g(s)\int_{\mathcal{O}} \eta_{N,s}^t\cdot(-\Delta)^{2\alpha}\eta_N^t dxds + \int_0^{\infty} g(s)\int_{\mathcal{O}} v\cdot(-\Delta)^{2\alpha}\eta_N^t dxds \\
&\quad + \int_0^{\infty} g(s)\int_{\mathcal{O}} z\cdot(-\Delta)^{2\alpha}\eta_N^t dxds \\
&\le -\frac{\delta}{4}\int_0^{\infty} g(s)\|(-\Delta)^{\alpha}\eta_N^t\|^2 ds - \int_0^{\infty} g(s)\int_{\mathcal{O}} (-\Delta)^{\alpha} v\cdot\Delta\eta_N^t dxds + c\|(-\Delta)^{\alpha} z\|^2.
\end{aligned}
\tag{4.17}
$$

Adding (4.16) and (4.17),

$$
\begin{aligned}
&\frac{d}{dt}\|\varphi_N\|_{\mathcal{V}}^2 + \frac{\delta}{2}\|(-\Delta)^{\alpha}\eta_N^t\|_{M_1}^2 + \|(-\Delta)^{\alpha} v_N\|^2 \\
&\quad \le \frac{\lambda_1}{2}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 + cp(\theta_t\omega) + c(1+\|(-\Delta)^{\frac{\alpha}{2}} u\|^2)(1+\|u\|^2) + c,
\end{aligned}
$$

Using Gagliardo-Nirenberg's inequality,

$$
\lambda_1\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 \le \frac{1}{2}\|(-\Delta)^{\alpha} v_N\|^2 + c\|v_N\|^2.
$$

Taking $\lambda = \min\{\frac{\lambda_1}{2}, \frac{\delta}{2}\}$, by the previous inequalities we have

$$
\frac{d}{dt}\|\varphi_N\|_{\mathcal{V}}^2 + \lambda\|\varphi_N\|_{\mathcal{V}}^2 + \frac{1}{2}\|(-\Delta)^{\alpha} v_N\|^2 \le cp(\theta_t\omega) + c(1+\|(-\Delta)^{\frac{\alpha}{2}} u\|^2)(1+\|u\|^2) + c\|v_N\|^2 + c, \tag{4.18}
$$

where $p(\theta_t\omega) = (1 + \|(-\Delta)^{\frac{\alpha}{2}} z(\theta_t\omega)\|^2 + \|(-\Delta)^{\alpha} z(\theta_t\omega)\|^2)$.

On the one hand, Lemma 4.5 and Lemma 4.6 ensure that there exists $T_{3B}(\omega) > T_{2B}(\omega)$ such that for all $t \geq T_{3B}(\omega)$,

$$\begin{aligned}&\int_t^{t+1} (1 + \|(-\Delta)^{\frac{\alpha}{2}} u(s, \theta_{s-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2)(1 + \|u(s, \theta_{s-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2) ds \\ &+ \int_t^{t+1} \|v_N(s, \theta_{s-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 ds + \int_t^{t+1} p(\theta_{s-t-1}\omega) ds \\ &\leq cR_0^2(\omega) + cR_0(\omega) + cr(\omega) + c.\end{aligned}$$

Then, by Lemma 4.2 and Lemma 4.3 we can prove that for all $t \geq T_{3B}(\omega)$,

$$\begin{aligned}&\|\phi_N(t, \theta_{-t}\omega, 0)\|_V^2 + \frac{1}{2}\int_0^t e^{\lambda(s-t)} \|(-\Delta)^{\alpha} v_N(s, \theta_{-t-1}\omega, 0)\|^2 ds \\ &\leq c\left(1 + C(\lambda)\left(R_0^2(\omega) + R_0(\omega) + r(\omega) + 1\right)\right) := R_1(\omega),\end{aligned}$$

as claimed. □

Remark 4.8. *Notice that, unlike the previous results, we are imposing now some restrictions on the values of* α *and* p *in Lemma 4.7. Indeed, the constant* $\zeta = \frac{3}{2\alpha}(\frac{p-2}{p-1})$*, appearing in* (4.15)*, must belong to the interval* $(0, 1)$*, and this implies that, for a given* $\alpha \in [\frac{1}{2}, 1)$*,* p *has to belong to the interval* $[2, 1 + \frac{3}{3-2\alpha})$ *(see Figure 1 below). We would like to emphasize that the statement in Lemma 4.7 also holds true for* $\alpha \in (0, \frac{1}{2})$*, but as we will need to impose* $\alpha \in [\frac{1}{2}, 1)$ *in Lemma 4.10 to ensure asymptotic compactness of our random dynamical system, we prefer to state it in this way.*

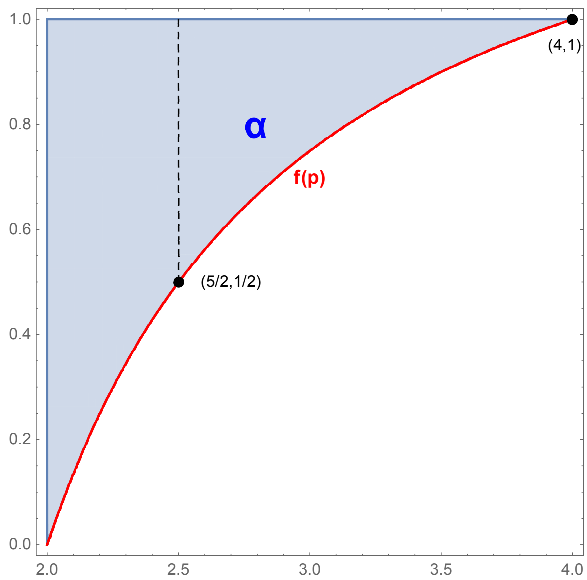

Figure 1: $\alpha > f(p) = \frac{3}{2}(1 - \frac{1}{p-1})$

Remark 4.9. *[6] obtained pullback attractor for random dynamical systems associated to* (1.1) *with* $\alpha = 1$ *and* $p \geq 1$*, while [9] investigated the deterministic version of* (1.1) *(i.e.* $h(x) = 0$*) with* $\alpha = 1$ *dealing with global attractors for the whole range* $p < 4$*. And in [27], authors considered* (1.1) *with* $\mu = 0$ *in the whole space* $\mathbb{R}^n$*, they assume that* $p > 1$ *and* $\alpha \in [\frac{1}{2}, 1)$ *hold, and proved random attractor in* $L^2(\mathbb{R}^n)$.

We now are in the position to finalize the proof of the existence of a random attractor.

Lemma 4.10. *Assume that* $(H1) - (H2)$ *hold,* $\alpha \in [\frac{1}{2}, 1)$ *and* $p \in [2, 1 + \frac{3}{3-2\alpha})$*. Denote by*

$$\mathcal{N} = \bigcup_{\eta_0 \in K(\theta_{-t}\omega)} \bigcup_{t \geq T_{3B}(\omega)} \bigcup_{\omega \in \Omega} \eta^t_N(\theta_{-t}\omega, \eta_0),$$

where $\{K(\omega)\}_{\omega\in\Omega}$ *is defined in Lemma 4.5 and* $T_{3B}(\omega)$ *is defined in Lemma 4.7.*

Then $\mathcal{N}$ *is relatively compact in* $L^2_g(\mathbb{R}^+, H^\alpha(\mathcal{O}))$.

Proof. It is clear from Lemma 4.7 that $\mathcal{N}$ is bounded in $L^2_g(\mathbb{R}^+, H^{2\alpha}(\mathcal{O}))$. Let $\eta^t_N \in \mathcal{N}$. The derivative of (4.13) yields

$$\frac{\partial}{\partial s}\eta^t_N = \begin{cases} u_N(t-s), & 0 < s \leq t, \\ 0, & s > t. \end{cases} \tag{4.19}$$

Thus

$$\begin{aligned} \int_0^\infty g(s) \left\| \frac{\partial}{\partial s}\eta^t_N \right\|^2 ds &= \int_0^t g(s)\|u_N(t-s)\|^2 ds = \int_0^t g(t-s)\|u_N(s)\|^2 ds \\ &\leq g(0) \int_0^t e^{\lambda(s-t)}\|u_N(s)\|^2 ds < \infty. \end{aligned} \tag{4.20}$$

We then conclude that $\mathcal{N}$ is bounded in $L^2_g(\mathbb{R}^+, H^{2\alpha}(\mathcal{O})) \cap H^1_g(\mathbb{R}^+, L^2(\mathcal{O}))$. Moreover, we can verify that, for every $\eta^t \in \mathcal{N}$,

$$\sup_{\eta^t \in \mathcal{N}, s \geq 0} \|\nabla\eta^t\|^2 = \begin{cases} s \cdot \int_{t-s}^t \|\nabla u_N(r)\|^2 dr, & 0 < s \leq t, \\ s \cdot \int_0^t \|\nabla u_N(r)\|^2 dr, & s > t. \end{cases}$$

By the embedding $H^{2\alpha}(\mathcal{O}) \hookrightarrow H^1_0(\mathcal{O})$, we find that

$$\sup_{\eta^t \in \mathcal{N}, s \geq 0} \|\nabla\eta^t\|^2 \leq s e^{\lambda s} \cdot \int_0^t e^{\lambda(r-t)}\|(-\Delta)^\alpha u_N(r)\|^2 dr := h(s), \quad t \geq 0.$$

Consequently, from Lemma 4.7 and the relation $u = v + z$, it is obvious that

$$\int_0^\infty g(s)\|\nabla\eta^t(s)\|^2 ds \leq \int_0^\infty s g(s) e^{\lambda s} ds \int_0^t e^{\lambda(r-t)}\|(-\Delta)^\alpha u_N(r)\|^2 dr < \infty,$$

which shows that $\mathcal{N} \subset L^2_g(\mathbb{R}^+, H^\alpha(\mathcal{O}))$ is a bounded set and $h(s) \in L^1_g(\mathbb{R}^+)$. Using Lemma 4.4, the proof can be completed immediately. □

Now we restate our main result about existence of random attractor for the RDS Φ:

Theorem 4.11. *Assume that* $(H1)-(H2)$ *hold,* $\alpha \in [\frac{1}{2}, 1)$ *and* $p \in [2, 1+\frac{3}{3-2\alpha})$*. Then for every* $\omega \in \Omega$*, the random dynamical system* Φ *associated with Eq.* (1.1) *possesses a compact random attracting set* $\tilde{K}(\omega) \subset \mathcal{H}$ *and possesses a random attractor* $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *with* $\mathcal{A}(\omega) = \tilde{K}(\omega) \cap K(\omega)$*, where* $K = \{K(\omega)\}_{\omega\in\Omega}$ *is defined in Lemma 4.5.*

Proof. Let $B_{\mathcal{V}}(\omega)$ be the closed ball in $\mathcal{V} = H^{\alpha}(\mathcal{O}) \times L^2_g(\mathbb{R}^+; H^{2\alpha}(\mathcal{O}))$ of radius $R_1(\omega)$. Setting $\tilde{K}(\omega) = B_{\mathcal{V}}(\omega) \times \overline{\mathcal{N}}$ with $\overline{\mathcal{N}}$ is the closure of $\mathcal{N}$, which is defined in Lemma 4.10. Since $H^{\alpha}(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ is compact and $\mathcal{N}$ is compact in $L^2_g(\mathbb{R}^+; H^{\alpha}(\mathcal{O}))$. Thus, $\tilde{K}(\omega)$ is compact in $\mathcal{H}$ with $\mathcal{H} = L^2(\mathcal{O}) \times L^2_g(\mathbb{R}^+; H^{\alpha}(\mathcal{O}))$. Now we show the following attracting property of $\tilde{K}(\omega)$ holds for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, i.e.,

$$\lim_{t\to+\infty} \operatorname{dist}_{\mathcal{H}}\left(\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \tilde{K}(\omega)\right) = 0. \tag{4.21}$$

By Lemma 4.5, there exits $t^* = t^*(B) > 0$ such that

$$\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega), \quad \forall t \geq t^*, \tag{4.22}$$

where $K = \{K(\omega)\}_{\omega\in\Omega}$ is the absorbing set for Φ in $\mathcal{H}$.

Setting $t = \tilde{t} + t^* + t_1 > 0$, and using the cocycle properties, we deduce that

$$\begin{aligned}\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) &= \Phi(t - t^* - t_1, \theta_{t^*+t_1}\theta_{-t}\omega) \circ \Phi(t^* + t_1, \theta_{-t}\omega)B(\theta_{-t}\omega) \\ &\subset \Phi(\tilde{t}, \theta_{-\tilde{t}}\omega)K(\omega).\end{aligned} \tag{4.23}$$

Pick any $\phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega)) \in \Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)$ for $t \geq t^* + t_1 > 0$. Applying now Lemma 4.7 with $T_{3B}(\omega) = t^* + t_1$ implies

$$\|(-\Delta)^{\frac{\alpha}{2}} u_N\|^2 \leq \|\phi_N\|^2_{\mathcal{V}} \leq c(R_1(\omega) + \|(-\Delta)^{\frac{\alpha}{2}} z(\omega)\|^2).$$

It is then clear that $\phi_N = (u_N, \eta^t_N) \in \tilde{K}(\omega)$. Therefore, from (4.11),

$$\inf_{m\in\tilde{K}(\omega)} \|\phi(t) - m\|_{\mathcal{H}} \leq \|\phi_L\|_{\mathcal{H}} \leq e^{-\frac{\lambda}{2}t}\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}, \quad \forall\ t > t^* + t_1.$$

We conclude that

$$\operatorname{dist}_{\mathcal{H}}\left(\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \tilde{K}(\omega)\right) \leq e^{-\frac{\lambda}{2}t}\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}} \to 0, \ \text{ as } \ t \to +\infty.$$

The proof follows immediately from Proposition 2.8. □

5 Hausdorff dimension

In this section, we prove that the random attractor $\mathcal{A}(\omega)$, whose existence has been proved in Section 4, has finite Hausdorff dimension. To this end, we need the following condition on f:

$$|f''(u)| \leq \beta_1, \ \text{ for some } \beta_1 > 0. \tag{5.1}$$

Set $\Phi(\omega) = \Phi(1,\omega)$ and consider the following first variant equation of equation (2.6),

$$\frac{d\tilde{W}}{dt} = L\tilde{W} + F'(\tilde{W},\theta_t\omega)\tilde{W}, \tag{5.2}$$

with

$$\tilde{W}(x,t) = \tilde{W}_0(x,t) = h,\ t \leq 0, \tag{5.3}$$

$$F^{'}(\tilde{W},\theta_t\omega)\tilde{W} = (-f^{'}(u)U(t),0) \tag{5.4}$$

and

$$L\tilde{W} = (-(-\Delta)^{\alpha}U - \int_0^{\infty} g(s)(-\Delta)^{\alpha}V(s)ds, U - V_s), \tag{5.5}$$

where $\tilde{W} = (U(t), V(t))$ with $U(t)$, $V(t)$ are the derivative of $u(t)$, η^t of problem (2.6), respectively.

Lemma 5.1. *Assume that $(H1)-(H2)$ hold, $\alpha \in [\frac{1}{2},1)$ and $p \in [2, 1+\frac{3}{3-2\alpha})$, and (5.1) is fulfilled. Then the mapping $\Phi(\omega)$ is almost surely uniformly differentiable on $\mathcal{A}(\omega)$: P-a.e. $\omega \in \Omega$, for every $w \in \mathcal{A}(\omega)$, there exists a bounded linear operator $D\Phi(\omega,w)$ such that if w and $w+h$ are in $\mathcal{A}(\omega)$, there holds*

$$\|\Phi(\omega)(w+h) - \Phi(\omega)(w) - D\Phi(\omega,w)h\|_{\mathcal{H}} \leq \overline{k}(\omega)\|h\|_{\mathcal{H}}^{1+\rho},$$

where $\rho > 0$ and $\overline{k}(\omega)$ is a random variable such that

$$\overline{k}(\omega) \geq 1, \quad E(\ln \overline{k}) < \infty, \quad \omega \in \Omega.$$

Moreover, for any $w \in \mathcal{A}(\omega)$, $D\Phi(\omega,w)h = \tilde{W}(1)$, where $\tilde{W}(t)$ is the solution of Eq.(5.2).

Proof. Let $w = (u(t),\eta^t)$, $\bar{w} = (\bar{u}(t),\bar{\eta}^t)$ be solutions to Eq.(2.6) with initial data $w(0) = w_0$, $\bar{w}(0) = \bar{w}_0$ and $w_0 - \bar{w}_0 = h$. Then $Y = w - \bar{w}$ satisfies the following problem

$$\frac{dY}{dt} = LY + F(w,\theta_t\omega) - F(\bar{w},\theta_t\omega) \tag{5.6}$$

with $F(w,\theta_t\omega) - F(\bar{w},\theta_t\omega) = (f(\bar{u}) - f(u), 0)$ and $Y_0 = w_0 - \bar{w}_0 = h$.

Taking the inner product of (5.6) with Y in $\mathcal{H}$, we obtain

$$\frac{d}{dt}\|Y\|_{\mathcal{H}}^2 = 2(LY,Y)_{\mathcal{H}} + 2(F(w,\theta_t\omega) - F(\bar{w},\theta_t\omega), Y)_{\mathcal{H}}. \tag{5.7}$$

Notice that

$$\begin{aligned} 2(F(w,\theta_t\omega) - F(\bar{w},\theta_t\omega), Y)_{\mathcal{H}} &= 2(f(\bar{u}) - f(u), u - \bar{u}) \\ &= -2(f'(u)(u-\bar{u}), u - \bar{u}) \\ &\leq 2\alpha_3\|u-\bar{u}\|^2 \\ &\leq 2\alpha_3\|Y\|_{\mathcal{H}}^2. \end{aligned} \tag{5.8}$$

and

$$2(LY,Y)_{\mathcal{H}} = -2\|(-\Delta)^{\frac{\alpha}{2}}(u-\bar{u})\|^2 + \int_0^{\infty} g'(s)\|(-\Delta)^{\frac{\alpha}{2}}(\eta^t - \bar{\eta}^t)\|^2 ds \leq 0. \tag{5.9}$$

Then, it follows from (5.7)-(5.9) that

$$\frac{d}{dt}\|Y\|_{\mathcal{H}}^2 \le 2\alpha_3\|Y\|_{\mathcal{H}}^2. \tag{5.10}$$

By Gronwall's lemma, we obtain

$$\|Y(t,\omega,Y_0)\|_{\mathcal{H}}^2 \le e^{2\alpha_3 t}\|h\|_{\mathcal{H}}^2, \quad \forall\, 0 \le t \le 1. \tag{5.11}$$

Now, set $Z = Y - \tilde{W}$, then

$$\frac{dZ}{dt} = LZ + F'(w,\theta_t\omega)Z + H(w,\bar{w}) \tag{5.12}$$

with

$$Z(x,t) = Z_0(x,t) = 0, \quad t \le 0, \tag{5.13}$$

where $Z = (u - \bar{u} - U, \eta^t - \bar{\eta}^t - V)$, $F'(w,\theta_t\omega)Z = (-f'(u)(u - \bar{u} - U), 0)$, while $H(w,\bar{w}) = (f'(u)(u - \bar{u}) - f(u) + f(\bar{u}), 0)$.

Take the inner product of (5.12) with Z in $\mathcal{H}$ to get

$$\frac{d}{dt}\|Z\|_{\mathcal{H}}^2 = 2(LZ,Z)_{\mathcal{H}} + 2(F'(w,\theta_t\omega)Z,Z)_{\mathcal{H}} + 2(H(w,\bar{w}),Z)_{\mathcal{H}}. \tag{5.14}$$

Note that

$$2(LZ,Z)_{\mathcal{H}} = -2\|(-\Delta)^{\frac{\alpha}{2}}(u - \bar{u} - U)\|_{\mathcal{H}}^2 + \int_0^\infty g'(s)\|(-\Delta)^{\frac{\alpha}{2}}(\eta^t - \bar{\eta}^t - V)\|^2 ds \le 0, \tag{5.15}$$

$$2(F'(w,\theta_t\omega)Z,Z)_{\mathcal{H}} \le 2\alpha_3\|u - \bar{u} - U\|^2 \le 2\alpha_3\|Z\|_{\mathcal{H}}^2, \tag{5.16}$$

and from (5.1) and Taylor's series, we derive

$$\begin{aligned} 2(H(w,\bar{w}),Z)_{\mathcal{H}} &= 2(f''(u)(u - \bar{u})^2, u - \bar{u} - U) \\ &\le c_1\|u - \bar{u}\|^4 + c\|u - \bar{u} - U\|^2 \le c_1\|u - \bar{u}\|^4 + c\|Z\|_{\mathcal{H}}^2. \end{aligned} \tag{5.17}$$

It follows from (5.14)-(5.17) that

$$\frac{d}{dt}\|Z\|_{\mathcal{H}}^2 \le c_2\|Z\|_{\mathcal{H}}^2 + c_1\|u - \bar{u}\|^4. \tag{5.18}$$

Therefore, by Gronwall's lemma, we find

$$\|Z\|_{\mathcal{H}}^2 \le c_1 e^{c_2 t}\int_0^t \|u(s) - \bar{u}(s)\|^4 ds, \tag{5.19}$$

which together with (5.11) gives that

$$\|Z(1)\|_{\mathcal{H}} \le C_1(\omega)\|h\|_{\mathcal{H}}^{1+\rho}, \tag{5.20}$$

where $C_1(\omega) = \sqrt{\frac{c_1 e^{c_2}}{4\alpha_3}(e^{4\alpha_3} - 1)}$ and $\rho = 1$. Choose $\bar{k}(\omega) = \max\{C_1(\omega), 1\}$. Hence, we obtain $E(\ln\bar{k}) < \infty$.

Therefore, $\Phi(\omega)$ is almost surely uniform differentiable on $\mathcal{A}(\omega)$. Furthermore, the differential of $\Phi(\omega)$ at w is $D\Phi(\omega,w)$. The proof is completed. □

Next, we check condition (iii) of Proposition 2.9. In fact, taking the inner product of (5.2) with $\tilde{W}$ in $\mathcal{H}$ and performing analogous calculations to those leading to (5.20), we obtain

$$\|\tilde{W}(1)\|_{\mathcal{H}}^2 \le e^{2\alpha_3+\delta}\|\tilde{W}_0\|_{\mathcal{H}}^2. \tag{5.21}$$

Since $\alpha_1(D\Phi(\omega, w))$ is equal to the norm of $D\Phi(\omega, w) \in L(\mathcal{H})$, we choose

$$\overline{\alpha_1(\omega)} = \max\left\{e^{\alpha_3+\frac{\delta}{2}}, 1\right\}.$$

Then one has

$$\alpha_1(D\Phi(\omega, w)) \le \overline{\alpha_1(\omega)},$$

and

$$E(\ln \overline{\alpha_1}) < \infty.$$

Theorem 5.2. *Assume that* $(H1)-(H2)$ *hold,* $\alpha \in [\frac{1}{2}, 1)$ *and* $p \in [2, 1+\frac{3}{3-2\alpha})$*, and* (5.1) *is fulfilled. Then the random attractor* $\mathcal{A}(\omega)$ *has finite Hausdorff dimension.*

Proof. Now, we only need to verify condition (ii) of Proposition 2.9.

To this end, let $\tilde{W} = (U, V)$ be a unitary vector belonging to the domain of $L+F'(\tilde{W}, \theta_t\omega)$ with $F'(\tilde{W}, \theta_t\omega)\tilde{W} = (-f'(u)U, 0)$. Then

$$\left(\left(L + F'(\tilde{W}, \theta_t\omega)\right)\tilde{W}, \tilde{W}\right)_{\mathcal{H}} = (L\tilde{W}, \tilde{W})_{\mathcal{H}} - (f'(u)U, U)_{L^2}. \tag{5.22}$$

By means of direct calculations

$$(L\tilde{W}, \tilde{W})_{\mathcal{H}} \le -\|(-\Delta)^{\frac{\alpha}{2}}U\|^2 - \frac{\delta}{2}\|V\|_M^2, \tag{5.23}$$

and

$$-(f'(u)U, U)_{L^2} \le \alpha_3\|U\|^2. \tag{5.24}$$

Thus,

$$\left((L + F'(\tilde{W}, \theta_t\omega))\tilde{W}, \tilde{W}\right)_{\mathcal{H}} \le -\|(-\Delta)^{\frac{\alpha}{2}}U\|^2 - \frac{\delta}{2}\|V\|_M^2 + \alpha_3\|U\|^2. \tag{5.25}$$

Therefore, we conclude that $L + F'(\tilde{W}, \theta_t\omega) \le A$, where A is the diagonal operator acting on $L^2(\mathcal{O}) \otimes L_g^2(\mathbb{R}^+, H^\alpha(\mathcal{O}))$ defined by

$$\begin{pmatrix} -(-\Delta)^\alpha + \alpha_3 I & 0 \\ 0 & -\frac{\delta}{2}(-\Delta)^\alpha \end{pmatrix}$$

From the definition of Tr_m (Definition 2.7), it is clear that $Tr_m(L + F'(\tilde{W}, \theta_t\omega)) \leq Tr_m(A)$. Since A is diagonal, it is easy to see that

$$Tr_m(A) = \sup_Q \sum_{j=1}^{m} (A\tilde{W}_j, \tilde{W}_j)_{\mathcal{H}},$$

where the supremum is taken over the projections Q of the form $Q_1 \otimes Q_2$. This amounts to consider vectors $\tilde{W}_j$ where only one of the two components is non-zero (and in fact of norm one in its space). Choose then $m > \max\{\beta_1, \beta_2\} > 0$, and let n_1, n_2 be the numbers of vectors $\tilde{W}_j$ of the form $(U, 0)$ and $(0, V)$, respectively. Using Sobolev-Lieb-Thirring's inequality, we have

$$Tr_m(A) \leq -\beta_1|O|^{\alpha} n_1^{1+\alpha} + n_1 - \frac{\delta\beta_2}{2}|O|^{\alpha} n_2^{1+\alpha} + \frac{\delta}{2} n_2 + \alpha_3 n_1, \tag{5.26}$$

and from [13] we can deduce that

$$\omega_m(D\Phi(\omega, w)) \leq \exp\left\{-\beta_1|O|^{\alpha} n_1^{1+\alpha} + (1 + \alpha_3)n_1 - \frac{\delta\beta_2}{2}|O|^{\alpha} n_2^{1+\alpha} + \frac{\delta}{2} n_2\right\}.$$

Denote

$$\overline{\omega}_m(\omega) = \exp\left\{-\beta_1|O|^{\alpha} n_1^{1+\alpha} + (1 + \alpha_3)n_1 - \frac{\delta\beta_2}{2}|O|^{\alpha} n_2^{1+\alpha} + \frac{\delta}{2} n_2\right\}.$$

On the other hand, (5.26) gives that

$$q_m \leq -\beta_1|O|^{\alpha} n_1^{1+\alpha} + (1 + \alpha_3)n_1 - \frac{\delta\beta_2}{2}|O|^{\alpha} n_2^{1+\alpha} + \frac{\delta}{2} n_2.$$

Since as m goes to infinity either n_1 or n_2 (or both) goes to infinity, it is clear that there exists m_0 such that $q_{m_0} < 0$. Then we have $\omega_{m_0}(D\Phi(\omega, w)) \leq \overline{\omega}_{m_0}(\omega)$ and $E(\ln \overline{\omega}_{m_0}) < 0$. Thus the desired conclusion follows from Proposition 2.9.

□

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