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# Pullback Exponential Attractors for Parabolic Equations with Dynamical Boundary Conditions

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Abstract. The existence of pullback exponential attractors for a nonautonomous semilinear parabolic equation with dynamical boundary condition is proved when the time-dependent forcing terms are translation bounded or even grow exponentially in the past and in the future.

## 1. Introduction

In this paper we consider the nonautonomous semilinear parabolic equation with dynamical boundary condition of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \kappa u + f_1(u) = h_1(t) & \text{in } \Omega \times (s,\infty), \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial \vec{n}} + f_2(u) = h_2(t) & \text{on } \partial\Omega \times (s,\infty), \\ u(x,s) = u_s(x) & \text{for } x \in \Omega, \\ u(x,s) = \varphi_s(x) & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary $\partial\Omega$, $\vec{n}$ is the outer normal unit vector to $\partial\Omega$, $s \in \mathbb{R}$ is an initial time, $u_s$, $\varphi_s$ are initial data, $\kappa > 0$, and the functions $f_1$, $f_2$, $h_1$, $h_2$ are given. Parabolic equations of the above type with dynamical boundary conditions serve as models in the heat transfer theory and in hydrodynamics, for example in the description of the heat transfer in a solid body in contact with a moving fluid. They have been investigated in many research articles (e.g., see [1–3, 11] and the references therein).

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We assume that $u_s \in L^2(\Omega)$, $\varphi_s \in L^2(\partial\Omega)$, $h_1 \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$, $h_2 \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\partial\Omega))$, and the functions $f_1, f_2 \in C(\mathbb{R})$ satisfy the following assumptions

$$(f_i(u) - f_i(v))(u - v) \geq -l(u - v)^2, \quad u, v \in \mathbb{R},\ i = 1, 2, \tag{1.2}$$

$$|f_i(u) - f_i(v)| \leq L\,|u - v| \left(1 + |u|^{p_i - 2} + |v|^{p_i - 2}\right), \quad u, v \in \mathbb{R},\ i = 1, 2, \tag{1.3}$$

$$f_i(u)u \geq \alpha\,|u|^{p_i} - \beta, \quad u \in \mathbb{R},\ i = 1, 2, \tag{1.4}$$

with some constants $p_i \geq 2$, $\alpha, l, L > 0$, $\beta \geq 0$.

The above conditions on the nonlinearities make that equations in problem (1.1) become a reaction-diffusion equation with dynamical boundary conditions. Note that, in particular, as $f_i$ we may take $f_i(u) = u\,|u|^{p_i - 2} - u$, $u \in \mathbb{R}$, with $p_i > 2$. We also see that (1.2) means that the functions $\mathbb{R} \ni u \mapsto f_i(u) + lu \in \mathbb{R}$, $i = 1, 2$, are nondecreasing. Moreover, we observe that there exists $C > 0$ such that

$$|f_i(u)| \leq C(1 + |u|^{p_i - 1}), \quad u \in \mathbb{R},\ i = 1, 2. \tag{1.5}$$

Finally, if $p_1 = p_2 = 2$, then (1.3) implies global Lipschitz continuity of $f_i$, $i = 1, 2$, i.e.,

$$|f_i(u) - f_i(v)| \leq \widetilde{L}\,|u - v|, \quad u, v \in \mathbb{R},\ i = 1, 2, \tag{1.6}$$

and the condition in (1.2) holds with $l = \widetilde{L} = 3L$.

*Remark* 1.1. If the system (1.1) does not contain the term with $\kappa$, but (1.4) holds, then by a suitable change of $f_1$, it can be considered in the form of (1.1) with any positive $\kappa$ for $p_1 > 2$ and $0 < \kappa < \alpha$ for $p_1 = 2$. Indeed, define $\widetilde{f_1}(u) = f_1(u) - \kappa u$ and note that (1.2) implies

$$\left(\widetilde{f_1}(u) - \widetilde{f_1}(v)\right)(u - v) \geq -(l + \kappa)(u - v)^2, \quad u, v \in \mathbb{R},$$

and, if $p_1 > 2$, for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\widetilde{f_1}(u)u \geq (\alpha - \varepsilon)\,|u|^{p_1} - \beta - c_\varepsilon, \quad u \in \mathbb{R},$$

whereas if $p_1 = 2$ we have

$$\widetilde{f_1}(u)u \geq (\alpha - \kappa)\,|u|^2 - \beta, \quad u \in \mathbb{R}.$$

Moreover, (1.3) implies

$$\left|\widetilde{f_1}(u) - \widetilde{f_1}(v)\right| \leq (L + \kappa)\,|u - v| \left(1 + |u|^{p_1 - 2} + |v|^{p_1 - 2}\right), \quad u, v \in \mathbb{R}.$$

In [1], under assumptions (1.2) and (1.4) for $\vec{f} = (f_1, f_2)$ and under some extra integrability condition for $\vec{h} = (h_1, h_2)$, the authors proved the existence of an evolution process for (1.1) on the space $H = L^2(\Omega) \times L^2(\partial\Omega)$, which possesses a minimal pullback attractor.

A minimal pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ for a process $\{U(t,s) : t \geq s\}$ on a Banach space $E$ is a family of nonempty compact subsets of $E$, which is invariant under the process, i.e., $U(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ for $t \geq s$, it pullback attracts all bounded subsets of $E$, i.e., for any bounded subset $D$ of $E$ and $t \in \mathbb{R}$

$$\lim_{s\to\infty} \operatorname{dist}_E(U(t,t-s)D, \mathcal{A}(t)) = 0,$$

where $\operatorname{dist}_E(A,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\|_E$ denotes the Hausdorff semidistance in $E$, and satisfies a minimality condition, which guarantees its uniqueness: if another family $\{C(t) : t \in \mathbb{R}\}$ of nonempty closed subsets of $E$ pullback attracts all bounded subsets of $E$, then $\mathcal{A}(t) \subset C(t)$ for $t \in \mathbb{R}$.

In the present article our aim is to prove the existence of a pullback exponential attractor for (1.1). This family $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ of nonempty compact subsets of $E$ is only positively invariant under the process, i.e., $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$ for $t \geq s$, but we require that the fractal dimension in $E$ (denoted by $\dim_f^E(\cdot)$) of the sets forming the family has a uniform bound, i.e., there exists $d \geq 0$ such that

$$\sup_{t\in\mathbb{R}} \dim_f^E(\mathcal{M}(t)) \leq d < \infty,$$

and the pullback attraction of bounded subsets of $E$ towards $\mathcal{M}(t)$ is at an exponential rate. This means that there exists $\omega > 0$ such that for every bounded subset $D$ of $E$ and $t \in \mathbb{R}$ we have

$$\lim_{s\to\infty} e^{\omega s} \operatorname{dist}_E(U(t,t-s)D, \mathcal{M}(t)) = 0.$$

Note that the existence of a pullback exponential attractor $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ implies the existence of the minimal pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ as its subset, that is, $\mathcal{A}(t) \subset \mathcal{M}(t)$ for $t \in \mathbb{R}$. In particular, the minimal pullback attractor also has a uniform bound of the fractal dimension.

The first constructions of pullback exponential attractors were presented in [8–10,14,16] and later in [5]. In this paper, however, we use the recent results of [7] to show the existence of pullback exponential attractors.

In Section 4 we prove the existence of a pullback exponential attractor for (1.1) in $H = L^2(\Omega) \times L^2(\partial\Omega)$ (cf. Theorem 4.5) if the forcing term $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ is translation bounded, i.e., there exists $K > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \left|\vec{h}(\tau)\right|_H^2 d\tau \leq K, \tag{1.7}$$

and the nonlinear terms $f_i$, $i = 1, 2$, have suitable exponents $p_i$ (see (4.5)) due to the available a priori estimate in $H$. If an additional condition (4.11) is satisfied, we are able to consider higher exponents $p_1 = p_2 = p$ given in (4.15). In particular, for $N = 2$ the nonlinearities $f_i(u) = u^3 - u$, $u \in \mathbb{R}$, among many others, are admitted.

In Section 5 we consider the Lipschitz case ($p_1 = p_2 = 2$) and show in Theorem 5.4 the existence of a pullback exponential attractor for (1.1) in $H$ even if the time-dependent forcing terms $h_1$ and $h_2$ may grow exponentially in the past and in the future, i.e., when the function $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ admits the exponential growth

$$\left|\vec{h}(t)\right|_H^2 \leq K e^{\theta|t|}, \quad t \in \mathbb{R} \tag{1.8}$$

for some $K > 0$ and $0 \leq \theta < 2(\lambda_1 + \alpha)$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $A_0$, specified in (2.5).

## 2. Evolution process of global weak solutions

We consider the problem (1.1) with

$$u_s \in L^2(\Omega),\ \varphi_s \in L^2(\partial\Omega),\ h_1 \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega)) \quad \text{and} \quad h_2 \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\partial\Omega))$$

given. Moreover, we assume that $f_i \in C(\mathbb{R})$, $i = 1, 2$, satisfy (1.2)–(1.4).

We denote by $|\cdot|_{p,\Omega}$ (respectively, $|\cdot|_{p,\partial\Omega}$) the norm in $L^p(\Omega)$ (respectively, in $L^p(\partial\Omega)$) and by $(\cdot\,,\cdot)_\Omega$ (respectively, $(\cdot\,,\cdot)_{\partial\Omega}$) the inner product in $L^2(\Omega)$ and $(L^2(\Omega))^N$, which defines the norm $|\cdot|_{2,\Omega} = |\cdot|_\Omega$, and the duality product between $L^{p'}(\Omega)$ and $L^p(\Omega)$ (respectively, the inner product in $L^2(\partial\Omega)$, which defines the norm $|\cdot|_{2,\partial\Omega} = |\cdot|_{\partial\Omega}$, and the duality product between $L^{p'}(\partial\Omega)$ and $L^p(\partial\Omega)$). The notation $|\cdot|$ will also be used for the Lebesgue measure of a set in both $\mathbb{R}^N$ or $\mathbb{R}^{N-1}$, without more indications since no confusion arises.

By $\|\cdot\|_\Omega$ we denote the norm in $H^1(\Omega)$, which is associated to the inner product $((\cdot\,,\cdot))_\Omega = (\nabla\cdot\,, \nabla\cdot)_\Omega + (\cdot\,,\cdot)_\Omega$. Furthermore, $\gamma_0$ will denote the trace operator

$$\gamma_0(u) = u|_{\partial\Omega}, \quad u \in C^\infty(\overline{\Omega}),$$

which belongs to $\mathcal{L}(H^1(\Omega), H^{1/2}(\partial\Omega))$ with norm $\|\gamma_0\|$ and is surjective. The norm in the subspace $H^{1/2}(\partial\Omega)$ of $L^2(\partial\Omega)$ is given by

$$\|u\|_{1/2,\partial\Omega} = \left(\int_{\partial\Omega} |u(x)|^2\, d\sigma_x + \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^N}\, d\sigma_x d\sigma_y\right)^{1/2}$$

and makes $H^{1/2}(\partial\Omega)$ a Hilbert space. Moreover, $H^1_0(\Omega) = \{u \in H^1(\Omega) : \gamma_0(u) = 0\}$, $H^{1/2}(\partial\Omega)$ is a dense subspace of $L^2(\partial\Omega)$ and $\gamma_0$ maps bounded subsets of $H^1(\Omega)$ into relatively compact subsets of $L^2(\partial\Omega)$ (for details see [12, Chapter 1], [13, Chapter 6] and [17, Chapter 2]). Finally, let us observe that throughout the paper $B^E_r(x)$ denotes the open ball in a metric space $E$ of center $x$ and radius $r$, and $\mathrm{cl}_E A$ denotes the closure in the topology of $E$ of a certain subset $A$ of $E$.

Following [1, 15] we will show existence and uniqueness of global weak solutions of (1.1).

**Definition 2.1.** A global weak solution of (1.1) is a pair of functions $(u, \varphi)$ satisfying

$$u \in C([s,\infty); L^2(\Omega)) \cap L^2(s,T; H^1(\Omega)) \cap L^{p_1}(s,T; L^{p_1}(\Omega)),$$
$$\varphi \in C([s,\infty); L^2(\partial\Omega)) \cap L^2(s,T; H^{1/2}(\partial\Omega)) \cap L^{p_2}(s,T; L^{p_2}(\partial\Omega))$$

for all $T > s$, $\gamma_0(u(t)) = \varphi(t)$ for a.e. $t \in (s,\infty)$, the following equality holds for all $v \in H^1(\Omega) \cap L^{p_1}(\Omega)$ such that $\gamma_0(v) \in L^{p_2}(\partial\Omega)$

$$\begin{aligned}
&\frac{d}{dt}(u(t), v)_\Omega + \frac{d}{dt}(\varphi(t), \gamma_0(v))_{\partial\Omega} + (\nabla u(t), \nabla v)_\Omega + \kappa(u(t), v)_\Omega \\
&\quad + (f_1(u(t)), v)_\Omega + (f_2(\gamma_0(u(t))), \gamma_0(v))_{\partial\Omega} \\
&= (h_1(t), v)_\Omega + (h_2(t), \gamma_0(v))_{\partial\Omega} \quad \text{for a.e. } t \in (s,\infty),
\end{aligned}$$

and $u(s) = u_s$ and $\varphi(s) = \varphi_s$.

As in the proof of [1, Theorem 5] we introduce the following spaces (with corresponding norms) and the following operators, which will be useful in the sequel. We define a Hilbert space

$$H = L^2(\Omega) \times L^2(\partial\Omega),$$

with the inner product $((u,\varphi),(v,\psi))_H = (u,v)_\Omega + (\varphi,\psi)_{\partial\Omega}$, which induces the norm $|\cdot|_H$ given by $|(u,\varphi)|_H^2 = |u|_\Omega^2 + |\varphi|_{\partial\Omega}^2$ for $(u,\varphi) \in H$, and the closed vector subspace of $H^1(\Omega) \times H^{1/2}(\partial\Omega)$ defined as

$$V_0 = \{(u, \gamma_0(u)) : u \in H^1(\Omega)\}$$

with the norm given by $\|(u,\gamma_0(u))\|_{V_0}^2 = \|u\|_\Omega^2 + \|\gamma_0(u)\|_{1/2,\partial\Omega}^2$ for $(u,\gamma_0(u)) \in V_0$. Observe that $V_0$ is a Hilbert space, which is densely and compactly embedded in $H$. We identify $H$ with its dual by the Riesz theorem and therefore we have the chain of inclusions $V_0 \subset H \subset V_0'$.

We consider the continuous linear operator $A_0 \colon V_0 \to V_0'$ defined through a symmetric continuous bilinear form $B \colon V_0 \times V_0 \to \mathbb{R}$ given as

$$B[\vec{u},\vec{v}] = \langle A_0\vec{u}, \vec{v}\rangle_{V_0',V_0} = (\nabla u, \nabla v)_\Omega + \kappa(u,v)_\Omega,$$

where $\vec{u} = (u, \gamma_0(u)), \vec{v} = (v, \gamma_0(v)) \in V_0$, since

$$|B[\vec{u},\vec{v}]| \le (1+\kappa)\, \|\vec{u}\|_{V_0} \|\vec{v}\|_{V_0}, \quad \vec{u},\vec{v} \in V_0. \tag{2.1}$$

Recall that $B$ is coercive (cf. (16) in [1]), i.e.,

$$B[\vec{u},\vec{u}] \geq \frac{1}{1+\|\gamma_0\|^2}\min\{1,\kappa\}\|\vec{u}\|^2_{V_0}, \quad \vec{u}\in V_0. \tag{2.2}$$

By Lax-Milgram lemma there exists the bounded inverse $A_0^{-1}\colon V_0' \to V_0$. Its restriction to $H$ is a bounded compact operator, which is the inverse of the unbounded linear operator $A_0\colon H \supset D(A_0) \to H$ with the domain $D(A_0) = \{\vec{u}\in V_0 : A_0\vec{u}\in H\}$. This operator is symmetric and surjective. Moreover, it is positive, since for $\vec{u} = (u,\gamma_0(u)) \in D(A_0)$ we have

$$(A_0\vec{u},\vec{u})_H = \langle A_0\vec{u},\vec{u}\rangle_{V_0',V_0} = |\nabla u|^2_\Omega + \kappa|u|^2_\Omega \geq \min\{1,\kappa/2\}\min\left\{1,\|\gamma_0\|^{-2}\right\}|\vec{u}|^2_H.$$

Hence there exists an orthonormal basis $\{\vec{w}_j = (w_j,\gamma_0(w_j))\} \subset D(A_0)$ in the Hilbert space $H$ consisting of eigenfunctions of $A_0$, with corresponding eigenvalues $\lambda_j$ such that $\lambda_{j+1} \geq \lambda_j > 0$, $j\in\mathbb{N}$, and $\lambda_j \to \infty$.

We define the linear subspaces $E_0 = \{\vec{0}\}$ and

$$E_n = \operatorname{span}\{\vec{w}_1,\ldots,\vec{w}_n\}, \quad n\in\mathbb{N}, \tag{2.3}$$

of $V_0$ and note that the bilinear form $B$ defines an inner product in $V_0$ and $\left\{\vec{w}_j/\sqrt{\lambda_j}\right\}$ is an orthonormal basis in $V_0$ with this inner product. Consequently, for any $\vec{u}\in V_0$ such that $\vec{u}\perp E_{n-1}$, we have

$$B[\vec{u},\vec{u}] = \sum_{j=1}^{\infty}\lambda_j^{-1}B[\vec{u},\vec{w}_j]^2 = \sum_{j=n}^{\infty}\lambda_j(\vec{u},\vec{w}_j)^2_H \geq \lambda_n|\vec{u}|^2_H, \quad n\in\mathbb{N}.$$

Hence we obtain

$$\lambda_n = \min_{\substack{\vec{u}\in V_0\setminus\{0\}\\ \vec{u}\perp E_{n-1}}}\frac{\langle A_0\vec{u},\vec{u}\rangle_{V_0',V_0}}{|\vec{u}|^2_H}, \quad n\in\mathbb{N}. \tag{2.4}$$

In particular, we have

$$\lambda_1 = \min_{\vec{u}\in V_0\setminus\{0\}}\frac{\langle A_0\vec{u},\vec{u}\rangle_{V_0',V_0}}{|\vec{u}|^2_H}. \tag{2.5}$$

Now, we introduce the nonlinear operators $A_1\colon V_1\to V_1'$ and $A_2\colon V_2\to V_2'$ given by

$$\begin{aligned} A_1(u,\varphi) &= (f_1(u),0), \qquad (u,\varphi)\in V_1 = L^{p_1}(\Omega)\times L^2(\partial\Omega),\\ A_2(u,\varphi) &= (0,f_2(\varphi)), \qquad (u,\varphi)\in V_2 = L^2(\Omega)\times L^{p_2}(\partial\Omega). \end{aligned}$$

The operators are well-defined by (1.5). Note that $V_i$, $i=0,1,2$, are separable, reflexive Banach spaces, densely embedded in $H$. We define

$$V = \bigcap_{i=0}^{2}V_i = V_0\cap(L^{p_1}(\Omega)\times L^{p_2}(\partial\Omega)) \quad \text{with} \quad \|\vec{u}\|^2_V = \sum_{i=0}^{2}\|\vec{u}\|^2_{V_i}.$$

We see that $V$ is a separable Banach space, densely embedded in $H$. Thus, we have

$$V \subset H \subset V' \quad \text{and} \quad V_i \subset H \subset V_i', \ i = 0, 1, 2.$$

Observe that from (1.3) it follows that each $A_i$, $i = 0, 1, 2$, is hemicontinuous, i.e., for every $\vec{u}, \vec{v}, \vec{w} \in V_i$ the function

$$\mathbb{R} \ni \mu \mapsto \langle A_i(\vec{u} + \mu\vec{v}), \vec{w}\rangle_{V_i', V_i} \in \mathbb{R}$$

is continuous. Moreover, by (1.5) we see that

$$\|A_i(\vec{u})\|_{V_i'} \leq C_i \left(1 + \|\vec{u}\|_{V_i}^{p_i - 1}\right), \quad \vec{u} = (u, \varphi) \in V_i, \ i = 1, 2.$$

We also have by (2.1)

$$\|A_0\vec{u}\|_{V_0'} \leq (1 + \kappa) \|\vec{u}\|_{V_0}, \quad \vec{u} \in V_0.$$

By (2.2) and (1.2) each operator is monotone, i.e.,

$$\langle A_0(\vec{u} - \vec{v}), \vec{u} - \vec{v}\rangle_{V_0', V_0} \geq 0, \quad \vec{u}, \vec{v} \in V_0,$$

$$\langle A_i(\vec{u}) - A_i(\vec{v}), \vec{u} - \vec{v}\rangle_{V_i', V_i} \geq -l\,|\vec{u} - \vec{v}|_H^2, \quad \vec{u}, \vec{v} \in V_i, \ i = 1, 2.$$

Finally, we have by (1.4)

$$\langle A_1(\vec{u}), \vec{u}\rangle_{V_1', V_1} \geq \alpha\,|u|_{p_1,\Omega}^{p_1} - \beta\,|\Omega|, \qquad \vec{u} = (u, \varphi) \in V_1,$$

$$\langle A_2(\vec{u}), \vec{u}\rangle_{V_2', V_2} \geq \alpha\,|\varphi|_{p_2,\partial\Omega}^{p_2} - \beta\,|\partial\Omega|, \qquad \vec{u} = (u, \varphi) \in V_2,$$

and by (2.2)

$$\langle A_0(\vec{u}), \vec{u}\rangle_{V_0', V_0} \geq \frac{1}{1 + \|\gamma_0\|^2} \min\{1, \kappa\} \|\vec{u}\|_{V_0}^2, \quad \vec{u} \in V_0.$$

Then by a modification of [15, Chapter 2, Theorem 1.4] for every $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, $s \in \mathbb{R}$, $T > s$ and $\vec{u}_s = (u_s, \varphi_s) \in H$ there exists a unique function

$$\vec{u} \in L^2(s, T; V_0) \cap L^{p_1}(s, T; V_1) \cap L^{p_2}(s, T; V_2) \cap C([0, T], H)$$

such that

$$\begin{cases} \dfrac{d\vec{u}}{dt} + \displaystyle\sum_{i=0}^{2} A_i(\vec{u}) = \vec{h}, \\ \vec{u}(s) = \vec{u}_s. \end{cases}$$

Moreover, we obtain the energy equality for a.e. $t > s$

$$\frac{1}{2}\frac{d}{dt}\,|\vec{u}(t)|_H^2 + \sum_{i=0}^{2} \langle A_i(\vec{u}(t)), \vec{u}(t)\rangle_{V_i', V_i} = (\vec{h}(t), \vec{u}(t))_H.$$

Thus we have proved (cf. also [1, Theorem 5]) the result on the existence and uniqueness of the global weak solutions to (1.1).

**Theorem 2.2.** *Under conditions* (1.2)–(1.4) *for any* $s \in \mathbb{R}$, $(u_s, \varphi_s) \in L^2(\Omega) \times L^2(\partial\Omega)$ *there exists a unique global weak solution* $(u, \varphi)$ *of problem* (1.1). *Moreover, this solution satisfies the energy equality*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|u(t)|_\Omega^2 + |\varphi(t)|_{\partial\Omega}^2\right) + |\nabla u(t)|_\Omega^2 + \kappa\,|u(t)|_\Omega^2 + (f_1(u(t)), u(t))_\Omega \\
&\quad + (f_2(\varphi(t)), \varphi(t))_{\partial\Omega} \\
&= (h_1(t), u(t))_\Omega + (h_2(t), \varphi(t))_{\partial\Omega}
\end{aligned} \tag{2.6}$$

*for a.e.* $t > s$.

Some conclusions from the above functional setting, abstract formulation and energy equality are given below. The first one is that the global weak solutions of (1.1) satisfy the following differential inequality.

**Proposition 2.3.** *Under the assumptions of Theorem* 2.2, *the solution* $\vec{u} = (u, \varphi)$ *of* (1.1) *satisfies with any* $\delta > 0$

$$\frac{d}{dt}\,|\vec{u}(t)|_H^2 + (2\lambda_1 - \delta)\,|\vec{u}(t)|_H^2 \le 2\beta(|\Omega| + |\partial\Omega|) + \delta^{-1}\left|\vec{h}(t)\right|_H^2 \tag{2.7}$$

*for a.e.* $t > s$.

*Proof.* We apply (1.4) and (2.5) to (2.6) to get

$$\frac{d}{dt}\,|\vec{u}(t)|_H^2 + 2\lambda_1\,|\vec{u}(t)|_H^2 \le 2\beta(|\Omega| + |\partial\Omega|) + 2\left[(h_1(t), u(t))_\Omega + (h_2(t), \varphi(t))_{\partial\Omega}\right]$$

for a.e. $t > s$. The Cauchy-Schwarz and Cauchy inequalities lead to (2.7). □

Another consequence, now from Theorem 2.2, is that the global weak solutions to (1.1) define an evolution process $\{U(t, s) : t \ge s\}$ in $H$, i.e.,

$$U(t, s)(u_s, \varphi_s) = (u(t), \varphi(t)), \quad (u_s, \varphi_s) \in H, \tag{2.8}$$

where $(u, \varphi)$ is the unique global weak solution of (1.1) with $(u(s), \varphi(s)) = (u_s, \varphi_s)$.

Observe that the process is Lipschitz continuous on $H$, which means that for each pair $(t, s)$, the map $U(t, s)$ is Lipschitz (and the Lipschitz constant is not supposed to be uniform for all the pairs).

**Proposition 2.4.** *Under the assumptions of Theorem* 2.2, *for every* $t \ge s$ *there exists a constant* $L_{t,s} = e^{(l-\lambda_1)(t-s)} > 0$ *such that*

$$|U(t, s)\vec{u}_s - U(t, s)\vec{v}_s|_H \le L_{t,s}\,|\vec{u}_s - \vec{v}_s|_H\,, \quad \vec{u}_s, \vec{v}_s \in H.$$

*Proof.* Consider a pair of initial data $\vec{u}_s, \vec{v}_s \in H$. Denoting the corresponding solutions by $\vec{u}$ and $\vec{v}$, we see that the difference $\vec{w} = \vec{u} - \vec{v}$ satisfies for a.e. $t > s$

$$\frac{1}{2}\frac{d}{dt}\,|\vec{w}|_H^2 + \langle A_0\vec{w}, \vec{w}\rangle_{V_0',V_0} + \langle A_1(\vec{u}) - A_1(\vec{v}), \vec{w}\rangle_{V_1',V_1} + \langle A_2(\vec{u}) - A_2(\vec{v}), \vec{w}\rangle_{V_2',V_2} = 0.$$

Using (1.2) and (2.5), we obtain

$$\frac{d}{dt}\,|\vec{w}(t)|_H^2 + 2(\lambda_1 - l)\,|\vec{w}(t)|_H^2 \le 0 \quad \text{for a.e. } t > s.$$

In particular, we conclude

$$|\vec{w}(t)|_H^2 \le e^{2(l-\lambda_1)(t-s)}\,|\vec{w}(s)|_H^2\,, \quad t \ge s,$$

which proves the claim. □

## 3. Existence of exponential pullback attractors

Our aim now is to prove the existence of a pullback exponential attractor for the process $\{U(t,s) : t \ge s\}$ in $H$ defined in (2.8). To achieve this goal we are going to apply [7, Corollaries 2.6 and 2.8], which we recall below.

**Theorem 3.1.** *Let $\{U(t,s) : t \ge s\}$ be a Lipschitz continuous process on a Hilbert space $H$. Assume that*

(H$_1$) *there exists a family of nonempty closed bounded subsets $B(t)$ of $H$, $t \in \mathbb{R}$, which is positively invariant under the process, i.e.,*

$$U(t,s)B(s) \subset B(t), \quad t \ge s,$$

(H$_2$) *there exist $t_0 \in \mathbb{R}$, $\gamma_0 \ge 0$ and $M > 0$ such that*

$$\mathrm{diam}_H(B(t)) < Me^{-\gamma_0 t}, \quad t \le t_0,$$

(H$_3$) *in the past the family $\{B(t) : t \in \mathbb{R}\}$ pullback absorbs all bounded subsets of $H$; that is, for every bounded subset $D$ of $H$ and $t \le t_0$ there exists $T_{D,t} \ge 0$ such that*

$$U(t, t-r)D \subset B(t), \quad r \ge T_{D,t},$$

*and, additionally, the function $(-\infty, t_0] \ni t \mapsto T_{D,t} \in [0,\infty)$ is nondecreasing for every bounded $D \subset H$.*

*Next, we assume that the semi-process $\{U(t,s) : t_0 \ge t \ge s\}$ can be represented as*

$$U(t,s) = C(t,s) + S(t,s),$$

*where $\{C(t,s) : t_0 \ge t \ge s\}$ and $\{S(t,s) : t_0 \ge t \ge s\}$ are families of operators satisfying the following properties:*

($\mathrm{H}_4$) *there exists* $\widetilde{t} > 0$ *such that* $C(t, t-\widetilde{t})$ *are contractions within the absorbing sets with the contraction constant independent of time, i.e.,*

$$\left|C(t, t-\widetilde{t})\vec{u} - C(t, t-\widetilde{t})\vec{v}\right|_H \leq \lambda \left|\vec{u} - \vec{v}\right|_H, \quad t \leq t_0,\ \vec{u}, \vec{v} \in B(t-\widetilde{t}),$$

*where* $0 \leq \lambda < \frac{1}{2}e^{-\gamma_0 \widetilde{t}}$,

($\mathrm{H}_5$) *for some* $\nu \in \left(0, \frac{1}{2}e^{-\gamma_0\widetilde{t}} - \lambda\right)$ *there exists* $N = N_\nu \in \mathbb{N}$ *such that for any* $t \leq t_0$, *any* $R > 0$ *and any* $\vec{u} \in B(t-\widetilde{t})$ *there exist* $\vec{v}_1, \ldots, \vec{v}_N \in H$ *such that*

$$S(t, t-\widetilde{t})\left(B(t-\widetilde{t}) \cap B^H_R(\vec{u})\right) \subset \bigcup_{i=1}^{N} B^H_{\nu R}(\vec{v}_i).$$

*Then there exists a pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^\nu(t) : t \in \mathbb{R}\}$ *in* $H$ *satisfying the properties:*

(a) $\mathcal{M}(t)$ *is a nonempty compact subset of* $B(t)$ *for* $t \in \mathbb{R}$,

(b) $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$, $t \geq s$,

(c) $\sup_{t\in\mathbb{R}} \dim^H_f(\mathcal{M}^\nu(t)) \leq -\ln N_\nu / \left[\ln\left(2(\nu+\lambda)\right) + \gamma_0\widetilde{t}\right]$,

(d) *for any* $t \in \mathbb{R}$ *there exists* $c_t > 0$ *such that for any* $s \geq \max\{t - t_0, 0\} + 2\widetilde{t}$

$$\operatorname{dist}_H(U(t, t-s)B(t-s), \mathcal{M}(t)) \leq c_t e^{-\omega_0 s},$$

*where* $\omega_0 = -\left(\ln\left(2(\nu+\lambda)\right) + \gamma_0\widetilde{t}\right)/\widetilde{t} > 0$,

(e) *for any* $0 < \omega < \omega_0$ *we have*

$$\lim_{s\to\infty} e^{\omega s} \operatorname{dist}_H(U(t, t-s)D, \mathcal{M}(t)) = 0, \quad t \in \mathbb{R},\ D \text{ bounded in } H.$$

*The process* $\{U(t,s) : t \geq s\}$ *has also the minimal pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$, *which is contained in the pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^\nu(t) : t \in \mathbb{R}\}$ *and thus has uniformly bounded fractal dimension.*

## 4. Translation bounded forcing terms

We consider (1.1) under assumptions (1.2), (1.3) and (1.4). The main ingredient of Theorem 3.1 is the pullback absorbing family $\{B(t) : t \in \mathbb{R}\}$. We will find a pullback absorbing family for the problem (1.1) when the function $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ is translation bounded, i.e., (1.7) holds.

By Proposition 2.3 we know that the global weak solutions $\vec{u} = (u, \varphi)$ of (1.1) satisfy (2.7). Setting $0 < \delta < 2\lambda_1$ we use (1.7) and apply a version of the Gronwall inequality from [6, Chapter II, Lemma 1.3] to (2.7) to get

$$|\vec{u}(t)|_H^2 \le |\vec{u}(s)|_H^2\, e^{-(2\lambda_1-\delta)(t-s)} + K_\delta, \quad t \ge s, \tag{4.1}$$

where $K_\delta = \left(2\beta(|\Omega| + |\partial\Omega|) + \delta^{-1}K\right)\left(1 + \frac{1}{2\lambda_1-\delta}\right)$.

We define

$$B_0 = \left\{\vec{u} \in H : |\vec{u}|_H^2 \le 2K_\delta\right\}. \tag{4.2}$$

From (4.1) and (4.2) it follows that for every bounded subset $D$ of $H$ there exists $r_D > 0$ such that

$$U(t, t-r)D \subset B_0, \quad r \ge r_D,\ t \in \mathbb{R}.$$

Moreover, there exists $r_0 > 0$ such that

$$U(t, t-r)B_0 \subset B_0, \quad r \ge r_0,\ t \in \mathbb{R}.$$

Thus, the family

$$B(t) = \mathrm{cl}_H \bigcup_{r \ge r_0} U(t, t-r)B_0, \quad t \in \mathbb{R}, \tag{4.3}$$

is positively invariant and pullback absorbing. Indeed, from above we see that $B(t) \subset B_0$ is a nonempty closed bounded subset of $H$ and by Proposition 2.4

$$U(t, s)B(s) \subset B(t), \quad t \ge s,$$

which shows $(\mathrm{H}_1)$. Moreover, we have

$$\mathrm{diam}_H(B(t)) < 2\,\mathrm{diam}_H(B_0), \quad t \in \mathbb{R},$$

so $(\mathrm{H}_2)$ holds with $M = 2\,\mathrm{diam}_H(B_0)$, $\gamma_0 = 0$ and $t_0 \in \mathbb{R}$ arbitrary. Furthermore, if $D$ is a bounded subset of $H$ and $t \le t_0$, then, setting $T_D = r_D + r_0$ and taking $s \ge T_D$, we get

$$U(t, t-s)D = U(t, t-r_0)U(t-r_0, t-r_0-(s-r_0))D \subset U(t, t-r_0)B_0 \subset B(t),$$

which shows that $(\mathrm{H}_3)$ is satisfied in this case.

We have proved the following

**Proposition 4.1.** *If $f_i$, $i = 1, 2$, satisfy* (1.2)–(1.4), *and $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies* (1.7), *then the family $B(t) \subset B_0$, $t \in \mathbb{R}$, defined by* (4.3) *is positively invariant and pullback absorbing for the process $\{U(t, s) : t \ge s\}$ in $H$ associated to problem* (1.1). *Moreover, this family satisfies the assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ *in Theorem* 3.1.

We consider the projections $P_n\colon H \to E_n$ given by

$$P_n\vec{u} = \sum_{j=1}^{n} (\vec{u}, \vec{w}_j)_H \vec{w}_j, \quad \vec{u} \in H, \tag{4.4}$$

where $E_n$ is defined in (2.3). We set $Q_n = I - P_n$.

**Proposition 4.2.** *Suppose that $f_i$, $i = 1, 2$, satisfy* (1.2), (1.3) *and* (1.4) *with the exponents*

$$\begin{aligned} &2 \le p_1 \le 2 + \frac{2}{N}, \qquad 2 \le p_2 \le 2 + \frac{1}{N-1} \qquad \text{for } N \ge 3,\\ &2 \le p_1 < 3, \qquad\qquad 2 \le p_2 < 3 \qquad\qquad \text{for } N = 2. \end{aligned} \tag{4.5}$$

*Assume further that $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies* (1.7). *Then the semi-process $\{U(t, s) : t_0 \ge t \ge s\}$ corresponding to problem* (1.1) *can be decomposed as*

$$U(t, s) = Q_n U(t, s) + P_n U(t, s)$$

*in such a way that for any $0 < \eta < 1$ and $0 < \varepsilon \le (1 - \eta)\frac{1}{1+\|\gamma_0\|^2} \min\{1, \kappa\}$ we have*

$$|Q_n(U(t, s)\vec{u}_s - U(t, s)\vec{v}_s)|_H^2 \le \left( e^{-2\eta\lambda_{n+1}(t-s)} + \frac{c_0}{4\varepsilon(\eta\lambda_{n+1} + l)} e^{2l(t-s)} \right) |\vec{u}_s - \vec{v}_s|_H^2 \tag{4.6}$$

*for all $t \ge s$ and $\vec{u}_s, \vec{v}_s \in B(s) \subset H$, with some constant $c_0 > 0$.*

*Proof.* Let us denote by $\vec{u} = (u, \varphi)$, $\vec{v} = (v, \psi)$ the global weak solutions of (1.1) corresponding to initial data $\vec{u}_s, \vec{v}_s \in B(s)$, respectively. By the positive invariance of $\{B(t) : t \in \mathbb{R}\}$ we infer that $\vec{u}(t), \vec{v}(t) \in B_0$ for every $t \ge s$. In particular, there exists $R_{B_0} > 0$ such that

$$|u(t)|_\Omega\,,\ |v(t)|_\Omega\,,\ |\varphi(t)|_{\partial\Omega}\,,\ |\psi(t)|_{\partial\Omega} \le R_{B_0}, \quad t \ge s. \tag{4.7}$$

Observe that $\vec{w} = \vec{u} - \vec{v}$ satisfies for a.e. $t > s$

$$\frac{d}{dt}(\vec{w}, \vec{z})_H + \langle A_0\vec{w}, \vec{z}\rangle_{V_0', V_0} + (f_1(u) - f_1(v), z)_\Omega + (f_2(\varphi) - f_2(\psi), \gamma_0(z))_{\partial\Omega} = 0$$

for any $\vec{z} = (z, \gamma_0(z)) \in V$.

Testing the above problem with $\vec{z} = Q_n\vec{w} = (I - P_n)\vec{w}$, we get for a.e. $t > s$

$$\frac{1}{2}\frac{d}{dt}|\vec{z}|_H^2 + \langle A_0\vec{z}, \vec{z}\rangle_{V_0', V_0} + (f_1(u) - f_1(v), z)_\Omega + (f_2(\varphi) - f_2(\psi), \gamma_0(z))_{\partial\Omega} = 0. \tag{4.8}$$

We fix $0 < \eta < 1$ and use (2.4) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\vec{z}|_H^2 + (1 - \eta)\langle A_0\vec{z}, \vec{z}\rangle_{V_0', V_0} + \eta\lambda_{n+1}|\vec{z}|_H^2 \\ &\le \|(f_1(u) - f_1(v), f_2(\varphi) - f_2(\psi))\|_{V_0'} \|(z, \gamma_0(z))\|_{V_0}\,. \end{aligned}$$

Taking $0 < \varepsilon \leq (1-\eta)\frac{1}{1+\|\gamma_0\|^2}\min\{1,\kappa\}$ we apply the Cauchy inequality and get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\vec{z}|_H^2 + (1-\eta)\langle A_0\vec{z},\vec{z}\rangle_{V_0',V_0} + \eta\lambda_{n+1}|\vec{z}|_H^2\\ &\leq \varepsilon\left(\|z\|_\Omega^2 + \|\gamma_0(z)\|_{1/2,\partial\Omega}^2\right) + \frac{1}{4\varepsilon}\|(f_1(u)-f_1(v), f_2(\varphi)-f_2(\psi))\|_{V_0'}^2 .\end{aligned}$$

Hence, by (2.2) it yields

$$\frac{1}{2}\frac{d}{dt}|\vec{z}|_H^2 + \eta\lambda_{n+1}|\vec{z}|_H^2 \leq \frac{1}{4\varepsilon}\|(f_1(u)-f_1(v), f_2(\varphi)-f_2(\psi))\|_{V_0'}^2 . \tag{4.9}$$

Since $L^{q_1}(\Omega)\times L^{q_2}(\partial\Omega)\hookrightarrow V_0'$ with $q_1 = 2N/(N+2)$, $q_2 = 2(N-1)/N$ for $N \geq 3$, and $q_1, q_2 > 1$ for $N = 2$, we estimate using (1.3) and the Hölder inequality

$$\begin{aligned}&\|(f_1(u)-f_1(v), f_2(\varphi)-f_2(\psi))\|_{V_0'}^2\\ &\leq c^2L^2|u-v|_\Omega^2\left(1+|u|^{p_1-2}_{\frac{2q_1}{2-q_1}(p_1-2),\Omega} + |v|^{p_1-2}_{\frac{2q_1}{2-q_1}(p_1-2),\Omega}\right)^2\\ &\quad + c^2L^2|\varphi-\psi|_{\partial\Omega}^2\left(1+|\varphi|^{p_2-2}_{\frac{2q_2}{2-q_2}(p_2-2),\partial\Omega} + |\psi|^{p_2-2}_{\frac{2q_2}{2-q_2}(p_2-2),\partial\Omega}\right)^2,\end{aligned} \tag{4.10}$$

for some constant $c > 0$. By (4.5) we have $\frac{2q_i}{2-q_i}(p_i-2) \leq 2$ for $i = 1, 2$. Thus, joining this estimate with (4.9) and using (4.7) we obtain

$$\frac{d}{dt}|\vec{z}|_H^2 + 2\eta\lambda_{n+1}|\vec{z}|_H^2 \leq \frac{c_0}{2\varepsilon}|\vec{w}|_H^2 \quad \text{for a.e. } t > s,$$

with some constant $c_0 > 0$. By Proposition 2.4, in particular, we have

$$\frac{d}{dt}\left(e^{2\eta\lambda_{n+1}t}|\vec{z}(t)|_H^2\right) \leq \frac{c_0}{2\varepsilon}e^{2\eta\lambda_{n+1}t+2l(t-s)}|\vec{w}(s)|_H^2 \quad \text{for a.e. } t > s.$$

Integrating and using $|\vec{z}(s)|_H \leq |\vec{w}(s)|_H$, we get (4.6). □

In [2] the authors proved the existence of a regular (i.e., in $D(A_0)\cap V$) minimal pullback attractor for (1.1) if $\partial\Omega$ is smooth enough and $f_1$, $f_2$, additionally to (1.2), (1.3) and (1.4), satisfy

$$|f_1(s) - f_2(s)| \leq C(1+|s|), \quad s \in \mathbb{R}, \tag{4.11}$$

which in particular implies $p = p_1 = p_2 \geq 2$. Although this seems a further restriction on $f_i$, $i = 1, 2$, it actually allows us to improve Proposition 4.2 in this case.

Denoting by $(u_n, \gamma_0(u_n))$ the Galerkin approximation of the global weak solution $\vec{u} = (u, \varphi)$ of (1.1) with $\vec{u}_s = (u_s, \varphi_s)$, we have (see [2, (18), (20)])

$$\begin{aligned}&|(u_n(t),\gamma_0(u_n(t)))|_H^2 + \frac{\min\{1,\kappa\}}{1+\|\gamma_0\|^2}\int_s^t\|(u_n(\tau),\gamma_0(u_n(\tau)))\|_{V_0}^2\,d\tau\\ &+ 2\alpha\int_s^t|u_n(\tau)|_{p,\Omega}^p\,d\tau + 2\alpha\int_s^t|\gamma_0(u_n(\tau))|_{p,\partial\Omega}^p\,d\tau\\ &\leq 2\beta(t-s)(|\Omega|+|\partial\Omega|) + \left(\frac{2}{\kappa} + \frac{\|\gamma_0\|^2}{\min\{1,\kappa/2\}}\right)\int_s^t\left|\vec{h}(\tau)\right|_H^2 + |\vec{u}_s|_H^2 ,\end{aligned} \tag{4.12}$$

$$(t-s)\left(\frac{\min\{1,\kappa\}}{1+\|\gamma_0\|^2}\|(u_n(t),\gamma_0(u_n(t)))\|_{V_0}^2+2\widetilde{\alpha}_1\left(|u_n(t)|_{p,\Omega}^p+|\gamma_0(u_n(t))|_{p,\partial\Omega}^p\right)\right)$$
$$\leq \max\{1,\kappa\}\int_s^t\|(u_n(\tau),\gamma_0(u_n(\tau)))\|_{V_0}^2\,d\tau+(t-s)\int_s^t\left|\vec{h}(\tau)\right|_H^2 d\tau \tag{4.13}$$
$$+2\widetilde{\alpha}_2\int_s^t\left(|u_n(\tau)|_{p,\Omega}^p+|\gamma_0(u_n(\tau))|_{p,\partial\Omega}^p\right)d\tau+(t-s)4\widetilde{\beta}(|\Omega|+|\partial\Omega|)$$

for all $t\geq s$ and any $n\in\mathbb{N}$, where $\widetilde{\alpha}_1,\widetilde{\alpha}_2,\widetilde{\beta}>0$ are such that

$$\widetilde{\alpha}_1|u|^p-\widetilde{\beta}\leq\int_0^u f_i(r)\,dr\leq\widetilde{\alpha}_2|u|^p+\widetilde{\beta},\quad u\in\mathbb{R},\ i=1,2.$$

From (1.7) and (4.12) it follows that if $\vec{u}_s\in B(t-1)\subset B_0$, $t\in\mathbb{R}$, we get uniform boundedness of

$$\int_{t-1}^t\|(u_n(\tau),\gamma_0(u_n(\tau)))\|_{V_0}^2\,d\tau,\quad \int_{t-1}^t|u_n(\tau)|_{p,\Omega}^p\,d\tau,\quad \int_{t-1}^t|\gamma_0(u_n(\tau))|_{p,\partial\Omega}^p\,d\tau$$

with respect to $t\in\mathbb{R}$. After passing to the limit (cf. [2, Corollary 8]) to get these estimates for the solutions, and applying them to (4.13), we obtain

$$U(t,t-1)B(t-1)\subset B_1=\left\{\vec{u}\in V_0:\|\vec{u}\|_{V_0}\leq R_{B_1}\right\},\quad t\in\mathbb{R}, \tag{4.14}$$

for some $R_{B_1}>0$. Arguing as in the proof of Proposition 4.2 with $s=t-1$ we obtain (4.9) and (4.10). Since $V_0\hookrightarrow L^{q_1'}(\Omega)\times L^{q_2'}(\partial\Omega)$ with $q_1'=2N/(N-2)$, $q_2'=2(N-1)/(N-2)$ for $N\geq 3$, and $q_1',q_2'\geq 1$ for $N=2$, we have

$$\frac{2q_i}{2-q_i}(p-2)\leq q_i',\quad i=1,2,$$

if

$$\begin{aligned} 2\leq p\leq 2+\frac{1}{N-2}\qquad &\text{for } N\geq 3,\\ p\geq 2\,\text{arbitrary}\qquad &\text{for } N=2,\end{aligned} \tag{4.15}$$

and we continue the proof of Proposition 4.2 using the uniform estimate (4.14) in $V_0$. Thus we have obtained

**Proposition 4.3.** *Suppose that $\partial\Omega$ is smooth enough and $f_i$, $i=1,2$, satisfy* (1.2), (1.3), (1.4) *and* (4.11) *with the exponents $p_1=p_2=p$ satisfying* (4.15). *Assume further that $\vec{h}=(h_1,h_2)\in L^2_{\mathrm{loc}}(\mathbb{R};H)$ satisfies* (1.7). *Then the semi-process $\{U(t,s):t_0\geq t\geq s\}$ corresponding to problem* (1.1) *can be decomposed as*

$$U(t,s)=Q_nU(t,s)+P_nU(t,s)$$

*in such a way that for any $0<\eta<1$ and $0<\varepsilon\leq(1-\eta)\frac{1}{1+\|\gamma_0\|^2}\min\{1,\kappa\}$ we have*

$$|Q_n(U(t,t-1)\vec{u}-U(t,t-1)\vec{v})|_H^2\leq\left(e^{-2\eta\lambda_{n+1}}+\frac{c_0}{4\varepsilon(\eta\lambda_{n+1}+l)}e^{2l}\right)|\vec{u}-\vec{v}|_H^2$$

*for all $\vec{u},\vec{v}\in B(t-1)\subset H$ and $t\in\mathbb{R}$ with some constant $c_0>0$.*

From the above result we conclude the following

**Corollary 4.4.** *Under the assumptions of Proposition* 4.2 *or Proposition* 4.3*, there exist two families of operators* $\{C(t,s) : t_0 \geq t \geq s\}$ *and* $\{S(t,s) : t_0 \geq t \geq s\}$ *with* $U(t,s) = C(t,s) + S(t,s)$ *satisfying hypotheses* ($\mathrm{H}_4$)–($\mathrm{H}_5$) *of Theorem* 3.1*.*

*Proof.* We put $\widetilde{t} = 1$ and $C = Q_n U$ and $S = P_n U$ with some $n \in \mathbb{N}$ large enough. ($\mathrm{H}_4$) follows from Propositions 4.2 and 4.3, while ($\mathrm{H}_5$) is a direct consequence of [4, Lemma 1] (see also [7, Lemma 4.2]) and Proposition 2.4. In particular, if $n \in \mathbb{N}$ is such that

$$\lambda = \left( e^{-2\eta\lambda_{n+1}} + \frac{c_0}{4\varepsilon(\eta\lambda_{n+1} + l)} e^{2l} \right)^{1/2} < \frac{1}{2}, \tag{4.16}$$

then, for $0 < \nu < \min\left\{\frac{1}{2} - \lambda, e^{l-\lambda_1}\right\}$, we have in ($\mathrm{H}_5$)

$$N_\nu \leq \left(1 + \frac{2e^{l-\lambda_1}}{\nu}\right)^n. \tag{4.17}$$

□

Collecting the above results, as an application of Theorem 3.1, we obtain

**Theorem 4.5.** *Suppose that functions* $f_i$*,* $i = 1, 2$*, satisfy* (1.2), (1.3) *and* (1.4) *with the exponents* $p_i$*,* $i = 1, 2$*, given in* (4.5) *or* (1.2), (1.3), (1.4) *and* (4.11) *with the exponents* $p_1 = p_2 = p$ *given in* (4.15) *and* $\partial\Omega$ *smooth enough. Assume further that* $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ *satisfies* (1.7)*. Then the process* $\{U(t,s) : t \geq s\}$ *on* $H = L^2(\Omega) \times L^2(\partial\Omega)$ *of global weak solutions of* (1.1) *possesses a pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^\nu(t) : t \in \mathbb{R}\}$ *in* $H$ *satisfying the properties:*

(a) $\mathcal{M}(t)$ *is a nonempty compact subset of* $B(t) \subset B_0$ *for* $t \in \mathbb{R}$*,*

(b) $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$*,* $t \geq s$*,*

(c) $\sup_{t\in\mathbb{R}} \dim_f^H(\mathcal{M}(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} N_\nu$*, where* $\lambda$ *is given in* (4.16) *and* $N_\nu$ *is given in* (4.17) *for* $0 < \nu < \min\left\{\frac{1}{2} - \lambda, e^{l-\lambda_1}\right\}$*,*

(d) *for any* $t \in \mathbb{R}$ *there exists* $c_t > 0$ *such that for any* $s \geq \max\{t - t_0, 0\} + 2$

$$\mathrm{dist}_H(U(t,t-s)B(t-s), \mathcal{M}(t)) \leq c_t e^{-\omega_0 s},$$

*where* $\omega_0 = -\ln(2(\nu + \lambda)) > 0$*,*

(e) *for any* $0 < \omega < \omega_0$ *we have*

$$\lim_{s\to\infty} e^{\omega s} \,\mathrm{dist}_H(U(t,t-s)D, \mathcal{M}(t)) = 0, \quad t \in \mathbb{R},\ D \text{ bounded in } H.$$

*The process possesses also the minimal pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *in* $H$*, which is contained in the pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^{\nu}(t) : t \in \mathbb{R}\}$ *and thus has uniformly bounded fractal dimension.*

Note that the above result holds for example for the nonlinearities $f_i$ of the form $f_i(u) = u^3 - a_i u$, $u \in \mathbb{R}$, for $N = 2$ under the assumption of the same order of $f_1$ and $f_2$, i.e., (4.11) and sufficiently smooth boundary. Actually, many other nonlinearities are allowed, like any polynomial of odd degree with positive leading coefficient. This also shows that the regular minimal pullback attractor obtained in [2] has uniformly bounded fractal dimension in $H$ if the forcing terms $\vec{h} = (h_1, h_2)$ are translation bounded.

## 5. Exponentially growing forcing terms

We consider now (1.1) under assumptions (1.6) and (1.4) with $p_1 = p_2 = 2$. Note that (1.2) holds with $l = \widetilde{L}$. We will find a pullback absorbing family for the problem (1.1) when the function $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ admits an exponential growth in the past and in the future by assuming (1.8) for some $K > 0$ and $0 \leq \theta < 2(\lambda_1 + \alpha)$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $A_0$.

Applying (1.4) and (2.5) to the energy equality (2.6), we see that the global weak solutions $\vec{u} = (u, \varphi)$ of (1.1) satisfy for a.e. $t > s$

$$\frac{d}{dt} |\vec{u}(t)|_H^2 + 2(\lambda_1 + \alpha) |\vec{u}(t)|_H^2 \leq 2\beta(|\Omega| + |\partial\Omega|) + 2\left[(h_1(t), u(t))_\Omega + (h_2(t), \varphi(t))_{\partial\Omega}\right].$$

Hence by the Cauchy inequality for $\delta > 0$ such that $0 < \theta + \delta < 2(\lambda_1 + \alpha)$ we have for a.e. $t > s$

$$\frac{d}{dt} |\vec{u}(t)|_H^2 + (2(\lambda_1 + \alpha) - \delta) |\vec{u}(t)|_H^2 \leq 2\beta(|\Omega| + |\partial\Omega|) + \delta^{-1} \left|\vec{h}(t)\right|_H^2. \tag{5.1}$$

Using (1.8) and applying the Gronwall inequality to (5.1) we get

$$\begin{aligned} |\vec{u}(t)|_H^2 \leq |\vec{u}(s)|_H^2 \, e^{-(2\lambda_1 + 2\alpha - \delta)(t-s)} &+ 2\beta(|\Omega| + |\partial\Omega|)(2\lambda_1 + 2\alpha - \delta)^{-1} \\ &+ \delta^{-1} K \int_s^t e^{-(2\lambda_1 + 2\alpha - \delta)(t-\tau)} e^{\theta|\tau|} \, d\tau, \quad t \geq s. \end{aligned}$$

Estimating the last term, we obtain

$$|\vec{u}(t)|_H^2 \leq |\vec{u}(s)|_H^2 \, e^{-(2\lambda_1 + 2\alpha - \delta)(t-s)} + K_1 + K_2 e^{\theta|t|}, \quad t \geq s, \tag{5.2}$$

where $K_1 = 2\beta(|\Omega| + |\partial\Omega|)(2\lambda_1 + 2\alpha - \delta)^{-1}$ and $K_2 = 2\delta^{-1}(2\lambda_1 + 2\alpha - \delta - \theta)^{-1} K$.

We define

$$\widetilde{B}(t) = \left\{\vec{u} \in H : |\vec{u}|_H^2 \leq 2K_1 + 2K_2 e^{\theta|t|}\right\}, \quad t \in \mathbb{R}.$$

It follows from (5.2) that for every bounded subset $D$ of $H$ there exists $r_D > 0$ such that

$$U(t,t-r)D \subset \widetilde{B}(t), \quad r \geq r_D,\ t \in \mathbb{R}.$$

Moreover, there exists $r_0 > 0$ such that

$$U(t,t-r)\widetilde{B}(t-r) \subset \widetilde{B}(t), \quad r \geq r_0,\ t \in \mathbb{R},$$

since, by using (5.2), it suffices to check that

$$2K_1 e^{-(2\lambda_1+2\alpha-\delta)r} + 2K_2 e^{\theta|t-r|} e^{-(2\lambda_1+2\alpha-\delta)r} \leq K_1 + K_2 e^{\theta|t|}, \quad t \in \mathbb{R},\ r \geq r_0.$$

Thus, the sets

$$B(t) = \mathrm{cl}_H \bigcup_{r \geq r_0} U(t,t-r)\widetilde{B}(t-r) \subset \widetilde{B}(t), \quad t \in \mathbb{R} \tag{5.3}$$

form a positively invariant family consisting of nonempty closed bounded subsets of $H$, which shows $(\mathrm{H}_1)$. Moreover, we have

$$\mathrm{diam}_H(B(t)) \leq 2\sqrt{2K_1 + 2K_2 e^{\theta|t|}} < 5 \max\left\{\sqrt{K_1}, \sqrt{K_2}\right\} e^{-\frac{\theta}{2}t}, \quad t \leq 0,$$

so $(\mathrm{H}_2)$ holds with $M = 5\max\{\sqrt{K_1}, \sqrt{K_2}\}$, $\gamma_0 = \theta/2$ and $t_0 \leq 0$ arbitrary. Furthermore, if $D$ is a bounded subset of $H$ and $t \leq t_0$, then setting $T_D = r_D + r_0$ and taking $s \geq T_D$ we get $U(t,t-s)D \subset B(t)$, which shows that $(\mathrm{H}_3)$ is satisfied in this case.

We have proved the following

**Proposition 5.1.** *Under assumptions* (1.6) *and* (1.4) *with* $p_1 = p_2 = 2$ *for* $f_i$, $i = 1,2$ *and* $\vec{h} = (h_1,h_2) \in L^2_{\mathrm{loc}}(\mathbb{R};H)$ *satisfying* (1.8), *the family* $B(t)$ *defined by* (5.3) *is positively invariant and pullback absorbing for the process* $\{U(t,s) : t \geq s\}$ *in* $H$. *Moreover, this family satisfies the assumptions* $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ *in Theorem* 3.1.

We consider the projections $P_n \colon H \to E_n$, $Q_n = I - P_n$ as in (4.4).

**Proposition 5.2.** *Suppose that* $f_i$, $i = 1,2$, *satisfy* (1.6) *and* (1.4) *with* $p_1 = p_2 = 2$. *Assume further that* $\vec{h} = (h_1,h_2) \in L^2_{\mathrm{loc}}(\mathbb{R};H)$ *satisfies* (1.8). *Then the semi-process* $\{U(t,s) : t_0 \geq t \geq s\}$ *corresponding to problem* (1.1) *can be decomposed as*

$$U(t,s) = Q_n U(t,s) + P_n U(t,s)$$

*in such a way that for every* $0 < \varepsilon < 2\lambda_1$ *we have*

$$\begin{aligned} &|Q_n(U(t,s)\vec{u}_s - U(t,s)\vec{v}_s)|_H^2 \\ &\leq \left(e^{-(2\lambda_{n+1}-\varepsilon)(t-s)} + \frac{\varepsilon^{-1}\widetilde{L}^2}{2\lambda_{n+1} - \varepsilon + 2\widetilde{L}} e^{2\widetilde{L}(t-s)}\right) |\vec{u}_s - \vec{v}_s|_H^2 \end{aligned} \tag{5.4}$$

*for all* $t \geq s$ *and* $\vec{u}_s, \vec{v}_s \in H$.

*Proof.* Let us denote by $\vec{u} = (u, \varphi)$, $\vec{v} = (v, \psi)$ the global weak solutions of (1.1) corresponding to initial data $\vec{u}_s, \vec{v}_s \in H$, respectively. Then, setting $\vec{w} = \vec{u} - \vec{v}$ and $\vec{z} = Q_n\vec{w} = (I - P_n)\vec{w}$, we obtain (4.8) as in the proof of Proposition 4.2. Using (2.4) and Cauchy-Schwarz and Cauchy inequalities to (4.8), we get for every $0 < \varepsilon < 2\lambda_1$ and for a.e. $t > s$

$$\frac{d}{dt} |\vec{z}|_H^2 + (2\lambda_{n+1} - \varepsilon) |\vec{z}|_H^2 \le \varepsilon^{-1} \left( |f_1(u) - f_1(v)|_\Omega^2 + |f_2(\varphi) - f_2(\psi)|_{\partial\Omega}^2 \right).$$

Since $f_i$, $i = 1, 2$, are globally Lipschitz continuous, it follows from (1.6) that

$$\frac{d}{dt} |\vec{z}|_H^2 + (2\lambda_{n+1} - \varepsilon) |\vec{z}|_H^2 \le \varepsilon^{-1} \widetilde{L}^2 |\vec{w}|_H^2 \quad \text{for a.e. } t > s.$$

By Proposition 2.4, in particular we have

$$\frac{d}{dt} \left( e^{(2\lambda_{n+1} - \varepsilon)t} |\vec{z}(t)|_H^2 \right) \le \varepsilon^{-1} \widetilde{L}^2 e^{(2\lambda_{n+1} - \varepsilon)t + 2\widetilde{L}(t-s)} |\vec{w}(s)|_H^2 \quad \text{for a.e. } t > s.$$

Integrating and using $|\vec{z}(s)|_H \le |\vec{w}(s)|_H$, we get (5.4). □

**Corollary 5.3.** *Under the assumptions of Proposition* 5.2*, there exist two families of operators* $\{C(t, s) : t_0 \ge t \ge s\}$ *and* $\{S(t, s) : t_0 \ge t \ge s\}$ *with* $U(t, s) = C(t, s) + S(t, s)$ *satisfying hypotheses* (H$_4$)–(H$_5$) *of Theorem* 3.1*.*

*Proof.* We put $\widetilde{t} > 0$ arbitrary and $C = Q_nU$ and $S = P_nU$ with some $n \in \mathbb{N}$ large enough. (H$_4$) follows from Proposition 5.2 and (H$_5$) follows from [4, Lemma 1] (see also [7, Lemma 4.2]) and Proposition 2.4. In particular, if $n \in \mathbb{N}$ is such that

$$\lambda = \left( e^{-(2\lambda_{n+1} - \varepsilon)\widetilde{t}} + \frac{\varepsilon^{-1}\widetilde{L}^2}{2\lambda_{n+1} - \varepsilon + 2\widetilde{L}} e^{2\widetilde{L}\widetilde{t}} \right)^{1/2} < \frac{1}{2} e^{-\frac{\theta}{2}\widetilde{t}}, \tag{5.5}$$

then, for $0 < \nu < \min\left\{ \frac{1}{2} e^{-\frac{\theta}{2}\widetilde{t}} - \lambda, e^{(\widetilde{L} - \lambda_1)\widetilde{t}} \right\}$, we have

$$N_\nu \le \left( 1 + \frac{2e^{(\widetilde{L} - \lambda_1)\widetilde{t}}}{\nu} \right)^n \tag{5.6}$$

in (H$_5$). □

Collecting the above results, as an application of Theorem 3.1, we obtain

**Theorem 5.4.** *If* $f_i$*,* $i = 1, 2$*, satisfy* (1.6) *and* (1.4) *with* $p_1 = p_2 = 2$*, whereas* $\vec{h} = (h_1, h_2) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ *satisfies* (1.8) *with some* $K > 0$ *and* $0 \le \theta < 2(\lambda_1 + \alpha)$*, then the process* $\{U(t, s) : t \ge s\}$ *on* $H = L^2(\Omega) \times L^2(\partial\Omega)$ *of global weak solutions of* (1.1) *possesses a pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^\nu(t) : t \in \mathbb{R}\}$ *in* $H$ *satisfying the properties:*

(a) $\mathcal{M}(t)$ *is a nonempty compact subset of* $B(t)$ *for* $t \in \mathbb{R}$*,*

(b) $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$, $t \geq s$,

(c) $\sup_{t\in\mathbb{R}} \dim_f^H(\mathcal{M}(t)) \leq -\ln N_\nu / \big[\ln(2(\nu+\lambda)) + \frac{\theta}{2}\widetilde{t}\big]$, *where* $\lambda$ *is given in* (5.5) *and* $N_\nu$ *is given in* (5.6) *for* $0 < \nu < \min\left\{\frac{1}{2}e^{-\frac{\theta}{2}\widetilde{t}} - \lambda, e^{(\widetilde{L}-\lambda_1)\widetilde{t}}\right\}$,

(d) *for any* $t \in \mathbb{R}$ *there exists* $c_t > 0$ *such that for any* $s \geq \max\{t - t_0, 0\} + 2\widetilde{t}$

$$\operatorname{dist}_H(U(t,t-s)B(t-s), \mathcal{M}(t)) \leq c_t e^{-\omega_0 s},$$

*where* $\omega_0 = -\frac{1}{\widetilde{t}}\left(\ln(2(\nu+\lambda)) + \frac{\theta}{2}\widetilde{t}\right) > 0$,

(e) *for any* $0 < \omega < \omega_0$ *we have*

$$\lim_{s\to\infty} e^{\omega s} \operatorname{dist}_H(U(t,t-s)D, \mathcal{M}(t)) = 0, \quad t \in \mathbb{R},\ D \text{ bounded in } H.$$

*The process possesses also the minimal pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *in* $H$, *which is contained in the pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^\nu(t) : t \in \mathbb{R}\}$ *and thus has uniformly bounded fractal dimension.*

It would be interesting to know if we may obtain the existence of pullback exponential attractors or minimal pullback attractors with uniformly bounded fractal dimension when the time-dependent forcing terms grow exponentially, but the nonlinearities have superlinear growth.

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