REDUCED BASIS METHOD FOR THE SMAGORINSKY MODEL

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INTRODUCTION

We present a reduced basis Smagorinsky model. This model includes a non-linear eddy diffusion term that we have to treat in order to solve efficiently our reduced basis model. We approximate this non-linear term using the Empirical Interpolation Method, in order to obtain a linearised decomposition of the reduced basis Smagorinsky model.

The reduced basis Smagorinsky model is decoupled in a Online/Offline procedure. First, in the Offline stage, we construct hierarchical bases in each iteration of the Greedy algorithm, by selecting the snapshots which have the maximum *a posteriori* error estimation value. To assure the Brezzi inf-sup condition on our reduced basis space, we have to define a *supremizer* operator on the pressure solution, and enrich the reduced velocity space. Then, in the Online stage, we are able to compute a speedup solution of our problem, with a good accuracy.

THE REDUCED BASIS MODEL

We define the steady reduced basis Smagorinsky model as follows:

$$\begin{cases} \mathbf{w}_N\cdot\nabla\mathbf{w}_N - \dfrac{1}{\mu}\Delta\mathbf{w}_N + \nabla p_N - \nabla\cdot(\nu_T(\mathbf{w}_N)\nabla\mathbf{w}_N) = \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{w}_N = 0 & \text{in } \Omega \\ \mathbf{w}_N = \mathbf{g}_D & \text{on } \Gamma_{D_{in}} \\ \mathbf{w}_N = 0 & \text{on } \Gamma_{D_w} \\ -p_N\mathbf{n} + \left(\dfrac{1}{\mu} + \nu_T(\mathbf{w}_N)\right)\dfrac{\partial\mathbf{w}_N}{\partial\mathbf{n}} = 0 & \text{on } \Gamma_{out} \end{cases} \tag{1}$$

where we are denoting $\nu_T(\mathbf{w}_N)_{|_K} = (C_S h_K)^2|\nabla\mathbf{w}_{N_{|_K}}(x)|$.

Defining the reduced space $X_N = Y_N \times M_N$, where Y_N is the reduced velocity space and M_N the reduced pressure space, we obtain the reduced variational problem

$$\begin{cases} \text{Given } \mu\in\mathcal{D}, \text{ find } U_N(\mu)\in X_N \text{ such that} \\ A(U_N(\mu), V_N;\mu) = F(V_N;\mu) \qquad \forall V_N\in X_N \end{cases} \tag{2}$$

EMPRICAL INTERPOLATION METHOD

We denote $g(\mu) := g(x;\mathbf{w}_h(\mu)) = |\nabla\mathbf{w}_h(\mu)|(x)$. The finality of using the EIM is decoupling the μ-dependence of the spatial dependence of the function $g(\mu)$, i.e.,

$$g(\mu) \approx \sum_{j=1}^{M}\sigma_j(\mu)q_j(x) \tag{3}$$

We define a reduced EIM-space $W_M = \text{span}\{q_1,\dots,q_M\}$ by a Greedy procedure. We also define a set of interpolation points $T_M = \{x_1,\dots,x_M\}$ that allows us, for each $\mu\in\mathcal{D}$, solve the linear system

$$\sum_{j=1}^{M}\sigma_j(\mu)q_j(x_i) = g(x_i;\mathbf{w}_h(\mu)) \qquad i = 1,\dots,M \tag{4}$$

Thanks to the Empirical Interpolation Method, we are able to approximate the non-linear Smagorinsky term in the following form

$$\sum_{K\in T}\int_K (C_S h_K)^2|\nabla\mathbf{w}|\nabla\mathbf{w}:\nabla\mathbf{v}\,d\Omega \approx \sum_{j=1}^{M}\sigma_j(\mu)\sum_{K\in T}\int_K (C_S h_K)^2 q_j\nabla\mathbf{w}:\nabla\mathbf{v}\,d\Omega$$

GREEDY ALGORITHM

For the startup of the Greedy algorithm, we choose an arbitrary parameter value $\mu_1\in\mathcal{D}$, and we compute the corresponding snapshot $(\mathbf{u}(\mu_1), p(\mu_1))$. We choose the (k+1)-th value of $\mu\in\overline{\mathcal{D}}$ as

$$\mu_{k+1} = \arg\max_{\mu\in\overline{\mathcal{D}}}\Delta_k(\mu), \qquad k = 1,\dots,N.$$

where $\Delta_k(\mu)$ is the *a posteriori* error estimator, which bounds the error between the FE solution and the RB solution.

The reduced velocity-pressure spaces are defined by

$$M_N = \text{span}\{\xi_k^p := p(\mu_k), \quad k = 1,\dots,N\}$$

$$Y_N = \text{span}\{\zeta_k^{\mathbf{v}} := \mathbf{u}(\mu_k),\ T_p^\mu\xi_k^p, \quad k = 1,\dots,N\}$$

where $T_p^\mu : M_h \to Y_h$ is the inner pressure *supremizer* operator defined as

$$(T_p^\mu, \mathbf{w}_h) = -\int_\Omega(\nabla\cdot\mathbf{w}_h)q\,d\Omega, \quad \forall\mathbf{w}\in Y_h \tag{5}$$

By denoting as $\partial_1 A(U_h, V_h;\mu)(Z_h)$ the directional derivative with respect the first variable, in the direction $Z_h\in X_h$, we define the positive constants ρ_T and β_N for the *a posteriori* error bound as

$$|\partial_1 A(U_h^1, V_h;\mu)(Z_h) - \partial_1 A(U_h^2, V_h;\mu)(Z_h)| \le \rho_T\|U_h^1 - U_h^2\|_X\|V_h\|_X\|Z_h\|_X$$

$$\beta_N(\mu) = \inf_{Z_h\in X_h}\sup_{V_h\in X_h}\frac{\partial_1 A(U_N, V_h;\mu)(Z_h)}{\|Z_h\|_X\|V_h\|_X}$$

The *a posteriori* error bound estimator is defined as

$$\Delta_N(\mu) = \frac{\beta_N(\mu)}{2\rho_T}\left[1 - \sqrt{1 - \tau_N(\mu)}\right]$$

where $\tau_N(\mu) = \dfrac{4\epsilon_N(\mu)\rho_T}{\beta_N(\mu)^2}$, and $\epsilon_N(\mu)$ is the dual norm of the residual.

NUMERICAL RESULTS

We solve the Smagorinsky RBM in a backward-facing step. We select $M_{\max} = 73$ basis for EIM, and $N_{\max} = 17$ basis for RBM.

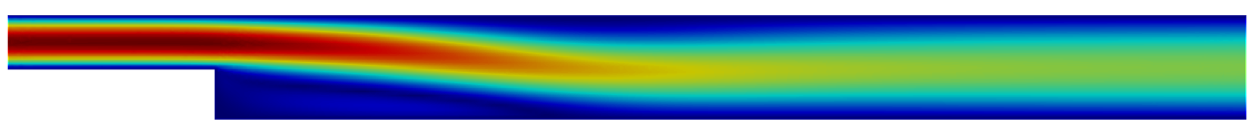
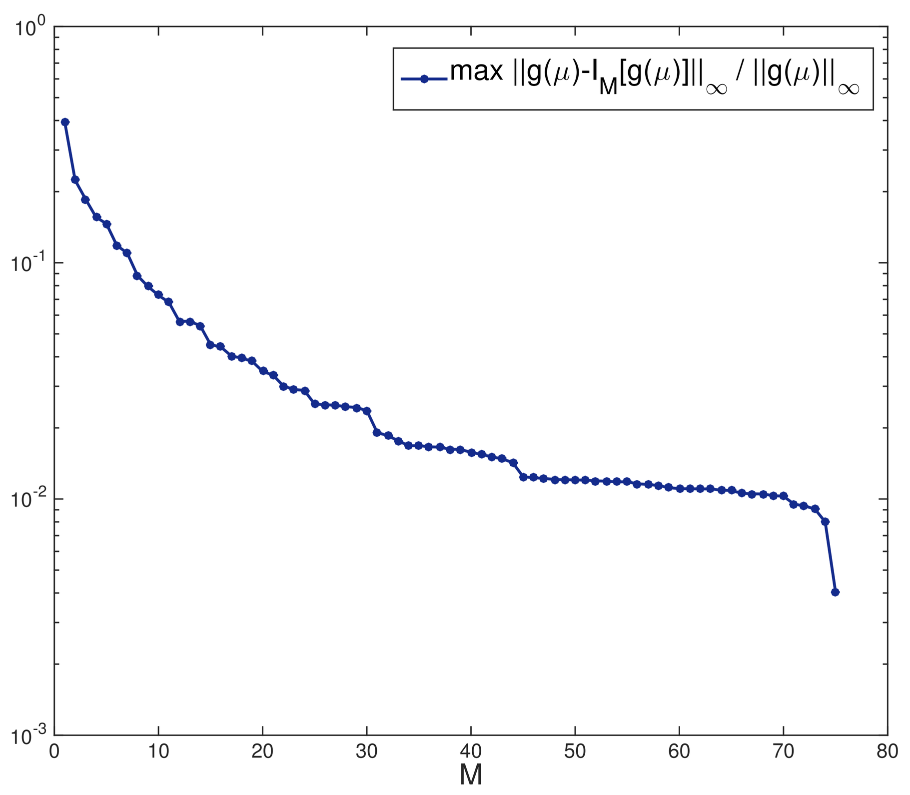


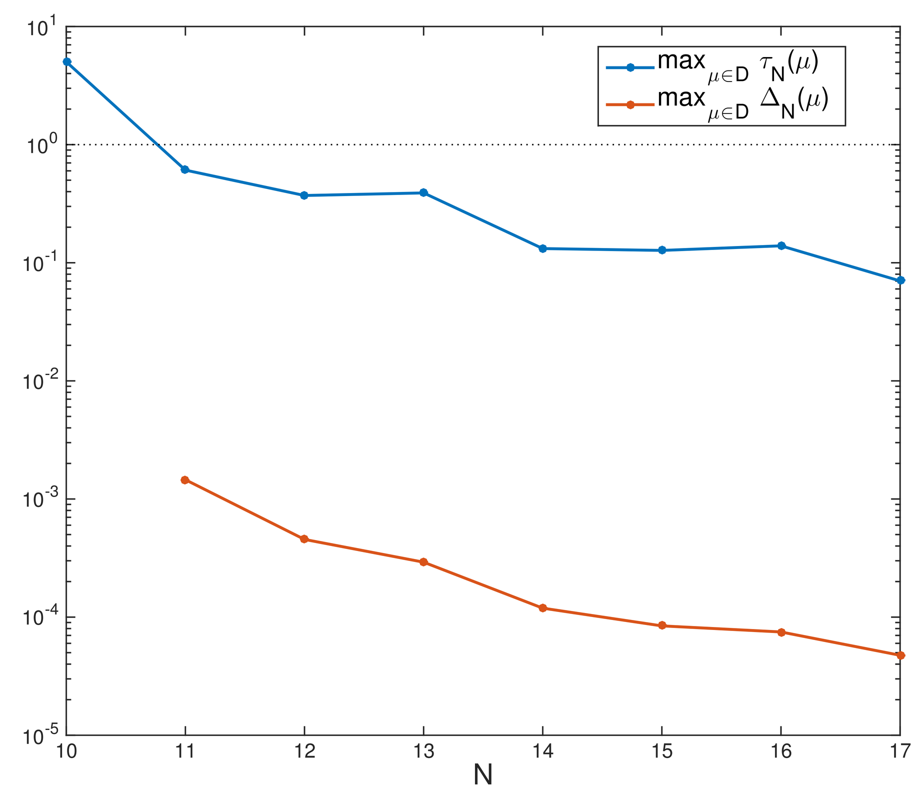
Figure 1: RB solution for $\mu = 320$

Data	$\mu = 56$	$\mu = 132$	$\mu = 236$	$\mu = 320$
T_{FE}	152.7s	508.7s	991.9s	1929.1s
T_{online}	1.37s	1.55s	1.60s	1.60s
speedup	111	326	626	1204
$\|\mathbf{u}_h - \mathbf{u}_N\|_1/\|\mathbf{u}_h\|_1$	$1.67\cdot10^{-7}$	$2.30\cdot10^{-6}$	$2.76\cdot10^{-6}$	$5.60\cdot10^{-6}$
$\|p_h - p_N\|_1/\|p_h\|_1$	$7.92\cdot10^{-8}$	$8.57\cdot10^{-7}$	$7.13\cdot10^{-6}$	$1.98\cdot10^{-5}$

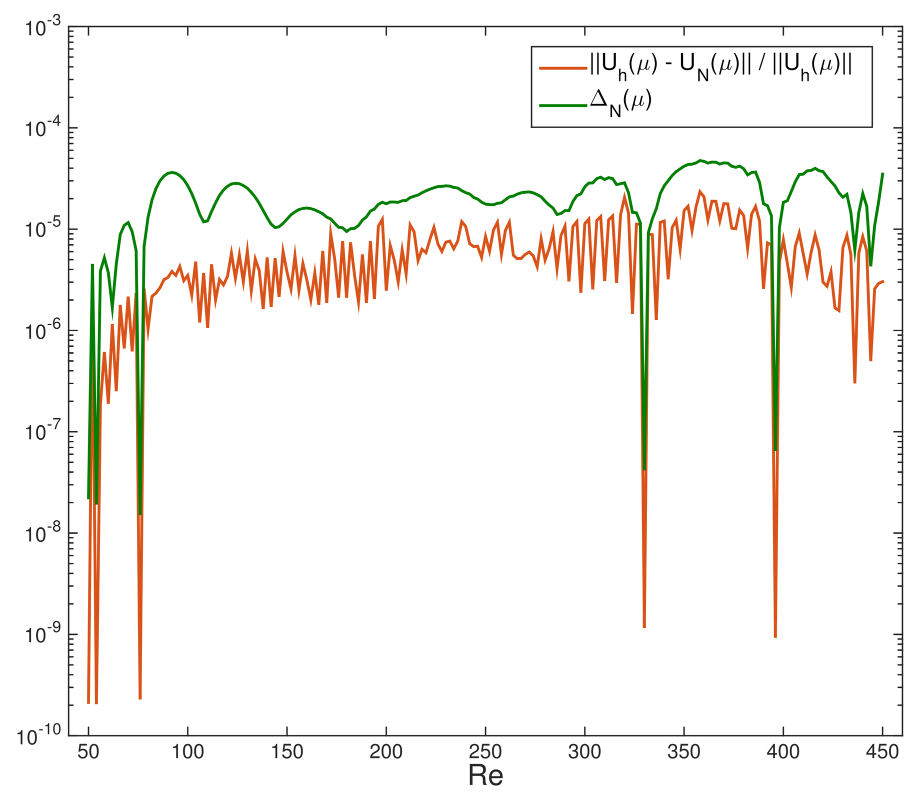
Table 1: Computational time for FE and RB solutions, with the speedup and the error.



(a) EIM Greedy convergence.



(b) RBM Greedy convergence.



(c) *A posteriori* error bound and relative error at $N = N_{\max}$

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