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Noise-induced forced synchronization of global variables in coupled bistable systems

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Abstract – We analyze the noise-induced synchronization between a collective variable characterizing a complex system with a finite number of interacting bistable units and time-periodic driving forces. A random phase process associated to the collective stochastic variable is defined. Its average phase frequency and average phase diffusion are used to characterize the phenomenon. Our analysis is based on numerical solutions of the corresponding set of Langevin equations.

Introduction. – Synchronization phenomena in nonlinear systems have been widely studied in very different contexts [1–3]. In complex nonlinear autonomous systems with many interacting subunits, nonlinear dynamics might lead to self-synchronization between the individual elements. Noisy environments and externally applied time-periodic forces might also influence synchronization. One might be interested on how noise and driving forces affect the synchronization between the different subunits, or the focus of interest might be the noise-induced forced synchronization to the external driving force of a suitably defined single collective variable characterizing the system as a whole.

In the case of systems characterized by just a single noisy dynamical variable subject to the action of a deterministic external time-periodic signal, noise-induced forced synchronization has been amply studied. With an appropriate definition of a random phase process associated to the response of the system, theories of noise-induced forced synchronization have been developed and analytical expressions for the quantities characterizing the forced-synchronization mechanism have been obtained [4–9]. The theory of forced synchronization in a single variable system has also been extended to the quantum regime of incoherent tunneling transitions in driven dissipative systems [10]. Synchronization effects in complex systems formed by arrays of excitable systems and stochastic resonators have been analyzed in [3,11]. A theory of synchronization in large arrays ($N = 10^3$–10^5) of non-interacting bistable systems driven by a small amplitude external force has been developed in [11] within the limits of linear response theory. It describes the strong enhancement of the synchronization mechanism for a collective variable with respect to that of a single unit. Numerical calculations by the same author shows the same effect in an array of 10^2 excitable systems. In this work, we find an analogous result for much smaller arrays of interacting bistable units driven by strong, but subthreshold forces, well beyond the limit where linear response theory can be faithfully applied.

In this work, we consider a model describing a *finite* set of N interacting *bistable* subsystems, each of them characterized by a single degree of freedom x_i $(i = 1, \ldots, N)$, whose dynamics is governed by the Langevin equations [12,13]

$$\dot{x}_i = x_i - x_i^3 + \frac{\theta}{N}\sum_{j=1}^{N}(x_j - x_i) + \xi_i(t) + F(t). \qquad (1)$$

Here, the $\xi_i(t)$'s are Gaussian white noises with zero average and $\langle \xi_i(t)\xi_j(s)\rangle = 2D\delta_{ij}\delta(t-s)$, θ is the parameter defining the strength of the interaction between subsystems, and $F(t) = F(t+T)$ is an external driving force of period T. The amplitude of the driving force is supposed to be large so that a linear response approximation of the dynamics is not adequate. Nonetheless, we are interested in noise-induced synchronization, so the driving amplitude is subthreshold in the sense that, in the absence of noise, the external driving by itself cannot induce the phenomenon.

We focus our interest on a single global variable, rather than on the variable characterizing an individual subsystem. In a recent work, we have analyzed the phenomenon of stochastic resonance associated to the global variable of the same system [13,14]. In this work, we analyze the phase process associated to the noise-induced jumps of the global variable between its dynamical attractors. Noise-induced forced synchronization between the global variable and the driving force is characterized by the behavior of the average phase frequency and the average phase diffusion constant with the noise strength.

The random phase process, its average phase frequency and the diffusion constant. – The variable of interest is the collective output $S(t)$ defined as

$$S(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t). \tag{2}$$

In the limit $N \to \infty$, an asymptotically valid Langevin equation for $S(t)$ can be obtained from eq. (1) [15]. For finite systems the validity of this Langevin equation is questionable and we rely on the solution of the Langevin equation for each x_i to obtain information about the global process $S(t)$. For low noise strengths and driving forces with sufficiently large periods, our numerical simulations show that a random trajectory of $S(t)$ contains essentially small fluctuations around two values (attractors) and random, sporadic transitions between them. For each realization of the noise term, we then introduce a random phase process, $\phi(t)$, associated to the stochastic variable $S(t)$ as follows. We refer to a "jump" of $S(t)$ along a trajectory, when a very large fluctuation takes the $S(t)$ trajectory from a value near an attractor to a value in the neighborhood of the other attractor. We count $N^{(\alpha)}(t)$, the number of jumps in the α trajectory of the process $S(t)$ within the interval $(0,t]$. A trajectory of the phase process is then constructed as

$$\phi^{(\alpha)}(t) = \pi N^{(\alpha)}(t), \tag{3}$$

so that $\phi(t)$ increases by 2π after every two consecutive jumps.

The first two moments of the phase process are estimated as

$$\langle \phi(t) \rangle = \frac{1}{\mathcal{M}}\sum_{\alpha=1}^{\mathcal{M}} \phi^{(\alpha)}(t), \tag{4}$$

$$\begin{aligned} v(t) &= \left\langle [\phi(t)]^2 \right\rangle - \langle \phi(t) \rangle^2 \\ &= \frac{1}{\mathcal{M}}\sum_{\alpha=1}^{\mathcal{M}} \left[\phi^{(\alpha)}(t)\right]^2 - \frac{1}{\mathcal{M}^2}\left[\sum_{\alpha=1}^{\mathcal{M}} \phi^{(\alpha)}(t)\right]^2, \end{aligned} \tag{5}$$

where, $\mathcal{M}$ is the number of generated random trajectories (typically, 3000 trajectories for the results presented in this work).

The instantaneous phase frequency is easily determined from the time derivative of $\langle \phi(t) \rangle$. After a sufficiently large number of periods of the driving force, n, the system forgets its initial preparation, but the instantaneous phase frequency is still a function of time. Then, we define a cycle average phase frequency $\overline{\Omega}_{\rm ph}$ by averaging the instantaneous phase frequency over a period of the external driving [7,10],

$$\overline{\Omega}_{\rm ph} = \frac{1}{T}\int_{nT}^{(n+1)T} {\rm d}t\, \frac{{\rm d}\langle \phi(t) \rangle}{{\rm d}t} = \frac{\langle \phi\left[(n+1)T\right] \rangle - \langle \phi(nT) \rangle}{T}. \tag{6}$$

Similarly, the cycle average phase diffusion coefficient is evaluated from the instantaneous slope of the variance $v(t)$ as [7,10],

$$\overline{D}_{\rm ph} = \frac{1}{T}\int_{nT}^{(n+1)T} {\rm d}t\, \frac{{\rm d}\langle v(t) \rangle}{{\rm d}t} = \frac{v\left[(n+1)T\right] - v\left(nT\right)}{T}. \tag{7}$$

In previous works, approximate anlytical expressions for these two quantities have been derived for the $N = 1$ problem in the classical [4,7,9,16] and quantum cases [10]. Those expressions cannot be applied to the collective variable of an N-particle problem, as a closed Langevin or Fokker-Planck equation for $S(t)$ for a finite-size system does not exist. Thus, we will rely on numerical solutions of the Langevin equations, eq. (1), for the estimation of the average phase frequency and diffusion coefficients in eqs. (6) and (7). The numerical method used to solve the Langevin dynamics has been detailed in [17].

Results. – We have analyzed the forced-synchronization phenomenon for the collective variable $S(t)$ for several values of the coupling strength, θ, the noise strength, D, and two types of periodic forces: sinusoidal forces ($F(t) = A\cos\Omega t$) and rectangular forces ($F(t) = A$ ($F(t) = -A$) if $t \in [nT/2, (n+1)T/2)$ with n even (odd)).

In fig. 1 we depict the behavior of the average phase frequency, $\overline{\Omega}_{\rm ph}$, and diffusion constant, $\overline{D}_{\rm ph}$, obtained from numerical simulations. The external driving force is taken to be sinusoidal with $A = 0.3$ and $\Omega = 0.01$. In the absence of noise, the variable $S(t)$ does not jump between the attractors. Thus, for these parameter values, the external driving is subthreshold and we have noise-induced effects. It is clear from the picture that, even for a single subsystem, $N = 1$, forced synchronization exists as there is a range of noise values for which $\overline{\Omega}_{\rm ph}$ matches the external frequency, Ω, and the diffusion constant is rather small, with a minimum around $D \approx 0.02$. As we discused previously in [7], these numerical results agree very well with the analytical expressions reported in [7]. For a set of $N = 10$ independent subsystems ($\theta = 0$), forced synchronization for the collective variable is very much enhanced with respect to that observed in a single-particle variable. The matching between the driving frequency and the average phase frequency extends over a very large range of noise values, for which the corresponding diffusion constant is very much reduced. It should be pointed out

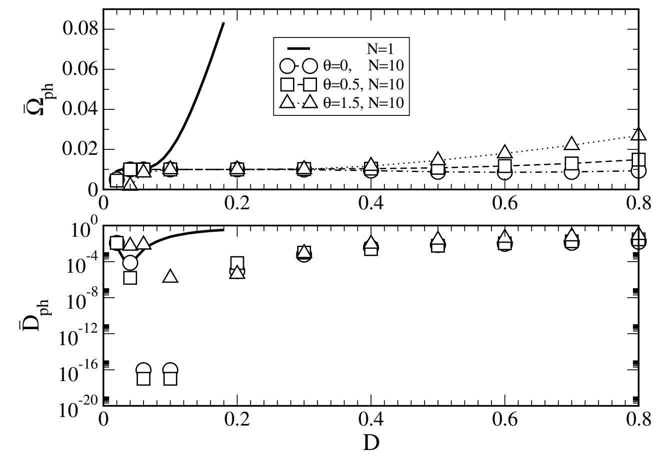

Fig. 1: The behavior of the average phase frequency, $\overline{\Omega}_{\rm ph}$, and the phase diffusion constant, $\overline{D}_{\rm ph}$, with the noise strength, D, for a sinusoidal driving term with $A = 0.3$ and $\Omega = 0.01$ and several values of the interaction strength θ.

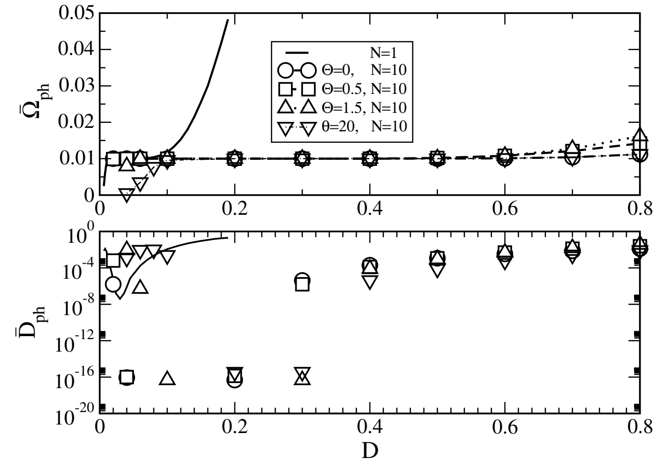

Fig. 2: The behavior of the average phase frequency, $\overline{\Omega}_{\rm ph}$, and the phase diffusion constant, $\overline{D}_{\rm ph}$, with the noise strength, D, for a rectangular driving term with $A = 0.3$ and fundamental frequency $\Omega = 0.01$ and several values of the interaction strength θ.

that, in this and subsequent figures, the fact that a "range" of D values shows up in the plots, for which $\overline{D}_{\rm ph}$ seems to be constant, is due to the lack of machine precision to compute very small numbers. It is safe to say that for the noise values in those ranges, the phase diffusion constant is zero, and that the collective variable jumps every half-period of the driver without skipping any change in sign of the applied force.

The results for a rectangular input force with an amplitude $A = 0.3$ and fundamental frequency $\Omega = 0.01$ are shown in fig. 2. The behaviors of $\overline{\Omega}_{\rm ph}$ and $\overline{D}_{\rm ph}$ with the noise strength for the different sets of parameter values is qualitatively similar to the one observed in fig. 1 for a sinusoidal driving. Quantitatively, for a rectangular driving, the noise-induced forced synchronization is enhanced with

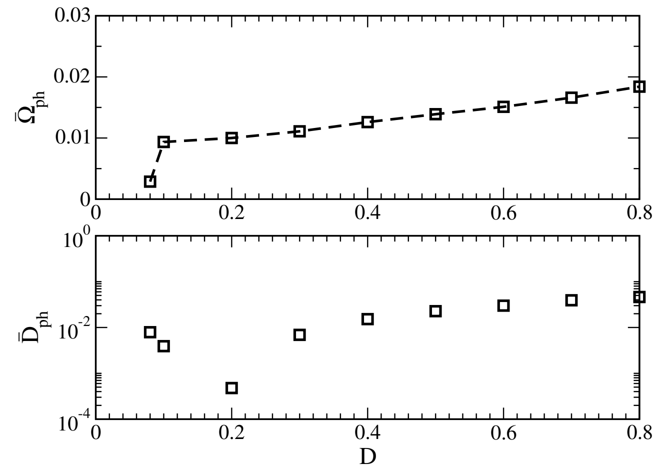

Fig. 3: The behavior of the average phase frequency, $\overline{\Omega}_{\rm ph}$, and the phase diffusion constant, $\overline{D}_{\rm ph}$, with the noise strength, D, for a rectangular driving term with $A = 0.1$ and $\Omega = 0.01$.

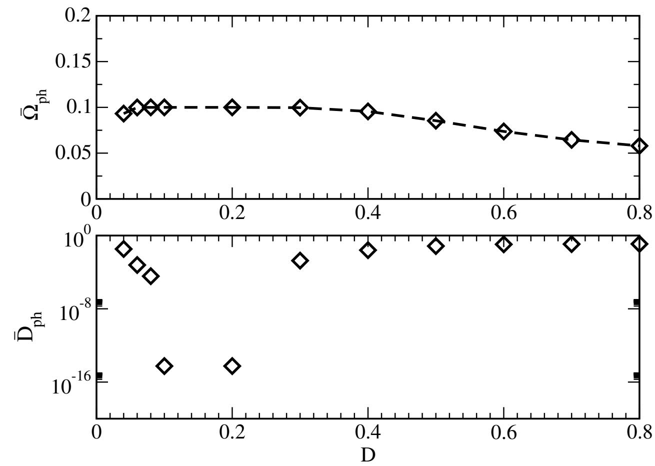

Fig. 4: The behavior of the average phase frequency, $\overline{\Omega}_{\rm ph}$, and the phase diffusion constant, $\overline{D}_{\rm ph}$, with the noise strength, D, for a rectangular driving term with $A = 0.3$ and $\Omega = 0.1$.

respect to that observed with a sinusoidal input with the same amplitude and fundamental frequency, as indicated by the wider range of D values for which frequencies match.

We have analyzed the influence of increasing the driving fundamental frequency or decreasing the external amplitude. In fig. 3 we depict the results obtained for $\overline{\Omega}_{\rm ph}$ and $\overline{D}_{\rm ph}$ for a rectangular input with $A = 0.1$ and $\Omega = 0.01$, with $N = 10$ interacting subsystems with $\theta = 0.5$. Comparison with the corresponding results in fig. 1 indicates that synchronization for this smaller-amplitude driving is much weaker than for the larger-amplitude case. Not only the noise range for which the average phase frequency matches the external frequency is reduced, but the dip in the phase diffusion constant plot is much less pronounced.

In the next figure (fig. 4), we plot the results of our simulations for a rectangular input with $A = 0.3$

(the same as in fig. 1) and $\Omega = 0.1$, with $N = 10$ interacting subsystems with $\theta = 0.5$. The increase in the fundamental driving frequency with respect to the value used in fig. 1 leads also to a reduction of the noise range for which noise-induced forced synchronization is optimal. Nonetheless, the dip in the $\overline{D}_{\mathrm{ph}}$ *vs.* the noise strength seems to be of the same order of magnitude as observed for the lower-frequency case depicted in fig. 1.

Noise-induced forced synchronization is due to the modification by the applied force of the intrinsic noisy dynamics of the set of bistable systems. Noise itself makes the collective variable jumps between attractors with a certain probability distribution. The external force tilts the potential surface such that every half-period, jumps towards one attractor are much favored against the ones towards the other one. Thus, the external driving greatly modifies this distribution by forcing the jumps to be coherent with the periodicity of the external driving.

In the absence of interactions between the subsystems, $\theta = 0$, the central limit theorem applies and, consequently, the collective variable fluctuations are expected to be much reduced with respect to those present in the $N = 1$ case. The probability distribution of the collective variable is very narrowly centered around the attractor favored by the sign of the driving force every half-period. Therefore, as long as the noise strength is large enough to induce jumps every half-period, the number of jumps in a given time interval will match almost exactly the number of times that $F(t)$ changes its sign during the same time interval. Increasing the noise beyond some value makes the jumps more and more independent of the periodic tilting induced by the driving force and destroys the matching between the number of jumps and the number of half-periods. Nonetheless, the matching will remain effective for a much wider range of D values than in the $N = 1$ case, as the effective noise for the dynamics of the global variable is not gauged simply by D but by $\frac{D}{N}$. For $\theta \neq 0$, the central limit theorem does not apply. One should expect a weakening of the forced synchronization mechanism with respect to the $\theta = 0$ case, due to the change in the potential energy relief brought up by the interaction term. As θ gets large compared with the driver amplitude, the tilting effect of the driving becomes relatively less important and there will be a slight mismatch between the number of jumps and the number of half-periods for noise values for which synchronization in the $\theta = 0$ case was almost perfect. As the value of θ is increased, this results in a small increase in the minimum noise value at which synchronization sets in, as observed in figs. 1 and 2. This reduces slightly the range of D values within which the matching of $\overline{\Omega}_{\mathrm{ph}}$ and Ω is optimal.

Large amplitudes and long driving periods provide potential surfaces which remain tilted during a time interval long enough for the noise-induced jumps of the subsystems variables to happen almost simultaneously every half-period of the driver. A weakening of noise-induced forced synchronization is then expected as the driving amplitude and/or the driving period decrease, as seen in figs. 3 and 4.

In conclusion, we have numerically demonstrated the phenomenon of noise-induced forced synchronization in finite arrays of bistable systems. The synchronization of the collective variable is much enhanced with respect to the one observed for a single bistable unit. Interactions between the subunits do not destroy this coherent effect. Driving forces with short periods and small amplitudes tend to weaken drastically the phenomenon.

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