Gamma-convergence of equi-coercive nonlinear energies defined on vector-valued functions, with non-uniformly bounded coefficients

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Homogenization of equi-coercive nonlinear energies defined on vector-valued functions, with non-uniformly bounded coefficients

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Abstract

The present paper deals with the asymptotic behavior of equi-coercive sequences $\{\mathscr{F}_n\}$ of nonlinear functionals defined over vector-valued functions in $W_0^{1,p}(\Omega)^M$, where $p > 1$, $M \geq 1$, and Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$. The strongly local energy density $F_n(\cdot, Du)$ of the functional $\mathscr{F}_n$ satisfies a Lipschitz condition with respect to the second variable, which is controlled by a positive sequence $\{a_n\}$ which is only bounded in some suitable space $L^r(\Omega)$. We prove that the sequence $\{\mathscr{F}_n\}$ Γ-converges for the strong topology of $L^p(\Omega)^M$ to a functional $\mathscr{F}$ which has a strongly local density $F(\cdot, Du)$ for sufficiently regular functions u. This compactness result extends former results on the topic, which are based either on maximum principle arguments in the nonlinear scalar case, or adapted div-curl lemmas in the linear case. Here, the vectorial character and the nonlinearity of the problem need a new approach based on a careful analysis of the asymptotic minimizers associated with the functional $\mathscr{F}_n$. The relevance of the conditions which are imposed to the energy density $F_n(\cdot, Du)$, is illustrated by several examples including some classical hyper-elastic energies.

1 Introduction

In this paper we study the asymptotic behavior of the sequence of nonlinear functionals, including some hyper-elastic energies (see the examples of Section 2.3), defined on vector-valued functions by

$$\mathscr{F}_n(v) := \int_\Omega F_n(x, Dv)\, dx \quad \text{for } v \in W_0^{1,p}(\Omega)^M, \quad \text{with } p \in (1,\infty),\ M \geq 1, \tag{1.1}$$

in a bounded open set Ω of $\mathbb{R}^N$, $N \geq 2$. The sequence $\mathscr{F}_n$ is assumed to be equi-coercive. Moreover, the associated density $F_n(\cdot, \xi)$ satisfies some Lipschitz condition with respect to $\xi \in \mathbb{R}^{M\times N}$, and its coefficients are not uniformly bounded in Ω.

The linear scalar case, *i.e.* when $F_n(\cdot, \xi)$ is quadratic with respect to $\xi \in \mathbb{R}^N$ ($M = 1$), with uniformly bounded coefficients was widely investigated in the seventies through G-convergence by Spagnolo [33], extended by Murat and Tartar with H-convergence [28, 35], and alternatively through Γ-convergence by De Giorgi [22, 23] (see also [21, 4]). The linear elasticity case was probably first derived by Duvaut (unavailable reference), and can be found in [32, 25]. In the nonlinear scalar case the first compactness results are due to Carbone, Sbordone [17] and Buttazzo, Dal Maso [14] by a Γ-convergence approach assuming the L^1-equi-integrability of the coefficients. More recently, these results were extended in [5, 9, 10] relaxing the L^1-boundedness of the coefficients but assuming that $p > N-1$ if $N \geq 3$, showing then the uniform convergence of the minimizers thanks to the maximum principle. In all these works the scalar framework combined with the condition $p > N-1$ if $N \geq 3$ and the equi-coercivity of the functionals, induce in terms of the Γ-convergence for the strong topology of $L^p(\Omega)$, a limit energy $\mathscr{F}$ of the same nature satisfying

$$\mathscr{F}(v) := \int_\Omega F(x, Dv)\, d\nu \quad \text{for } v \in W, \tag{1.2}$$

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where $C_c^1(\Omega)^M \subset W$ is some suitable subspace of $W_0^{1,p}(\Omega)^M$, and ν is some Radon measure on Ω. Removing the L^1-equi-integrability of the coefficients in the three-dimensional linear scalar case (note that $p = N-1 = 2$ in this case), Fenchenko and Khruslov [24] (see also [26]) were, up to our knowledge, the first to obtain a violation of the compactness result due to the appearance of local and nonlocal terms in the limit energy $\mathscr{F}$. This seminal work was also revisited by Bellieud and Bouchitté [2]. Actually, the local and nonlocal terms in addition to the classical strongly local term come from the Beurling-Deny [3] representation formula of a Dirichlet form, and arise naturally in the homogenization process as shown by Mosco [27]. The complete picture of the attainable energies was obtained by Camar-Eddine and Seppecher [15] in the linear scalar case. The elasticity case is much more intricate even in the linear framework, since the loss of uniform boundedness of the elastic coefficients may induce the appearance of second gradient terms as Seppecher and Pideri proved in [30]. The situation is dramatically different from the scalar case, since the Beurling-Deny formula does not hold in the vector-valued case. In fact, Camar-Eddine and Seppecher [16] proved that any lower semi-continuous quadratic functional vanishing on the rigid displacements, can be attained. Compactness results were obtained in the linear elasticity case using some (strong) equi-integrability of the coefficients in [11], and using various extensions of the classical Murat-Tartar [28] div-curl result in [7, 13, 12, 29] (which were themselves initiated in the former works [6, 9] of the two first authors).

In our context the vectorial character of the problem and its nonlinearity prevent us from using the uniform convergence of [10] and the div-curl lemma of [12], which are (up to our knowledge) the more recent general compactness results on the topic. We assume that the nonnegative energy density $F_n(\cdot, \xi)$ of the functional (1.1) attains its minimum at $\xi = 0$, and satisfies the following Lipschitz condition with respect to $\xi \in \mathbb{R}^{M\times N}$:

$$\begin{cases} \left|F_n(x,\xi) - F_n(x,\eta)\right| \le \left(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\right)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\, |\xi - \eta| \\ \forall\, \xi, \eta \in \mathbb{R}^{M\times N},\ \text{a.e. } x \in \Omega, \end{cases}$$

which is controlled by a positive function $a_n(\cdot)$ (see the whole set of conditions (2.1) to (2.6) below). The sequence $\{a_n\}$ is assumed to be bounded in $L^r(\Omega)$ for some $r > (N-1)/p$ if $1 < p \le N-1$, and bounded in $L^1(\Omega)$ if $p > N-1$. Note that for $p > N-1$ our condition is better than the L^1-equi-integrability used in the scalar case of [17, 14], but not for $1 < p \le N-1$. Under these assumptions we prove (see Theorem 2.3) that the sequence $\{\mathscr{F}_n\}$ of (1.1) Γ-converges for the strong topology of $L^p(\Omega)^M$ (see Definition 1.1) to a functional of type (1.2) with

$$W \subset \begin{cases} W^{1,\frac{pr}{r-1}}(\Omega)^M, & \text{if } 1 < p \le N-1 \\ C^1(\overline{\Omega})^M, & \text{if } p > N-1, \end{cases} \qquad \text{and} \qquad \nu = \begin{cases} \text{Lebesgue measure}, & \text{if } 1 < p \le N-1 \\ \mathscr{M}(\Omega)* - \lim\limits_{n\to\infty} a_n, & \text{if } p > N-1. \end{cases}$$

Various types of boundary conditions can be taken into account in this Γ-convergence approach.

A preliminary result (see Theorem 2.2) allows us to prove that the sequence of energy density $\{F_n(\cdot, Du_n)\}$ converges in the sense of Radon measures to some strongly local energy density $F(\cdot, Du)$, when u_n is an asymptotic minimizer for $\mathscr{F}_n$ of limit u (see definition (2.15)). The proof of this new compactness result is based on an extension (see Lemma 2.5) of the fundamental estimate for recovery sequences in Γ-convergence (see, *e.g.*, [21], Chapters 18, 19), which provides a bound (see (2.24)) satisfied by the weak-$*$ limit of $\{F_n(\cdot, Du_n)\}$ with respect to the weak-$*$ limit of any sequence $\{F_n(\cdot, Dv_n)\}$ such that the sequence $\{v_n - u_n\}$ converges weakly to 0 in $W_0^{1,p}(\Omega)^M$. Rather than using fixed smooth cut-off functions as in the classical fundamental estimate, here we need to consider sequences of radial cut-off functions φ_n whose gradient has support in n-dependent sets on which $u_n - u$ satisfies some uniform estimate with respect to the radial coordinate (see Lemma 2.10 and its proof). This allows us to control the zero-order term $\nabla\varphi_n(u_n - u)$, when we put the trial function $\varphi_n(u_n - u)$ in the functional $\mathscr{F}_n$ of (1.1). The uniform estimate is a consequence of the Sobolev compact embedding for the $(N-1)$-dimensional sphere, and explains the role of the exponent $r > (N-1)/p$ if $1 < p \le N-1$. A similar argument was used in the linear case [12] to obtain a new div-curl lemma which is the key-ingredient for the compactness of quadratic elasticity functionals of type (1.1).

Notations

- $\mathbb{R}_s^{N\times N}$ denotes the set of the symmetric matrices in $\mathbb{R}^{N\times N}$.
- For any $\xi \in \mathbb{R}^{N\times N}$, ξ^T is the transposed matrix of ξ, and $\xi^s := \frac{1}{2}(\xi + \xi^T)$ is the symmetrized matrix of ξ.
- I_N denotes the unit matrix of $\mathbb{R}^{N\times N}$.
- $\cdot$ denotes the scalar product in $\mathbb{R}^N$, and $:$ denotes the scalar product in $\mathbb{R}^{M\times N}$ defined by

$$\xi : \eta := \mathrm{tr}\,(\xi^T \eta) \quad \text{for } \xi, \eta \in \mathbb{R}^{M\times N},$$

where tr is the trace.

- $|\cdot|$ denotes both the euclidian norm in $\mathbb{R}^N$, and the Frobenius norm in $\mathbb{R}^{M\times N}$, *i.e.*
$$|\xi| := \big(\mathrm{tr}\,(\xi^T \xi)\big)^{\frac{1}{2}} \quad \text{for } \xi \in \mathbb{R}^{M\times N}.$$

- For a bounded open set $\omega \subset \mathbb{R}^N$, $\mathscr{M}(\omega)$ denotes the space of the Radon measures on ω with bounded total variation. It agrees with the dual space of $C^0_0(\omega)$, namely the space of the continuous functions in $\bar\omega$ which vanish on $\partial\omega$. Moreover, $\mathscr{M}(\bar\omega)$ denotes the space of the Radon measures on $\bar\omega$. It agrees with the dual space of $C^0(\bar\omega)$.

- For any measures $\zeta, \mu \in \mathscr{M}(\omega)$, with $\omega \subset \mathbb{R}^N$, open, bounded, we define $\zeta^\mu \in L^1_\mu(\Omega)$ as the derivative of ζ with respect to μ. When μ is the Lebesgue measure, we write ζ^L.

- C is a positive constant which may vary from line to line.

- O_n is a real sequence which tends to zero as n tends to infinity. It can vary from line to line.

Recall the definition of the De Giorgi Γ-convergence (see, *e.g.*, [21, 4] for further details).

Definition 1.1. *Let V be a metric space, and let $\mathscr{F}_n, \mathscr{F} : V \to [0,\infty]$, $n \in \mathbb{N}$, be functionals defined on V. The sequence $\{\mathscr{F}_n\}$ is said to Γ-converge to $\mathscr{F}$ for the topology of V in a set $W \subset V$ and we write*

$$\mathscr{F}_n \xrightarrow{\Gamma} \mathscr{F} \quad \text{in } W,$$

if

- *the Γ-liminf inequality holds*

$$\forall\, v \in W, \ \ \forall\, v_n \to v \quad \text{in } V, \quad \mathscr{F}(v) \le \liminf_{n\to\infty} \mathscr{F}_n(v_n),$$

- *the Γ-limsup inequality holds*

$$\forall v\, \in W, \ \ \exists\, \overline{v}_n \to v \quad \text{in } V, \quad \mathscr{F}(v) = \lim_{n\to\infty} \mathscr{F}_n(\overline{v}_n).$$

Any sequence $\overline{v}_n$ satisfying (1.1) is called a recovery sequence *for $\mathscr{F}_n$ of limit v.*

2 Statement of the results and examples

2.1 The main results

Consider a bounded open set $\Omega \subset \mathbb{R}^N$ with $N \ge 2$, M a positive integer, a sequence of nonnegative Carathéodory functions $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$, and $p > 1$ with the following properties:

- There exist two constants $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\int_\Omega F_n(x, Du)\, dx \ge \alpha \int_\Omega |Du|^p\, dx + \beta, \quad \forall\, u \in W^{1,p}_0(\Omega)^M, \tag{2.1}$$

and

$$F_n(\cdot, 0) = 0 \ \text{ a.e. in } \Omega. \tag{2.2}$$

- There exist two sequences of measurable functions $h_n, a_n \ge 0$, and a constant $\gamma > 0$ such that

$$h_n \text{ is bounded in } L^1(\Omega), \tag{2.3}$$

$$a_n \text{ is bounded in } L^r(\Omega) \text{ with } \begin{cases} r > \dfrac{N-1}{p}, & \text{if } 1 < p \le N-1 \\ r = 1, & \text{if } p > N-1, \end{cases} \tag{2.4}$$

$$\begin{cases} \big|F_n(x,\xi) - F_n(x,\eta)\big| \le \big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\, |\xi - \eta| \\ \forall\, \xi, \eta \in \mathbb{R}^{M\times N}, \ \text{a.e. } x \in \Omega, \end{cases} \tag{2.5}$$

and

$$F_n(x, \lambda\xi) \le h_n(x) + \gamma\, F_n(x,\xi), \quad \forall\, \lambda \in [0,1], \ \forall\, \xi \in \mathbb{R}^{M\times N}, \ \text{a.e. } x \in \Omega. \tag{2.6}$$

Remark 2.1. *From (2.5) and Young's inequality, we get that*

$$\begin{aligned} F_n(x,\xi) &\le F_n(x,\eta) + \big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\,|\xi-\eta| \\ &\le F_n(x,\eta) + \frac{p-1}{p}\big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big) + \frac{1}{p}\,a_n(x)\,|\xi-\eta|^p, \end{aligned}$$

and then

$$F_n(x,\xi) \le (p-1)\,h_n(x) + (2p-1)\,F_n(x,\eta) + (p-1)\big(|\xi|^p+|\eta|^p\big) + a_n(x)\,|\xi-\eta|^p, \quad \forall\,\xi,\eta\in\mathbb{R}^{M\times N},\ \text{a.e. } x\in\Omega. \tag{2.7}$$

In particular, taking $\eta = 0$, we have

$$F_n(x,\xi) \le (p-1)\,h_n(x) + \big(p-1+a_n(x)\big)\,|\xi|^p, \quad \forall\,\xi\in\mathbb{R}^{M\times N},\ \text{a.e. } x\in\Omega, \tag{2.8}$$

where the right-hand side is a bounded sequence in $L^1(\Omega)$.

From now on, we assume that

$$a_n^r \overset{*}{\rightharpoonup} \mathrm{A} \ \text{ in } \mathscr{M}(\Omega) \quad \text{and} \quad h_n \overset{*}{\rightharpoonup} h \ \text{ in } \mathscr{M}(\Omega). \tag{2.9}$$

The paper deals with the asymptotic behavior of the sequence of functionals

$$\mathscr{F}_n(v) := \int_\Omega F_n(x,Dv)\,dx \quad \text{for } v\in W^{1,p}(\Omega)^M. \tag{2.10}$$

First of all, we have the following result on the convergence of the energy density $F_n(\cdot,Du_n)$, where u_n is an asymptotic minimizer associated with functional (2.10).

Theorem 2.2. *Let $F_n:\Omega\times\mathbb{R}^{M\times N}\to[0,\infty)$ be a sequence of Carathéodory functions satisfying (2.1) to (2.6). Then, there exist a function $F:\Omega\times\mathbb{R}^{M\times N}\to\mathbb{R}$ and a subsequence of n, still denoted by n, such that for any $\xi,\eta\in\mathbb{R}^N$,*

$$\begin{cases} F(\cdot,\xi) \ \text{ is Lebesgue measurable,} & \text{if } 1<p\le N-1 \\ F(\cdot,\xi) \ \text{ is A-measurable,} & \text{if } p>N-1, \end{cases} \tag{2.11}$$

$$\big|F(x,\xi)-F(x,\eta)\big| \le \begin{cases} C\big(h^L + F(x,\xi) + F(x,\eta) + (1+(\mathrm{A}^L)^{\frac{1}{r}})(|\xi|^p+|\eta|^p)\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|\xi-\eta| \ \text{ a.e. in } \Omega, & \text{if } 1<p\le N-1 \\ C\big(1+h^{\mathrm{A}} + F(x,\xi) + F(x,\eta) + |\xi|^p+|\eta|^p\big)^{\frac{p-1}{p}}|\xi-\eta| \ \ \mathrm{A}\text{-a.e. in } \Omega, & \text{if } p>N-1, \end{cases} \tag{2.12}$$

and

$$F(\cdot,0) = 0 \ \text{ a.e. in } \Omega. \tag{2.13}$$

For any open set $\omega\subset\Omega$, and any sequence $\{u_n\}$ in $W^{1,p}(\omega)^M$ which converges weakly in $W^{1,p}(\omega)^M$ to a function u satisfying

$$u\in\begin{cases} W^{1,\frac{pr}{r-1}}(\omega)^M, & \text{if } 1<p\le N-1 \\ C^1(\omega)^M, & \text{if } p>N-1, \end{cases} \tag{2.14}$$

and such that

$$\exists\lim_{n\to\infty}\int_\omega F_n(x,Du_n)\,dx = \min\left\{\liminf_{n\to\infty}\int_\omega F_n(x,Dw_n)\,dx:\ w_n-u_n\rightharpoonup 0 \text{ in } W_0^{1,p}(\omega)^M\right\}<\infty, \tag{2.15}$$

we have

$$F_n(\cdot,Du_n) \overset{*}{\rightharpoonup} \begin{cases} F(\cdot,Du), & \text{if } 1<p\le N-1 \\ F(\cdot,Du)\,\mathrm{A}, & \text{if } p>N-1 \end{cases} \quad \text{in } \mathscr{M}(\omega). \tag{2.16}$$

From Theorem 2.2 we may deduce the Γ-limit (see Definition 1.1) of the sequence of functionals (2.10) with various boundary conditions.

Theorem 2.3. *Let $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a sequence of Carathéodory functions satisfying (2.1) to (2.6). Let ω be an open set such that $\omega \subset\subset \Omega$, and let V be a subset of $W^{1,p}(\omega)^M$ such that*

$$\forall\, u \in V,\ \forall\, v \in W_0^{1,p}(\omega)^M, \quad u + v \in V. \tag{2.17}$$

Define the functional $\mathscr{F}_n^V : V \to [0,\infty)$ by

$$\mathscr{F}_n^V(v) := \int_\omega F_n(x, Dv)\, dx \quad \text{for } v \in V. \tag{2.18}$$

Assume that the open set ω satisfies

$$\begin{cases} |\partial\omega| = 0, & \text{if } 1 < p \le N-1 \\ \text{A}(\partial\omega) = 0, & \text{if } p > N-1. \end{cases} \tag{2.19}$$

Then, for the subsequence of n (still denoted by n) obtained in Theorem 2.2 we get

$$\begin{cases} \mathscr{F}_n^V \xrightarrow{\Gamma} \mathscr{F}^V := \displaystyle\int_\omega F(x, Dv)\, dx & \text{in } V \cap W^{1,\frac{pr}{r-1}}(\omega)^M, \quad \text{if } 1 < p \le N-1 \\ \mathscr{F}_n^V \xrightarrow{\Gamma} \mathscr{F}^V := \displaystyle\int_\omega F(x, Dv)\, dx & \text{in } V \cap C^1(\overline{\omega})^M, \quad \text{if } p > N-1, \end{cases} \tag{2.20}$$

for the strong topology of $L^p(\omega)^M$, where F is given by convergence (2.16).

Remark 2.4. *The condition (2.19) on the open set ω is not so restrictive. Indeed, for any family $(\omega)_{i\in I}$ of open sets of Ω with two by two disjoint boundaries, at most a countable subfamily of $(\partial\omega)_{i\in I}$ does not satisfy (2.19).*

2.2 Auxiliary lemmas

The proof of Theorem 2.2 is based on the following lemma which provides an estimate of the energy density for asymptotic minimizers. In our context it is equivalent to the fundamental estimate for recovery sequences (see Definition 1.1) in Γ-convergence theory (see, *e.g.*, [21], Chapters 18, 19).

Lemma 2.5. *Let $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a sequence of Carathéodory functions satisfying (2.1) to (2.6). Consider an open set $\omega \subset \Omega$, and a sequence $\{u_n\} \subset W^{1,p}(\omega)^M$ converging weakly in $W^{1,p}(\omega)^M$ to a function u satisfying (2.14), and such that*

$$F_n(\cdot, Du_n) \xrightarrow{*} \mu \quad \text{in } \mathscr{M}(\omega),$$
$$|Du_n|^p \xrightarrow{*} \varrho \quad \text{in } \mathscr{M}(\omega).$$

Then, the measure ϱ satisfies

$$\varrho \le \begin{cases} C\big(|Du|^p + |Du|^p(\text{A}^L)^{\frac{1}{r}} + h + \mu + \text{A}^L\big) & \text{a.e. in } \omega, \quad \text{if } 1 < p \le N-1 \\ C\big(|Du|^p\text{A} + h + \mu + \text{A}\big) & \text{A-a.e. in } \omega, \quad \text{if } p > N-1. \end{cases} \tag{2.21}$$

Moreover if u_n satisfies

$$\exists \lim_{n\to\infty} \int_\omega F_n(x, Du_n)\, dx = \min\left\{ \liminf_{n\to\infty} \int_\omega F_n(x, Dw_n)\, dx : \ w_n - u_n \rightharpoonup 0 \text{ in } W_0^{1,p}(\omega)^M \right\}, \tag{2.22}$$

then for any sequence $\{v_n\} \subset W^{1,p}(\omega)^M$ which converges weakly in $W^{1,p}(\omega)^M$ to a function

$$v \in \begin{cases} W^{1,\frac{pr}{r-1}}(\omega)^M, & \text{if } 1 < p \le N-1 \\ C^1(\omega)^M, & \text{if } p > N-1, \end{cases}$$

and such that

$$F_n(\cdot, Dv_n) \xrightarrow{*} \nu \quad \text{in } \mathscr{M}(\omega), \tag{2.23}$$
$$|Dv_n|^p \xrightarrow{*} \varpi \quad \text{in } \mathscr{M}(\omega),$$

we have

$$\mu \le \begin{cases} \nu + C\big(h^L + \nu^L + \varpi^L + (1 + (\text{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}} (\text{A}^L)^{\frac{1}{pr}} |D(u-v)| \ \text{a.e. in } \omega, & \text{if } 1 < p \le N-1 \\ \nu + C\big(1 + h^\text{A} + \nu^\text{A} + \varpi^\text{A} + |D(u-v)|^p\big)^{\frac{p-1}{p}} \text{A}\, |D(u-v)| \ \text{A-a.e. in } \omega, & \text{if } p > N-1. \end{cases} \tag{2.24}$$

We can improve the statement of Lemma 2.5 if we add a non-homogeneous Dirichlet boundary condition on $\partial\omega$.

Lemma 2.6. *Let ω be an open set such that $\omega \subset\subset \Omega$, and let u be a function satisfying*

$$u \in \begin{cases} W^{1,\frac{pr}{r-1}}(\Omega)^M, & \text{if } 1 < p \le N-1 \\ C^1(\overline{\Omega})^M, & \text{if } p > N-1. \end{cases} \tag{2.25}$$

Let $\{u_n\}$ and $\{v_n\}$ be two sequences in $W^{1,p}(\omega)^M$, such that u_n satisfies condition (2.22) and

$$u_n - u,\ v_n - u \in W_0^{1,p}(\omega)^M,$$

$$F_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad F_n(\cdot, Dv_n) \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathscr{M}(\overline{\omega}), \tag{2.26}$$

$$|Du_n|^p \overset{*}{\rightharpoonup} \varrho \quad \text{and} \quad |Dv_n|^p \overset{*}{\rightharpoonup} \varpi \quad \text{in } \mathscr{M}(\overline{\omega}). \tag{2.27}$$

Then, estimates (2.21) and (2.24) hold in $\overline{\omega}$.

Remark 2.7. *Condition (2.22) means that u_n is a recovery sequence in ω for the functional*

$$w \in W^{1,p}(\omega)^M \mapsto \int_\omega F_n(x, Dw)\, dx, \tag{2.28}$$

with the Dirichlet condition $w - u_n \in W_0^{1,p}(\omega)^M$. Since $w = u_n$ clearly satisfies $w - u_n \in W_0^{1,p}(\omega)^M$, this makes u_n a recovery sequence without imposing any boundary condition. In particular, condition (2.22) is fulfilled if for a fixed $f \in W^{-1,p}(\omega)^M$, u_n satisfies

$$\int_\omega F_n(x, Du_n)\, dx = \min \left\{ \int_\omega F_n\big(x, D(u_n + v)\big)\, dx - \langle f, v\rangle :\ v \in W_0^{1,p}(\omega)^M \right\}.$$

Assuming the differentiability of F_n with respect to the second variable, it follows that u_n satisfies the variational equation

$$\int_\omega D_\xi F_n(x, Du_n) : Dv\, dx - \langle f, v\rangle = 0, \quad \forall\, v \in W_0^{1,p}(\omega)^M,$$

i.e. u_n is a solution of

$$-\operatorname{Div}\big(D_\xi F_n(x, Du)\big) = f \quad \text{in } \omega,$$

where no boundary condition is imposed.

Assumption (2.22) allows us to take into account very general boundary conditions. For example, if u_n is a recovery sequence for (2.28) with (non necessarily homogeneous) Dirichlet or Neumann boundary condition, then it also satisfies (2.22).

Remark 2.8. *Condition (2.22) is equivalent to the asymptotic minimizer property satisfied by u_n:*

$$\int_\omega F_n(x, Du_n)\, dx \le \int_\omega F_n(x, Dw_n)\, dx + O_n, \quad \forall\, w_n \ \text{with}\ w_n - u_n \rightharpoonup 0 \ \text{in } W_0^{1,p}(\omega)^M.$$

We can check that if u_n satisfies this condition in ω, then u_n satisfies it in any open subset $\hat{\omega} \subset \omega$. To this end, it is enough to consider for a sequence $\hat{w}_n$ with $\hat{w}_n - u_n \in W_0^{1,p}(\hat{\omega})^M$, the extension

$$w_n := \begin{cases} \hat{w}_n & \text{in } \hat{\omega} \\ u_n & \text{in } \omega \setminus \hat{\omega}. \end{cases}$$

Corollary 2.9. *Let $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ be a sequence of Carathéodory functions satisfying (2.1) to (2.6). Consider two open sets $\omega_1, \omega_2 \subset \Omega$ such that $\omega_1 \cap \omega_2 \neq \emptyset$, a sequence u_n converging weakly in $W^{1,p}(\omega_1)^M$ to a function u and a sequence v_n converging weakly in $W^{1,p}(\omega_2)^M$ to a function v, such that*

$$u, v \in \begin{cases} W^{1,\frac{pr}{r-1}}(\omega_1 \cap \omega_2)^M, & \text{if } 1 < p \le N-1 \\ C^1(\omega_1 \cap \omega_2)^M, & \text{if } p > N-1, \end{cases}$$

$$|Du_n|^p \overset{*}{\rightharpoonup} \varrho, \quad F_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathscr{M}(\omega_1),$$

$$|Dv_n|^p \overset{*}{\rightharpoonup} \varpi, \quad F_n(\cdot, Dv_n) \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathscr{M}(\omega_2),$$

$$\exists \lim_{n\to\infty} \int_{\omega_1} F_n(x, Du_n)\, dx = \min \left\{ \liminf_{n\to\infty} \int_{\omega_1} F_n(x, Dw_n)\, dx : \ w_n - u_n \rightharpoonup 0 \ \textit{in } W_0^{1,p}(\omega_1)^M \right\},$$

$$\exists \lim_{n\to\infty} \int_{\omega_2} F_n(x, Dv_n)\, dx = \min \left\{ \liminf_{n\to\infty} \int_{\omega_2} F_n(x, Dw_n)\, dx : \ w_n - v_n \rightharpoonup 0 \ \textit{in } W_0^{1,p}(\omega_2)^M \right\}.$$

Then, we have

$$|\mu - \nu| \leq \begin{cases} C\big(h^L + \mu^L + \nu^L + \varrho^L + \varpi^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |D(u-v)| \ \textit{a.e. in } \omega_1 \cap \omega_2, & \textit{if } 1 < p \leq N-1 \\ C\big(1 + h^{\mathrm{A}} + \mu^{\mathrm{A}} + \nu^{\mathrm{A}} + \varrho^{\mathrm{A}} + \varpi^{\mathrm{A}} + |D(u-v)|^p\big)^{\frac{p-1}{p}} \mathrm{A}\, |D(u-v)| \ \mathrm{A}\textit{-a.e. in } \omega_1 \cap \omega_2, & \textit{if } p > N-1. \end{cases} \tag{2.29}$$

Lemma 2.5 is itself based on the following compactness result.

Lemma 2.10. *Let $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ be a sequence of Carathéodory functions satisfying (2.1) to (2.6), and let ω be an open subset of Ω. Consider a sequence $\{\xi_n\} \subset L^p(\omega)^{M\times N}$ such that*

$$F_n(\cdot, \xi_n) \overset{*}{\rightharpoonup} \Lambda \quad \textit{and} \quad |\xi_n|^p \overset{*}{\rightharpoonup} \Xi \quad \textit{in } \mathscr{M}(\omega). \tag{2.30}$$

- *If $1 < p \leq N-1$ and the sequence $\{\rho_n\}$ converges strongly to ρ in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$, then there exist a subsequence of n and a function $\vartheta \in L^1(\omega)$ such that*

  $$F_n(\cdot, \xi_n + \rho_n) - F_n(\cdot, \xi_n) \rightharpoonup \vartheta \quad \textit{weakly in } L^1(\omega), \tag{2.31}$$

  where ϑ satisfies

  $$|\vartheta| \leq C\big(h^L + \Lambda^L + \Xi^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|\rho|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |\rho| \quad \textit{a.e. in } \omega. \tag{2.32}$$

- *If $p > N-1$ and the sequence $\{\rho_n\}$ converges strongly to ρ in $C^0(\overline{\omega})^{M\times N}$, then there exist a subsequence of n and a function $\vartheta \in L^1_{\mathrm{A}}(\omega)$ such that*

  $$F_n(\cdot, \xi_n + \rho_n) \overset{*}{\rightharpoonup} \Lambda + \vartheta\, \mathrm{A} \quad \textit{in } \mathscr{M}(\omega),$$

  where ϑ satisfies

  $$|\vartheta| \leq C\big(1 + h^{\mathrm{A}} + \Lambda^{\mathrm{A}} + \Xi^{\mathrm{A}} + |\rho|^p\big)^{\frac{p-1}{p}} |\rho| \quad \mathrm{A}\textit{-a.e. in } \omega. \tag{2.33}$$

2.3 Examples

In this section we give three examples of functionals $\mathscr{F}_n$ satisfying the assumptions (2.1) to (2.6) of Theorem 2.2.

1. The first example illuminates the Lipschitz estimate (2.5). It is also based on a functional coercivity of type (2.1) rather than a pointwise coercivity.
2. The second example deals with the Saint Venant-Kirchhoff hyper-elastic energy (see, *e.g.*, [18] Chapter 4).
3. The third example deals with an Ogden's type hyper-elastic energy (see, *e.g.*, [18] Chapter 4).

Let Ω be a bounded set of $\mathbb{R}^N$, $N \geq 2$. We denote for any function $u : \Omega \to \mathbb{R}^N$,

$$e(u) := \frac{1}{2}\left(Du + Du^T\right), \quad E(u) := \frac{1}{2}\left(Du + Du^T + Du^T Du\right), \quad C(u) := (I_N + Du)^T (I_N + Du). \tag{2.34}$$

Example 1

Let $p \in (1, \infty)$, and let A_n be a symmetric tensor-valued function in $L^\infty\big(\Omega; \mathscr{L}(\mathbb{R}^{N\times N}_s)\big)$. We consider the energy density function defined by

$$F_n(x, \xi) := \left|A_n(x)\xi^s : \xi^s\right|^{\frac{p}{2}} \quad \text{a.e. } x \in \Omega, \ \forall\, \xi \in \mathbb{R}^{N\times N}.$$

We assume that there exists $\alpha > 0$ such that

$$A_n(x)\xi : \xi \geq \alpha\, |\xi|^2, \quad \text{a.e. } x \in \Omega, \ \forall\, \xi \in \mathbb{R}^{N\times N}_s, \tag{2.35}$$

and that

$$|A_n|^{\frac{p}{2}} \text{ is bounded in } L^r(\Omega) \text{ with } r \text{ defined by (2.4).} \tag{2.36}$$

Then, the density F_n and the associated functional

$$\mathscr{F}_n(u) := \int_\Omega \left|A_n e(u) : e(u)\right|^{\frac{p}{2}} dx \quad \text{for } u \in W_0^{1,p}(\Omega)^N,$$

satisfy the conditions (2.1) to (2.6) of Theorem 2.2.

Proof. Using successively (2.35) and the Korn inequality in $W_0^{1,p}(\Omega)^N$ for $p > 1$ (see, *e.g.*, [34]), we have for any $u \in W_0^{1,p}(\Omega)^N$,

$$\mathscr{F}_n(u) = \int_\Omega \left|A_n e(u) : e(u)\right|^{\frac{p}{2}} dx \geq \alpha \int_\Omega |e(u)|^p\, dx \geq \alpha\, C \int_\Omega |Du|^p\, dx,$$

which implies (2.1). Conditions (2.2) and (2.6) are immediate. It remains to prove condition (2.5) with estimate (2.4). Taking into account that

$$|D_\xi F_n(x,\xi)| = p\left|(A_n(x)\xi^s : \xi^s)^{\frac{p-2}{2}} A_n(x)\xi^s\right| \leq p\left|A_n(x)\xi^s : \xi^s\right|^{\frac{p-1}{2}} |A_n(x)|^{\frac{1}{2}}, \quad \forall\, \xi \in \mathbb{R}^{N\times N}, \text{ a.e. } x \in \Omega,$$

then using the mean value theorem and Hölder's inequality, we get

$$\begin{aligned} \left|F_n(x,\xi) - F_n(x,\eta)\right| &\leq p\left((A_n\xi^s : \xi^s)^{\frac{1}{2}} + (A_n\eta^s : \eta^s)^{\frac{1}{2}}\right)^{p-1} |A_n|^{\frac{1}{2}} |\xi^s - \eta^s| \\ &\leq p\, 2^{\frac{(p-1)^2}{p}} \left(F_n(x,\xi) + F_n(x,\eta)\right)^{\frac{p-1}{p}} |A_n|^{\frac{1}{2}} |\xi - \eta|, \end{aligned}$$

for every $\xi, \eta \in \mathbb{R}^{N\times N}$ and a.e. $x \in \Omega$. This implies estimate (2.5) with $h_n = 0$ and $a_n = |A_n|^{\frac{p}{2}}$ bounded in $L^r(\Omega)$. ☐

The two next examples belong to the class of hyper-elastic materials (see, *e.g.*, [18], Chapter 4).

Example 2

For $N = 3$, we consider the Saint Venant-Kirchhoff energy density defined by

$$F_n(x,\xi) := \frac{\lambda_n(x)}{2}\left[\operatorname{tr}\big(\tilde{E}(\xi)\big)\right]^2 + \mu_n(x)\left|\tilde{E}(\xi)\right|^2, \quad \text{a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^{3\times 3}, \tag{2.37}$$

where $\tilde{E}(\xi) := \frac{1}{2}\left(\xi + \xi^T + \xi^T\xi\right)$, and λ_n, μ_n are the Lamé coefficients.

We assume that there exists a constant $C > 1$ such that

$$\lambda_n, \mu_n \geq 0 \text{ a.e. in } \Omega, \quad \operatorname*{ess-inf}_\Omega (\lambda_n + \mu_n) > C^{-1}, \quad \int_\Omega (\lambda_n + \mu_n)\, dx \leq C. \tag{2.38}$$

Then, the density F_n and the associated functional (see definition (2.34))

$$\mathscr{F}_n(u) := \int_\Omega \left(\frac{\lambda_n}{2}\left[\operatorname{tr}\big(E(u)\big)\right]^2 + \mu_n\left|E(u)\right|^2\right) dx \quad \text{for } u \in W_0^{1,4}(\Omega)^3, \tag{2.39}$$

satisfy the conditions (2.1) to (2.6) of Theorem 2.2.

Proof. There exists a constant $C > 1$ such that we have for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^{3\times 3}$,

$$C^{-1}(\lambda_n + \mu_n)\,|\xi|^4 - C\,(\lambda_n + \mu_n) \leq F_n(x,\xi) \leq C\,(\lambda_n + \mu_n)\,|\xi|^4 + C\,(\lambda_n + \mu_n). \tag{2.40}$$

Hence, we deduce that for a.e. $x \in \Omega$ and any $\xi, \eta \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned} \left|F_n(x,\xi) - F_n(x,\eta)\right| &\leq C\,(\lambda_n + \mu_n)\left(1 + |\xi|^2 + |\eta|^2\right)^{\frac{3}{2}} |\xi - \eta| \\ &= C\left((\lambda_n + \mu_n)^{\frac{1}{2}} + (\lambda_n + \mu_n)^{\frac{1}{2}}|\xi|^2 + (\lambda_n + \mu_n)^{\frac{1}{2}}|\eta|^2\right)^{\frac{3}{2}} (\lambda_n + \mu_n)^{\frac{1}{4}} |\xi - \eta| \\ &\leq C\left((\lambda_n + \mu_n)^{\frac{1}{2}} + F_n(x,\xi)^{\frac{1}{2}} + F_n(x,\eta)^{\frac{1}{2}}\right)^{\frac{3}{2}} (\lambda_n + \mu_n)^{\frac{1}{4}} |\xi - \eta| \\ &\leq C\left(\lambda_n + \mu_n + F_n(x,\xi) + F_n(x,\eta)\right)^{\frac{3}{4}} (\lambda_n + \mu_n)^{\frac{1}{4}}\, |\xi - \eta|, \end{aligned}$$

which implies estimate (2.5) with $p = 4$ and $h_n = a_n = \lambda_n + \mu_n$, while (2.3) and (2.4) are a straightforward consequence of (2.38). Moreover, by the first inequality of (2.40) combined with (2.38) we get that the functional (2.39) satisfies the coercivity condition (2.1). Condition (2.2) is immediate. Finally, since we have

$$\left[\operatorname{tr}\big(\tilde{E}(\lambda\xi)\big)\right]^2 + \left|\tilde{E}(\lambda\xi)\right|^2 \le C\big(1 + |\xi|^4\big), \quad \forall\, \lambda \in [0,1],\ \forall\, \xi \in \mathbb{R}^{3\times 3},$$

condition (2.6) follows from the first inequality of (2.40), which concludes the proof of the second example. □

Remark 2.11. *The default of the Saint Venant-Kirchhoff model is that the function $F_n(x,\cdot)$ of (2.37) is not* polyconvex *(see [31]). Hence, we do not know if it is* quasiconvex, *or equivalently, if the functional $\mathscr{F}_n$ of (2.39) is lower semi-continuous for the weak topology of $W^{1,4}(\Omega)^3$ (see,* e.g. *[20], Chapter 4, for the notions of polyconvexity and quasiconvexity).*

Example 3

For $N = 3$ and $p \in [2,\infty)$, we consider the Ogden's type energy density defined by

$$F_n(x,\xi) := a_n(x)\Big[\operatorname{tr}\big(\tilde{C}(\xi)^{\frac{p}{2}} - I_3\big)\Big]^+ \quad \text{a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^{3\times 3}, \tag{2.41}$$

where $\tilde{C}(\xi) := (I_3 + \xi)^T(I_3 + \xi)$, and $t^+ := \max(t, 0)$ for $t \in \mathbb{R}$. We assume that there exists a constant $C > 1$ such that

$$\operatorname*{ess-inf}_{\Omega} a_n > C^{-1} \quad \text{and} \quad \int_\Omega a_n^r\, dx \le C \quad \text{with} \begin{cases} r > 1, & \text{if } p = 2 \\ r = 1, & \text{if } p > 2. \end{cases} \tag{2.42}$$

Then, the density F_n and the associated functional (see definition (2.34))

$$\mathscr{F}_n(u) := \int_\Omega a_n(x)\Big[\operatorname{tr}\big(C(u)^{\frac{p}{2}} - I_3\big)\Big]^+ dx \quad \text{for } u \in W_0^{1,p}(\Omega)^3, \tag{2.43}$$

satisfy the conditions (2.1) to (2.6) of Theorem 2.2.

Proof. There exists a constant $C > 1$ such that we have for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^{3\times 3}$,

$$C^{-1} a_n\,|\xi|^p - C\,a_n \le F_n(x,\xi) \le C\,a_n\,|\xi|^p + C\,a_n. \tag{2.44}$$

This combined with the fact that the (well-ordered) eigenvalues of a symmetric matrix are Lipschitz functions (see, *e.g.*, [19], Theorem 2.3-2), implies that for a.e. $x \in \Omega$ and any $\xi, \eta \in \mathbb{R}^N$, we have

$$\begin{aligned} \big|F_n(x,\xi) - F_n(x,\eta)\big| &\le C\,a_n(1 + |\xi| + |\eta|)^{p-1}|\xi - \eta| \\ &\le C\left(a_n + a_n|\xi|^p + a_n|\eta|^p\right)^{\frac{p-1}{p}} a_n^{\frac{1}{p}}\,|\xi - \eta| \\ &\le C\left(a_n + F_n(x,\xi) + F_n(x,\eta)\right)^{\frac{p-1}{p}} a_n^{\frac{1}{p}}\,|\xi - \eta|, \end{aligned}$$

which implies estimate (2.5) with $h_n = a_n$, while (2.3) and (2.4) are a straightforward consequence of (2.42). Moreover, by the first inequality of (2.44) combined with (2.42) we get that the functional (2.43) satisfies the coercivity condition (2.1). Condition (2.2) is immediate. Finally, since we have

$$\operatorname{tr}\big(\tilde{C}(\lambda\xi)^{\frac{p}{2}}\big) \le C\big(1 + |\xi|^p\big), \quad \forall\, \lambda \in [0,1],\ \forall\, \xi \in \mathbb{R}^{3\times 3},$$

condition (2.6) follows from the first inequality of (2.44), which concludes the proof of the third example. □

Remark 2.12. *Contrary to Example 2, the function $F_n(x,\cdot)$ of (2.41) is polyconvex since it is the composition of the Ogden density energy defined for a.e. $x \in \Omega$, by*

$$W_n(x,\xi) := a_n(x)\Big[\operatorname{tr}\big(\tilde{C}(\xi)^{\frac{p}{2}} - I_3\big)\Big]^+ \quad \text{for } \xi \in \mathbb{R}^{3\times 3}, \tag{2.45}$$

which is known to be polyconvex (see [1]), by the non-decreasing convex function $t \mapsto t^+$. However, in contrast with (2.45) the function (2.41) does attain its minimum at $\xi = 0$, namely in the absence of strain.

3 Proof of the results

3.1 Proof of the main results

Proof of Theorem 2.2. The proof is divided into two steps. In the first step we construct the limit functional F and we prove the properties (2.11), (2.12), (2.13) satisfied by the function F. The second step is devoted to convergence (2.16).

First step: Construction of F.
Let $\mathscr{F}_n : W^{1,p}(\Omega)^M \to [0,\infty]$ be the functional defined by

$$\mathscr{F}_n(v) = \int_\Omega F_n(x, Dv)\, dx \quad \text{for } v \in W^{1,p}(\Omega)^M.$$

By the compactness Γ-convergence theorem (see *e.g.* [21], Theorem 8.5), there exists a subsequence of n, still denoted by n, such that $\mathscr{F}_n$ Γ-converges for the strong topology of $L^p(\Omega)^M$ to a functional $\mathscr{F} : W^{1,p}(\Omega)^M \to [0,\infty]$ with domain $\mathscr{D}(\mathscr{F})$.

Let ξ be a matrix of a countable dense subset D of $\mathbb{R}^{M\times N}$ with $0 \in D$. Since the linear function $x \mapsto \xi x$ belongs to $\mathscr{D}(\mathscr{F})$ by (2.8), up to the extraction of a new subsequence, for any $\xi \in D$ there exists a recovery sequence w_n^ξ in $W^{1,p}(\Omega)^M$ which converges strongly to ξx in $L^p(\Omega)^M$ and such that

$$F_n(\cdot, Dw_n^\xi) \overset{*}{\rightharpoonup} \mu^\xi \quad \text{and} \quad |Dw_n^\xi|^p \overset{*}{\rightharpoonup} \varrho^\xi \quad \text{in } \mathscr{M}(\Omega).$$

In particular, since $F_n(\cdot, 0) = 0$ we have $\mu^0 = 0$. Moreover, by estimates (2.21) and (2.29) we have for any $\xi, \eta \in D$,

$$\varrho^\xi \le \begin{cases} C\big(|\xi|^p + |\xi|^p(\mathrm{A}^L)^{\frac{1}{r}} + h + \mu^\xi + \mathrm{A}^L\big) & \text{a.e. in } \omega, \quad \text{if } 1 < p \le N-1 \\ C\big(|\xi|^p \mathrm{A} + h + \mu^\xi + \mathrm{A}\big) & \mathrm{A}\text{-a.e. in } \omega, \quad \text{if } p > N-1, \end{cases} \tag{3.1}$$

$$|\mu^\xi - \mu^\eta| \le$$
$$\begin{cases} C\big(h^L + (\mu^\xi)^L + (\mu^\eta)^L + (\varrho^\xi)^L + (\rho^\eta)^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|\xi - \eta|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |\xi - \eta| \text{ a.e. in } \Omega, & \text{if } 1 < p \le N-1 \\ C\left(1 + h^{\mathrm{A}} + (\mu^\xi)^{\mathrm{A}} + (\mu^\eta)^{\mathrm{A}} + (\varrho^\xi)^{\mathrm{A}} + (\varrho^\eta)^{\mathrm{A}} + |\xi - \eta)|^p\right)^{\frac{p-1}{p}} \mathrm{A}\, |\xi - \eta| \ \mathrm{A}\text{-a.e. in } \Omega, & \text{if } p > N-1. \end{cases} \tag{3.2}$$

Hence, by a continuity argument we can define a function $F : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ satisfying (2.11), (2.13) and such that

$$\mu^\xi = \begin{cases} F(\cdot, \xi), & \text{if } 1 < p \le N-1 \\ F(\cdot, \xi)\, \mathrm{A}, & \text{if } p > N-1, \end{cases} \qquad \forall\, \xi \in D, \tag{3.3}$$

where the property (2.12) is deduced from (3.1), (3.2).

Second step: Proof of convergence (2.16).
Let ω be an open set of Ω, let $\{u_n\}$ be a sequence fulfilling (2.15), which converges weakly in $W^{1,p}(\omega)^M$ to a function u satisfying (2.14), and let $\xi \in D$. Since $F_n(\cdot, Du_n)$ is bounded in $L^1(\Omega)$, there exists a subsequence of n, still denoted by n, such that

$$F_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |Du_n|^p \overset{*}{\rightharpoonup} \varrho \quad \text{in } \mathscr{M}(\Omega). \tag{3.4}$$

Applying Corollary 2.9 to the sequences u_n and $v_n = w_n^\xi$, we have

$$|\mu - \mu^\xi| \le$$
$$\begin{cases} C\big(h^L + \mu^L + (\mu^\xi)^L + \varrho^L + (\varrho^\xi)^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|Du - \xi|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |Du - \xi| \text{ a.e. in } \omega, & \text{if } 1 < p \le N-1 \\ C\big(1 + h^{\mathrm{A}} + \mu^{\mathrm{A}} + (\mu^\xi)^{\mathrm{A}} + \varrho^{\mathrm{A}} + (\varrho^\xi)^{\mathrm{A}} + |Du - \xi|^p\big)^{\frac{p-1}{p}} \mathrm{A}\, |Du - \xi| \quad \mathrm{A}\text{-a.e. in } \omega, & \text{if } p > N-1. \end{cases}$$

Using (3.1), (3.3) and the continuity of $F(x, \xi)$ with respect to ξ, we get that

$$\mu = \begin{cases} F(\cdot, Du), & \text{if } 1 < p \le N-1 \\ F(\cdot, Du)\, \mathrm{A}, & \text{if } p > N-1. \end{cases} \tag{3.5}$$

Note that since the limit μ is completely determined by F, the first convergence of (3.4) holds for the whole sequence, which concludes the proof. $\square$

Proof of Theorem 2.3. The proof is divided into two steps.

First step: The case where $V = \{\hat{u}\} + W_0^{1,p}(\omega)^M$.
Fix a function $\hat{u}$ satisfying (2.25), and define the set $V := \{\hat{u}\} + W_0^{1,p}(\omega)^M$. Let $u \in V$ such that

$$u \in \begin{cases} W^{1,\frac{pr}{r-1}}(\omega)^M, & \text{if } 1 < p \le N-1 \\ C^1(\overline{\omega})^M, & \text{if } p > N-1. \end{cases}$$

which is extended by $\hat{u}$ in $\Omega \setminus \omega$, and consider a recovery sequence $\{u_n\}$ for $\mathscr{F}_n^V$ of limit u. There exists a subsequence of n, still denoted by n, such that the first convergences of (2.26) and (2.27) hold. By Theorem 2.2 convergences (2.16) are satisfied in ω, which implies (3.5). Now, applying the estimate (2.24) of Lemma 2.6 with u_n and $v_n = u$, it follows that

$$\mu \le \nu \ \text{ in } \overline{\omega} \quad \text{with} \quad F_n(\cdot, Dv_n) \overset{*}{\rightharpoonup} \nu \ \text{ in } \mathscr{M}(\Omega),$$

where the convergence holds up to a subsequence. Then, using estimate (2.5) with $\eta = 0$ and Hölder's inequality, we have for any $\varphi \in L^\infty\big(\Omega; [0,1]\big)$ with compact support in Ω,

$$\int_\Omega \varphi\, F_n(x, Du)\, dx \le$$
$$\begin{cases} \left(\displaystyle\int_\Omega \varphi\left(h_n + F_n(x,Du) + |Du|^p\right) dx\right)^{\frac{p-1}{p}} \left(\displaystyle\int_\Omega \varphi\, a_n^r\, dx\right)^{\frac{1}{pr}} \left(\varphi\, |Du|^{\frac{pr}{r-1}}\, dx\right)^{\frac{r-1}{pr}}, & \text{if } 1 < p \le N-1 \\ \left(\displaystyle\int_\Omega \varphi\left(h_n + F_n(x,Du) + |Du|^p\right) dx\right)^{\frac{p-1}{p}} \left(\displaystyle\int_\Omega \varphi\, a_n\, dx\right)^{\frac{1}{p}} \|Du\|_{L^\infty(\Omega)^M}, & \text{if } p > N-1, \end{cases}$$

which implies that ν is absolutely continuous with respect to the Lebesgue measure if $1 < p \le N-1$, and absolutely continuous with respect to measure A if $p > N-1$. Due to condition (2.19) in both cases the equality $\nu(\partial\omega) = 0$ holds, so does with μ. This combined with (2.16) and (3.5) yields

$$\lim_{n\to\infty} \int_\omega F_n(x, Du_n)\, dx = \begin{cases} \displaystyle\int_\omega F(x, Du)\, dx, & \text{if } 1 < p \le N-1 \\ \displaystyle\int_\omega F(x, Du)\, d\mathrm{A}, & \text{if } p > N-1, \end{cases}$$

which concludes the first step.

Second step: The general case.
Let V be a subset of $W^{1,p}(\omega)^M$ satisfying (2.17). Let u be a function such that

$$u \in \begin{cases} V \cap W^{1,\frac{pr}{r-1}}(\Omega)^M, & \text{if } 1 < p \le N-1 \\ V \cap C^1(\overline{\Omega})^M, & \text{if } p > N-1, \end{cases}$$

and define the set $\tilde{V} := \{u\} + W_0^{1,p}(\omega)^M$. Consider a recovery sequence $\{u_n\}$ for $\mathscr{F}_n^V$ given by (2.18) of limit u, and a recovery sequence $\{\tilde{u}_n\}$ for $\mathscr{F}_n^{\tilde{V}}$ of limit u. By virtue of Theorem 2.2 the convergences (2.16) hold for both sequences $\{u_n\}$ and $\{\tilde{u}_n\}$. Hence, since ω is an open set, and $F_n(x, Du_n)$ is non-negative, we have

$$\left.\begin{array}{ll} \text{if } 1 < p \le N-1, & \displaystyle\int_\omega F(x, Du)\, dx \\ \text{if } p > N-1, & \displaystyle\int_\omega F(x, Du)\, d\mathrm{A} \end{array}\right\} \le \liminf_{n\to\infty} \int_\omega F_n(x, Du_n)\, dx. \tag{3.6}$$

Moreover, since $\tilde{u}_n - u_n \rightharpoonup 0$ in $W_0^{1,p}(\omega)^M$, $\tilde{u}_n \in V$ by property (2.17) and because $\{u_n\}$ is a recovery sequence for $\mathscr{F}_n^V$, $\{\tilde{u}_n\}$ is an admissible sequence for the minimization problem (2.15), which implies that

$$\exists \lim_{n\to\infty} \int_\omega F_n(x, Du_n)\, dx \le \liminf_{n\to\infty} \int_\omega F_n(x, D\tilde{u}_n)\, dx. \tag{3.7}$$

On the other hand, by the first step applied with $\tilde{u} = u$ and the set $\tilde{V}$, we have

$$\lim_{n\to\infty} \int_\omega F_n(x, D\tilde{u}_n)\,dx = \begin{cases} \displaystyle\int_\omega F(x, Du)\,dx, & \text{if } 1 < p \le N-1 \\ \displaystyle\int_\omega F(x, Du)\,d\mathrm{A}, & \text{if } p > N-1. \end{cases} \tag{3.8}$$

Therefore, combining (3.6), (3.7), (3.8), for the sequence n obtained in Theorem 2.2, the sequence $\{\mathscr{F}_n^V\}$ Γ-converges to some functional $\mathscr{F}^V$ satisfying (2.20) with $v = u$, which concludes the proof of Theorem 2.3. □

3.2 Proof of the lemmas

Proof of Lemma 2.10. Assume that $1 < p \le N-1$. Using (2.7), we have

$$F_n(x, \xi_n + \rho_n) \le (p-1)h_n + (2p-1)F_n(x, \xi_n) + (p-1)\big(|\xi_n + \rho_n|^p + |\xi_n|^p\big) + a_n|\rho_n|^p \quad \text{a.e. in } \omega.$$

From this we deduce that $\{F_n(\cdot, \xi_n + \rho_n)\}$ is bounded in $L^1(\omega)$. Moreover, by (2.5), we have

$$\big|F_n(x, \xi_n + \rho_n) - F_n(x, \xi_n)\big| \le \big(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + |\xi_n + \rho_n|^p + |\xi_n|^p\big)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n| \quad \text{a.e. in } \omega,$$

where, thanks to the strong convergence of $\{\rho_n\}$ in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$, we can show that the right-hand side is bounded in $L^1(\omega)$ and equi-integrable. Indeed, taking into account

$$\frac{p-1}{p} + \frac{1}{pr} + \frac{r-1}{pr} = 1,$$

we have the boundedness in $L^1(\omega)$, while the strong convergence of $\{\rho_n\}$ in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$ implies that $\{|\rho_n|^{\frac{pr}{r-1}}\}$ is equi-integrable and therefore, the equi-integrability of the right-hand side. By the Dunford-Pettis theorem, extracting a subsequence if necessary, we conclude (2.31), which, together with (2.30), in particular implies

$$F_n(\cdot, \xi_n + \rho_n) \overset{*}{\rightharpoonup} \Lambda + \vartheta \quad \text{in } \mathscr{M}(\omega).$$

Moreover, for any ball $B \subset \omega$, we have

$$\begin{aligned}
&\int_B \big|F_n(x, \xi_n + \rho_n) - F_n(x, \xi_n)\big|\,dx \\
&\le \int_B \big(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + |\xi_n + \rho_n|^p + |\xi_n|^p\big)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n| dx \\
&\le \left(\int_B \big(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + C|\xi_n|^p + C|\rho_n|^p\big)\,dx\right)^{\frac{p-1}{p}} \left(\int_B a_n^r dx\right)^{\frac{1}{pr}} \left(\int_B |\rho_n|^{\frac{pr}{r-1}} dx\right)^{\frac{r-1}{pr}},
\end{aligned}$$

which, passing to the limit, implies

$$\int_B |\vartheta| dx \le \left((h + 2\Lambda + \vartheta + C\Xi)(\overline{B}) + C\int_B |\rho|^p dx\right)^{\frac{p-1}{p}} \mathrm{A}(\overline{B})^{\frac{1}{pr}} \left(\int_B |\rho|^{\frac{pr}{r-1}} dx\right)^{\frac{r-1}{pr}},$$

and then, dividing by $|B|$, the measures differentiation theorem shows that

$$|\vartheta| \le \big(h^L + 2\Lambda^L + \vartheta + C\Xi + C|\rho|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |\rho| \quad \text{a.e. in } \omega. \tag{3.9}$$

Using Young's inequality in (3.9)

$$|\vartheta| \le \frac{p-1}{p}\big(h^L + 2\Lambda^L + \vartheta + C\Xi^L + C|\rho|^p\big) + \frac{1}{p}(\mathrm{A}^L)^{\frac{1}{r}}|\rho|^p \quad \text{a.e. in } \omega,$$

and then

$$|\vartheta| \le C\big(h^L + \Lambda^L + \Xi^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|\rho|^p\big) \quad \text{a.e. in } \omega,$$

which substituted in (3.9) shows (2.32).

Assume now that $p > N-1$. Again, using (2.7) we deduce that $\{F_n(\cdot, \xi_n + \rho_n)\}$ is bounded in $L^1(\omega)$, and thanks to (2.5) we get

$$\big|F_n(x,\xi_n+\rho_n) - F_n(x,\xi_n)\big| \le \big(h_n + F_n(x,\xi_n+\rho_n) + F_n(x,\xi_n) + |\xi_n+\rho_n|^p + |\xi_n|^p\big)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n| \quad \text{a.e. in } \omega.$$

Consequently, the sequence $\{F_n(\cdot,\xi_n+\rho_n) - F_n(\cdot,\xi_n)\}$ is bounded in $L^1(\omega)$. Extracting a subsequence if necessary, the sequence $\{F_n(\cdot,\xi_n+\rho_n) - F_n(\cdot,\xi_n)\}$ weakly-$*$ converges in $\mathscr{M}(\omega)$ to a measure Θ, which, together with (2.30), implies

$$F_n(\cdot,\xi_n+\rho_n) \overset{*}{\rightharpoonup} \Lambda + \Theta \quad \text{in } \mathscr{M}(\omega).$$

Furthermore, if E is a measurable subset of ω, then, using Hölder's inequality, we have

$$\begin{aligned}
&\int_E \big|F_n(x,\xi_n+\rho_n) - F_n(x,\xi_n)\big|\,dx \\
\le& \int_E \big(h_n + F_n(x,\xi_n+\rho_n) + F_n(x,\xi_n) + |\xi_n+\rho_n|^p + |\xi_n|^p\big)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n|\,dx \\
\le& \|\rho_n\|_{L^\infty(\omega)^{M\times N}} \left(C\|\rho_n\|^p_{L^\infty(\omega)^{M\times N}} + \int_E \big(h_n + F_n(x,\xi_n+\rho_n) + F_n(x,\xi_n) + C|\xi_n|^p\big)\,dx\right)^{\frac{p-1}{p}} \left(\int_E a_n dx\right)^{\frac{1}{p}},
\end{aligned}$$

which, passing to the limit, shows that Θ is absolutely continuous with respect to A. By the Radon-Nikodym theorem, there exists $\vartheta \in L^1_{\mathrm{A}}(\omega)$ such that

$$\Theta = \vartheta \mathrm{A} \quad \text{in } \mathscr{M}(\omega).$$

From the previous expression and using the measures differentiation theorem, we get (2.33). $\square$

Proof of Lemma 2.5. Let $x_0 \in \omega$ and two numbers $0 < R_1 < R_2$ with $B(x_0,R_2) \subset \omega$. Lemma 2.6 in [12] gives the existence of a sequence of closed sets

$$U_n \subset [R_1,R_2], \text{ with } |U_n| \ge \frac{1}{2}(R_2 - R_1),$$

such that defining

$$\bar u_n(r,z) = u_n(x_0 + rz), \quad \bar u(r,z) = u(x_0+rz), \quad r \in (0,R_2),\ z \in S_{N-1},$$

we have

$$\|\bar u_n - \bar u\|_{C^0(U_n;X)} \to 0, \tag{3.10}$$

where X is the space defined by

$$X := \begin{cases} L^s(S_{N-1})^M, \text{ with } 1 \le s < \frac{(N-1)p}{N-1-p}, & \text{if } 1 < p < N-1, \\ L^s(S_{N-1})^M, \text{ with } 1 \le s < \infty, & \text{if } p = N-1, \\ C^0(S_{N-1})^M, & \text{if } p > N-1. \end{cases}$$

For the rest of the prove we assume $1 < p \le N-1$ because the case $p > N-1$ is quite similar.

We define $\bar\varphi_n \in W^{1,\infty}(0,\infty)$ by

$$\bar\varphi_n(r) = \begin{cases} 1, & \text{if } 0 < r < R_1, \\ \dfrac{1}{|U_n|}\displaystyle\int_r^{R_2} \chi_{U_n} ds, & \text{if } R_1 < r < R_2, \\ 0, & \text{if } R_2 < r, \end{cases} \tag{3.11}$$

and

$$\varphi_n(x) = \bar\varphi_n(|x - x_0|).$$

Applying the coercivity inequality (2.1) to the sequence $\varphi_n(u_n - u)$ and using $F_n(\cdot,0) = 0$, $\varphi_n = 1$ in $B(x_0,R_1)$, we get

$$\begin{aligned}
&\alpha\int_{B(x_0,R_1)} |Du_n - Du|^p\,dx \le \alpha\int_{B(x_0,R_2)} \big|D\big(\varphi_n(u_n-u)\big)\big|^p\,dx \\
&\le \int_{B(x_0,R_2)} F_n\big(x, D(\varphi_n(u_n-u))\big)\,dx = \int_{B(x_0,R_2)} F_n\big(x, \varphi_n Du_n - \varphi_n Du + (u_n-u)\otimes\nabla\varphi_n\big)\,dx.
\end{aligned}$$

By the convergence (2.31) with $\xi_n := \varphi_n Du_n$, $\rho_n := -\varphi_n Du + (u_n - u)\otimes\nabla\varphi_n$, and by estimate (2.6) we obtain up to a subsequence

$$\begin{aligned}
&\lim_{n\to\infty}\int_{B(x_0,R_2)} F_n\big(x, \varphi_n Du_n - \varphi_n Du + (u_n - u)\otimes\nabla\varphi_n\big)\,dx \\
&\le \lim_{n\to\infty}\int_{B(x_0,R_2)} F_n\big(x,\varphi_n Du_n\big)\,dx + \int_{B(x_0,R_2)}\vartheta\,dx \\
&\le C(h+\mu)\big(\overline{B}(x_0,R_2)\big) + \int_{B(x_0,R_2)}\vartheta\,dx,
\end{aligned}$$

with

$$|\vartheta| \le C\big(h^L + \mu^L + \varrho^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|Du| \text{ a.e. in } \omega.$$

Indeed, thanks to (3.10) the sequence $(u_n - u)\otimes\nabla\varphi_n$ converges strongly to 0 in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$ taking into account the inequality

$$\frac{(N-1)p}{N-1-p} \ge \frac{pr}{r-1}.$$

Hence, we deduce from the previous estimates that

$$\begin{aligned}
\varrho\big(B(x_0,R_1)\big) \le\ & C(h+\mu)\big(\overline{B}(x_0,R_2)\big) + C\int_{B(x_0,R_1)}|Du|^p\,dx \\
&+ C\int_{B(x_0,R_2)}\Big(\big(h^L + \mu^L + \varrho^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|Du|\Big)\,dx.
\end{aligned}$$

Taking R_2 such that

$$(h+\mu)\big(\{|x-x_0| = R_2\}\big) = 0,$$

which holds true except for a countable set $E_{x_0}\subset\big(0,\mathrm{dist}(x_0,\partial\omega)\big)$, and making R_1 tend to R_2, we get that

$$\begin{aligned}
\varrho\big(B(x_0,R_2)\big) \le\ & C(h+\mu)\big(B(x_0,R_2)\big) + C\int_{B(x_0,R_2)}|Du|^p\,dx \\
&+ C\int_{B(x_0,R_2)}\Big(\big(h^L + \mu^L + \varrho^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|Du|\Big)\,dx,
\end{aligned}$$

for any $R_2\in\big(0,\mathrm{dist}(x_0,\partial\omega)\big)\setminus E_{x_0}$. Then, by the measures differentiation theorem it follows that

$$\varrho \le C\left(|Du|^p + h + \mu\right) + C\left(\big(h^L + \mu^L + \varrho^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}\right)(\mathrm{A}^L)^{\frac{1}{pr}}|Du|.$$

Finally, the Young inequality yields the desired estimate (2.21).

Now consider $\{u_n\}$ and $\{v_n\}$ as in the statement of the lemma. Let $x_0\in\omega$ and $0 < R_0 < R_1 < R_2$ with $B(x_0,R_2)\subset\omega$. Again using Lemma 2.6 in [12] there exist two sequences of closed sets

$$V_n\subset[R_0,R_1],\quad U_n\subset[R_1,R_2],$$

with

$$|V_n| \ge \frac{1}{2}(R_1-R_0),\quad |U_n|\ge\frac{1}{2}(R_2-R_1),$$

such that defining

$$\begin{aligned}
&\bar u_n(r,z) = u_n(x_0+rz), && \bar v_n(r,z) = v_n(x_0+rz), && r\in(0,R_2),\ z\in S_{N-1}, \\
&\bar u(r,z) = u(x_0+rz), && \bar v(r,z) = v(x_0+rz), && r\in(0,R_2),\ z\in S_{N-1},
\end{aligned}$$

we have

$$\|\bar u_n - \bar u\|_{C^0(U_n;X)}\to 0,\quad \|\bar v_n - \bar v\|_{C^0(V_n;X)}\to 0.$$

Then, consider the function $\bar\varphi_n$ defined by (3.11) and the function $\bar\psi_n\in W^{1,\infty}(0,\infty)$ defined by

$$\bar\psi_n(r) = \begin{cases} 1, & \text{if } 0<r<R_0, \\ \dfrac{1}{|V_n|}\displaystyle\int_r^{R_1}\chi_{V_n}ds, & \text{if } R_0<r<R_1, \\ 0, & \text{if } R_1<r. \end{cases}$$

From these sequences we define $w_n \in W^{1,p}(\omega)^M$ by

$$w_n = \psi_n(v_n - v + u) + \varphi_n(1-\psi_n)u + (1-\varphi_n)u_n,$$

with

$$\varphi_n(x) = \bar\varphi_n(|x-x_0|), \quad \psi_n(x) = \bar\psi_n(|x-x_0|),$$

i.e.

$$w_n = \begin{cases} v_n - v + u, & \text{if } |x-x_0| < R_0, \\ \psi_n(v_n - v) + u, & \text{if } R_0 < |x-x_0| < R_1, \\ \varphi_n u + (1-\varphi_n)u_n, & \text{if } R_1 < |x-x_0| < R_2, \\ u_n, & \text{if } R_2 < |x-x_0|, x \in \omega. \end{cases} \tag{3.12}$$

It is clear that, for a subsequence, w_n converges a.e. to u. Using then that $w_n - u_n$ is in $W_0^{1,p}(\omega)^M$ and that, thanks to φ_n, ψ_n bounded in $W^{1,\infty}(\Omega)$, w_n is bounded in $W^{1,p}(\omega)^M$, we get

$$w_n - u_n \rightharpoonup 0 \quad \text{weakly in } W_0^{1,p}(\omega).$$

Thus, from (2.22) we deduce

$$\begin{aligned}\int_\omega F_n(x, Du_n)\,dx &\le \int_\omega F_n(x, Dw_n)\,dx + O_n \\ &= \int_{B(x_0,R_0)} F_n\big(x, D(v_n - v + u)\big)\,dx + \int_{\{R_2<|x-x_0|\}\cap\omega} F_n(x, Du_n)\,dx \\ &\quad + \int_{\{R_0<|x-x_0|<R_1\}} F_n\big(x, \psi_n D(v_n - v) + Du + (v_n - v)\otimes\nabla\psi_n\big)\,dx \\ &\quad + \int_{\{R_1<|x-x_0|<R_2\}} F_n\big(x, \varphi_n Du + (1-\varphi_n)Du_n + (u-u_n)\otimes\nabla\varphi_n\big)\,dx + O_n,\end{aligned}$$

what implies, in particular

$$\begin{aligned}\int_{B(x_0,R_2)} F_n(x, Du_n)\,dx &\le \int_{B(x_0,R_0)} F_n\big(x, D(v_n - v + u)\big)\,dx \\ &\quad + \int_{\{R_0<|x-x_0|<R_1\}} F_n\big(x, \psi_n D(v_n - v) + Du + (v_n - v)\otimes\nabla\psi_n\big)\,dx \\ &\quad + \int_{\{R_1<|x-x_0|<R_2\}} F_n\big(x, \varphi_n Du + (1-\varphi_n)Du_n + (u-u_n)\otimes\nabla\varphi_n\big)\,dx + O_n.\end{aligned} \tag{3.13}$$

To estimate the first term on the right-hand side of this inequality, we use Lemma 2.10 with $\xi_n = Dv_n$, $\rho_n = D(-v+u)$, which take into account (2.23), gives

$$\begin{aligned}&\int_{B(x_0,R_0)} F_n\big(x, D(v_n - v + u)\big)\,dx \\ &\le \nu\big(\overline{B}(x_0,R_0)\big) + C\int_{B(x_0,R_0)} \big(h^L + \nu^L + \varpi^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |D(u-v)|dx + O_n.\end{aligned} \tag{3.14}$$

For the second term, we use again Lemma 2.10 with $\xi_n = \psi_n Dv_n$ and $\rho_n = -\psi_n Dv + Du + (v_n - v)\otimes\nabla\psi_n$. Therefore, up to subsequence it holds

$$\begin{aligned}&\int_{\{R_0<|x-x_0|<R_1\}} F_n\big(x, \psi_n D(v_n - v) + Du + (v_n - v)\otimes\nabla\psi_n\big)\,dx \\ &\le C(h+\nu+\varpi)\big(\{R_0 \le |x-x_0| \le R_1\}\big) \\ &\quad + C\int_{\{R_0<|x-x_0|<R_1\}} \big(h^L + \nu^L + \varpi^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})(|Dv|^p + |Du|^p)\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} (|Du| + |Dv|)\,dx + O_n.\end{aligned} \tag{3.15}$$

The third term is analogously estimated by Lemma 2.10 with $\xi_n = (1-\varphi_n)Du_n$ and $\rho_n = \varphi_n Du + (u-u_n)\otimes\nabla\varphi_n$. Extracting a subsequence if necessary, it yields

$$\begin{aligned}
&\int_{\{R_1<|x-x_0|<R_2\}} F_n\big(x,\varphi_n Du + (1-\varphi_n)Du_n + (u-u_n)\otimes\nabla\varphi_n\big)\,dx \\
&\leq C(h+\mu+\varrho)\big(\{R_1\leq|x-x_0|\leq R_2\}\big) \\
&\quad + C\int_{\{R_1<|x-x_0|<R_2\}}\big(h^L+\mu^L+\varrho^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|Du|dx + O_n.
\end{aligned} \tag{3.16}$$

From (3.13), (3.14), (3.15) and (3.16) we deduce that

$$\begin{aligned}
\mu\big(B(x_0,R_2)\big) &\leq \nu\big(\overline{B}(x_0,R_0)\big) + C\int_{B(x_0,R_0)}\big(h^L+\nu^L+\varpi+(1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|dx \\
&\quad + C(h+\nu+\varpi)\big(\{R_0\leq|x-x_0|\leq R_1\}\big) \\
&\quad + C\int_{\{R_0<|x-x_0|<R_1\}}\big(h^L+\nu^L+\varpi^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})(|Dv|^p+|Du|^p)\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}(|Du|+|Dv|)\,dx \\
&\quad + C(h+\mu+\varrho)\big(\{R_1\leq|x-x_0|\leq R_2\}\big) \\
&\quad + C\int_{\{R_1<|x-x_0|<R_2\}}\big(h^L+\mu^L+\varrho^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|Du|dx.
\end{aligned} \tag{3.17}$$

Taking R_0 such that

$$(h+\nu+\varpi+\mu+\varrho)\big(\{|x-x_0|=R_0\}\big)=0,$$

which holds true except for a countable set $E_{x_0}\subset\big(0,\mathrm{dist}(x_0,\partial\omega)\big)$, and making R_1, R_2 tend to R_0, from (3.17) we deduce that

$$\begin{aligned}
\mu\big(B(x_0,R_0)\big) &\leq \nu\big(B(x_0,R_0)\big) \\
&\quad + C\int_{B(x_0,R_0)}\big(h^L+\nu^L+\varpi^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|dx,
\end{aligned}$$

for any $R_0\in\big(0,\mathrm{dist}(x_0,\partial\omega)\big)\setminus E_{x_0}$ (observe that the right term in the integral is well defined as an element of $L^1(\omega)$). Therefore, the measures differentiation theorem shows (2.24).

□

Proof of Lemma 2.6. The proof is the same as the proof of Lemma 2.5 choosing any point x_0 in Ω rather than ω, extending the functions u_n, v_n by u in $\Omega\setminus\omega$, and then noting that the function w_n defined by (3.12) in Ω is also equal to u in $\Omega\setminus\omega$. □

Proof of Corollary 2.9. Assume that $1<p\leq N-1$. Applying Lemma 2.5 with $\omega=\omega_1$ (see also Remark 2.7 about the subsets of ω) we obtain

$$\mu\leq\nu+C\big(h^L+\nu^L+\varpi^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)^p|\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|\quad\text{in }\omega_1\cap\omega_2.$$

Analogously with $\omega=\omega_2$, we get

$$\nu\leq\mu+C\big(h^L+\mu^L+\varrho^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)^p|\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|\quad\text{in }\omega_1\cap\omega_2.$$

These two expressions prove the first estimate of (2.29). The proof of the second estimate is similar. □

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