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The sextic oscillator as a γ-independent potential

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The sextic oscillator is proposed as a two-parameter solvable γ-independent potential in the Bohr Hamiltonian. It is shown that closed analytical expressions can be derived for the energies and wave functions of the first few levels and for the strength of electric quadrupole transitions between them. Depending on the parameters this potential has a minimum at $\beta=0$ or at $\beta>0$, and might also have a local maximum before reaching its minimum. A comparison with the spectral properties of the infinite square well and the β^4 potential is presented, together with a brief analysis of the experimental spectrum and $E2$ transitions of the ^{134}Ba nucleus.

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I. INTRODUCTION

In the last few years there has been considerable interest in looking for analytic solutions of the Bohr Hamiltonian which describes the collective motion in nuclei in terms of shape variables (β, γ) [1]. This was initiated with the introduction of the interacting boson model (IBM) [2]. This model presents four different dynamical symmetries, each one associated with a well-defined nuclear shape. The model Hamiltonian provides a natural way of going from a phase to another one by changing systematically few parameters and, consequently, allows one to study shape phase transitions. Thus, the IBM phase diagram has been analyzed from different points of view [3–7]. Specially important are the critical points since in these situations structural changes occur rapidly and it is difficult to design appropriate theoretical models. Recently, Iachello has proposed a new dynamical symmetry for describing the critical point at the transition from spherical to deformed γ-unstable shapes [8]. This symmetry has been called $E(5)$ and is expected to occur in nuclei in which the $V(\beta, \gamma)$ potential depends only on the β variable, and it has a relatively flat shape in the β variable. Around this critical point of the phase transition the potential appearing in the Bohr Hamiltonian can be approximated by an infinite square well in the β variable. In this case the Bohr Hamiltonian can be solved exactly in terms of Bessel functions, and various quantitative predictions can be obtained for various spectroscopic properties [ratios of the excitation energies and $B(E2)$ transitions], on the basis of which one can search for candidates for the $E(5)$ symmetry among nuclei. Lately, two other dynamical symmetries $X(5)$ [9] and $Y(5)$ [10] to describe the critical point at the transition from spherical to axially deformed shapes and from axially deformed to triaxial shapes, respectively, have been proposed.

The introduction of the $E(5)$ symmetry renewed the interest in studying exactly solvable $V(\beta, \gamma)$ potentials. Most efforts have concentrated on γ-independent potentials, for which the most well-known example is the harmonic oscillator in the five-dimensional space [11] and its extension which contains a term proportional to β^{-2} [12]. More recently two more exactly solvable γ-unstable potentials have been discussed: the Coulomb and the Kratzer potentials [13]. Here again the latter potential is an extension of the former in the sense that it contains a term proportional to β^{-2}, which formally changes the τ variable, the analog of the l angular momentum of radial potentials in three spatial dimensions. This formal changing of τ introduces a minimum of the potentials at $\beta>0$ in both cases [12,13]. The bound solutions of the harmonic oscillator and the Coulomb potentials (and their extensions) are given in terms of generalized Laguerre polynomials [14].

Similar to the three-dimensional case, these examples, together with the infinite square well, practically exhaust those exactly solvable potentials that contain a β^{-2} term. Here we propose another potential which has this property, and although it is not exactly solvable in the classical sense, it has a number of features that make it an ideal potential to be used in the Bohr Hamiltonian. This is the sextic oscillator, which belongs to the class of quasi-exactly solvable potentials [15]. These potentials have the property that their solutions can be obtained in closed form for a number of energy eigenvalues, i.e., for the lowest few values of the n principal quantum number, which are also the lowest in energy. This is clearly sufficient for potentials appearing in the Bohr Hamiltonian. It is rarely necessary to consider more than a few levels with the same angular momentum, and these can be obtained from the lowest few solutions of the sextic oscillator. Furthermore, the sextic oscillator has a more flexible shape than other solvable potentials, as depending on its parameters, it can have a minimum at $\beta=0$ or at $\beta=\beta_{\min}>0$, and in addition, it can also have a local maximum at $\beta_{\max}<\beta_{\min}$.

The paper is structured as follows. In Sec. II the Bohr Hamiltonian is revised together with its solutions for γ-independent potentials. In Sec. III the lowest energy solu-

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tions of the Bohr Hamiltonian for the sextic oscillator potential are worked out. Section IV is devoted to show the simple use of the sextic oscillator as a flexible γ-independent potential. Finally, in Sec. V we present preliminary applications and discuss future extensions.

II. THE BOHR HAMILTONIAN FOR γ-INDEPENDENT POTENTIALS

Let us first consider that the Bohr Hamiltonian describing the collective motion of a deformed nucleus in the five-dimensional space determined by the θ_i Euler angles ($i=1,2,3$) and the intrinsic β and γ variables is [1]

$$H=-\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}-\frac{1}{4\beta^2}\sum_k\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right)+V(\beta,\gamma). \tag{1}$$

In what follows we assume that the potential in Eq. (1) depends only on β, i.e., $V(\beta,\gamma)=U(\beta)$. For these γ-independent potentials the wave functions can be separated into two parts,

$$\Psi(\beta,\gamma,\theta_i)=f(\beta)\Phi(\gamma,\theta_i), \tag{2}$$

which satisfy the following differential equations:

$$\left(-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}+\frac{1}{4}\sum_k\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right)\Phi(\gamma,\theta_i)=\Lambda\Phi(\gamma,\theta_i),$$

$$\Lambda=\tau(\tau+3),\qquad \tau=0,1,2,\ \dots, \tag{3}$$

$$\left(-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{\Lambda}{\beta^2}+u(\beta)\right)f(\beta)=\epsilon f(\beta). \tag{4}$$

Here we have introduced $\epsilon=(2B/\hbar^2)E$ and $u(\beta)=(2B/\hbar^2)U(\beta)$. Note that the τ values determine the allowed angular momenta J, too [16]. By setting $\phi(\beta)=\beta^2 f(\beta)$ we obtain an equation which has the form of a radial Schrödinger equation

$$-\frac{d^2\phi}{d\beta^2}+\left(\frac{(\tau+1)(\tau+2)}{\beta^2}+u(\beta)\right)\phi=\epsilon\phi. \tag{5}$$

Note that this is different from Eq. (6) in Ref. [8], in that it contains no linear derivative term due to the different definition of $\phi(\beta)$. This choice also implies that the factor corresponding to the β volume element in the integration of functions of the type $f(\beta)$ in Eq. (4) has been transferred to the solutions of the type $\phi(\beta)$ in Eq. (5). Thus, in the integration of these no factor arising from the volume element has to be included. The complete solution of the problem implies the solution of Eq. (3), too; this was solved in Ref. [16].

III. THE SEXTIC OSCILLATOR

The sextic oscillator with a centrifugal barrier is defined [15] as

$$H=-\frac{d^2}{dx^2}+\frac{(2s-1/2)(2s-3/2)}{x^2}+\left[b^2-4a\left(s+\frac{1}{2}+M\right)\right]x^2+2abx^4+a^2x^6, \tag{6}$$

where $x\in[0,\infty)$ and M is a non-negative integer. This potential is quasi-exactly solvable, which means that for any non-negative integer value of M, $M+1$ of its solutions can be obtained in an algebraic way. The (unnormalized) solutions are written in the form

$$\phi_n(x)=P_n(x^2)(x^2)^{s-1/4}\exp\left(-\frac{a}{4}x^4-\frac{b}{2}x^2\right),\qquad n=0,1,2,\ \dots, \tag{7}$$

where P_n is a polynomial of order n. Obviously, normalizability requires $a\geqslant 0$, while $a=0$ reduces the problem to the exactly solvable harmonic oscillator.

The simplest solutions are obtained for $M=0$ and $M=1$ [15]. For $M=0$ only one nodeless (i.e., ground state) solution appears at $E_0^{(M=0)}=4bs$, with the corresponding wave function being

$$\phi_0^{(M=0)}(x)\sim(x^2)^{s-1/4}\exp\left(-\frac{a}{4}x^4-\frac{b}{2}x^2\right). \tag{8}$$

For $M=1$ two solutions appear, one nodeless and another with one node for $x>0$. These correspond to the ground state and the first excited state, respectively, at energies $E_0^{(M=1)}=4bs+\lambda_-(s)$ and $E_1^{(M=1)}=4bs+\lambda_+(s)$, where

$$\lambda_\pm(s)=2b\pm 2(b^2+8as)^{1/2} \tag{9}$$

are the roots of the equation $\lambda^2-4b\lambda-32as=0$. The corresponding wave functions are

$$\phi_n^{(M=1)}(x)\sim\left(1-\frac{\lambda}{8s}x^2\right)(x^2)^{s-1/4}\exp\left(-\frac{a}{4}x^4-\frac{b}{2}x^2\right), \tag{10}$$

and the $\lambda=\lambda_-(s)$ and $\lambda=\lambda_+(s)$ choice has to be made for $n=0$ and $n=1$, respectively [15]. [Note that $a\geqslant 0$ and $s\geqslant 0$ imply $\lambda_-(s)\leqslant 0$, so the polynomial part of Eq. (10) is nodeless.] It has to be mentioned that the solutions for $M=0$ and $M=1$ belong to *different* sextic potentials if s is the same, as the coefficient of the quadratic term is different then. We shall see, however, that with appropriate combinations of s and M it is possible to solve sextic potentials that differ only in the strength of the centrifugal term.

The normalization of the wave functions can also be given in closed form. For this one has to evaluate integrals of the type

$$I^{(A)} \equiv \int_0^\infty x^A \exp\left(-\frac{a}{2}x^4 - bx^2\right)$$
$$= \frac{1}{2}\Gamma\left(\frac{A+1}{2}\right)a^{-(A+1)/4}\exp\left(\frac{b^2}{4a}\right)D_{-(A+1)/2}\left(\frac{b}{a^{1/2}}\right) \tag{11}$$

$$= \frac{1}{2}\Gamma\left(\frac{A+1}{2}\right)(2a)^{(-A+1)/4}U\left(\frac{A+1}{4},\frac{1}{2};\frac{b^2}{2a}\right), \tag{12}$$

where $D_p(z)$ is the parabolic cylinder function and $U(\alpha,\beta;z)$ is one of the forms of the confluent hypergeometric function [14].

Larger values of M can also be considered (e.g., for $M=2$ three different solutions are obtained for the three roots of a cubic algebraic equation for λ), but $M=0$ and $M=1$ are sufficient for our purposes in this paper. A complete study including more solutions, explicit closed forms for the normalization factors, and applications to actual nuclei is underway.

IV. APPLICATION AS A γ-INDEPENDENT POTENTIAL

In order to cast Eq. (6) in a form similar to Eq. (5) we have to write $x=\beta$ and $s=(\tau/2)+5/4$ (remember that $\tau \geqslant 0$). In order to keep the quadratic term at a constant value (once the a,b parameters are fixed) we also have to prescribe

$$s+M+\tfrac{1}{2}=\tfrac{1}{2}\left(\tau+2M+\tfrac{7}{2}\right)\equiv c=\text{const.} \tag{13}$$

With this the sextic oscillator Hamiltonian can be brought to the form of Eq. (4) with $u(\beta)$ being

$$u^{\pi}(\beta)=(b^2-4ac^{\pi})\beta^2+2ab\beta^4+a^2\beta^6+u_0^{\pi}, \tag{14}$$

where the index $\pi=\pm$ is included to distinguish the potential for even/odd τ's, which is slightly different as explained below. In Eq. (14) c^{π} are the constants obtained in Eq. (13) for even/odd values of τ and we have introduced a constant u_0^{π} for convenience, as will be discussed below.

Equation (13) implies that increasing/decreasing M by one unit has to come with decreasing/increasing τ by *two* units. Thus, once the values of the (a,b) parameters are fixed, the sequence of (M,τ) values $(K,0),(K-1,2),(K-2,4),\ldots$ correspond to solutions of Eq. (14) with $c^{+}=\frac{7}{4}+K$. In the same way, the sequence of (M,τ) values $(K,1),(K-1,3),(K-2,5),\ldots$ correspond to solutions of Eq. (14) with $c^{-}=\frac{9}{4}+K$. Consequently, the potential for τ-even and τ-odd states is slightly different due to the fact that the coupling coefficient $b^2-4ac^{\pm}$ of the quadratic term is different in the two cases due to the choices for c^{+} and c^{-} that are necessary to separate the $(\tau+1)(\tau+2)\beta^{-2}$ term in a uniform way. This situation can be handled using different strategies. One possibility is setting $u_0^{+}=u_0^{-}=0$ and considering $b^2>10a$, which minimizes the deviation of the quadratic terms compared to the quartic and sextic terms. Another possibility is introducing a relative energy shift between the τ-even and τ-odd potentials by setting u_0^{+} and u_0^{-} such that the potential minima are at the same energy. We shall discuss

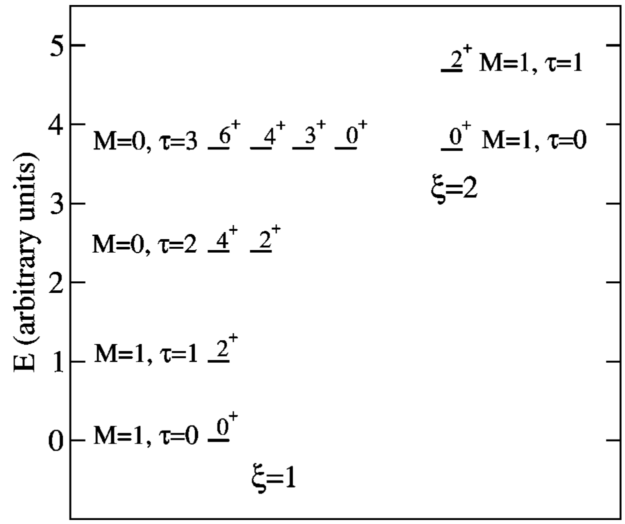

FIG. 1. Schematic typical spectrum for the sextic oscillator with indication of the relevant quantum numbers.

this possibility after analyzing qualitatively the spectrum and the potential shapes.

Let us analyze the spectrum obtained in a simple case. Taking $M=1$ we obtain, as explained in the preceding section, two solutions for $\tau=0$ (one with no nodes, $n=0$, and the other one with one node, $n=1$). In the notation introduced in Ref. [8] the label ξ is our $n+1$. Thus the two solutions of Eq. (10) with $s=5/4$ are $\phi_n^{(M=1)}(\beta)$ with $n=0$ and 1 and correspond to $\phi_{\xi,\tau}=\phi_{1,0}$ and $\phi_{\xi,\tau}=\phi_{2,0}$, respectively, in Ref. [8] notation. Inspecting the energy eigenvalues, the corresponding energies are $E_1(M=1,\tau=0)=E_{1,0}=7b-2(b^2+10a)^{1/2}+u_0^{+}$ and $E_2(M=1,\tau=0)=E_{2,0}=7b+2(b^2+10a)^{1/2}+u_0^{+}$, respectively. The same potential is obtained by taking $M=0$ and $\tau=2$. In this case we have a single solution, Eq. (8) with $s=9/4$, $\phi_0^{(M=0)}(\beta)$ with no nodes, and with energy $E_1(M=0,\tau=2)=9b+u_0^{+}$. This corresponds to $\phi_{\xi,\tau}=\phi_{1,2}$ in the notation of Ref. [8]. A similar analysis can be performed for the solutions with odd-τ values. For $M=1$ there are two solutions with $\tau=1$, Eq. (10) with $s=7/4$, which corresponds to $\phi_{\xi,\tau}=\phi_{1,1}$ and $\phi_{\xi,\tau}=\phi_{2,1}$ for $n=0$ and $n=1$, respectively. The corresponding energy eigenvalues are $E_{1,1}=9b-2(b^2+14a)^{1/2}+u_0^{-}$ and $E_{2,1}=9b+2(b^2+14a)^{1/2}+u_0^{-}$. Again the same potential is obtained for $M=0$ and $\tau=3$. In this case there is a single solution with no nodes, Eq. (8) with $s=11/4$, which corresponds to $\phi_{1,3}$ and has an energy $E_{1,3}=11b+u_0^{-}$. In Fig. 1 a schematic spectrum is shown with indication of the relevant quantum numbers. In Fig. 2 the corresponding wave functions with the notation $\phi_{\xi,\tau}$ are presented.

Now we analyze the different potential shapes that can be produced by different election of parameters in Eq. (14). From Eq. (14) we find that the shape of the potential $u^{\pi}(\beta)$ depends on the sign of b^2-4ac^{π} and b, which sets the coefficients of the quadratic and quartic terms. (The coefficient of the leading sextic term is always positive.) When $b^2>4ac^{\pi}$ and $b>0$ hold [i.e., for $b>2(ac^{\pi})^{1/2}$], the potential has a minimum at $\beta=0$ and it increases monotonously with β. When $b^2<4ac^{\pi}$, irrespective of the sign of b [i.e., for

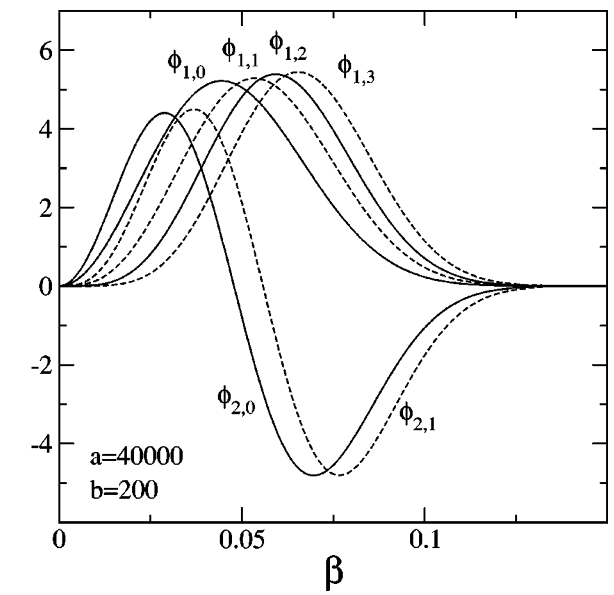

FIG. 2. Wave functions with the notation $\phi_{\xi,\tau}$ for the case of potential parameters $a=40\,000$ and $b=200$.

$-2(ac^{\pi})^{1/2}<b<2(ac^{\pi})^{1/2}$], a minimum appears for $\beta>0$, while for $b^2>4ac^{\pi}$ and $b<0$ [i.e., for $b<-2(ac^{\pi})^{1/2}$], first a maximum appears and then a minimum as β increases. In all three cases the exact location of the extremal point(s) can be obtained from the real and positive solutions of

$$(\beta_0^{\pi})^2=\frac{1}{3a}[-2b\pm(b^2+12ac^{\pi})^{1/2}]. \qquad (15)$$

Due to the relatively small difference in c^+ and c^-, the τ-even and τ-odd potentials have the same types of extrema at about the same β, except for some peculiar combinations of a and b. Assuming that there are no complications of this kind, we can now return to the question of renormalizing the minima of the τ-even and τ-odd potentials. For $b>2(ac^{\pi})^{1/2}$, $\pi=+,-$ the minima of the two potentials will be u_0^+ and u_0^- at

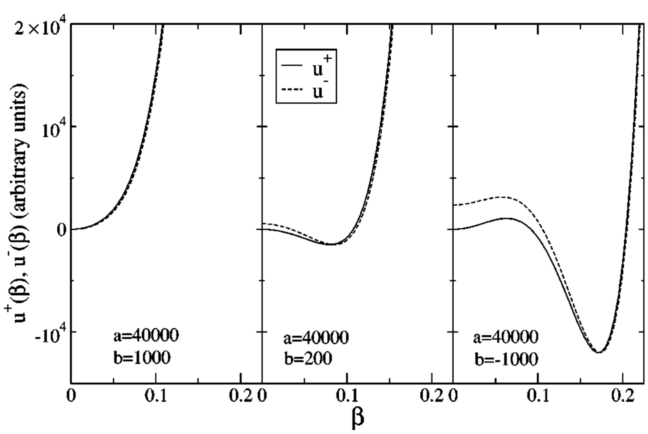

FIG. 3. Potentials $u^+(\beta)$ (full line) and $u^-(\beta)$ (broken line) for $a=40\,000$, and $b=1000$ (left panel), $b=200$ (middle panel), and $b=-1000$ (right panel). The lowest energy level appears in these potentials at $E_{1,0}=4633.57$, 73.35, and -9366.43, respectively.

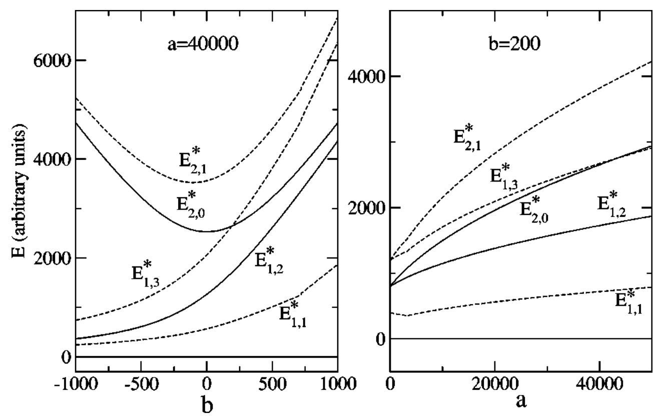

FIG. 4. Excitation energies $E^*_{\xi,\tau}=E_{\xi,\tau}-E_{1,0}$ with $a=40\,000$ fixed as a function of b (left panel) and with $b=200$ fixed as a function of a (right panel).

$\beta=0$, so they coincide if $u_0^+=u_0^-$ holds. For $b<2(ac^{\pi})^{1/2}$ we can equate the minima of $u^+(\beta)$ and $u^-(\beta)$ if we set $u_0^+=0$ and

$$u_0^-=(b^2-11a)(\beta_0^+)^2-(b^2-13a)(\beta_0^-)^2+2ab[(\beta_0^+)^4-(\beta_0^-)^4]+a^2[(\beta_0^+)^6-(\beta_0^-)^6], \qquad (16)$$

where β_0^{π} are obtained from Eq. (15) with the choice of the "+" sign. With this the two potentials have their minima at the same energy, but they take on different values at the origin. Illustrations of the possible potential shapes are displayed in Fig. 3. Obviously, u_0^- in Eq. (15) also has to be added to the energies of the τ-odd potential. Figure 4 shows the relative position of the energy levels $E_{\xi,\tau}$ as either a or b is varied and the other parameter is kept at a fixed value.

The electric quadrupole transition rates can also be determined analytically by calculating the matrix elements of the transition operator [8,11]

$$T^{(\mathrm{E2})}=t\alpha_{2\mu}=t\beta[D^{(2)}_{\mu,0}\cos\,\gamma+2^{-1/2}(D^{(2)}_{\mu,2}+D^{(2)}_{\mu,-2})\sin\,\gamma]. \qquad (17)$$

The radial integrals that appear in the β variable in the matrix elements of $T^{(\mathrm{E2})}$ can again be determined using Eq. (11).

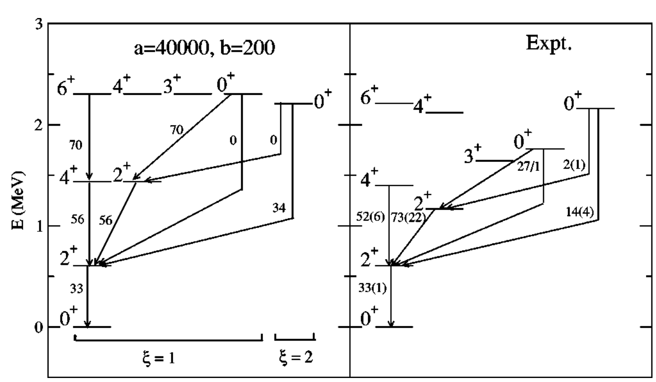

FIG. 5. The energy spectrum and the strength of some electric quadrupole transitions calculated with $a=40\,000$ and $b=200$ (left panel) and the corresponding data for ^{134}Ba (right panel).

TABLE I. Ratios of some energy eigenvaluess and electric quadrupole transition strengths from the sextic oscillator with $a=40\,000$, $b=200$, the infinite square well [8], and the β^4 potential [19], together with the experimentally observed quantities for ^{134}Ba.

	$\frac{E(4^+_{1,2})}{E(2^+_{1,1})}$	$\frac{E(0^+_{2,0})}{E(2^+_{1,1})}$	$\frac{E(6^+_{1,3})}{E(2^+_{1,1})}$	$\frac{B(E2;4^+_{1,2}\to 2^+_{1,1})}{B(E2;2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(E2;2^+_{2,0}\to 2^+_{1,1})}{B(E2;2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(E2;0^+_{1,3}\to 2^+_{1,2})}{B(E2;2^+_{1,1}\to 0^+_{1,0})}$
Sextic oscillator	2.39	3.68	3.70	1.70	1.03	2.12
$E(5)$	2.20	3.03	3.59	1.68	0.86	2.21
β^4	2.09	2.39	3.27	1.82	1.41	2.52
^{134}Ba (expt.)	2.31	3.57	3.65	1.56 (18)	0.42 (12)	

In order to obtain the *total* matrix elements, one has to calculate also the components depending on γ and the Euler angles θ_i. This can be done following the techniques described in Ref. [16]. These parts introduce certain selection rules not only for the angular momenta, but also for τ.

V. DISCUSSION

In order to compare the main characteristics of the sextic oscillator as a γ-unstable potential with those of other potentials of this kind, we present calculations for a particular value of the parameters, $a=40\,000$ and $b=200$. These numbers were chosen such that the resulting energy spectrum approximates that of the ^{134}Ba nucleus, the first candidate for $E(5)$ symmetry [17]. We stress that our aim is not to reproduce the experimental data, rather to get a qualitative picture about the general performance of the model. The potentials $u^{\pm}(\beta)$ are displayed in the middle panel of Fig. 3, while the energy eigenvalues are shown in Fig. 5, together with the corresponding experimental energy levels. Figure 5 also shows the calculated and the experimental $B(E2)$ values for transitions between the energy levels. Note that electric quadrupole transitions which change τ by more than one unit are zero if we use the transition operator (17), but finite $B(E2)$ strengths can be obtained if we apply terms of the next order (see, e.g., Ref. [18]).

In Table I we summarize the ratio of the most important energy eigenvalues and those of the most characteristic $B(E2)$ transition rates obtained from the sextic oscillator with parameters $a=40\,000$, $b=200$, the infinite square well potential [8], and the numerically solved β^4 potential [19] together with the corresponding experimental values for ^{134}Ba, whenever available. It is seen that the energy ratios corresponding to the $E(5)$ symmetry systematically fall between the values of the β^4 potential and the sextic oscillator. The situation is less obvious for the ratio of the $B(E2)$ values: here the sextic oscillator and the infinite square well seem to yield similar ratios, while the numbers obtained from the β^4 potential are systematically higher. This might be due to the fact that the sextic oscillator potential goes to infinity steeper than the β^4 potential, so the asymptotic behavior of its wave functions can be closer to that of the wave functions of the infinite square well. Comparing the results with the experimental data for ^{134}Ba we can conclude that, at least in this case, the sextic oscillator allows a better approximation than the other essentially parameter-free potentials. We expect that this conclusion will be general due to the flexible nature of the sextic potential whose shape is governed by two parameters. In fact this potential can be used not just at the critical point but it can be useful to model the full shape phase transition from spherical to deformed γ-unstable nuclei by changing the parameters a and b.

Before closing, we mention some aspects of the sextic oscillator that might give further help in the analysis of nuclei near critical points. First, we note that with $M=2$ in Eq. (6) the analysis can be extended to further states, such as $\phi_{1,4}$, $\phi_{1,5}$, $\phi_{2,2}$, $\phi_{2,3}$, $\phi_{3,0}$, and $\phi_{3,1}$. Second, the potential shape which contains both a local maximum and a minimum at $\beta>0$ might be useful in the description of nuclei with the so-called $X(5)$ symmetry, which is thought to occur in the shape phase transition between the spherical and the axially deformed domain [9]. Third, there are further quasi-exactly solvable potentials both with confining and nonconfining nature [15], which can also be considered in the Bohr Hamiltonian.

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